Joris Potier

April 10, 2007

1 The finite Morley rank case

$(G, . , 1 , . . .)$ whose theory is of finite MR.

Example 1.1

1. *Algebraic groups over algebraically closed fields.*
2. $\mathbb{Z}_{p^\infty}$ *: the Prufer p-group (* $\aleph_1$*-categorical)*

Cherlin Zilber conjecture : an infinite simple group of finite MR is isomorphic to an algebraic group over an algebraically closed field.

Definition 1.2 $X \subseteq G$ *definable is said to be* indecomposable *if*

$$\forall H \leqslant G \quad definable, \ \ |XH/H| = 1 \ or \geqslant \aleph_0$$

Theorem 1.3 (Zilber's indecomposable theorem)
Let G *be a group of finite MR and* $(X_i)_{i \in I}$ *an arbitrary collection of definable indecomposable subsets such that* $1 \in X_i$, $\forall i$ *.Then*

$$H := < X_i | i \in I >$$

is a definable and connected *subgroup of* G*.*
In fact, there is $X_1, ..., X_m$*,* $m \leq MR(G)$*, such that*

$$H = X_1 ... X_m$$

Remark 1.4

1. $1 \in X_i \ \forall i$ *is needed:*
   otherwise, as singletons are definables, any subgroup would be definable. But

   $$(\mathbb{Z}, +) \leqslant (\mathbb{Q}, +)$$

   and the first one is not ω*-stable while the second one is of finite MR, so could not be a definable subgroup !*

2. *G needs to be of finite MR :*
   Let G *be an infinite elementary abelian group of exponant 2 (i.e. every element has order 2, or equivalentaly* G *is a vector space over* $\mathbb{F}_2$*), with a predicate* A *for an inifinite independant subset such that* $< A >$ *is a proper subset of* G*.* $MR(G) = \omega$*, and every proper definable subgroup is finite. Hence every infinite definable subset is indecomposable. So* A *is indecomposable but* $< A >$ *is not a definable subgroup !*

Proof : (theorem)
$\exists X_{i_1},...,X_{i_m}$ such that if $B := X_{i_1}...X_{i_m}$ then $MR(X_iB) = MR(B)$ for all $i \in I$. Let p be a type of maximal rank in B (i.e. "$x \in B$" is in p). $H := stab(p)$.

Claim : $\forall i \in I$, $|X_iH/H| < \aleph_0$.
otherwise, X_iB contains infinitely many translates of p, which contradicts maximality of MR of types in X_iB.

Thus X_i is in only one coset of H. But $1 \in X_i$, so $X_i \subseteq H$. $< X_i : i \in I > \leqslant H$, hence $B \subseteq H$ and p contains "$x \in H$". But now we have the following :

Lemma 1.5 *Let G be a group of finite MR, and $p \in S_1(G)$. Then p is generic iff $stab(p) = G^0$* □

$H = stab(p)$ implies that p is the unique generic type of H, $H = H^0$, B generic in H, so $H = B^2$. □

Applications of Zilber's theorem: they are endless ! for example :

1. Generation :
   If G is a group of finite MR, $(H_i : i \in I)$ a family of definable subgroups, then $< H_i : i \in I >$ is a definable and connected subgroup.
   Indeed, for $H \leqslant_{def} G$, H is connected iff it is indecomposable.

2. Commutators :
   Let $H \leqslant G$ definable and connected, $X \subseteq G$ any subset. Then $[H, X]$ is a definable connected subgroup.
   Indeed it can be checked out that $\forall x \in G$, x^H is indecomposable, so $(x^{-1})^H$ is indecomposable, so $(x^{-1})^H x$ is indecomposable and contains 1.

