Intermediate Model Theory

(Notes by Josephine de la Rue and Marco Ferreira)

1 Monday 12th December 2005

1.1 Ultraproducts

Let L be a first order predicate language. Then $\mathcal{M} =< M, c^{\mathcal{M}}, f^{\mathcal{M}}, R^{\mathcal{M}} >$, an L-structure is a nonempty set M with interpretations for function, relation and constant symbols. Here we use a one-sorted structure, but this could be extended to a many-sorted structure.

Definition 1. *Suppose we have a set $\mathcal{M}_i$ $(i \in I)$ of L-structures, then the product of these structures is $\mathcal{M} = \prod_i \mathcal{M}_i$, which has the natural (pointwise) L-structure, that is, $c^{\mathcal{M}} = (c^{\mathcal{M}_i})_i$, $f^{\mathcal{M}}((a_i^1)_i, \cdots, (a_i^n)_i) = (f^{\mathcal{M}_i}(a_i^1, \cdots, a_i^n))_i$ and $((a_i^1)_i, \cdots, (a_i^n)_i) \in R^{\mathcal{M}}$ iff $(a_i^1, \cdots, a_i^n) \in R^{\mathcal{M}_i}$ for all $i \in I$.*

In general, a product has a different theory to that of the structures in the set $\mathcal{M}_i$. For example a product of fields is just a ring. Note also that if φ is an atomic formula then $\prod_i \mathcal{M}_i \models \varphi((a_i^1)_i, \cdots, (a_i^n)_i)$ iff $\{i \in I : \mathcal{M}_i \models \varphi(a_i^1, \cdots, a_i^n)\} = I$. In particular this shows that if $J \subseteq I$ then the restiction function $(a_i)_{i \in I} \to (a_i)_{i \in J}$ is a surjective homomorphism from $\prod_{i \in I} \mathcal{M}_i$ to $\prod_{i \in J} \mathcal{M}_i$. We will define next the notion of reduced product which generalizes the notion of product in a sense that if φ is atomic then the reduced product satisfies $\varphi((a_i^1)_i, \cdots, (a_i^n)_i)$ iff $\{i \in I : \mathcal{M}_i \models \varphi(a_i^1, \cdots, a_i^n)\}$ is "large" inside I. Let us first define a notion of "large" inside I.

Definition 2. *A filter on a set I is a subset $\mathcal{F} \subseteq \mathcal{P}(I)$ such that $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, if $J, K \in \mathcal{F}$ then $J \cap K \in \mathcal{F}$ and if $J \subseteq K \subseteq I$ and $J \in \mathcal{F}$ then $K \in \mathcal{F}$. So, a filter is non-empty, does not contain the empty set, the intersection of two elements in the filter is in the filter and anything above an element in the filter is in the filter.*

We sometimes refer to the elements of a filter as "large" sets.

Definition 3. *An ultrafilter, $\mathcal{U}$ is a maximal filter on I with respect to inclusion. Equivalently, for every $J \subseteq I$ either $J \in \mathcal{U}$ or $J^c \in \mathcal{U}$ (where J^c is the complement of J), equivalently if $J \cup K \in \mathcal{U}$ then $J \in \mathcal{U}$ or $K \in \mathcal{U}$.*

Every filter can be extended to an ultrafilter; but you should not expect to be able to specify any non-principal ultrafilter completely since existence of such is just a bit weaker than the Axiom of Choice/Zorn's Lemma.

Definition 4. *An ultrafilter is principal if it is of the form* $\{J \subseteq I : i_0 \in J\}$ *for some* $i_0 \in I$.

Note that an ultrafilter is nonprincipal if and only if it contains the Fréchet filter $\mathcal{F} = \{J \subseteq I : I \setminus J \text{ is finite}\}$.

Define a relation $\sim\, = \,\sim_{\mathcal{F}}$ (for a given filter $\mathcal{F}$) on $\prod_i \mathcal{M}_i$ by $(a_i)_i \sim (b_i)_i$ if and only if $\{i \in I : a_i = b_i\} \in \mathcal{F}$, where $(a_i)_i, (b_i)_i \in \prod_i \mathcal{M}_i$ (i.e. two elements are equivalent if their coordinates are equal on a "large" set). Let $\prod_i \mathcal{M}_i/\mathcal{F}$ be the quotient $\prod_i \mathcal{M}_i/\!\sim$. Note that $\sim$ is indeed an equivalence relation. There is a natural induced structure on $\mathcal{M}^* = \prod_i \mathcal{M}_i/\!\sim$, more precisely: $c^{\mathcal{M}^*} = (c^{\mathcal{M}_i})_i/\sim$; $f^{\mathcal{M}^*}((a_i^1)_i/\sim, \cdots, (a_i^n)_i/\sim) = (f^{\mathcal{M}_i}(a_i^1, \cdots, a_i^n))_i/\sim$; $((a_i^1)_i/\sim, \cdots, (a_i^n)_i/\sim) \in R^{\mathcal{M}^*}$ iff $\{i \in I : (a_i^1, \cdots, a_i^n) \in R^{\mathcal{M}_i}\} \in \mathcal{F}$. Note that this is well defined. So we get a well defined L-structure on $\mathcal{M}^* = \prod_i \mathcal{M}_i/\mathcal{F}$.

Definition 5. *The structure* $\mathcal{M}^*$ *defined above is the reduced product of the* $\mathcal{M}_i$ *with respect to the filter* $\mathcal{F}$. *If* $\mathcal{F}$ *is an ultrafilter then this* $\mathcal{M}^*$ *is the ultraproduct of the* $\mathcal{M}_i$. *If these* $\mathcal{M}_i$ *are all the same structure* $\mathcal{M}$ *then this is called a reduced power of* $\mathcal{M}$ *and if* $\mathcal{F}$ *is an ultrafilter then* $\mathcal{M}^*$ *is called an ultrapower of* $\mathcal{M}$.

Definition 6. *Let the formula* $\varphi = \varphi(x_1, \ldots, x_n)$ *of* L *be a formula with free variables among* $x_1, \ldots, x_n = \bar{x}$. *It is said to be positive primitive (pp) if it has the form* $\exists \bar{y} \bigwedge_{i=1}^n \theta_i(\bar{x}, \bar{y})$ *where each* θ_i *is an atomic formula.*

Theorem 1 (Łos' Theorem)**.** *Take structures* $\mathcal{M}_i (i \in I)$ *and a filter* $\mathcal{F}$ *on* I. *Let* $\mathcal{M}^* = \prod_i \mathcal{M}_i/\mathcal{F}$ *and take a positive primitive formula* $\varphi(x_1, \ldots, x_n)$. *Let* $a^1, \ldots, a^n \in M^*$ $(a^j = (a_i^j)_i/\sim)$.
i) $\mathcal{M}^* \models \varphi(a^1, \ldots, a^n)$ *if and only if* $\{i \in I : \mathcal{M}_i \models \varphi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F}$. *(Note that this is not true with negations in* φ*).*
ii) if $\mathcal{F}$ *is an ultrafilter then the first part is true for all formulas* φ

Proof. First check that it works for atomic formulas and then use induction on the complexity of the formula φ. For i) use the connective $\wedge$ and existential quantifier $\exists$ and the proof works using only the properties of filters. For ii) the induction step also works for the conective $\neg$ but it requires the fact that $\mathcal{F}$ is an ultrafilter. □

If $\mathcal{U}$ is principal, say $\mathcal{U} = \{J \subseteq I : i_0 \in J\}$, then $\prod_i \mathcal{M}_i/\mathcal{U} \simeq M_{i_0}$ so, even though we usually forget to say so, the ultrafilters we use will be non principal.

