A survey on groups definable in *o*-minimal structures

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1 Introduction

Groups definable in *o*-minimal structures have been studied for the last twenty years. The starting point of all the development is Pillay's theorem that a definable group is a definable group manifold (see Section 2). This implies that when the group has the order type of the reals, we have a real Lie group. The main lines of research in the subject so far have been the following:

(1) Interpretability, motivated by an *o*-minimal version of Cherlin's conjecture on groups of finite Morley rank (see Sections 4 and 3).

(2) The study of the Euler characteristic and the torsion, motivated by a question of Y.Peterzil and C.Steinhorn and results of A.Strzebonski (see Sections 6 and 5).

(3) Pillay's conjectures (see Sections 8 and 7).

On interpretability, we have a clear view, with final results in Theorems 4.1 and 4.3 below. Lines of research (2) and (3) can be seen as a way of comparing definable groups with real Lie groups (see Section 2). The best results on the Euler characteristic are those of Theorems 6.3 and 6.5. The study of the torsion begins the study of the algebraic properties of definable groups, and the best result about the algebraic structure of the torsion

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subgroups is Theorem 5.9. On the other hand, the cases in which Pillay's conjectures are proved are stated in Theorems 8.3 and 8.8. In my opinion, what is most beautiful about the proofs of the conjectures is that they make an essential use of most of the results obtained for definable groups; they also bring into the game properties of *o*-minimal structures, such as NIP and properties of Keisler measures, which have not been used so far, for the study of definable groups; and they introduce the notion of compact domination which opens a new line of research.

For the rest of the paper, I fix an o*-minimal structure $\mathcal{M}$ and 'definable' means 'definable in $\mathcal{M}$'. Any other assumption on $\mathcal{M}$ is stated either at the beginning of a section or within the statements of the results. (See Section 2 for more conventions.)*

A group G is *definable* if both the set and the graph of the group operation are sets definable (with parameters) in $\mathcal{M}$. Here there are some examples of definable groups when $\mathcal{M}$ is an *o*-minimal expansion of a real closed field: the additive group of M; the multiplicative group of M; the algebraic subgroups of $GL(n, M)$ and $H(M)$, the M-rational points of an (abstract) algebraic group H defined over the field M. Since every compact Lie group is isomorphic, as a Lie group, to a (linear) real algebraic group (see [9]), compact Lie groups are also examples of groups definable over the ordered real field. In [47] A. Strzebonski gives a good list of examples of semialgebraic groups. More examples can be found in [40] (see comments after Theorem 5.2 below). In [35] Peterzil, Pillay and Starchenko give examples of definable subgroups of $GL(n, M)$ – for certain *o*-minimal expansions $\mathcal{M}$ of the real field – which are not definably isomorphic to semialgebraic groups (compare with Theorem 4.6 below).

When the order type of $\mathcal{M}$ is $(\mathbb{R}, <)$, Theorem 2.1 (see below) implies that a definable group G is a Lie group. It is because of this last fact that Lie groups have been a good source for the study of definable groups. Much research in the field have been done comparing both situations: Lie and definable. I refer the reader to [8] and [9] for results about Lie groups. Properties of definable groups which correspond to analogous properties of Lie groups can be found below (*e.g.*, Theorems 2.1, 4.8, 5.9 and 6.7). Sometimes definable groups behave better than Lie groups (*e.g.*, definable subgroups are closed), but some others existence results in Lie groups (*e.g.*, the existence of subgroups or isomorphisms) cannot be ensured in the definable context. Other sources for the study of definable groups have been groups of finite Morley rank or groups definable in a strongly minimal structures. We can find good examples of such results in Sections 3 and 4.

If $\mathcal{M}$ is the ordered real field then a definable group G is a Nash group. That is, G is equipped with a semialgebraic manifold structure in which the maps involved in the manifold structure are Nash (real-analytic and semialgebraic), and multiplication and inversion are also Nash maps. E. Hrushovski and A. Pillay study in [19] groups definable in local fields and they prove that if G is a (Nash) group definable in the real field, then there is an algebraic group H defined over $\mathbb{R}$, and a Nash isomorphism between neighbourhoods of the identity of G and $H(\mathbb{R})$ (see Theorem A in [19]). This local isomorphism is the best possible because it is not always possible to lift a local Nash isomorphism to a global one. In affine groups they avoid this obstacle (note that the embedding obtained by Robson's theorem – see Section 2 – does not need to be a Nash map). More precisely, they prove that if G is a connected Nash group over $\mathbb{R}$, for which there is a Nash embedding of G into some $\mathbb{R}^l$, then there is a a Nash surjective homomorphism with finite kernel between G and the real connected component of the set of real points of an algebraic group defined over $\mathbb{R}$ (see Theorem B in [19]). The proofs of these two results make use of the study of geometric structures over the reals, also developed in [19].

I have written this paper as a collection of known results on definable groups with hopefully precise references, and there are also some open problems scattered through the text. I have tried to give the taste of the proofs or at least to give some information about what one can expect to find if going to the original papers. For basic properties of o-minimal structures I refer the reader to [10] and [45].

I started working on this paper during my visit to Cambridge within the Programme Model Theory and Applications to Algebra and Analysis in the spring 2005. I would like to thank both the Newton Institute for their hospitality, and Sergei Starchenko for many instructive comments on some of the results stated here. I also thank Alessandro Berarducci for helpful conversations.

2 The manifold structure and basic properties

I begin by stating the following essential result on definable groups due to A.Pillay.

Theorem 2.1. *Let G be a definable group. Then, G can be equipped with a definable manifold structure making G a topological group.*

This is Proposition 2.5 in [42]. For a detailed description of the manifold structure see Section 1.1 in [33]. The ingredients of the proof are the following. Firstly, Pillay proves that the (model theoretic) dimension of a definable set coincides with its geometrical dimension (via cells) – Lemma 1.4 in [42]. Then, he proves the following key lemma.

Lemma 2.2. *Let G be a definable group. Let X be a large subset of G. Then finitely many translates of X cover G.*

This is Lemma 2.4 in [42], where X *large* means that $dim(G \setminus X) < dim\, G$. For the rest of the proof of the theorem, A.Pillay adapts to the *o*-minimal context a proof by E.Hrushovski of Weil's theorem that an algebraic group over an algebraic closed field can be defined from *birational data.*

It is easy to see that the topology of our definable group $G \subset M^n$, say, obtained in Theorem 2.1 does not coincide in general with the topology induced by the ambient space M^n: take for instance $[0, 1) \subset \mathbb{R}$ and let the operation be addition *mod* 1 (with the definable manifold topology $1 - x$ tends to 0 when x tends to 0). However, as G is a topological group it is a (Hausdorff) regular manifold (regular as a topological space) and when $\mathcal{M}$ is an expansion of a real closed field, G can be embedded (as a manifold) in some M^l; this is due to Robson's theorem (see Theorem 1.8 in Section 10 in [10]). Therefore, the following convention can be used for the rest of the paper:

When $\mathcal{M}$ is an expansion of a real closed field all the topological concepts about definable groups refer to the topology induced by the ambient space. Otherwise, the concepts mentioned refer to the manifold topology.

In particular, the notion of *definable compactness*, introduced by Y. Peterzil and C. Steinhorn – Definition 1.1 in [40] – is equivalent to that of being closed and bounded, when we work over an expansion of a real closed field. This is because in Theorem 2.1 in [40], they prove that both concepts coincide when the topology is the one induced by the ambient space. Note also that when $\mathcal{M}$ is an expansion of a real closed field the usual definition of derivative makes sense and we can speak of C^p–*maps* for $p \geq 0$. In this case, the same proof of Theorem 2.1 (which is the case $p = 0$) gives that a definable group carries on a definable C^p–manifold structure. See Section 1.1 and 1.2 in [33] for details. Also in [33], Y.Peterzil, A.Pillay and S.Starchenko adapt the proof of Theorem 2.1 above to prove – Theorem 2.12 there – the following result.