3. Simplicity :

   Definition 1.6 *A group is said to be* definably simple *if it has no proper normal definable subgroups.*

   Corollary 1.7 *A non abelian group G of finite MR is simple iff it is definably simple.*

   Proof : suppose G is definably simple, in particular $G = G^0$. Let N be a proper normal subgroup of G, then $[G, N]$ is a definable normal subgroup of G, and $[G, N] \leqslant N < G$, so that $[G, N] = 1$. So $N \leqslant Z(G) = 1$ □

   Remark 1.8 *Free groups are definably simple.*

4. Categoricity :

   Theorem 1.9 *Let $< G, ., ^{-1}, 1, ... >$ be an infinite simple group of finite MR. Then G is $\aleph_1$-categorical.*

   Theorem 1.10 *Same for a field.*

Proof : (group case)
Equivalently, we will check that G is ω-stable with no Vaughtian pair:

- G ω-stable : ok.
- no Vaughtian pair : Take G^* an elementary extension of G. Let A be an infinite definable subset of G, then $A = A_1 \coprod ... \coprod A_n$, with A_i indecomposable. $a_1 \in A_1$, $a_1^{-1}A_1$, $1 \in a_1^{-1}A_1$. $\forall g \in G$, $a_1^{-1}A_1{}^g$ is indecomposable and contains 1. By Zilber's theorem, $< a_1^{-1}A_1{}^g : g \in G >= G$ (G simple) $=a_1^{-1}A_1{}^{g_1}...a_1^{-1}A_1{}^{g_m}$. As $G^* \succ G$, $G^* = G$.

□

2 Strongly minimal sets

Definition 2.1 *$\mathfrak{M}$ $\mathcal{L}$-structure. $D \subseteq M^n$ infinite definable set.*

- *D, or the formula $\phi(\overline{x}, \overline{a})$ defining it, is* minimal *if every definable subset $Y \subseteq D$ is finite or cofinite in D.*
- *D, or the formula $\phi(\overline{x}, \overline{a})$ defining it, is* strongly minimal *if ϕ is minimal in every elementary extension of $\mathfrak{M}$.*

Definition 2.2 *a theory T is strongly minimal if the formula $x = x$ is strongly minimal (i.e. every model $\mathfrak{M} \models T$ is (strongly) minimal).*

Example 2.3

1. *$\mathcal{L} = \{=\}$, theory of infinite sets.*
2. *Theory of vector space over a division ring :*
   F a division ring (i.e. a non necessarily commutative field), V an infinite vector space over F. V is a structure in the langage $\mathcal{L} = \{+, 0, \lambda_a : a \in F\}$ where $\lambda_a(x) = a.x$ for $a \in F$ and $x \in V$. Then V is a strongly minimal set (thanks to quantifier elimination of the theory of vector spaces over F in the previous langage).
3. *$\mathcal{L} = \{+, 0\}$, theory of torsion free division abelian groups (quantifier elimination)*
4. *$\mathcal{L} = \{+, -, ., 0, 1\}$, theory of algebraicly closed fields (quantifier elimination)*

Remark 2.4 *a set D is strongly minimal iff $MR(D) = 1$ and $deg_M = 1$.*

Algebraic closure :

Definition 2.5 *b is* algebraic over A *if there is a formula $\phi(x, \overline{a})$, with $\overline{a} \in A$, such that $\phi(\mathfrak{M}, \overline{a})$ is finite and $\models \phi(b, \overline{a})$.*

For $A \subseteq D$, let $\mathrm{acl}_D(A) := \{b \in D$: b is algebraic over $A\}$.
acl_M will be denoted simply acl.