Examples

1. Let L_0 be the language containing only the equality relation. In this case an L_0-structure is just a set. Let $\mathcal{M}_n = \{0, 1, \ldots, n-1\}$ $(n \geq 1)$. So

$I = \mathbb{P}$ (the set of positive integers). Take $\mathcal{F}$ to be the Fréchet filter (the set of all cofinite sets of $\mathbb{P}$). Form $\prod_{n\geq 1} \mathcal{M}_n/\mathcal{F}$, a set of cardinality $2^{\aleph_0}$ (because all equivalence classes are countable). Clearly $(a_n)_n \sim (b_n)_n$ if they agree on all but finitely many coordinates. Now extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ on $\mathbb{P}$. The equivalence relation defined by $\mathcal{U}$ is coarser than the one defined by $\mathcal{F}$ so there is a natural surjection $\prod_n \mathcal{M}_n/\mathcal{F} \to \prod \mathcal{M}_n/\mathcal{U}$. This map is a homomorphism. Note $\mathcal{U}$ must be non principal. Also note that $\prod_n \mathcal{M}_n/\mathcal{U}$ is infinite. To see this let $\sigma_{\geq n}$ be the sentence (i.e. no free variables) which says that there are at least n elements. We see that $\{i : \mathcal{M}_i \models \sigma_{\geq n}\} \in \mathcal{F} \subseteq \mathcal{U}$. So, by Łos' Theorem $\prod_i \mathcal{M}_i/\mathcal{U} \models \sigma_{\geq n}$. This is true for every n, so $\prod_i \mathcal{M}_i/\mathcal{U}$ is infinite.

2. Let $F^* = \prod_n \mathbb{F}_{p^n}/\mathcal{U}$ for some prime p, where $\mathbb{F}_{p^n}$ is the finite field with p^n elements, and some ultrafilter $\mathcal{U}$ on $\mathbb{P}$. Then F^* is an infinite field of characteristic p (as each $\mathbb{F}_{p^n}$ has characteristic p) and F^* is not algebraically closed. To see this let σ be the sentence $\exists y_0, y_1 \neg\exists x(x^2 + y_1 x + y_0 = 0)$. Then $\mathbb{F}_{p^n} \models \sigma$ for each $\mathbb{F}_{p^n}$ ($\mathbb{F}_{p^n}$ has a proper quadratic extension), so $F^* \models \sigma$ and hence F^* is not algebraically closed.

3. Let $F^* = \prod_p \mathbb{F}_p/\mathcal{U}$ where $\mathbb{F}_p$ is the prime field of characteristic p. Then, by Łos' Theorem, F^* does not have characteristic p for any p, hence it has characteristic 0.
   Question - Does -1 have a square root in F^*?
   Answer - It depends on $\mathcal{U}$ (going from a filter to an ultrafilter uses the axiom of choice, so we do not know exactly what is in the ultrafilter from its construction).

4. Let $F^* = \prod_p \widetilde{\mathbb{F}}_p/\mathcal{U}$ (where $\widetilde{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$). Then F^* is an algebraically closed field of characteristic 0 (hence can be identified with $\mathbb{C}$ since it has, in fact, cardinality $2^{\aleph_0}$).

1.2 Functoriality

Consider (*L-Struct*) the category in which objects are L-structures and morphisms are homomorphisms. Consider also the category $(L\text{-}Struct)^I$ in which objects are I-indexed families of L-structures and morphisms are I-indexed families of homomorphisms between the coordinates of the objects. The reduced power with respect to a given filter $\mathcal{F}$ on I induces a functor from $(L\text{-}Struct)^I$ to (*L-Struct*). More precisely the functor assigns to each family $(\mathcal{M}_i)_i$ the reduced power $\prod_i \mathcal{M}_i/\mathcal{F}$ and to each morphism $(\alpha_i : \mathcal{M}_i \to \mathcal{N}_i)_i$ between the objects $(\mathcal{M}_i)_i$ and $(\mathcal{N}_i)_i$ is assigned the homomorphism $\prod_i \alpha_i/\mathcal{F}$ between $\prod_i \mathcal{M}_i/\mathcal{F}$ and $\prod_i \mathcal{N}_i/\mathcal{F}$ defined by $(a_i)_i/\sim \mapsto (\alpha_i(a_i))_i/\sim$. We need to check that $\prod_i \alpha_i/\mathcal{F}$ is in fact a homomorphism, but that is a consequence of Łos' Theorem, in fact by Łos' Theorem if φ is an atomic formula we have $\prod_i \mathcal{M}_i/\mathcal{F} \models \varphi((a_i^1)_i/\sim, \cdots, (a_i^n)_i/\sim)$ iff $\{i \in I : \mathcal{N}_i \models \varphi(a_i^1, \cdots, a_i^n)\} \in \mathcal{F}$ and $\prod_i \mathcal{N}_i/\mathcal{F} \models \varphi((\alpha_i(a_i^1))_i/\sim, \cdots, (\alpha_i(a_i^n))_i/\sim)$ iff $\{i \in I : \mathcal{N}_i \models \varphi(\alpha_i(a_i^1), \cdots, \alpha_i(a_i^n))\} \in \mathcal{F}$ thus we get what we need because $\mathcal{F}$

is closed under upwards inclusion. Note that the same argument also shows that embeddings are sent to embeddings and if $\mathcal{F}$ is an ultrafilter then elementary embeddings are also preserved. It is trivial to check that the functor preserves composition of morphisms.
We can also trivially define a functor in the reverse direction by sending $\mathcal{M}$ to $(\mathcal{M})_i$ and $(\alpha : \mathcal{M} \to \mathcal{N})$ to $(\alpha : \mathcal{M} \to \mathcal{N})_i$ and composing these two functors we get a functor from (*L-Struct*) to (*L-Struct*) that assigns to each L-structure the corresponding reduced power with respect to $\mathcal{F}$. Thus homomorphisms and embeddings between L-structures induce homomorphisms and embeddings between the corresponding reduced powers and the same for elementary embeddings in the case of ultrapowers.

1.3 Direct Limits

These are also referred to as directed colimits and are denoted $\varinjlim$.

Definition 7. *An upwards directed poset is a poset such that for any two elements, x, y in the poset there is another element, z in the poset greater than or equal to both x and y.*

Definition 8. *Let P be a(n upwards directed) poset. P can be regarded as a category, with objects the elements of P and a (single) arrow from x to y exactly if $x \leq y$. A P-diagram in a category $\mathcal{C}$ is a functor $P \to \mathcal{C}$. The term is also used to refer to the image of such a functor.*

For example : $C_x \to C_z \leftarrow C_y$ is a directed diagram in a category, as is $C_0 \to C_1 \to \dots$ (indexed by the natural numbers).