Theorem 2.3. *Let $p \geq 0$ and let $\mathcal{M}$ be an expansion of a real closed field if $p > 0$. Let G be a definable group, A a definable set and α a definable transitive action of G on A. Then, A can be equipped with a definable C^p-manifold structure making α a definable C^p-action.*

Note that when $\mathcal{M}$ is an expansion of a real closed field and N is a definable normal subgroup of a definable group G, we have three topologies on G/N: the topology as a definable group (from Theorem 2.1), the quotient topology (where G gets the topology from Theorem 2.1) and the topology obtained via Theorem 2.3 (for the action of G on G/N by left multiplication); A.Beraducci proves in Theorem 4.3 in [2] – using the trivialization theorem – that the three topologies indeed coincide.

Next I state some corollaries of Theorem 2.1.

Corollary 2.4. *Let G be a definable group, and let H be a definable subgroup of G. Then,*

(i) H is closed;

(ii) if G is infinite then it has an infinite definable abelian sugbroup;

(iii) the following three are equivalent: H open, H has finite index, and $dim\, H = dim\, G$;

(iv) the definably connected component of the identity in G, G^0, is the smallest subgroup of finite index and moreover it is normal in G, and

(v) G has the descending chain condition on definable subgroups.

Once we have an explicitely defined topology on G, (i) and (ii) are proved by A. Pillay in [41] (Propositions 2.7 and 5.6 respectively, noting that the group topology makes G topologically totally transcendental); (iii) is Lemma 2.11 in [42] and (iv) and (v) can be easily obtained from (iii). S.Strzebonski gave a different proof of (iv) and (v) – Theorem 2.6 in [47] – provided $\mathcal{M}$ is an expansion of a group. For more on descending chain conditions see Theorem 8.3 below.

Corollary 2.5. *Let G be a definable group, $\mathbb{G} = (G, \cdot)$ and A an abelian subgroup of G (definable or not). Then, the centre of the centralizer $C_G(A)$ is a $\mathbb{G}$-definable abelian subgroup containing A, which is normal if A is normal.*

This follows from (v) above (see *e.g.* Cor. 1.17 in [33]). It is because of this last corollary (and thinking of linear algebraic groups) that we say that a definable group is *semisimple* if it has no infinite abelian normal definable subgroup.

In several papers on definable groups in (general) o-minimal structures the extra assumption of having definable choice is added just to be able to speak about *definable* quotient groups or having a *definable* set of representatives of the cosets. The following result due to M.Edmundo – Theorem 7.2 in [13] – allow us to eliminate this extra assumption.

Theorem 2.6. *Let G be a definable group and let $\{T(x) : x \in X\}$ be a definable family of non-empty definable subsets of G. Then, there is a definable function $t\colon X \to G$ such that for $x, y \in X$ we have $t(x) \in T(X)$ and if $T(X) = T(Y)$ then $t(x) = t(y)$.*

3 Small dimension

In this section I consider definable groups of dimension less than or equal to three. I begin with dimension one and the work of V. Razenj in [46].

Proposition 3.1. *Let G be a definably connected 1-dimensional group. Then,*

(i) G is abelian;

(ii) G does not have bounded exponent, and

(iii) either, G is definably compact and the torsion subgroups $G[m]$ are isomorphic to $\mathbb{Z}/m\mathbb{Z}$ (for each $m > 0$), or G is torsion–free.

Corollary 2.4(ii) above gives (i). For (ii) see Proposition 1, and for (iii) Proposition 3 and 4, all of them in [46].

Razenj makes use of Theorem 2.1 and adapts the classical classification of 1-dimensional topological Hausdorff manifolds to obtain that $G \setminus \{pt.\}$ has either, one definably connected component ($\mathbb{S}^1$-type), or two components ($\mathbb{R}$-type). Then, using the Reineke classification of strongly minimal groups (note that G as a pure group is strongly minimal, see Remark 6.5 in [41]), he obtains the following (see Theorem p.272 in [46]).

Theorem 3.2. *Let G be a definably connected 1-dimensional group. Then, (as an abstract group) G is isomorphic to either $\bigoplus_p \mathbb{Z}_{p^\infty} \oplus \bigoplus_\delta \mathbb{Q}$ or $\bigoplus_\delta \mathbb{Q}$, $\delta \geq 0$.*

For more on 1-dimensional groups definable in an arbitrary o-minimal structure see Theorem 8.1 below. When $\mathcal{M}$ is the real field our definable group is a Nash group (see Section 1), and J.Madden and C.Stanton classify in [24] (see also [25]) the 1-dimensional Nash groups, up to Nash isomorphism. When $\mathcal{M}$ is an expansion of a real closed field, Strzebonski proves in Theorem p.204 in [48] that a definably connected 1-dimesional group is definably isomorphic to either an abelian group on either the set (0,1) or the set $[0,1)$. On the other hand, when $\mathcal{M}$ is an o-minimal expansion of a real closed field, there are two natural torsion-free 1-dimensional definably connected groups: M_a (the additive group of $\mathcal{M}$) and M_m (the multiplicative group of the positive elements of $\mathcal{M}$). C. Miller and S. Starchenko prove – Theorem C in [26] – the following result as a consequence of their Growth Dichotomy Theorem (Theorem A in [26]) and a result in [29].

Theorem 3.3. *Let $\mathcal{M}$ be a polynomially bounded expansion of a real closed field. Up to definable isomorphism, there are exactly two definably connected torsion-free 1-dimensional groups: M_a and M_m.*

Miller and Starchenko posed in [26] the following question.

Question 3.4. *Let $\mathcal{M}$ be an expansion of a real closed field. Is every definably connected torsion-free definable 1-dimensional group definably isomorphic to M_a or M_m?*

Definable groups of dimension two and three are analysed by A.Nesin, A.Pillay and V. Razenj in [28]. Their work is a continuation of that of V.Razenj above.

Theorem 3.5. *Let G be a definably connected 2-dimensional group. Then, G is solvable. Moreover, either G is abelian, or $Z(G) = 1$ and G is definably isomorphic to a semidirect product of R_a and R_m, where R is a definable real closed field.*

This is Theorem 2.6 in [28], noting that firstly, by Theorem 2.6 we can eliminate the assumption on strong elimination of imaginaries, and secondly, by J. Johns Theorem in [20], the open mapping theorem holds in any o-minimal structure.

The situation is not so clear for abelian definable 2-dimensional groups. Peterzil and Steinhorn give in [40] examples of definably compact 2-dimensional abelian groups definable in an o-minimal expansion of a real closed field which are not a direct sum of definable 1-dimensional subgroups (see the comments after Theorem 5.2 below). On the other hand, by Corollary 6.6 below, a torsion-free group G definable in an o-minimal expansion of a real closed field cannot be definably compact, so by Theorem 5.2 G has a definable torsion-free 1-dimensional subgroup H which must be divisible. If G is commutative, H abstractly has a complement, but we do not know what happens in the definable context. The following question was posed in [40] and in [38].

Question 3.6. *Let $\mathcal{M}$ be an expansion of a real closed field. Is every abelian torsion-free 2-dimensional group a direct sum of definable 1-dimensional subgroups?*

I end this section with a result of Nesin, Pillay and Razenj in [28].