Example 2.6

- *theory of infinite set :* $\mathrm{acl}(A) = A$.
- *vector spaces :* $\mathrm{acl}(A) = spam(A)$
- *theory of algebraicaly closed fields :* $\mathrm{acl}(A) =$ *algebraic closure of the subfield generated by A.*

Basic properties :

1. $A \subseteq \mathrm{acl}(A)$
2. $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.
3. if $A \subseteq B$, then $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$.
4. if $a \in \mathrm{acl}(A)$, then $a \in \mathrm{acl}(A_0)$ for some finite $A_0 \subseteq A$.
5. (Steinitz exchange principle)
suppose D strongly minimal, $A \subseteq D$, $a, b \in D$.
If $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$, then $b \in \mathrm{acl}(A \cup \{a\})$

Proof : suppose $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$. $\phi(a, b)$ where ϕ is a formula with parameters in A and $| \{x \in D \mid \phi(x, b)\} |= n$ finite. Let $\psi(y)$ be the formula asserting that $| \{x \in D \mid \phi(x, y)\} |= n$. If $\psi(y)$ defines a finite subset of D, then $b \in \mathrm{acl}(A)$ and $a \in \mathrm{acl}(A, b) \subseteq \mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$, contradiction.
Hence $\psi(y)$ defines a cofinite subset of D.
If $\{y \in D \mid \psi(a, y) \wedge \psi(y)\}$ is finite, then we are done because then $b \in \mathrm{acl}(A, a)$.
suppose toward a contradiction that : $| D \setminus \{y \in D \mid \phi(a, y) \wedge \psi(y)\} |= l$ for some $l < \omega$. Let $\chi(x)$ be the formula asserting that $| D \setminus \{y \in D \mid \phi(x, y) \wedge \psi(y)\} |= l$. If $\chi(x)$ defines a finite subset of D, then $a \in \mathrm{acl}(A)$, a contradiction. Thus $\chi(x)$ defines a cofinite subset of D. Take $a_1, ..., a_{n+1}$ elements of D such that $\models \chi(a_i)$. $\forall i \in \{1, ..., n\}$ $B_i := \{y \in D \mid \phi(a_i, y) \wedge \psi(y)\}$ is cofinite. Chose $\hat{b}$ in $\bigcap_{1 \leq i \leq n+1} B_i$. Then $\phi(a_i, \hat{b})$ for each i, so $| \{x \in D \mid \phi(x, \hat{b})\} | \geqslant n+1$, a contradiction to $\psi(\hat{b})$. □

Definition 2.7 *$A \subseteq D$.*
A is said to be independant *if $a \notin \mathrm{acl}(A \setminus \{a\})$, $\forall a \in A$. (or equivalently $a \notin \mathrm{acl}_D(A \setminus \{a\})$, $\forall a \in A$)*
If $C \subseteq D$, then A is independant over *C if $a \notin \mathrm{acl}(C \cup A \setminus \{a\})$, $\forall a \in A$.*

Definition 2.8 *A is a* basis *for $Y \subseteq D$ if $A \subseteq Y$ is independant and $\mathrm{acl}(A) = \mathrm{acl}(Y)$.*
One can easily check that basis are maximal independant subsets.

Lemma 2.9 *Let $A, B \subseteq D$ independant with $A \subseteq \mathrm{acl}(B)$.*

1. *Suppose $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cup B_0$ basis for $\mathrm{acl}(B)$ and $a \in A \setminus A_0$. Then there is $b \in B_0$ such that $A_0 \cup \{a\} \cup (B_0 \setminus \{b\})$ is a basis for $\mathrm{acl}(B)$.*
2. *$|A| \leq |B|$.*
3. *If A and B are basis for $Y \subseteq D$, then $|A| = |B|$.*

Proof :

1. Let $C \subseteq B_0$ of minimal cardinality such that $a \in \mathrm{acl}(A_0 \cup C)$. A is independant, so $|C| \geq 1$. Let $b \in C$. The exchange principle implies that $b \in \mathrm{acl}(A_0 \cup \{a\} \cup (C \setminus \{b\}))$. Thus $\mathrm{acl}(A_0 \cup \{a\} \cup (C \setminus \{b\})) = \mathrm{acl}(B)$. If $a \in \mathrm{acl}(A_0 \cup (B_0 \setminus \{b\}))$ then $b \in \mathrm{acl}(A_0 \cup (B_0 \setminus \{b\}))$, contradicting that $A_0 \cup B_0$ is a basis. Thus $A_0 \cup \{a\} \cup (B_0 \setminus \{b\})$ is independant.