Definition 9. *A cocone on such a diagram is an object C together with a map $C_x \to C$ for every C_x in the diagram, and such that all triangles in the resulting diagram commute (see below). The directed colimit, if there is one, is a universal cocone on the diagram and is denoted $\varinjlim_{i \in P}(C_i)$. It is unique up to isomorphism.*

The category of L-structures has direct limits. The reduced power $\prod_i \mathcal{M}_i/\mathcal{F}$ is the direct limit of the diagram, indexed by $\mathcal{F}^{op}$, which takes $J \in \mathcal{F}$ to $\prod_{i \in J} \mathcal{M}_i$ and which takes the (opposite of the) inclusion $J \subseteq J'$ to the projection from $\prod_{i \in J'} \mathcal{M}_i$ to $\prod_{i \in J} \mathcal{M}_i$. This diagram is directed since $\mathcal{F}$ is closed under intersection: $\prod_{i \in J} \mathcal{M}_i \to \prod_{i \in J \cap K} \mathcal{M}_i \leftarrow \prod_{i \in K} \mathcal{M}_i$ where both maps are projection maps and $J, K \in \mathcal{F}$.

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2.1 Direct Limits

Definition 10. *In a category $\mathcal{C}$ a directed diagram is a collection of objects $(C_i)_i$, indexed by a poset $(I, \leq)$, and a collection of maps between them $(C_i \xrightarrow{f_{ij}}$*

$C_j)_{i\leq j}$ such that for each pair i,j of elements of I there is $k \in I$ with $i,j \leq k$, hence for every pair, C_i, C_j, of objects in the diagram there is an object C_k in the diagram and there are the morphisms f_{ik} and f_{jk}. A cocone is given by an object D and maps $g_i : C_i \to D$ such that $g_j = g_i f_{ij}$ whenever $i \leq j$. This cocone is a direct limit if it has the property that for every cocone, given by data D', $(g'_i)_i$, there is a unique map $h : D \to D'$ such that $hg_i = g'_i$ for every i.

If $\mathcal{C}$ is the category of L-structures (or a suitable subcategory thereof), the direct limit is, as a set, $\varinjlim(C_i) = \dot{\cup}_i C_i/\sim$ where, if $c \in C_i$ and $d \in C_k$ then $c \sim d$ if these elements are identified in C_l for some $l \geq i,k$ (i.e. if $f_{il}c = f_{kl}d$). The L-structure is defined on this set in the obvious way: every element in the direct limit is the image of an element in the diagram, indeed, any finitely many elements have, since the diagram is directed, pre-images in some C_i. For example, if $\mathcal{C}$ is the category of $\mathbb{Z}$-modules, then, $\varinjlim(\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\times 4} \mathbb{Z} \to \dots) = \mathbb{Q}$. The category of L-structures has direct limits, but the full subcategory of models of T, $Mod(T)$, need not be closed under direct limits in the category of L-structures. For example, $Mod(\{\exists x, y(x \neq y)\})$ i.e. sets with at least two elements.

$$\begin{array}{ccccccccc}
\{0,1\} & \to & \{0,1,2\} & \to & \{0,1,2,3\} & \to & \{0,1,2,3,4\} & \to & \dots \\
0,1 & \mapsto & 1 & & & & & & \\
 & & 0,1,2 & \mapsto & 2 & & & & \\
 & & & & 0,1,2,3 & \mapsto & 3 & & \dots
\end{array}$$

Then the direct limit is a one point set, which is outside the definable subcategory. However, if T is a set of formulas of the form $\forall\exists \wedge \theta_i$ with the θ_i atomic then $Mod(T)$ is closed under direct limits. For example consider the formula $\forall \bar{x} \exists \bar{y} \theta_1(\bar{x}, \bar{y})$ with θ_1 an atomic formula. Take $\bar{a}$ from $\varinjlim C_i$. There will be a preimage in the system, say $\bar{a'}$ from C_i. There will be $\bar{b'}$ in C_j for some $j \geq i$ such that $\theta_1(\bar{a'}, \bar{b'})$ holds in C_j so, since morphisms preserve atomic formulas, $\theta_1(\bar{a}, Im(\bar{b'}))$ holds in the direct limit, i.e. we have preservation under direct limits.

Ultraproducts are direct limits of products. Take $(\mathcal{M}_i)_{i\in I}$ (L-structures) and a filter $\mathcal{F}$ on I. Form the direct system of products over $J \in \mathcal{F}^{op}$, $\prod_{i\in J} \mathcal{M}_i$, where $\mathcal{F}^{op}$ is $\mathcal{F}$ directed by reverse inclusion. Then $\prod_J \mathcal{M}_i \to \prod_K \mathcal{M}_i$ if $K \subseteq J$ gives a directed system and $\varinjlim_{J\in\mathcal{F}^{op}}(\prod_{i\in J} \mathcal{M}_i) = \prod_i \mathcal{M}_i/\mathcal{F}$.

In the other direction we have the following proposition.

Definition 11. *We say that an embedding $f : \mathcal{M} \to \mathcal{N}$ is pure if for every pp-formula φ and $a^1, \cdots, a^n \in M$ we have $\mathcal{M} \models \varphi(a^1, \cdots, a^n)$ iff $\mathcal{N} \models \varphi(f(a^1), \cdots, f(a^n))$.*

Proposition 1. *Let $(\mathcal{M}_i)_{i\in I}$ be a directed diagram of L-structures and let $\mathcal{F}$ be a filter on I containing the Fréchet filter generated by $\{\{j \in I : j \geq i\} : i \in I\}$. (Note that this set has the finite intersection property, so extends to a filter). Then there is a pure embedding from the direct limit of the diagram to the reduced product with respect to $\mathcal{F}$.*

Proof. Let $\mathcal{M}^l$ be the direct limit of the diagram and let $\mathcal{M}^* = \prod_i \mathcal{M}_i/\mathcal{F}$. First we will show that $\mathcal{M}^*$ is a cocone for the diagram. Let f_{ij} denote the homomorphism between $\mathcal{M}_i$ and $\mathcal{M}_j$ if $i \leq j$ and let f_i denote the limit homomorphism from $\mathcal{M}_i$ to $\mathcal{M}^l$. Define $g_i : \mathcal{M}_i \to \mathcal{M}^*$ by $g_i(a) = (f_{ij}(a))_{j \geq i}/\sim$ (if $j < i$ define $g_i(a)_j$ arbitrarily, this will not change the equivalence class of the image because $\{j \in I : j \geq i\} \in \mathcal{F}$). To see that these form a cocone we need to check that $g_k = g_l f_{kl}$, in fact $g_l(f_{kl}(a)) = (f_{lj}(f_{kl}(a)))_{j \geq l}/\sim = (f_{kj}(a))_{j \geq l}/\sim = (f_{kj}(a))_{j \geq k}/\sim = g_k(a)$. Thus by definition of direct limit there is a unique homomorphism $h : \mathcal{M}^l \to \mathcal{M}^*$ such that the resulting system is comutative. Therefore, if φ is a *pp*-formula then:

(*) $\mathcal{M}^l \models \varphi(\bar{a}) \Rightarrow \mathcal{M}^* \models \varphi(h(\bar{a}))$ for every $\bar{a} \in M^l$.

