AN EQUIVARIANT DEFINABLE VERSION OF A THEOREM OF J.H.C. WHITEHEAD

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Abstract. Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field R. We consider an equivariant definable version of a theorem of J.H.C. Whitehead.

1. Introduction

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field R. General references on o-minimal structures are [2], [4], see also [14]. Examples and constructions of them can be seen in [3], [5], [11].

J.H.C. Whitehead proves a weak homotopy equivalence between CW complexes is a homotopy equivalence ([15]). Its equivariant version of it is proved by T. Matumoto ([10]) and its definable version of it proved by [1].

In this paper, we consider an equivariant definable version of the theorem of J.H.C. Whitehead.

Everything is considered in $\mathcal{N}$ and a definable map is assumed to be continuous unless otherwise stated.

Theorem 1.1. *Let G be a definably compact definable group and $\phi : (X, A) \to (Y, B)$ a definable G map between definable G CW complex pairs. If X^H, A^H and B^H are nonempty and the induced maps $\phi_* : \pi_n(X^H) \to \pi_n(Y^H)$ and $\phi_* : \pi_n(A^H) \to \pi_n(B^H)$ are bijective for $1 \le n \le \max(\dim X, \dim Y)$ and each definable subgroup H which appears as an isotropy subgroup in X or Y, then $\phi : (X, A) \to (Y, B)$ is a definable G homotopy equivalence map.*

2. Preliminaries

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset R^n \times R^m$) is a definable set. A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A definable subset X of R^n is *definably compact* if for every definable map $f : (a, b)_R \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in X, where $(a, b)_R = \{x \in R | a \le x < b\}, -\infty \le a < b \le \infty$. A definable subset X of R^n is definably compact if and only if X is closed and bounded ([13]). Note that if X is a definably compact definable set and $f : X \to Y$ is a definable map, then $f(X)$ is definably compact.

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If R is the field of real numbers $\mathbb{R}$, then for any definable subset X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general, a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1\}$ is definably compact but not compact.

Note that every definable subgroup of a definable group is closed ([12]) and a closed subgroup of a definable group is not necessarily definable. For example $\mathbb{Z}$ is a closed subgroup of $\mathbb{R}$ but not a definable subgroup of $\mathbb{R}$.

Let G be a definable group. A pair (X, ϕ) is a *definable G set* if X is a definable set and the G action $\phi : G \times X \to X$ is definable. We simply write X instead of (X, ϕ).

Let X, Y be a definable G sets. A definable map $f : X \to Y$ is a *definable G map* if for any $g \in G, x \in X, f(gx) = gf(x)$. A definable G map $f : X \to Y$ is a *definable G homeomorphism* if there exists a definable G map $h : Y \to X$ such that $f \circ h = id_Y$ and $h \circ f = id_X$. Two definable G maps $f, h : X \to Y$ are *definably G homotopic* if there exists a definable G map $H \times [0,1]_R \to Y$ such that $H(x,0) = f(x), H(x,1) = h(x)$ for all $x \in X$, where $[0,1]_R = \{x \in R | 0 \leq x \leq 1\}$. A definable G map $f : X \to Y$ is a *definably G homotopy equivalence* if there exists a definable G map $h : Y \to X$ such that $f \circ h$ is definably G homotopic to id_Y and $h \circ f$ is definably G homotopic to id_X.

Recall existence of definable quotient.

Theorem 2.1. *(Existence of definable quotient (e.g. 10. 2.18* [2]*)). Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.*

Using Theorem 2.1, if H is a definable subgroup of a definably compact definable group G, then G/H is a definable set, and the standard action $G \times G/H \to G/H$ defined by $(g, g'H) \mapsto gg'H$ of G on G/H makes G/H a definable G set.

Recall definable G CW complexes and a result on them ([6], [7]).

Definition 2.2 ([7]). *Let G be a definably compact definable group.*

(1) A definable G CW complex is a finite G CW complex $\{X, \{c_i | i \in I\}\}$ satisfying the the following three conditions.

(a) The underlying set $|X|$ of X is a definable G set.

(b) The characteristic map $f_{c_i} : G/H_{c_i} \times \Delta \to \overline{c_i}$ of each open G cell c_i is a definable G map and $f_{c_i}|G/H \times Int\ \Delta : G/H \times Int\ \Delta \to c_i$ is a deifnable G homeomorphism, where H_{c_i} is a definable subgroup of G, Δ denote a standard closed simplex, c_i is the closure of C_i in X and $Int\ \Delta$ means the interior of Δ.

(c) For each c_i, $\overline{c_i} - c_i$ is a finite union of open G cells.