2. suppose B inifinite.
$A \subseteq \cup_{B_0 \subseteq B finite}(A \cap \mathrm{acl}(B_0))$, and $A \cap \mathrm{acl}(B_0)$ is finite, so $|A| \leq |B|$.
Suppose now B finite. We will make the proof by induction on $|B|$. Suppose $|B| = n$ and $a_1, ..., a_{n+1}$ are distinct elements of A. The first point implies that there is $b_1, ..., b_n$ in B distincts such that $\{a_1, ..., a_i\} \cup (B \setminus \{b_1, ..., b_i\})$ is a basis for $\mathrm{acl}(B)$, $\forall i \leq n$. Then $\mathrm{acl}(a_1, ..., a_n) = \mathrm{acl}(B)$. But $a_{n+1} \in \mathrm{acl}(B)$ implies that A is not independant.

3. follows trivially from point 2.

□

Definition 2.10 *Let $Y \subseteq D$. The* dimension *of Y (dim(Y)) is the cardinal number of any basis of Y.*

Theorem 2.11 *(see [1] p.211) T a strongly minimal theory, $\mathfrak{M}, \mathfrak{N} \models T$. Then $\mathfrak{M} \cong \mathfrak{N}$ iff dim(M) = dim(N).*

Corollary 2.12 *T a stongly minimal theory. Then T is κ-categorical for all $\kappa \geq \aleph_1$. Moreover $I(T, \aleph_0) \leq \aleph_0$.*

Proof : $\kappa > \aleph_0 \Rightarrow$ basis of M of cardinality κ.
$\kappa = \aleph_0 \Rightarrow$ basis of M of cardinality $\leq \aleph_0$. □

<u>Back to $\aleph_1$-categorical theories :</u>

T ω-stable + no Vaughtian pair $\Rightarrow$ T κ-categorical for all $\kappa \geq \aleph_1$.

Lemma 2.13 *T ω-stable.*

1. *If $\mathfrak{M} \models T$, there is a minimal formula ϕ in $\mathfrak{M}$.*
2. *If $\mathfrak{M}$ is $\aleph_0$-saturated, then ϕ is strongly minimal.*

Proof : See [1] p. 212. □

More generally we have the following :

Definition 2.14 *T does not have the finite cover property (fcp) if for every formula $\varphi(x, y)$ without parameters there exists $n_\varphi < \omega$ such that for all a the set defined by $\varphi(x, a)$ is either infinite or has at most n_φ elements (this is axiom 4 of groups of finite Morley rank).*

Lemma 2.15 *A minimal formula in a theory without fcp is strongly minimal.*

Corollary 2.16 *T uncountably categorical $\Rightarrow I(T, \aleph_0) \leq \aleph_0$.*

Proof : Let M_0 be a prime model of T, $\phi(v)$ a stongly minimal formula with parameters from $A \subseteq M_0$ finite. $M, N \models T$.
$dim(\phi(M)/A) = dim(\phi(N)/A)$, so $M \cong N$.
Only $\aleph_0$ possibilities for that, hence $I(T, \aleph_0) \leq \aleph_0$. □

Indeed : (rather easy for strongly minimal sets alone)

Theorem 2.17 *(Baldwin-Lachlan)*
T uncountably categorical $\Rightarrow I(T, \aleph_0) = 1$ or $\aleph_0$.

3 Geometric stability theory

Definition 3.1 *X a set,* $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$.
(X, cl) is a pregeometry *if:*

1. *(closure)* $\forall A \subseteq X$, $A \subseteq \mathrm{cl}(A)$ *and* $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
2. *(monotonicity)* $\forall A \subseteq B \subseteq X$, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
3. *(finite character)* $\forall A \subseteq X$, $\mathrm{cl}(A) = \cup_{A_0 \subseteq A finite} \mathrm{cl}(A_0)$.
4. *(Steinitz exchange principle)* $\forall A \subseteq X$, $\forall a, b \in X$, *if* $a \in \mathrm{cl}(A \cup \{b\})$ *then either* $a \in \mathrm{cl}(A)$, *or* $b \in \mathrm{cl}(A \cup \{a\})$.