Theorem 3.7. *Let G be a definably connected nonsolvable 3-dimensional group. Then, $G/Z(G)$ is definably isomorphic to either $PSL_2(R)$ or $SO_3(R)$, for some definable real closed field R.*

The existence of two types depends on whether a 2-dimensional subgroup of G exists ($PSL_2(R)$), or not (see Propositions 3.5 and 3.1 in [28], noting that again as above we can remove the extra assumptions). The proof in the first case is done in the spirit of both Cherlin's analysis of groups of Morley rank 3 and [27]. This result also proves an *o*-minimal version of Cherlin's conjecture for nonsolvable 3-dimensional groups (see Section 4 below).

4 Interpretability and linear groups

Classical results tell us that in a compact simple Lie group, the Lie structure is implicitly defined from the abstract group structure, that is, any group automorphism is an (analytic) homeomorphism; see Corollary 3.7 in [43] for a model-theoretic version of this fact. A. Nesin and A. Pillay prove in [27] that this Lie structure is actually (explicitely) definable from the abstract group structure. Namely, if $\mathbb{G} = (G, \cdot)$ is a simple (centreless) compact Lie group (definable in the real field) then, there is an isomorphic copy K of the real field interpretable in $\mathbb{G}$, and there is also a $\mathbb{G}$-definable isomorphism between G and a Nash group over K (see Theorem 0.1 in [27]). This last result and Theorems 3.5 and 3.7 above, give us examples of definable groups in which a field is interpretable. Y.Peterzil, A.Pillay and S.Starchenko prove in [33] and [34] (see also [32] for a survey), that an infinite definably simple group $\mathbb{G} = \langle G, \cdot \rangle$ interprets a field R and G is $\mathbb{G}$–definably isomorphic to a linear group over R. The full strength of their analysis of definably simple groups is the following result, which is also a strong version of an *o*-minimal analogue to Cherlin's conjecture. I recall that an algebraic group defined over a field k is said to be *k-simple* if it has no nontrivial normal algebraic subgroups defined over k.

Theorem 4.1. *Let $\mathbb{G} = (G, \cdot)$ be an infinite definably simple (centreless) group. Then, there is a real closed field k such that one and only one of the following holds:*

(1) $\mathbb{G}$ and the field $k[\sqrt{-1}]$ are bi-interpretable, and G is $\mathbb{G}$-definably isomorphic to $H(k[\sqrt{-1}])$, where H is a linear algebraic group defined over $k[\sqrt{-1}]$.

(2) $\mathbb{G}$ and the field k are bi-interpretable, and G is $\mathbb{G}$-definably isomorphic to the semialgebraic connected component of a group $H(k)$, where H is a k-simple algebraic group defined over k.

This is Theorem 1.1 in [34]. Note that it also proves Cherlin's conjecture for groups of finite Morley rank which happen to be definable in an *o*-minimal structure. The proof is divided in two main parts. Firstly, given G as in the hypothesis of the theorem they find a real closed field R *definable in* $\mathcal{M}$, such that G is definably isomorphic to a semialgebraic subgroup of some $GL(n, R)$ (this is carried out in [33]). To find a field they make use of the Trichotomy theorem (proved by Y. Peterzil and S. Starchenko in [36]) to prove that a definably connected centreless group G is a direct product of (unidimensional) groups, each one defined over a definable real closed field (actually, they get that each one of the groups – in the direct product – is a linear group over the corresponding field). See Theorems 3.1 and 3.2 in [33]. Once we have a field, we may suppose that our *o*-minimal structure is an expansion of a real closed field, and in this situation any definable centreless group is definably isomorphic to a subgroup of $GL(n, M)$ (see below in this Section). If moreover the group is semisimple we have the following result.

Theorem 4.2. *Let $\mathcal{M}$ be an expansion of a real closed field. Let G be a definably connected centreless semisimple group. Then, G is definably isomorphic to a semialgebraic linear group over M.*

This is Theorem 2.37 in [33]. For its proof Peterzil, Pillay and Starchenko develop Lie algebra machinery, and they study the Lie algebra of G and the adjoint representation. A key result there is that the dimension of a semisimple Lie algebra coincides with the dimension of its automorphism group. The latter is obtained by transfer from the reals.

Then, for the rest (and second part) of the proof of Theorem 4.1 above, we may suppose that our G is a semialgebraic subgroup of some $GL(n, R)$ where R is a definable (in $\mathcal{M}$) real closed field (this second part is carried out in [34]). The first goal is to find a field K interpretable in $\mathbb{G}$. The latter is first proved in the solvable not nilpotent case (see Theorem 2.12 in [34]) through an o-minimal version of a theorem by Zil'ber (see Theorem 2.6 in [34]). Now, going back to G in our general hypothesis, the last part of the proof is based on the study of the geometric structures partly developed in [19].

Y.Peterzil and S.Starchenko make use of the above results to give the following characterization of definable groups which interpret a field.

Theorem 4.3. *Let $\mathbb{G} = (G, \cdot)$ be a definable infinite group. Then, $\mathbb{G}$ interprets an infinite field if and only if G is not abelian-by-finite.*

This is Corollary 5.1 in [37] noting that by Theorem 2.6 the quotient groups are definable (see also Corollary 6.3 in [13]). Firstly, note that to avoid local modularity, for G to interpret a field, we need G not to be abelian-by-finite. They prove that this condition is also sufficient. By the results in [33] and [34], stated above, they reduce to the case in which G is not semisimple. Then, they obtain an infinite definable family of homomorphisms between two abelian definable groups (either subgroups or quotients of subgroups of G). In this situation they have already proved – Theorem 4.4 in [37] – that a field is interpretable in $\mathbb{G}$ (the field is actually defined in a quotient of subgroups of G). In turn, the proof of Theorem 4.4 in [37] is based on the study of $\bigvee$-definable groups. (For more on $\bigvee$-definable groups see section 2.1 in [34] and also [15] by M.Edmundo.)

The following theorem on definable permutation groups is proved by D. Macpherson, A. Mosley and K.Tent (Theorem 1.1 in [23]), and in particular gives one more example of how to find a field via a definable group. They consider definable groups G equipped with a definable faithful transitive action on a set Ω. Such a permutation group (G, Ω) is *definably primitive* if there is no proper nontrivial definable G-invariant equivalence relation on Ω. A transitive action on Ω is said to be *regular* if the stabilizer of a point is the identity.

Theorem 4.4. *Let (G, Ω) be a definably primitive permutation group. In the case when G has a non-trivial abelian normal subgroup, assume also that G is definably connected and not regular. Then, (G, Ω) is definably isomorphic to a semialgebraic permutation group over a definable real closed field.*

Firstly, note that the extra assumption when G has a nontrivial abelian normal subgroup is needed. The proof of Theorem 4.4 is based on a fine structure theorem for definably primitive permutation groups (Theorem 1.2

in [23]) which is an analogue of the O'Nan-Scott theorem for *finite* primitive permutation groups. Macpherson, Mosley and Tent apply Theorem 4.4 to give a description of definable permutation groups (G, Ω) where $dim\, \Omega = 1$ – Theorem 1.5 in [23]. They propose as an application of their results the following problem (for more problems on definable permutation groups see p.669 in [23]).

Problem 4.5. *Classify definably primitive permutation groups* (G, Ω) *such that* G *has finitely many orbits on* Ω^2.

In [49] K. Tent classifies sharply n-transitive infinite definable permutation groups for $n = 2, 3$ (there are no infinite 4-transitive groups definable in an *o*-minimal structure), where *sharply* n-transitive means that the stabilizer of n distinct points is trivial. Note that Problem 4.5 is a generalization of this last result.

By the above results on (semi)simple groups, the study of definable linear groups can be seen as a continuation of them. On the other hand, any definably connected centreless group defined over an expansion of a real closed field is definably isomorphic to a linear group (see Corollary 3.3 in [29], proved by Y.Peterzil, A.Pillay and myself). Y. Peterzil, A. Pillay and S. Starchenko study in [35] linear groups defined over an expansion of a real closed field. They obtain the following results.