But the reverse implication is also true, making h a pure embedding. To see this, let $\bar{a}$ be a tuple in $\mathcal{M}^l$, so there exist i and $a_1, \cdots, a_n \in \mathcal{M}_i$ such that $\bar{a} = (f_i(a_1), \cdots, f_i(a_n))$. Recall that we have $hf_i = g_i$ thus $h(\bar{a}) = h(f_i(a_1), \cdots, f_i(a_n)) = (g_i(a_1), \cdots, g_i(a_n)) = ((f_{ij}(a_1))_{j \geq i}/\sim, \cdots, (f_{ij}(a_n))_{j \geq i}/\sim)$. Now if we assume the right side of $(*)$, for j large enough we get $\mathcal{M}_j \models \varphi(f_{ij}(a_1), \cdots, f_{ij}(a_n))$ and applying the homomorphism f_j we get
$\mathcal{M}^l \models \varphi(f_j f_{ij}(a_1), \cdots, f_j f_{ij}(a_n)) \Leftrightarrow \mathcal{M}^l \models \varphi(f_i(a_1), \cdots, f_i(a_n)) \Leftrightarrow$
$\mathcal{M}^l \models \varphi(\bar{a})$. □

Note that in this proposition we can choose $\mathcal{F}$ to be an ultrafilter.

2.2 Boolean Algebras and Stone Spaces

Definition 12. *A Boolean algebra, B is a lattice (with meet, $\wedge$, and join, $\vee$. operations) with $0, 1$ (a bottom and a top element) in which every element has a complement and in which each of the lattice operations is distributive over the other.*

For example the power set, $\mathcal{P}(I)$, of a set I, with the usual union, intersection and complementation and with top element I and bottom element $\emptyset$ is a Boolean algebra.

Definition 13. *A filter, $\mathcal{F}$ in a boolean algebra B is a subset of B such that if $a, b \in \mathcal{F}$ then $a \wedge b \in \mathcal{F}$ (this corresponds to $a \cap b$), if $a \leq b \in B$ then $b \in \mathcal{F}$ and also $1 \in \mathcal{F}$ and $0 \notin \mathcal{F}$. (The ordering on a boolean algebra is defined by $a \leq b$ iff $a \wedge b = a$, equivalently if $a \vee b = b$.)*

This definition generalises the previous definition of a filter.

Definition 14. *An ultrafilter is a maximal filter. Equivalently, $\mathcal{F}$ is an ultrafilter if and only if for all $b \in B$ either $b \in \mathcal{F}$ or $b^c \in \mathcal{F}$. Alternatively, for all $a, b \in B$ if $a \vee b \in \mathcal{F}$ then $a \in \mathcal{F}$ or $b \in \mathcal{F}$.*

Definition 15. *The Stone space of B, $\mathcal{S}(B)$ is a topological space with points the ultrafilters, $\mathcal{F}$, in B and a basis of open sets $\mathcal{O}_b = \{\mathcal{F} : b \in \mathcal{F}\}$ for $b \in B$, (alternatively, $\{\mathcal{F} : b \notin \mathcal{F}\}$ is the same collection of sets).*

Since $\mathcal{S}(B) = \mathcal{O}_b \dot\cup \mathcal{O}_{b^c}$, $\mathcal{O}_b$ is clopen and so $\mathcal{S}(B)$ is a totally disconnected space. Clearly $\mathcal{S}(B)$ is Hausdorff (given two distinct filters we can find an element in the one which is not in the other) and compact. For, if $\mathcal{S}(B) = \bigcup_i \mathcal{O}_{b_i}$ (an open cover with no finite subcover) consider the set of b_i^c. This set has the finite intersection property (i.e. non-zero meet) since $b_1^c \wedge \cdots \wedge b_n^c = 0$ implies that $b_1 \vee \cdots \vee b_n = 1$ so then $\mathcal{S}(B) = \mathcal{O}_{b_1} \cup \cdots \cup \mathcal{O}_{b_n}$. Contradiction - this would then be a finite subcover. So there is a filter and hence an ultrafilter, $\mathcal{F}$, (by Zorn's Lemma) containing all the b_i^c, so $\mathcal{F} \in \mathcal{S}(B) \setminus \bigcup_i \mathcal{O}_{b_i}$. Contradiction. Therefore $\mathcal{S}(B)$ is compact.
We have therefore that the Stone space, $\mathcal{S}$, is a compact Hausdorff space with a basis of clopen sets.

Let $\mathcal{B}(\mathcal{S})$ be the Boolean algebra of compact open sets of $\mathcal{S}$. Then $\mathcal{B}(\mathcal{S}(B)) \simeq B$ and $\mathcal{S}(\mathcal{B}(\mathcal{S})) \simeq \mathcal{S}$. This is called Stone duality - it is a contravariant equivalence between the category of boolean algebras and a certain category of topological spaces.

The Stone space is homeomorphic to the maximal ideal space of a ring when we consider Boolean algebras as rings.

Fix variables $x_1, \ldots, x_n$, fix T, a set of sentences of L, and apply the above to the formulas of L with free variables among $x_1, \ldots, x_n$. The corresponding *Lindenbaum algebra* of T is the set of formulas $\varphi(x_1, \ldots, x_n) = \varphi(\bar{x})$ of L factored by $\varphi \sim \psi$ if $T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$. This is a Boolean algebra with the natural operations. For example, take an L'-structure $\mathcal{M}$ and a subset $A \subseteq M$. Let $L = L'_A$ and take $T_A = Th(\mathcal{M}, A)$. The ultrafilters in the Lindenbaum algebra of L_A mod T_A are the (complete) types (in $x_1, \ldots, x_n$) of $Th(\mathcal{M})$ over A.
Given an L-structure $\mathcal{M}$ and $A \subseteq M$, each formula $\varphi(x_1, \ldots, x_n) \in L_A$ (a formula with parameters from A) defines a subset of M^n, $\varphi(M) = \{\bar{c} : \mathcal{M} \models \varphi(\bar{c})\}$, a subset of M^n *definable in* $\mathcal{M}$. Let $n = 1$. Then any type p defines a potential element of $\mathcal{M}$, but we could have $p(M) = \bigcap_{\varphi \in p} \varphi(M) = \emptyset$. In which case the type p is not realised in $\mathcal{M}$ (i.e. $\mathcal{M}$ omits p). For fixed n, the subsets of M^n definable in M form a Boolean algebra and the types in n free variables correspond to the ultrafilters in this boolean algebra.

Example

$\mathcal{M} = \langle \mathbb{R}, +, \cdot, 0, 1, \leq \rangle$. Let $\chi_n(x)$ be the formula which says x is in $(0, \frac{1}{n})$. These sets have the finite intersection property (and are definable) therefore they generate a filter, which can be extended to an ultrafilter (i.e. a type). So there is a type p (here $A = \emptyset$) containing all these (in fact, by quantifier elimination for $\mathcal{M}$, there is a unique such p). This p is a description of an infinitesimal. Note that there are no such elements in $\mathbb{R}$, but if we enlarge $\mathbb{R}$ (i.e. take an elementary extension) then we will get infinitesimals.