$A \subseteq X$ *is said to be* closed *if* $A = \mathrm{cl}(A)$.

Definition 3.2 (X, cl) *a pregeometry.*
$A \subseteq X$ *is* independant *if* $a \notin \mathrm{cl}(A \setminus \{a\})$, $\forall a \in A$.
B is a basis for $Y \subseteq X$ if $B \subseteq Y$ is independant and $Y \subseteq \mathrm{cl}(B)$ (or equivalently $\mathrm{cl}(Y) = \mathrm{cl}(B)$ *thanks to axioms 1 and 2).*

Lemma 3.3 (X, cl) *a pregeometry, $B_1, B_2 \subseteq Y \subseteq X$. Suppose that B_1, B_2 are both bases for Y. Then $|B_1| = |B_2|$. $|B_i| := dim(Y)$, the dimension of Y.*

Definition 3.4 *$A \subseteq X$. We define another closure operator by setting* $\mathrm{cl}_A(B) = \mathrm{cl}(A \cup B)$.

Now the following is is easily checked :

Lemma 3.5 *If (X, cl) is a pregeometry, then (X, cl_A) is a pregeometry.*

Definition 3.6 *$Y \subseteq X$ is said to be independant over A if Y is independant in (X, cl_A). Similarly, the dimension of Y over A is $dim(Y/A)$ = dimension of Y in (X, cl_A).*

Definition 3.7 *A pregeometry (X, cl) is a geometry if* $\mathrm{cl}(\emptyset) = \emptyset$ *and* $\mathrm{cl}(\{x\}) = \{x\}$ *for all* $x \in X$.

Starting from a pregeometry (X, cl), we can always deduce a geometry $(\hat{X}, \hat{\mathrm{cl}})$ as follows (in the same manner one goes from a vector space to the corresponding projective space, see example 3 below) :

First define $X_0 = X \setminus \mathrm{cl}(\emptyset)$. Then define an equivalence relation $\sim$ on X_0 by $a \sim b$ iff $\mathrm{cl}(\{a\}) = \mathrm{cl}(\{b\})$. By exchange principle, $a \sim b$ iff $a \in \mathrm{cl}(\{b\})$. Let $\hat{X} := X_0/\sim$. Define on $\hat{X}$ a closure operation $\hat{\mathrm{cl}}$ by $\hat{\mathrm{cl}}(A/\sim) = \{b/\sim : b \in \mathrm{cl}(A)\}$. So we get :

Lemma 3.8 *If (X, cl) is a pregeometry, then $(\hat{X}, \hat{\mathrm{cl}})$ is a geometry, which is called the* canonical geometry associated to (X, cl).

Definition 3.9 *Let (X, cl) be a pregeometry.*

1. (X, cl) *is* trivial *if* $\mathrm{cl}(A) = \cup_{a \in A} \mathrm{cl}(\{a\})$, *for all* $A \subseteq X$.
2. (X, cl) *is* modular *if for all* $A, B \subseteq X$ *finite dimensional,* $dim(A \cup B) = dim(A) + dim(B) - dim(A \cap B)$.
3. (X, cl) *is* locally modular *if* (X, cl_a) *is modular for some* $a \in X$.