Theorem 4.6. *Let* $\mathcal{M}$ *an expansion of a real closed field. Let* G *be a definably connected subgroup of some* $GL(n, M)$. *Then* G *is semialgebraic, provided it is either nilpotent, semisimple or definably compact. If* G *is not semialgebraic then there are definable funcions* $f_1, \ldots, f_k$ *such that* G *is defined over* $(M, +, \cdot, f_1, \ldots, f_k)$ *and either* $k = 1$ *and* f_1 *is some exponential function or the* f_i*'s are power functions.*

The nilpotent case is Proposition 3.10 in [35]. The rest of the cases are treated in Theorems 4.3 and 4.6 and Corollary 4.2 in [35]. Their proofs are based on Lie algebra machinery, together with Theorem 4.7 below. The latter – Theorem 4.1 in [35] – also makes use of results on Lie algebras, especially of the classical fact that the commutator of a Lie subalgebra of the general Lie algebra (over an algebraic closed field) is an *algebraic* Lie algebra.

Theorem 4.7. *Let* $\mathcal{M}$ *an expansion of a real closed field. Let* G *be a definably connected subgroup of some* $GL(n, M)$. *Then, there are semial-*

*gebraic groups G_1, G_2 of $GL(n, M)$ such that $G_2 < G < G_1$ with G_2 normal in G_1 and G_1/G_2 abelian (*i.e. *G is an extension of a definable subgroup of a abelian semialgebraic group by a semialgebraic group). Moreover, there are abelian definably connected subgroups $A_1, \ldots, A_k$ of G such that $G = G_2 \cdot A_1 \cdots A_k$.*

Peterzil, Pillay and Starchenko prove – Theorem 5.1 in [35] – the following result. They make use of their results in this section – in particular of Theorem 4.1– and the classification of the simple Lie algebras over $\mathbb{R}$.

Theorem 4.8. *Let G be a definably simple (centreless) group. Then,*

(i) $(G, \cdot)$ is elementary equivalent to $(H, \cdot)$ for some simple Lie group H, and

(ii) G is definably isomorphic to some semialgebraic linear group, defined over $\mathbb{R}_{alg}$ (= the real algebraic numbers).

5 Algebraic aspects

In this section I collect various properties of different nature which give some information about the algebraic structure of definable groups (see also Section 6 below). However, most of the time the hypotheses on the definable group are not algebraic.

Theorem 5.1. *Let G be an infinite definable group. Then,*

(i) G does not have bounded exponent, and

(ii) if G is abelian then the torsion subgroup $G[m]$ is finite, for each $m > 0$.

In dimension one we get the result by Proposition 3.1. The general case is proved by A.Strzebonski in Proposition 6.1 in [47] (once we remove the extra assumption via Theorem 2.6). Clearly (ii) is obtain from (i), and (i) is reduced to the abelian case via Corollary 2.4 (2). For (i) in the abelian case Strzebonski makes use of his results on Euler characteristic (see Section 6).

Theorem 5.2. *Let G be a definable not definably compact group. Then, there is a 1-dimensional torsion-free definable subgroup of G.*

This result is proved by Y.Peterzil and C.Steinhorn (Theorem 1.2 in [40]). The assumption on G ensures the existence of a noncompletable curve in G. This allows them to equip G with an equivalence relation based on *infinitesimal* neighbourhoods and tangency at infinity (inspired on a concept of tangency due to Zil'ber). See Definition 3.2 in [40]. They prove that the equivalence class of the identity element of G is the required subgroup. Peterzil and Steinhorn also give examples of both *(a)* a definably compact abelian group with no definable infinite subgroups, and *(b)* a definable group which is not definably compact and which has no definably compact subgroup; both are over the real field (see Examples 5.2 and 5.6 in [40], based on some examples in [47]). Example (a) indicates that the assumption on G in Theorem 5.2 cannot be avoided. On the other hand, any connected Lie group is homeomorphic to $K \times \mathbb{R}^l$, where K is a maximal compact subgroup. Example (b) tell us that we cannot hope for this in the definable context.

The analysis of definably (semi)simple groups and linear groups in Section 4 gives the following results.

Theorem 5.3. *Let G be a definably connected semisimple centreless group. Then, G is definably isomorphic to a direct product of definably connected definably simple groups.*

This is Theorem 4.1 in [33], observing (as is done in Remark 4.2 there) that by the results on linear groups (Theorem 5.1 in [35]), the factor groups are definably connected and hence definably simple.

Theorem 5.4. *Let $\mathcal{M}$ be an expansion of a real closed field. Let G be a definably connected subgroup of some $GL(n, M)$. Then G is an almost semidirect product of a normal solvable definable subgroup N and a semialgebraic semisimple subgroup H (*i.e.*, $G = NH$ where $N \cap H$ is finite).*

This is Theorem 4.5 in [35]. The ingredients of the proof are: the analysis of the associated Lie algebra and Levi's decomposition theorem of real Lie algebras.

Question 5.5. *Can we eliminate the assumption: "$G < GL(n, M)$" in Theorem 5.4?*

Theorem 5.6. *Let G be a definably compact group. Then, either G is abelian-by-finite or $G/Z(G)$ is semisimple. In particular, if G is solvable then it is abelian-by-finite.*

This is Corollary 5.4 in [37], observing, as usual, that we can remove the extra hypothesis via Theorem 2.6. For the proof, Peterzil and Starchenko consider two cases depending on whether $Z(G)$ is finite or not. The proof follows the lines of the proof of Theorem 4.3 noting (as observed in [40]) that an infinite field cannot be definably compact.

In [13] M.Edmundo studies group extensions and solvable groups. He proves the following results.

Theorem 5.7. *Let G be a definable group and let N be a definable normal subgroup of G. Then there is a definable extension $1 \to N \to G \to H \to 1$, with definable section $s\colon H \to G$.*

Theorem 5.8. *Let G be a definably connected solvable group. Then, there is a definable normal subgroup N of G such that G/N is definably compact and N is $K \times H$, where K is the definably connected definably compact maximal dimensional subgroup of G, and H is a direct product of subgroups, each of which is defined over either a definable semibounded* o*-minimal expansion of a group or a definable real closed field, where 'semibounded' means generated by sets which are bounded together with sets which are linear.*

Theorem 5.7 is Corollary 3.11 in [13] and it is a special case of Theorem 2.6. The ingredients of this case are the Trichotomy Theorem from [36] and non-orthogonality as in [33]. The proof of the description of the definable solvable groups in Theorem 5.8 (see Theorem 5.8 in [13]) is based on the study of definable extensions also developed in [13] and also make use of Theorem 4.2 in [12] and Theorem 5.2.

Continuing with the resemblance between definable groups and Lie groups, one could expect that a definably compact definably connected abelian group should resemble a torus of $\mathcal{M}$. Even though we know, by examples in [40] (see example (a), above in this section), that such a group cannot be a direct product of $SO(2, M)$'s, one could expect that the structure of the torsion is that of a real torus. That this is the case is proved in [16] by M. Edmundo and myself. However the result does not give information of how the torsion points lie within the group (see Question 7.4).