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3.1 Boolean Algebras and Stone Spaces

Let L be a language, T be a complete theory, $\mathcal{M} \models T$ and $A \subseteq M$. Denote by $S_n^T(A)$ the set of all n-types in $x_1, \ldots, x_n$ (i.e. with free variables among $x_1, \ldots, x_n$) with parameters from A - the Stone space of the Lindenbaum algebra of L_A mod T_A. Take the category of models of T with elementary embeddings. Consider $\mathcal{M} \nearrow$, the subcategory with objects the elementary extensions of $\mathcal{M}$. Note that each formula $\varphi(x_1, \ldots, x_n)$ defines a functor on $\mathcal{M} \nearrow$, taking an elementary extension $\mathcal{N}$ of $\mathcal{M}$ to $\varphi(\mathcal{N}) \subseteq N^n$. Also note that if $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, and if $A \subseteq M$, $B \subseteq N$ and $A \subseteq B$ then we have a homomorphism of Lindenbaum algebras $L_A \to L_B$, and the corresponding restriction map $S_n^T(B) \to S_n^T(A)$. This is the (Stone) dual map and it is continuous.

Definition 16. *Let T be a topological space. A point $p \in T$ is isolated if $\{p\}$ is open.*

Let $T' = T \backslash \{p : p \text{ is isolated}\}$ (T' is called the Cantor-Bendixson derivative of T). Note that T' is a closed subset of T as we have removed a union of open sets. Now repeat this. Having defined $T^{(\alpha)}$ where $T^{(0)} = T$ and $T^{(1)} = T'$, let $T^{(\alpha+1)} = T^{(\alpha)} \backslash \{p : p \in T^{(\alpha)} \text{ and } p \text{ is isolated}\}$ for a sucessor ordinal $\alpha + 1$. If λ is a limit ordinal set $T^{(\lambda)} = \bigcap_{\alpha<\lambda} T^{(\alpha)}$. This is an intersection of closed sets, therefore each $T^{(\alpha)}$ is closed. Let $T^{(\infty)} = \bigcap_{\alpha} T^{(\alpha)}$. If this process does stabilize with $T^{(\infty)} = \emptyset$ (i.e. every type becomes isolated at some stage) then T *has Cantor-Bendixson rank*. If T is compact then so is each $T^{(\gamma)}$, and then if $T^{(\infty)} = \emptyset$ the least β such that $T^{(\beta)} = \emptyset$ is not a limit ordinal, so $\beta = \alpha + 1$ for some α. Then write $CB(T) = \alpha$ - this is the *Cantor-Bendixson rank* of T. For example every point in T is isolated if and only if $CB(T) = 0$. For $p \in T$ set $CB(p) = \alpha$ such that $p \in T^{(\alpha)}$ and $p \notin T^{(\alpha+1)}$, or $CB(p) = \infty$ if $p \in T^{(\infty)}$. If $T^{(\infty)} \neq \emptyset$ set $CB(T) = \infty$ (i.e. undefined).

If B is a Boolean algebra and if B is countable then either
1) $CB(\mathcal{S}(B)) < \infty$, in which case $\mathcal{S}(B)$ is countable, or
2) $CB(\mathcal{S}(B)) = \infty$, in which case $|\mathcal{S}(B)| = 2^{\aleph_0}$.
In case 1, where rank is defined, recall a basis of open sets $\mathcal{O}_b$. Each point p in $T = \mathcal{S}(B)$ is isolated in some $T^{(\alpha)}$, so there is some open set such that $\mathcal{O}_b \cap T^{(\alpha)} = \{p\}$. Each $\mathcal{O}_b$ is only used once to isolate a point. There are countably many $\mathcal{O}_b$, so there are only countably many p's that can be isolated. As the rank is defined each p gets isolated at some point.
In case 2, $T^{(\infty)} \neq \emptyset$ and $T^{(\infty)}$ has no isolated points (but it is still Hausdorff). Since there are no isolated points $T^{(\infty)}$ must have more than one point. So split the space into two closed non-empty subsets. Both of these are open and have no isolated points. Repeat this. Take any decreasing sequence and it is, by compactness of T, hence of $T^{(\infty)}$, non-empty. There are $2^{\aleph_0}$ of these sequences/nests of closed sets, all inhabited.

For any space T if $CB(T) < \infty$ (i.e. it is defined) then the set of isolated points of T is dense in T (an open set which contains no isolated point remains thus throughout the Cantor-Bendixson process).

If B is a Boolean algebra then $CB(\mathcal{S}(B)) < \infty$ if and only if B is *superatomic*. Superatomic means that every quotient Boolean algebra of B is atomic (B is atomic if every element of B, except the bottom element, is above an atom, that is, for all $b \in B$, $b > 0$, there is $a \in B$ such that $b \geq a > 0$ with a an atom - i.e. a minimal element, meaning that there does not exist z such that $a > z > 0$). In contrast, we say that B is *atomless* if for all $b > 0$ there exists a such that $b > a > 0$. For example, $B = \mathcal{P}(\mathbb{N})/\sim$ where $x \sim y$ ($x, y \subseteq \mathbb{N}$) if $x \triangle y$ (their symmetric difference) is finite, is atomless.
If B_1 and B_2 are countable atomless Boolean algebras then $B_1 \simeq B_2$. This is proved by a back and forth argument.

Note that if $p \in S_n^T(A)$ then p is isolated if and only if there is a formula $\varphi(\bar{x}) \in L_A$ such that $p = \{\psi(\bar{x}) \in L_A : \varphi \to \psi\}$ if and only if p as an ultrafilter of A-definable sets is principal. Also note that such a p must be realised in (every $\mathcal{N} \succ$)$\mathcal{M}$ because $\mathcal{M} \models \exists \bar{x} \varphi(\bar{x})$, say $\mathcal{M} \models \varphi(\bar{b})$. Since φ generates p it must be that $tp^{\mathcal{M}}(b/A) = \{\psi(\bar{x}) : \psi \in L_A, \mathcal{M} \models \psi(\bar{b})\} = p$ (i.e. p is realised in $\mathcal{M}$).

3.2 The Omitting Types Theorem

Theorem 2 (Omitting Types Theorem)**.** *If L is a countable language, T is a complete theory and if $p \in S_n^T(\emptyset)$ is non-isolated then there is a model $\mathcal{M}_0 \models T$ which omits p.*
By the downward Löwenheim-Skolem theorem we can assume that $\mathcal{M}_0$ is countable.