Examples :

1. Pure sets :
   D an infinite set with no structure. Then D is a mininal structure, and for all $a \in D$, $\mathrm{acl}(a) = \{a\}$, and $\mathrm{acl}(\emptyset) = \emptyset$. (D, acl) is a trivial geometry.
2. $D \models Th(\mathbb{Z}, s)$ where s is the successor function. Then $\mathrm{acl}(\emptyset) = \emptyset$, $\mathrm{acl}(A) = \cup_{a \in A}\{s^n(a) : n \in \mathbb{Z}\}$, for all $A \subseteq D$. (D, acl) is a trivial pregeometry.
3. Projective geometry :
   F a division ring (i.e. a non necessarily commutative field), V an infinite **vector space** over F. We have already seen that V is a strongly minimal structure, with $acl(A) =$ span of A. Moreover the pregeometry is modular, but this is not a geometry because $\mathrm{acl}(\emptyset) = \{0\}$.

   Now consider the **projective space** $\mathbb{P}(V)$ constructed over V : a point of this space is a line of V throught the origin, i.e. $\mathbb{P}(V)$ is the quotient of $V \setminus \{O\}$ by the equivalence relation $a \sim b$ iff there is $\lambda \neq 0$ in F such that $a = \lambda.b$. Then $\mathbb{P}(V)$ is a geometry if we take for the closure of a set of lines the quotient of their linear span. Note that this is the canonical geometry associated to V, and that as a result of a general fact easily proven, the modularness of V passes to $\hat{V} = \mathbb{P}(V)$. Note also that the algebraic notion of dimension and the notion of dimension of a pregeometry coincide in the case of the vector space, but not in the case of $\mathbb{P}(V)$; but the translation between the two notions of dimensions in this last case is harmless : for any closed $W \subseteq \mathbb{P}(V)$, the dinension of W as a subspace of the projective space $\mathbb{P}(V)$ is equal to $dim(W) - 1$. The reason why we use in the geometrical context the first notion of dimension is just that we expect a point to have dimension 0 and not 1 !
4. Affine geometry :
   F and V as above, but V is equipped with his canonical structure of **affine space**

over F. On V define $\mathrm{cl}(A)$ = smallest affine space containing A. Then it is easily checked that this defines a pregeometry on V which is a geometry; also that the notion of dimension from affine geometry and from the notion of pregeometry does not coincide, but there is no harm because, just as in the projective case, for all closed $W \subseteq V$ the dimension of W as an affine subspace is equal to $dim(W) - 1$. Now take two parallel lines D_1 and D_2 : $dim(D_1) + dim(D_2) = 2 + 2 = 4$, and on the other hand $dim(D_1 \cap D_2) + dim(D_1 \cup D_2) = 0 + 3 = 3$; so this is a not a modular geometry, but locally modular, because pointing an affine space gives rise to a bijection with the associated vector space, and this bijection in our case is an isomorphism of pregeometries (i.e. a bijection which respect the closure operation).