Theorem 5.9. *Let $\mathcal{M}$ be an expansion of a real closed field. Let G be a definably compact definably connected abelian n-dimensional group. Then, the torsion subgroups $G[m] \cong (\mathbb{Z}/m\mathbb{Z})^n$, for each $m > 0$.*

This is Theorem 1.1(b) in [16]; the 1-dimensional case is Proposition 3.1(iii). Here are the ingredients of the proof: Firstly, we obtain some properties of definable covering maps and making use of Corollary 2.10 in [4] (proved by A. Berarducci and myself), we prove that if H is a definably connected abelian group then, there is an $s \geq 0$ such that both the *o*-minimal fundamental group of H is $\mathbb{Z}^s$, and the torsion subgroups $H[m]$ are $(\mathbb{Z}/m\mathbb{Z})^s$, for each $m > 0$. Hence it remains to prove that if H is definably compact (as is G) then $s = dim\, H$. This is done developing some *o*-minimal cohomology machinery – Theorem 6.7 below – based on *o*-minimal homology (developed by A.Woerheide in [50]), and making use of Theorem 5.2 in [5], the latter proved by A. Berarducci and myself. Along the way, we prove that both the o-minimal fundamental group and the o-minimal cohomology $\mathbb{Q}$-algebra of G are isomorphic to the (classical) fundamental group and cohomology $\mathbb{Q}$-algebra of a real torus with Lie dimension n.

We have the following corollary to the proof of Theorem 5.9.

Corollary 5.10. *Let $\mathcal{M}$ be an expansion of a real closed field. Let G be a definably connected abelian group. Then,*

(i) G is torsion-free if and only if G is definably simply-connected, and

(ii) if there is a $k > 1$ for which there is a $g \in G$ of order k, then for every $k > 1$ there is $g \in G$ of order k.

I finish this section with two recent results on algebraic properties of definably compact groups which again are o-minimal versions of results on Lie groups.

Theorem 5.11. *Let G be a definably compact definably connected group. Then,*

(i) there is a definably connected abelian subgroup T of G such that $G = \bigcup_{g \in G} T^g$, and

*(ii) G is divisible (*i.e.* the map $x \mapsto x^m$ on G is surjective for each $m > 0$).*

Firstly, note that (ii) follows from (i) and the fact that any definably connected abelian group is divisible – divisibility is obtained by Theorem 5.1(ii) (see *e.g.* Proof of Theorem 2.1 p.170 in [16]). When $\mathcal{M}$ is an expansion of a real closed field (i) is proved by A. Berarducci in Theorem 6.12 in [2] making use of Theorem 3.3 in [5]. The general case is proved by M.Edmundo in Proposition 1.2 in [14]. His proof is based on Theorem 5.6 and transfer from the reals together with Theorem 4.2. Note that the definable compactness assumption is needed because of examples such as $SL_2(\mathbb{R})$. Both proofs of the divisibility – via (i) – simplily an unpublished proof by myself.

6 Euler characteristic and torsion

In an o-minimal structure, besides the dimension we have another definable invariant: the o-minimal Euler characteristic (E). If X is definable $E(X) := \sum_{C \in \mathcal{D}} (-1)^{\dim C}$, where $\mathcal{D}$ is a cell decomposition of X. The map E is well defined, invariant under definable bijections and $E(X \times Y) = E(X)E(Y)$, for X and Y definable; see Chapter 4.2 in [10]. When $\mathcal{M}$ is an expansion of a real closed field, we can triangulate X and, if X is definably compact, $E(X) = \chi(X(\mathbb{R}))$ where χ is the classical Euler characteristic and $X(\mathbb{R})$ denotes the realization over $\mathbb{R}$ of a (closed) simplicial complex determined by X; the definable compactness asssumption is essential because, even over the real field, $E((0,1)) = -1 \neq 1 = \chi((0,1))$, where $(0,1)$ denotes the unit interval.

A.Strzebonski in [47] makes use of the o-minimal Euler characteristic to study groups definable in o-minimal structures. (He has the extra assumption of $\mathcal{M}$ having definable choice. However, he uses this to have definable quotients – in this case we can avoid it via Theorem 2.6 – and to give another proof of the descending chain condition on definable subgroups.) Strzebonski develops the nice idea of making the Euler characteristic play the role (in definable groups) that cardinality plays in finite groups. Of course, $E(G) = card(G)$ if G is finite, so his results are generalizations of (basic) results on finite groups. For p a prime *or zero* and G a definable group, he defines G to be a p-group if for any proper definable subgroup H of G we have $E(G/H) \equiv 0 \,(mod\, p)$ (with equality if p is zero), where G/H denotes a definable choice of representatives for left cosets of H in G. Note that, in general, both $E(G/H)$ and $E(H)$ divide $E(G)$, for any definable subgroup H of G. Strzebonski proves the three Sylow theorems, with p-Sylow subgroups as maximal definable p-subgroups. As a basis case, he has the following generalization of Cauchy's theorem on finite groups, which links the o-minimal Euler characteristic with the torsion of a definable group.

Theorem 6.1. *Let G be a definable group and p a prime number. If p divides $E(G)$ then G has an element of order p. In particular, if $E(G) = 0$ then G has nontrivial torsion for each prime p.*

This is Lemma 2.5 in [47]. He considers the usual action of $\mathbb{Z}/p\mathbb{Z}$ on the set $\{(g_1, \dots, g_p) \in G \times \cdots \times G \mid g_1 \cdots g_p = 1\}$ and works with the Euler characteristic instead of cardinality.

For the proof of the following result see Propositions 4.1 and 4.2, both in [47].

Proposition 6.2. *Let G be a definable group. Then,*

*(i) if $E(G) = \pm 1$, then G is uniquely divisible (*i.e.*, the map $x \mapsto x^m$ is a bijection for each $m > 0$), and*

(ii) if G is abelian and definably connected then $E(G) = 0, \pm 1$.

The most interesting cases of p-groups are the 0-groups. Note that if G is a p-group and $E(G) \neq 0$ then G is finite (see Remark 2.15 in [47]). On the other hand, Strzebonski proves – Corollary 5.17 in [47] – that 0-groups are abelian. (See [2], by A.Berarducci, for more on 0-groups.)

A.Strzebonski conjectured – first section in [47] – that we do not need G to be abelian in Proposition 6.2 (ii). In [33] (see the end of the introduction therein) it is observed that the results in [33] prove this conjecture.

Theorem 6.3. *Let G be a definably connected group. Then $E(G) = 0, \pm 1$. Moreover, if G is semisimple then $E(G) = 0$.*

I thank Sergei Starchenko for giving the following details to me. We work by induction on the dimension of G. Either, there is an abelian normal definable subgroup N of G with $dim(G/N) < dim\, G$, or G is semisimple. In the first case, since $E(G) = E(G/N)E(N)$, the result follows by induction and Proposition 6.2(ii). For the semisimple case, we will show that $E(G) = 0$. We may clearly suppose that $Z(G) = 1$. Now, by Theorem 3.1 in [33] (see the comments after Theorem 4.1, above) we may also suppose G is unidimensional and hence, by Theorem 3.2 in [33], that $\mathcal{M}$ is an expansion of a real closed field. Then, our semisimple centreless definably connected group is the definably connected component of $Aut(g)$, the automorphism group of the Lie algebra g of G (see proof of Theorem 2.37 in [33]). Again, it suffices to prove that $E(Aut(g)) = 0$, but now $Aut(g)$ is an algebraic subgroup of a $GL(n, M)$. For $M = \mathbb{R}$, the automorphism group of a semisimple Lie algebra has elements of order p for each p, and transfering this to M, we get $E(Aut(g)) = 0$.

As a corollary of Proposition 6.2 and Theorem 6.3 we have the following result.

Corollary 6.4. *Let G be a definably connected group. If G is torsion-free then it is uniquely divisible.*

In connection with Theorem 5.2 above, Y.Peterzil and C.Steinhorn ask in [40] the question if every definably compact group has at least nontrivial torsion (see Question 5.8 there, and a survey by Y.Peterzil on this torsion problem in [30]). By Theorem 6.1 above, one way of giving an affirmative answer to the above question is proving that the Euler characteristic of such a group is zero. In March 2000 M.Edmundo announced the following result.