Proof. (Sketch) For convenience take $n = 1$. Let L_1 be L with new constants c_n ($n \in \omega$). Enumerate the sentences of L_1 as $\sigma_0, \sigma_1, \ldots$. Let $T_0 = T$ and define T_m inductively as follows. At each stage we will have $T_m \setminus T_0$ finite and T_m consistent. At the m^{th} stage, say $T_m \setminus T_0 = \{\tau_1 \wedge \cdots \wedge \tau_k\}$. Denote by τ^* the conjunction of the τ_i. Choose n such that all c_j appearing in τ^* are among $c_0, \ldots, c_n$ and replace each occurrence of c_i in τ^* by a new variable y_i, to get $\tau(x_0, \ldots, x_n) \in L$, such that τ^* is $\tau(c_0, \ldots, c_m)$. Then show that T proves $\tau(x_0, \ldots, x_n) \not\twoheadrightarrow p(x_m)$, say $\phi \in p$ is such that $\tau(x_0, \ldots, x_n) \not\twoheadrightarrow \phi(x_m)$. It follows that $T'_m = T_m \cup \{\neg\phi(c_m)\}$ is consistent. If σ_m is consistent with T'_m then add it to T'_m to obtain T''_m; otherwise set $T''_m = T'_m \cup \{\neg\sigma_m\}$. If $T''_m = T'_m \cup \{\sigma_m\}$ and if σ_m has the form $\exists x \psi(x)$ where x is the (arbitrary but fixed) variable designated to be the free variable for p then set $T_{m+1} = T''_m \cup \{\psi(c_k)\}$ where c_k is the lowest-indexed c_i not appearing in T''_m, otherwise set $T_{m+1} = T''_m$. Finally let T_ω be the union of all these T_m: consistent and complete by construction. Then show that a model of T_ω yields a model of T omitting p. □

3.3 Large Structures

Let L be a language, T an L theory and $\mathcal{M} \models T$.

Definition 17. *We say that $\mathcal{M}$ is weakly saturated if it realises every type in $\bigcup_n S_n^T(\emptyset)$. If κ is an infinite cardinal then $\mathcal{M}$ is κ-saturated if it realises every type in each $S_n^T(A)$ where A is any subset of $\mathcal{M}$ with $|A| < \kappa$.*

For the latter, it is enough that every type in $S_1^T(A)$ for $|A| < \kappa$ be realised in $\mathcal{M}$. To prove this we show by induction on n that every type in $S_n^T(A)$ is realised. Given the result for n take $p \in S_{n+1}^T(A)$, $p = p(x, y_1, \ldots, y_n) = p(x, \bar{y})$. Let $q(x) = \exists \bar{y} p(x, \bar{y})$ i.e. $q(x) = \{\exists \bar{y} \varphi(x, \bar{y}) : \varphi(x, \bar{y}) \in p\}$. This is consistent. For, take $\varphi_1, \ldots, \varphi_t \in p$, so $\bigwedge_{i=1}^t \varphi_i \in p$, then, by the induction hypothesis, $\mathcal{M} \models \exists x \exists \bar{y} \bigwedge_{i=1}^t \varphi_i(x, \bar{y}) \equiv \exists x \bigwedge_{i=1}^t \exists \bar{y} \varphi_i(x, \bar{y})$. This being consistent implies, by the induction hypothesis, that p is realised, by c say in $\mathcal{M}$. Consider $p(c, \bar{y}) = \{\varphi(c, \bar{y}) : \varphi(x, \bar{y}) \in p\}$. This is consistent by a similar argument. So this is realised, by $\bar{b}$ say in $\mathcal{M}$ (by the inductive hypothesis) i.e. $\mathcal{M} \models p(c, \bar{b})$, as required.

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4.1 Large Structures

Recall that $\mathcal{M}$ is λ-saturated if for all $A \subseteq M$, $|A| < \lambda$, for all $p \in S_n^T(A)$, p is realised in $\mathcal{M}$ (and it is enough to take $n = 1$).

Definition 18. *Call $\mathcal{M}$ $\aleph_0$-homogeneous if for every finite tuple $\bar{a}, \bar{b}$ from $\mathcal{M}$, if $tp(\bar{a}) = tp(\bar{b})$ then for all $a' \in \mathcal{M}$ then there is some $b' \in \mathcal{M}$ such that $tp(\bar{a}a') = tp(\bar{b}b')$. More generally, $\mathcal{M}$ is κ-homogeneous if the same is true for tuples $\bar{a}, \bar{b}$ of cardinality less than κ. Say that $\mathcal{M}$ is strongly κ-homogeneous if for all tuples $\bar{a}, \bar{b} \in \mathcal{M}$ of length less than κ if $tp(\bar{a}) = tp(\bar{b})$ then there is an automorphism $f \in Aut(\mathcal{M})$ with $f(\bar{a}) = \bar{b}$.*

Clearly, strongly κ-homogeneous implies κ-homogeneous (as automorphisms preserve types, take $b' = f(a')$). Also, if $\mathcal{M}$ is $|\mathcal{M}|$-homogeneous then $\mathcal{M}$ is strongly $|\mathcal{M}|$-homogeneous. To see this take $\bar{a}, \bar{b}$ of length less than κ with $tp(\bar{a}) = tp(\bar{b})$. We build an automorphism of $\mathcal{M}$ which takes $\bar{a}$ to $\bar{b}$. Enumerate $\mathcal{M} \backslash \bar{a}$ as $c_0, \ldots, c_\alpha, \ldots$ for $\alpha < |\mathcal{M}|$. Then set the automorphism to take c_0 to d_0 where d_0 is such that $tp(\bar{a}c_0) = tp(\bar{b}d_0)$. Such a d_0 exists by the homogeneity of $\mathcal{M}$. Continue in this way to obtain a map from $\mathcal{M}$ to $\mathcal{M}$ which preserves types. We realise that, in order to ensure that we have an automorphism, we should interleave a 'back' argument with this 'forth' argument.

Definition 19. *Define $\mathcal{M}$ to be κ-universal if for all $\mathcal{N} \equiv \mathcal{M}$, $|\mathcal{N}| < \kappa$ there is an elementary embedding from $\mathcal{N}$ to $\mathcal{M}$.*

Theorem 3. *If $\mathcal{M}$ is κ-saturated then $\mathcal{M}$ is κ-homogeneous and κ^+-universal.*

Proof. For κ-homogeneous : Let $\bar{a}, \bar{b}$ be from $\mathcal{M}$ of cardinality less than κ, let $a' \in \mathcal{M}$ and assume that $tp(\bar{a}) = tp(\bar{b})$. Let $p(\bar{y}, x) = tp(\bar{a}a')$ and consider $p(\bar{b}, x) = \{\varphi(\bar{b}, x) : \varphi(\bar{a}, x) \in tp(\bar{a}a')\} = \{\varphi(\bar{b}, x) : \mathcal{M} \models \varphi(\bar{a}, a')\}$. Then $p(\bar{b}, x)$ is consistent. For, if $\varphi_1(\bar{b}, x), \dots, \varphi_k(\bar{b}, x) \in p(\bar{b}, x)$ then by definition $\mathcal{M} \models \bigwedge_{i=1}^{k} \varphi_i(\bar{a}, a')$. So $\exists x \bigwedge_{i=1}^{k} \varphi_i(\bar{y}, x) \in tp(\bar{a}) = tp(\bar{b})$ i.e. $\mathcal{M} \models \exists x \bigwedge_{i=1}^{k} \varphi_i(\bar{b}, x)$. So $p(\bar{b}, x)$ is consistent and hence is realised, by b' say. Note that $tp(\bar{b}b') = tp(\bar{a}a')$ as required.
For κ^+-universal : Let $\mathcal{N} \equiv \mathcal{M}$ with $|\mathcal{N}| \leq \kappa$, say $|\mathcal{N}| = \lambda$. Enumerate $\mathcal{N}$ as $b_0, \dots, b_\alpha, \dots$ for $\alpha < \lambda$. Consider b_0 and $tp(b_0)$, which is realised by some a_0 in $\mathcal{M}$ (as it is a type over $\emptyset$). Inductively take b_0 to a_0, $\dots$, b_α to a_α, $\dots$ such that $tp(b_0 \dots b_\alpha) = tp(a_0 \dots a_\alpha)$. Now consider $b_{\alpha+1}$ and let $tp(b_{\alpha+1}/b_0 \dots b_\alpha) = p(x, b_0, \dots, b_\alpha)$ which has less than κ parameters. Consider $p(x, a_0, \dots, a_\alpha)$: it is consistent (as above) and so it is realised by $a_{\alpha+1}$ say in $\mathcal{M}$ since $|\{a_0, \dots, a_\alpha\}| < \kappa$. Then send $b_{\alpha+1}$ to $a_{\alpha+1}$. Continue in this way (similarly at limit ordinals). In the end we define $f : N \to M$ by $b_i \mapsto a_i$. Then f is an elementary embedding because, by the inductive hypothesis $tp(b_0 \dots) = tp(a_0 \dots)$. □