5. Fields :
   Let K be a field ; we define a closure operation by $\mathrm{cl}(X)$ = the algebraic closure of the subfield generated by X (i.e. the subfield of the algbraic elements – in the sense of field theory – over the subfield generated by X). In order to prove that this defines a pregeometry, the only non obvious axioms to check out are that cl is idempotent and the exchange principle.
   The first one comes from a standard (but not immediatly obvious) fact from ring theory, namely that in any ring the notion of integral closure is idempotent.
   As to the exchange principle, let's take A a subset of K, $b \in K$, $< A >$ the subfield generated by A, and a an algebraic element over $k :=< A \cup \{b\} >$ such that a is not algebraic over $< A >$. Let $P_a(X)$ be the minimal polynomial of a over k; then for some $c \neq 0$, some integer n, some $Q \in \mathbb{Z}[Y_1, ..., Y_n, T, X]$ and some $\alpha_1, ..., \alpha_n$ we have:
   $$c.P_a(X) = Q(\alpha_1, ..., \alpha_n, b, X),$$
   with $deg_X(Q) > 0$. As a is not algebraic over $< A >$, we also have $m = deg_T(Q) > 0$. Take such a Q with minimal m. Then consider the polynomial $R(T) \in \mathbb{Z}(\alpha_1, ..., \alpha_n, a)[T]$ given by $R(T) = Q(\alpha_1, ..., \alpha_n, T, a) = d_0 + d_1T + ... + d_mT^m$. By minimality of m, $d_m \neq 0$, and so $R \neq 0$; but $c.P_a(a) = 0 = Q(\alpha_1, ..., \alpha_n, b, a) = R(b)$, so that $b \in \mathrm{cl}(A \cup \{a\})$ □
   Now we want to check that the notion of independance from this pregeometry and the notion of beeing "algebraically independant over a subfield" from classical field theory coincide : let us fix a subfield $M \subseteq K$, and a finite tuple $(a_1, ..., a_n)$ from K. First suppose that $(a_1, ..., a_n)$ is dependant over M in the pregeometry ; then without loss of generality a_n belongs to the algebraic closure of $M(a_1, ..., a_{n-1}) := N$, and the same argument as above (considering the minimal polynomial of a_n over N) shows that $Q(a_1, ..., a_n) = 0$, for some $Q \in M[X_1, ..., X_n]$ with $Q \neq 0$, so that $a_1, ..., a_n$ is algebraic over M.
   Conversaly suppose $a_1, ..., a_n$ is algebraic over M : then $Q(a_1, ..., a_n) = 0$ for some $Q \in M[X_1, ..., X_n]$ with $Q \neq 0$; we can assume $m = deg_{X_n}(Q) > 0$, and again choose one such polynomial with minimal m. By minimality of m, $Q(a_1, ..., a_{n-1}, X_n) \neq 0$, and so a_n is algebraic over $M(a_1, ..., a_{n-1})$ □
   So a **base** of K over a subfield M is nothing more than a **transcendental base** of K over M in the sense of field theory.

6. Algebraically closed fields :
   What is the peculiarity of algebraically closed fields (ACF) with respect to this notion of pregeometry defined for fields ? essentially that thanks to quantifier elimination for ACF in the langage of rings, we can define here the closure operation by means of model theoretic notions : indeed we have $\mathrm{acl}(A)$ = the algebraic closure (in the sense of field theory) of the subfield generated by A, i.e. $\mathrm{cl}(A) = \mathrm{acl}(A)$ and so the

two notions of closure agree in this case. Note that in a field in general we only have $\mathrm{cl}(A) \subseteq \mathrm{acl}(A)$, and moreover an unspecified field has no reason to be a minimal structure (in the language of rings) as in the case of ACF, so that acl does not even define a pregeometry in the general case.
The case ACF presents another peculiarity, namely that none of the localizations at any subset is modular, and in particular neither is the pregeometry modular, nor is it locally modular. To show that , let K be an algebraically closed field of infinite transcendance degree over its prime field. We claim that (K, acl) is not locally modular. Let k be an algebraically closed subfield of finite transcendance degree. We will show that even localizing at k the pregeometry is not modular. Let a, b, x be albraically independant over k. Let $y = ax + b$; then $dim(k(x, y, a, b)/k = 3$ and $dim(k(x, y)/k) = dim(k(a, b)/k)02$. We contradict modularity by showing that $\mathrm{acl}(k(x, y)) \cap \mathrm{acl}(k(a, b)) = k$. To see that, suppose for purpose of contradiction that $d \in (\mathrm{acl}(k(x, y)) \cap \mathrm{acl}(k(a, b))) \setminus k$; because $k(x, y)$ has transcendance degree 2 over k, we may without lost of generality assume that y is algebraic over $k(d, x)$. Let $k_1 = \mathrm{acl}(k(d))$. Then there is $p(X, Y) \in k_1[X, Y]$ an irreducible polynomial such that $p(x, y) = 0$. By model completeness, $p(X, Y)$ is still irreducible over $\mathrm{acl}(k(a, b))$. Thus $p(X, Y)$ is $\alpha(Y - aX - b)$ for some $\alpha \in \mathrm{acl}(k(a, b))$ which is impossible as then $\alpha \in k_1$ and $a, b \in k_1$.

References

[1] D. Marker. *Model theory : an introduction,* Springer, 2002