Theorem 6.5. *Let $\mathcal{M}$ be an expansion of a real closed field. Let G be an infinite definably compact group. Then $E(G) = 0$*

Corollary 6.6. *Let $\mathcal{M}$ be an expansion of a real closed field. Let G be definable group. Suppose there is a definable normal subgroup H of G such that G/H is an infinite definably compact group (in particular this holds if G is definably compact). Then G has an element of order p, for each prime p*

Edmundo's nice idea was to consider *o*-minimal cohomology (dualizing *o*-minimal homology developed by Woerheide in [50]), and then follow a classical proof. Once we have proved that the relevant cohomology algebra is not trivial, the classical proof is just a bit of linear algebra. I give the ingredients. The basic properties of the *o*-minimal cohomology are developed in section 3 in [16] by M. Edmundo and myself. In particular we prove the following result.

Theorem 6.7. *Let $\mathcal{M}$ be an expansion of a real closed field. Let G be an infinite definably connected n-dimensional group. Then, the* o*-minimal cohomology $\mathbb{Q}$-vector space $H^*(G;\mathbb{Q}) = \bigoplus_{m=0}^{n} H^m(G;\mathbb{Q})$ can be equipped with a structure of graded $\mathbb{Q}$-algebra generated by $y_1,\dots,y_r$ $(r \geq 0)$ such that*

(i) $H^0(G;\mathbb{Q}) \cong \mathbb{Q}$ (1 say, generates $H^0(G;\mathbb{Q})$ as $\mathbb{Q}$-vector space);

(ii) each $y_j \in H^{m_j}(G;\mathbb{Q})$ with $deg\, y_j = m_j$ odd $(1 \leq j \leq r)$, and

(iii) $B = \{y_{j_1}\cdots y_{j_l} \colon 1 \leq j_1 < \cdots < j_l \leq r\}$ together with 1 (from (i)) form a basis of the $\mathbb{Q}$-vector space $H^*(G;\mathbb{Q})$ (for a monomial $x = y_{j_1}\cdots y_{j_l}$ we say $len\, x = l$).

This is Theorem 3.4 and its corollaries in [16]. The ingredients of the proof are the following. Dualizing the results in [50] we get that $H^*(G;\mathbb{Q})$ is a graded $\mathbb{Q}$-vector space, $H^m(G;\mathbb{Q}) = 0$ for each $m > n$ and (i) (the latter because G is definably connected); the $\mathbb{Q}$-algebra structure is obtained via a Hopf-algebra structure and a classical classification of Hopf algebras.

We now go back to the proof of Theorem 6.5. We may suppose G is definably connected. Since G is definably compact, the top homology group is nontrivial (this is proved in [5] Theorem 5.2, and also independently by M.Edmundo); dualizing we get $H^*(G;\mathbb{Q}) \not\cong \mathbb{Q}$ and hence $r > 0$, in Theorem 6.7. On the other hand, by the triangulation theorem and after dualizing, the results in [50] (linking simplicial and singular *o*-minimal homology) give $E(G) = \sum_{m=0}^{n}(-1)^m dim\, H^m(G;\mathbb{Q})$. For $m > 0$, $dim\, H^m(G;\mathbb{Q}) = card\{x \in B \colon deg\, x = m\}$, hence $E(G) = 1 + \sum_{x\in B}(-1)^{deg\, x}$. Now, by (ii) in Theorem 6.7 above, the graded $\mathbb{Q}$-algebra product makes $deg\, x \equiv len\, x\ (mod\, 2)$ for any monomial $x \in B$, so we can substitute $deg\, x$ by $len\, x$ in the last equality.

Finally, note that there are $\binom{r}{l}$ monomials of length l in B, hence $E(G) = \sum_{l=0}^{r}\binom{r}{l}(-1)^l$, which is 0, since $r > 0$.

There are other proofs of Theorem 6.5. In [5] A.Berarducci and myself proved the result (see the proof of Corollary 3.4 there) via a Lefschetz fixed point theorem for *o*-minimal expansions of real closed fields, which in turn is proved by transfer from the reals; the transfering can be done after we prove that the top homology group of a definably compact definably connected group is $\mathbb{Z}$ (see Theorem 5.2 in [5]). Using a differential topology approach, in [3], we define the Lefschetz number of id_G, $\Xi(G)$, as a self intersection number of the diagonal in $G \times G$ (see Definition 9.11, there) and we prove in Theorem 11.4 also there, that $\Xi(G) = 0$; our hope was – as in the classical case and as we conjectured – that $\Xi(G) = E(G)$. This conjecture has recently been proved by Y.Peterzil and S.Starchenko in [39]; they also give there – Corollary 4.6 – another proof of Theorem 6.5. They extend further the results in [3] introducing definable Morse functions and prove that $E(G)$ is the degree of a map, a degree that they have shown is 0. To finish with the different proofs of Theorem 6.5, let me note that we can also get one via the structure of the torsion – Theorem 5.9 – which gives $E(G) = 0$, for G abelian, and then make use of either Theorem 6.3 and Theorem 5.6, or the existence of an infinite abelian subgroup – Corollary 2.4(ii). If we think only of the torsion, the best result is via Theorem 5.9 and Corollary 2.4(ii), which gives an element of order m for each $m > 0$. Another proof of the existence of torsion is in [18] (Remark 2 after Corollary 8.4); this one is based on the facts (see Section 8, below) that a definable group has DCC for type-definable subgroups of bounded index, that a definably compact group has *fsg* (see Definiton 8.5 below) and also on the classical fact that a compact Lie group has torsion.

Question 6.8. *Can we eliminate the assumption: "$\mathcal{M}$ is an expansion of a real closed field" in Theorem 6.5 or in Corollary 6.6?*

Y.Peterzil and S.Starchenko have recently been studying (see [38]) torsion-free groups definable in *o*-minimal expansions of real closed fields. Note that by Corollary 6.6 such a group cannot be definably compact, and by Theorem 6.1, it must have *o*-minimal Euler characteristic ± 1; moreover, it must also be solvable (see Claim 2.11 in [38]). They develop in [38] definable group extensions and prove in Corollary 5.8 the following result.

Theorem 6.9. *Let $\mathcal{M}$ be an* o*-minimal expansion of a real closed field. Let G be a torsion-free n-dimensional definable group. Then $E(G) = (-1)^n$. Moreover, G is definably diffeomorphic to M^n.*

When $\mathcal{M}$ is $\mathbb{R}_{an}$ (or any polynomially bounded *o*-minimal expansion of a real closed field in which any definable function $f\colon M \to M$ has a definable Puiseux-like expansion at infinity) Peterzil and Starchenko get the following structure theorem (see Theorem 4.14 together with Theorem 4.9 in [38]).