Theorem 4. *a) If $\mathcal{M} \equiv \mathcal{N}$, both have cardinality κ and are κ-saturated then $\mathcal{M} \simeq \mathcal{N}$ (i.e. there is at most one κ-saturated model of each cardinality up to isomorphism).*
b) If $\mathcal{M} \equiv \mathcal{N}$ both are of cardinality κ, are κ-homogeneous and realise the same types over $\emptyset$ then $\mathcal{M} \simeq \mathcal{N}$.

Proof. Each part is proved using a back and forth argument. □

Theorem 5. *Every $\mathcal{M}$ has a κ-saturated elementary extension.*

Proof. Let A range over all subsets of $\mathcal{M}$ of cardinality less than κ, and for each such A consider $S_1^T(A)$ where $T = Th(\mathcal{M})$. Enumerate $\bigcup_A S_1^T(A)$ as $p_0, \dots, p_\alpha, \dots$. By compactness we can realise a type in an elementary extension. Realise p_0 in say $\mathcal{M}_0 \succ \mathcal{M}$, realise p_1 in $\mathcal{M}_1 \succ \mathcal{M}_0, \dots$, realise p_α in $\mathcal{M}_{\alpha+1}$. Let $\mathcal{M}_\lambda$, for λ a limit ordinal, be an elementary extension of $\bigcup_{\alpha<\lambda} \mathcal{M}_\alpha$ which realises p_λ. By the elementary chain theorem $\mathcal{M}^1 = \bigcup_\alpha \mathcal{M}_\alpha \succ \mathcal{M}_1$. Note that $\mathcal{M}^1$ realises every 1-type over any subset A of M with $|A| < \kappa$. However, $\mathcal{M}^1$ is not necessarily κ-saturated (there are lots of new subsets), so repeat this to get $\mathcal{M}^2 \succ \mathcal{M}^1$. Repeat inductively over ordinals β for $\beta < \mu$ where μ is a cardinal of cofinality κ. So we have a chain $\mathcal{M} \prec \mathcal{M}^0 \prec \mathcal{M}^1 \prec \dots \prec \mathcal{M}^\beta \prec \dots$. Let $\mathcal{M}^* = \bigcup_\beta \mathcal{M}^\beta$: this is κ-saturated. For let $A \subseteq \mathcal{M}^*$ with $|A| < \kappa$, then since $cf(\mu) = \kappa$ there is $\beta < \mu$ such that $|A| \subseteq \mathcal{M}^\beta$ and so, by construction, every 1-type over A is realised in $\mathcal{M}^{\beta+1}$, and hence in $\mathcal{M}^*$. □

4.2 κ-Saturated Ultraproducts

Definition 20. *An ultrafilter, $\mathcal{U}$ on a set I is ω_1-incomplete (or countably incomplete) if there are $J_n \subseteq I$ for $n \in \omega$ with $J_n \notin \mathcal{U}$ for all n, but $\bigcup_n J_n \in$*

$\mathcal{U}$. Equivalently, $\mathcal{U}$ is ω_1-incomplete if there are sets $J'_n \in \mathcal{U}$ $(n \in \omega)$ (the complements of the J_n) and with $\bigcap_n J'_n \notin \mathcal{U}$.

For example, every non principal ultrafilter on a countable set I is ω_1-incomplete since $I = \bigcup_{i\in I}\{i\}$ and, on any infinite set I, there is an ω_1-incomplete ultrafilter. For, let $I = \dot{\bigcup}_n J_n$ with $J_n \neq \emptyset$, then $\{I\backslash J_n : n \in \omega\}$ has the finite intersection property and so is contained in an ultrafilter $\mathcal{U}$ which must be ω_1-incomplete.

Theorem 6. *Assume that L is a countable language. Let $\mathcal{U}$ be an ω_1-incomplete ultrafilter on a set I. Let $\mathcal{M}_i$ be any L-structures $(i \in I)$. Then we have that $\prod_{i\in I} \mathcal{M}_i/\mathcal{U} = \mathcal{M}^*$ is $\aleph_0$-saturated.*

Proof. Let $p \in S_1^T(A)$ for some finite $A \subseteq \mathcal{M}^*$. Enumerate p as $\varphi_0, \ldots, \varphi_n, \ldots$. Suppose that $I = J'_0 \supseteq J'_1 \supseteq \ldots$ are such that $J'_n \in \mathcal{U}$ but $\bigcap_n J'_n \notin \mathcal{U}$. Define $K_n \subseteq I$ $(n \in \omega)$ by $K_0 = I$ and inductively define $K_n = J'_n \cap \{i \in I : \mathcal{M}_i \models \exists x \bigwedge_{j=0}^{n} \varphi_j(x)\}$. We know that $\{i \in I : \mathcal{M}_i \models \exists x \bigwedge_{j=o}^{n} \varphi_j(x)\}$ is in $\mathcal{U}$, and therefore, note, this means $K_n \in \mathcal{U}$. Also we see that $\bigcap_n K_n \notin \mathcal{U}$. For $i \in I\backslash \bigcap_n K_n$ define $n(i)$ to be the greatest n such that $i \in K_n$. Define $a_i \in \mathcal{M}_i$ as follows. Fix i: if $n(i) = 0$ take any value for $a_i \in \mathcal{M}_i$; if $n(i) > 0$ let $a_i \in \mathcal{M}_i$ be such that $\mathcal{M}_i \models \bigwedge_{j=0}^{n(i)} \varphi_j(a_i)$ (such an a_i exists since $n(i) \geq n$); if $n(i)$ is not defined i.e. if $i \in \bigcap_n K_n$ then take any value for a_i. We claim that $a^* = (a_i)_i/\sim$ realises p. For, given φ_n, we have for every $i \in K_n$ that $n(i) \geq n$ (by definition). So $\mathcal{M}_i \models \varphi_n(a_i)$, but $K_n \in \mathcal{U}$. So by Łos' theorem $\mathcal{M}^* \models \varphi(a^*)$ i.e. a^* is a realisation of p in $\mathcal{M}^*$. □

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5.1 κ-Saturated Ultraproducts

Theorem 7. *Suppose $|L| \leq \kappa$ and take $|I| = \kappa$. Let $\mathcal{U}$ be a suitable ultrafilter (ω_1-incomplete, 'κ-good' - these exist). Then $\prod_{i\in I} \mathcal{M}_i/\mathcal{U}$ is κ-saturated.*

See for example Chang and Keisler 6.1.8, alternatively Hodges, Marker, Bell and Slomson.