Theorem 6.10. *Let G be a connected abelian torsion-free group definable in $\mathbb{R}_{an}$. Then, G is definably isomorphic to a direct sum of $(\mathbb{R}, +)^k$ and $(R^{>0}, \cdot)^m$, for some $k, m \geq 0$.*

7 Genericity and measure

In this section $\mathcal{M}$ is a sufficiently saturated o*-minimal expansion of a real closed field,* small *or* bounded *means small with respect to the degree of saturation of $\mathcal{M}$ and* model *means elementary substructure of $\mathcal{M}$.*

The motivation for considering measures and generic sets in the *o*-minimal context was Pillay's conjecture (see Section 8). A definable subset X of a definable group H is *left* (resp. *right*) *generic* (in H) if finitely many left (resp. right) translates of X cover H. And, X is *generic* (in H) if it is both left and right generic. A first result on generic sets already appeared in Lemma 2.4 in [42], where A. Pillay proves that large sets are generic (see Lemma 2.2 above). Since X generic in H implies $dim\, X = dim\, H$, generic sets lie between large and having the same dimension. Note also that if p is a complete type in H and X is (left) generic in H, then a (resp. left) translate of X must be in p. If a definable group H is not definably compact, then the complement of a nongeneric set does not need to be generic (take $G = (M, +)$ and $X = (0, +\infty)$). However, this is the only obstruction in the abelian case:

Theorem 7.1. *Let G be an abelian definably compact group. If $X \subset G$ is definable and not generic then $G \setminus X$ is generic.*

This is proved when G is a torus – Proposition 5.6 of [6] by A. Berarducci and myself – using the existence of a measure on definable sets (see below in this section). The general case is proved by Y.Peterzil and A.Pillay – Corollary 3.9 in [31] – as a corollary of the following result of A. Dolich in [11] (see also Appendix in [31]).

Theorem 7.2. *Let X be a definably compact set and $\mathcal{M}_0$ a small model. Then the following are equivalent:*

(a) the set of $\mathcal{M}_0$-conjugates of X is finitely satisfiable, and

(b) X has a point in $\mathcal{M}_0$.

We have the following corollary to Theorem 7.1, see Lemma 3.12 and Corollary 3.10 in [31].

Corollary 7.3. *Let G be an abelian definably compact group. Then,*

(i) the set $\mathcal{I} = \{X \subset G\colon X$ is definable and not generic$\}$ is an ideal of $Def(G)$ (= the Boolean algebra of definable subsets of G);

(ii) for every $X \subset G$ definable and left-generic, the stabilizer $Stab_{\mathcal{I}}(X) = \{g \in G\colon gX \triangle X \in \mathcal{I}\}$ is a type-definable subgroup of G ($\triangle$ = symmetric difference), and

(iii) there is a complete generic type (= every formula in the type defines a generic set) in G.

In Section 8 the commutativity assumption – in both Theorem 7.1 and Corollary 7.3 – will be substituted by *fsg* (see Definition 8.5). An answer to the following question will give us some information about how the torsion points lie in a definably compact group (see the comments just before Theorem 5.9).

Question 7.4. *Let G be a definably connected definably compact group. Let X be a generic subset of G. Does X contain a torsion point of G?*

We do not know the answer even if we assume $\mathcal{M}$ an expansion of a real closed field and hence the positive solution to Pillay's conjectures (see Theorem 8.8 below) or if we assume G commutative. We have the same open questions for X large in G.

A *Keisler measure* μ on a definable set X is a finitely additive probability measure on $Def(X)$; hence $\mu\colon Def(X) \to [0,1]$, where $[0,1]$ is the unit real interval. For instance, a type is a 0-1 valued measure on any definable set. Keisler measures were introduced by Keisler in [21]. If μ is a Keisler measure on X, we have the following equivalence relation: $Y \sim_\mu Z$ if $\mu(Y \triangle Z) = 0$, for $Y, Z \subset X$ definable. The following result was obtained in [18] as a corollary of Keisler's work in [21].

Theorem 7.5. *Let X be definable and μ a Keisler measure on X. Then, there are bounded many $\sim_\mu$-classes. In particular, there is a small model $\mathcal{M}_0$ such that every definable subset Y of X is $\sim_\mu$-equivalent to some M_0-definable subset of X.*

For a proof see Corollary 3.4 in [18] noting that by Corollary 3.10 in [45], the theory of an o-minimal structure has NIP.

Observe that if μ is a left-invariant Keisler measure on a definably compact group H, then any left generic subset of H must have positive measure. Over the reals, there exist (nondefinable) subsets of the two-dimensional torus with positive measure and empty interior (hence, *non-generic*). We will see below that, in the definable context, generics are the only subsets of a definably compact group with positive Keisler measure (see Corollary 8.9). With the results we have so far we can use the existence of a measure to prove the following property of stabilizers (defined in Corollary 7.3 above).

Corollary 7.6. *Let G be a definably compact abelian group. Then, for every $X \subset G$ definable and generic, $Stab_{\mathcal{I}}(X)$ is a type-definable subgroup of G of bounded index.*

This is Proposition 6.3(ii) in [18]. By the fact that every abelian group is amenable (*i.e.*, it has a (left)-invariant finitely additive measure over *all* subsets), G – being abelian – has in particular an invariant Keisler measure μ. By Theorem 7.5 there are bounded many $\sim_\mu$-classes. Since generic subsets of G have positive measure, the type-definable subgroup $Stab_{\mathcal{I}}(X)$ must have bounded index.

8 Pillay's conjectures

The fact that we have a definable manifold topology (Theorem 2.1), the results on definably simple groups (Theorems 4.2 and 4.8) and the structure of the torsion of definably compact abelian groups (Theorem 5.9) induce us to think that a definably compact group must be a nonstandard version of a real Lie group; that is, if we quotient out by the right subgroup of infinitesimals (see *e.g.*, Proposition 3.8 in [43]), a definably compact group becomes a Lie group. The definable compactness assumption is needed (see below) to ensure that we do not collapse the group when we take the quotient. More formally, the following conjectures were formulated by A. Pillay around 1998, and formally stated during the Problem Session of the ECMTA in honour of A.J. Macintyre. See Problem 19 p.464 in [1], and Conjecture 1.1 in [44].

Pillay's conjectures. *Let $\mathcal{M}$ be sufficiently saturated and let G be a definably connected group. Then,*

(C1) *G has a smallest type-definable subgroup of bounded index, G^{00};*

(C2) *G/G^{00} is a compact connected Lie group, when equipped with the logic topology;*

(C3) *if moreover, G is definably compact then the Lie dimension of G/G^{00} is equal to the (*o*-minimal) dimension of G, and*

(C4) *if moreover, G is definably compact and abelian, then G^{00} is divisible and torsion-free.*

A type-definable equivalence relation Q on a definable set is *bounded* if X/Q is small. The *logic topology* was introduced by D. Lascar and A. Pillay in [22]. With X and Q as above, we say that $C \subset X/Q$ is closed if its preimage under the natural projection is type-definable in X (see Definition 3.1 in [22] or Definition 2.3 in [44]).

In the rest of this section I fix a sufficiently saturated o*-minimal structure $\mathcal{M}$ and 'small' means small with respect to the degree of saturation of $\mathcal{M}$.*

If H is any type-definable normal subgroup of bounded index of a definable group G, then G/H with the logic topology is a compact Hausdorff topological group (see Lemma 3.3 in [22] and Lemmas 2.5, 2.6 and 2.10 in [44]). This, in particular also means that the projection map from G to G/H is *definable* over some small model, in the sense of Definition 2.1 in [18].

The rest of this section is dedicated to stating the cases in which the conjectures are proved to be true and to giving the ingredients of the proofs.

Theorem 8.1. *Conjectures* **C1**–**C4** *are true when G has dimension one.*

This is Proposition 3.5 of [44]. The proof is based in Razenj's analysis of 1-dimensional definable groups (see Section 3). In the definably compact case Pillay makes use of the torsion subgroup of G which is abstractly isomorphic to the torsion subgroup of the circle group of $\mathbb{S}^1$ (see Proposition 3.1(iii)), to define G^{00}, and then he proves that G/G^{00} is $\mathbb{S}^1$. In the non-definably compact case he gets $G^{00} = G$.