Thus there is a functor (*L-Struct*) $\to$ (*L-Struct*), $\mathcal{M} \mapsto \mathcal{M}^I/\mathcal{U}$, which takes each L-structure to a κ-saturated elementary extension.

5.2 Monster Models

Let $\mathcal{M}$ be an infinite structure and let $\mathcal{M}^* \succ \mathcal{M}$. A monster model is a large very saturated elementary extension.

See for example Hodges Chapter 10, 'λ-big' extensions, for existence and properties/usefulness of these.

5.3 Small Models

Definition 21. *A structure $\mathcal{M}$ is atomic if every type realised in $\mathcal{M}$ is isolated.*

Theorem 8. *If $\mathcal{M} \equiv \mathcal{N}$ are atomic countable structures then $\mathcal{M} \simeq \mathcal{N}$.*

Proof. Use a back and forth argument □

Theorem 9. *Suppose that L is a countable language and T is a complete L-theory. Then T has a (countable) atomic model if and only if for all n the isolated points of $S_n^T(\emptyset)$ are dense.*

(This relates back to CB rank.)

Proof. See for example Marker 4.2.10 □

Theorem 10. *Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures. Then $\mathcal{M} \equiv \mathcal{N}$ if and only if there exists I and there is an ultrafilter $\mathcal{U}$ on I such that $\mathcal{M}^I/\mathcal{U} \simeq \mathcal{N}^I/\mathcal{U}$.*

5.4 $\aleph_0$-Categoricity

Definition 22. *Let κ be an infinite cardinal. Then T is κ-categorical if there is just one model of T of cardinality κ up to isomorphism.*

Theorem 11 (Morley's Theorem)**.** *If T is a countable theory and it is κ-categorical for some uncountable κ then T is κ-categorical for every uncountable κ.*

There are no implications between $\aleph_0$-categorical and $\aleph_1$-categorical - all four combinations of these properties and their negations may be illustrated by examples.

Theorem 12 ((Engeler) Ryll-Nardzewski (Svenonius) Theorem)**.** *If L is a countable language and T is complete with no finite models then the following are equivalent:*

1. *T is $\aleph_0$-categorical,*
2. *for all n, $S_n^T(\emptyset)$ is finite,*
3. *for all n, every $p \in S_n^T(\emptyset)$ is isolated,*
4. *for all n, there are only finitely many formulas $\phi(x_1, \ldots, x_n)$ up to equivalence mod T, and*
5. *for all countable $\mathcal{M} \models T$, $Aut(\mathcal{M})$ has only finitely many orbits in its diagonal action on M^n.*

Proof. $1 \Rightarrow 3$: By the Omitting Types Theorem and the downward Löwenheim Skolem theorem.
$3 \Rightarrow 2$: The Stone space $S_n^T(\emptyset)$ is compact.
$2 \Rightarrow 4$: Given $\varphi(x_1, \dots x_n)$ let $\mathcal{O}_\varphi = \{p \in S_n^T(\emptyset) : \varphi \in p\}$. Check that $\varphi \leftrightarrow \psi$ mod T if and only if $\mathcal{O}_\varphi = \mathcal{O}_\psi$. There are finitely many possibilities for φ because the Stone space is finite.
$4 \Rightarrow 3$: Given $p \in S_n^T(\emptyset)$ let $\psi_p = \bigwedge\{\psi_i : \psi_i \in p\}$ where $\{\psi_1, \dots, \psi_k\}$ is a representative set of formulas mod T. Then $p \leftrightarrow \psi_p$.
$3 \Rightarrow 1$: 3 implies that every model is atomic and countable and, with elementarily equivalent, this implies isomorphic.
$5 \Rightarrow 2$: If there are infinitely many types choose countably many and realise these in the countable $\mathcal{M}$. $Aut(\mathcal{M})$ cannot map n-tuples to tuples realizing different types, so $Aut(\mathcal{M})$ has infinitely many orbits. Also we know that $Aut(\mathcal{M})$ preserves types.
$2,3 \Rightarrow 5$: Build an isomorphism with a given base. There are only finitely many types. If two n-tuples have the same type then build an automorphism between them (as in the proof of $3 \Rightarrow 1$), a back and forth argument. □

5.5 Imaginaries

Suppose that $\mathcal{M}$ is an L-structure. We usually consider $\mathcal{M}^2, \dots, \mathcal{M}^n, \dots$ i.e. taking tuples from $\mathcal{M}$.

Definition 23. *A definable equivalence relation, E on M^n is an equivalence relation which is defined by a formula, say $\varphi(\bar{x}, \bar{y})$ (with $2n$ free variables) which is such that $\varphi(M) \subseteq M^{n+n}$ is an equivalence relation on M^n i.e. $\mathcal{M} \models \forall \bar{x} \varphi(\bar{x}, \bar{x})$ and it is symmetric and transitive.*

We can incorporate $\mathcal{M}, \mathcal{M}^2, \dots, \mathcal{M}^n, \dots, \mathcal{M}^n/E$ (n and, given n, E varies) etc. into a single many-sorted structure denoted $\mathcal{M}^{eq}$. The corresponding enriched language is denoted L^{eq}. Assume that $\mathcal{M}$ was one-sorted. The sorts of L^{eq} are $\sigma_{(n,E)}$ (one sort for each n and each E). For each such there is a function symbol: $\pi_{(n,E)} : \sigma^n \to \sigma^n/E$. If you like, a function symbol for each definable function between sorts can be added. We have a category of sorts with definable morphisms between them.
We have a functor Mod $(T) \to$ Mod (T^{eq}) which takes $\mathcal{M} \mapsto \mathcal{M}^{eq}$. For example $(\sigma^n/E)(M) = M^n/E$. If T is complete then T^{eq} is complete, where $T^{eq} = Th_{L^{eq}}(\mathcal{M}^{eq})$ with $\mathcal{M}$ any model of T.

If we repeat this i.e. we do $L \rightsquigarrow L^{eq} \rightsquigarrow (L^{eq})^{eq}$, we have added more sorts to get the last of these, but every new sort is definably isomorphic to one in L^{eq}.

Elimination of imaginaries : take a subset of L^{eq}; we can ask whether we have elimination of imaginaries to this collection of sort, i.e. whether every sort in L^{eq} is definably isomorphic to a definable subset of a finite product of these sorts.

In this above we can allow $\mathcal{M}$ itself to be a many-sorted structure. In which case we use arbitrary finite products of sorts of $\mathcal{M}$ in place of powers of the home sort.

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