Theorem 8.2. *Conjectures* **C1**–**C4** *are true when G is definably simple (centreless).*

This is Proposition 3.6 in [44] and its proof is obtained using the properties of definably simple groups. By Theorem 4.8 we can assume that G is a semi-algebraic subgroup of $GL(n, R)$, where R is a saturated real closed field (containing the reals) and G is defined over $\mathbb{R}$. Then making use of some properties of Lie groups, Pillay proves that the Kernel of the standard part map $St\colon G \to G(\mathbb{R})$ is G^{00} and the logic topology in $G/Ker(st)$ coincide with the standard topology on $G(\mathbb{R})$.

Theorem 8.3. *Conjectures* **C1** *and* **C2** *are true. Moreover, if G is abelian then G^{00} is divisible.*

See Theorem 1.1 in [7] by A.Berarducci, Y.Peterzil, A.Pillay and myself. Firstly, we make use of Proposition 2.12 in [44] in which Pillay establishes the equivalence between *(a)* **C1** and **C2** are true for G, and *(b)* G has the DCC (descending chain condition on type-definable subgroups of bounded index). With the aim of proving (b) note first that if N is a normal definable subgroup of G, and N and G/N have DCC, then so does G – Lemma 1.10 in [7]; making use of the latter, we can reduce the situation to the abelian and the semisimple cases. The semisimple case reduces to the simple case via Theorem 5.3 – in which we know that **C1** and **C2** are true. It hence remains to prove (b) in the abelian case. Associated to a type-definable set we have a concept of *definably connected* (= intersection of a small directed family of definable connected sets; see Definition 2.1 and Lemma 2.2 in [7]), and any type-definable set is a disjoint union of a small number of maximal definably connected type-definable subsets – Theorem 2.3 in [7]. Now we argue by contradiction. If there were a descending chain of type-definable subgroups of G then this would also happen in a countable language with everything defined over a countable model $\mathcal{M}_0$. Since $\mathcal{M}_0$ is small, there is a smallest M_0-type-definable subgroup of G of bounded index H, say. This H must be normal and definably connected, and also divisible (see Claim 3 in p.311 in [7], a proof which also yields that G^{00} is divisible, once we have proved it exists). This implies that the compact group G/H is connected and locally connected – Lemma 2.6 and Theorem 3.9 in [7] – and has finite m-torsion, for each $m > 0$. These facts together with a classical characterization of abelian compact Lie groups makes G/H a compact Lie group. But now the descending chain of subgroups of G induces a descending chain of closed subgroups in G/H, a contradiction.

Note that conjectures **C1** and **C2** do not ensure $G^{00} \neq G$. In fact, for G not definably compact we have already encountered the case $G^{00} = G$ in dimension one, and in general if G is commutative and torsion-free then $G^{00} = G$ (Corollary 1.2 in [7]). When $\mathcal{M}$ is a saturated model of *any* o-minimal structure the rest of the conjectures are still unproved, and we do not even know if $G^{00} \neq G$. Recently, the rest of the conjectures have been proved by E. Hrushovski, Y. Peterzil and A. Pillay in [18] provided $\mathcal{M}$ is an expansion of a real closed field, and even more recently when $\mathcal{M}$ is an ordered vector space in [17] by P.Eleftherion and S.Starchenko. I will give the ingredients of the proof in [18]. The key idea in their proof comes with the property *fsg* (see Definition 8.5, below). Hrushovski, Peterzil and Pillay prove the abelian case by nicely putting together some previous results and then reducing to the abelian and semisimple cases using the fact that definably compact groups have *fsg*. The ingredients follow.

Theorem 8.4. *Conjectures* **C3** *and* **C4** *are true if both $\mathcal{M}$ is an expansion of a real closed field, and G is abelian.*

This is in Lemma 8.2 in [18]. By Theorem 8.3 we know that G^{00} exists and is divisible. By the hypothesis on $\mathcal{M}$ we can apply the results from Section 7. By Corollary 7.6, for every definable $X \subset G$ (generic), $Stab_{\mathcal{I}}(X)$ is a type-definable subgroup of G of bounded index, and hence it contains G^{00}. By Proposition 3.13 in [31], for each $m > 0$, there is a definable generic subset X of G such that $Stab_{\mathcal{I}}(X) \cap G[m] = 0_G$. Therefore, the divisible group G^{00} is also torsion-free (and hence, **C4** is true) and so $(G/G^{00})[m] \cong G[m]$, for each $m > 0$. But G/G^{00} is an abelian compact Lie group, hence a torus and so $(G/G^{00})[m] \cong (\mathbb{Z}/m\mathbb{Z})^l$, where l is the Lie dimension of G/G^{00}. On the other hand, again because we are over an expansion of a real closed field, we can apply Theorem 5.9, and get $G[m] \cong (\mathbb{Z}/m\mathbb{Z})^n$ where n is the dimension of G. From these three isomorphisms, we obtain that **C3** is also true.

For the next results (in [18]) we need the following.

Definition 8.5. *Let G be a definable group. We say that G has fsg (finitely satisfiable generics) if there is some global type $p(x)$ in G and some small model $\mathcal{M}_0$ such that every left translate $gp = \{\varphi(x) \colon \varphi(g^{-1}) \in p\}$ of p with $g \in G$, is finitely satisfiable in $\mathcal{M}_0$.*

Proposition 8.6. *Let G have fsg, witnessed by p and $\mathcal{M}_0$, and let X be a definable subset of G. Then,*

(a) If X is left (resp. right) generic then X is generic, and

(b) Conditions (i), (ii) and (iii) of Corollary 7.3 hold for G. Moreover, in (iii), the generic type can be taken to be p.

Theorem 8.7. *Let $\mathcal{M}$ be an expansion of a real closed field. Let G be a definable group and N be a definable normal subgroup of G. Then,*

(1) if N and G/N have fsg, then so does G, and

(2) if G is definably compact then G has fsg.

See Proposition 4.2 in [18] for Proposition 8.6. Theorem 8.7 (1) is Proposition 4.5 in [18]; the proof of (2) in the commutative case is based on Theorem 7.5 and Theorem 7.2 (see Lemma 8.2 in [18]); the proof of (2) in the general case is reduced – by (1) and (2) in the commutative case – to the semisimple case, and this in turn is reduced to the simple case via Theorem 5.3; the simple case is solved via Theorem 4.8 in Proposition 4.6 of [31], the latter is based on Theorem 4.6 in [6] (see Theorem 8.1(i) in [18]).

Finally, we can state the following.

Theorem 8.8. *Conjectures* **C1**–**C4** *are true when* $\mathcal{M}$ *is an expansion of a real closed field.*

This is Theorem 8.1(ii) in [18] (taking into account the comments after Remark 2.3 there). By Theorems 8.3 and 8.4, it remains to prove conjecture **C3**, and this is done by induction on $dim\, G (> 0)$. Either G is semisimple or there is a normal definable abelian subgroup N of G with $dim\, G/N < dim\, G$. The semisimple case is reduced – via Theorem 5.3 – to the definably simple case, for which we know – Theorem 8.2 – the conjecture is true. On the other hand – by Theorem 8.4 – **C3** is true for N, and by induction for G/N. Knowing the latter, we still have to prove **C3** for G, but now we also know – Theorem 8.7(2) – that G has *fsg*. For the rest of the proof one makes use of Proposition 8.6 and the fact that N has finite n-torsion for each $n > 0$.

We have the following result which is partly a corollary to the proof of the conjectures.

Corollary 8.9. *Let* $\mathcal{M}$ *be an expansion of a real closed field. Let* G *be a definably compact definably connected group. Then, there is a left invariant Keisler measure* μ *on* G*. Moreover, for any definable* $X \subset G$*,* X *is generic if and only if* $\mu(X) > 0$*.*

This is Proposition 6.2 in [18] (including its proof) noting that by Corollary 3.10 in [45] the theory of an *o*-minimal structure has NIP, and by the proof of Theorem 8.8 G has *fsg*.

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