(2) Let X and Y be definable G CW complexes. A cellular G map $f : X \to Y$ is definable if $f : |X| \to |Y|$ is definable.

Since G and every standard closed simplex are definably compact, every definable G CW complex is definably compact.

Let G be a definably compact definable group. A group homomorphism from G to some $O_n(R)$ is a *representation* if it is definable, where $O_n(R)$ means the nth orthogonal group of R. A *representation space* of G is R^n with the orthogonal action induced from a representation of G.

Theorem 2.3 ([6]). *Let G be a definably compact definable group. Let X be a G invariant definable subset of some representation space of G and Y a definable closed G subset of*

X. Then there exist a definable G CW complex Z in a representation space Ξ of G, a G CW subcomplex W of Z, and a definable G map $f : X \to Z$ such that:

1. *f maps X and Y definably G homeomorphically onto G invariant definable subsets Z_1 and W_1 of Z and W obtained by removing some open G cells from Z and W, respectively.*
2. *The orbit map $p : Z \to Z/G$ is a definable cellular map.*
3. *The orbit space Z/G is a finite simplicial complex compatible with $p(Z_1)$ and $p(W_1)$.*
4. *For each open G cell c of Z, $p|\bar{c} : \bar{c} \to p(\bar{c})$ has a definable section $s : p|(\bar{c}) \to \bar{c}$, where $\bar{c}$ denotes the closure of c in Z.*

Moreover, if X is definably compact, then $Z = f(X)$ and $W = f(Y)$.

Corollary 2.4. *Let G be a definably compact definable group and X a G invariant definably compact definable subset of some representation space of G. Then X is a definable G CW complex.*

Let G be a definably compact definable group, X a definable G set and Y a definable G subset of X. We say that a pair (X, Y) *admits a definable G homotopy extension* if for any definable G map from X to a definable G set Z and any definable G homotopy $F : Y \times [0, 1]_R \to Z$ with $F(y, 0) = f(y)$ for all $y \in Y$, there exists a definable G homotopy $H; X \times [0, 1]_R \to Z$ such that $H(x, 0) = f(x)$ for all $x \in X$ and $H|Y \times [0, 1]_R = F$.

Theorem 2.5 ([8])**.** *Let G be a definably compact definable group. If X is a definable G set and Y a definable close G subset of X, then (X, Y) admits a definable G homotopy extension.*

3. Proof of Theorem 1.1

O-minimal homotopy groups are defined in [1]. We use there groups instead of the classical homotopy groups

Proposition 3.1. *Let Z be a definable G set and $Y \subset X$ be a definable G CW pair such that the dimensions of whose cells do not exceed N. If for each definable subgroup H of G, Z^H is nonempty, definably connected and $\pi_n(Z^H)$ vanishes for $n < N$, then any definable G map of Y into Z is extended equivariantly on X.*

Let $\emptyset = Z_{-1} \subset Z_0 \subset \dots$ be a sequence of definable G subsets of a definable G set Z such that any definable G map $(G/H \times \Delta^n, G/H \times \partial\Delta^n) \to (Z, Z_{n-1})$ is definably G homotopic rel. G/H to a definable G map $G/H \times \Delta^n \to Z_n$ $(n = 0, 1, 2, \dots)$, where H is any definable subgroup of G.

Let $Y \subset X$ be a definable G CW subcomplex and $f_0 : X \to Z$ be a definable G map such that $f_0(Y^n) \subset Z_n$ for each $n = 0, 1, \dots$.

Lemma 3.2. *There exists a definable G homotopy $f_t : X \to Z$ rel. Y such that $f_1(X^n) \subset Z_n$, for each $n = 0, 1, 2, \dots$.*

Proof. We proceed by induction on n. We may assume that there exists a definable G homotopy $f_t^{n-1} : X^{n-1} \to Z$ rel $Y \cap X^{n-1}$ such that $f_0^{n-1} = f_0|X^{n-1}$ and $f_1^{n-1}(X^{n-1}) \subset Z_{n-1}$. Let e^n be an n cell of X which is not contained in Y and has the G characteristic map $G\sigma : G/He \times \Delta^n \to \overline{Ge} \subset X$. We define a definable G map $F'_s : (G/He) \times \Delta^n \times \{0\} \cup (G/He) \times \partial\Delta^n \times [0, 1]_R \to Z$ by $F'_s(g, s, 0) = f_0(\sigma(g, s)), s \in \Delta$

and $F_s(g,s,t) = f_t^{n-1}(\sigma(g,s)), s \in \partial\Delta^n$. By the inductive hypothesis, $F'(G/He \times \partial\Delta^n \times \{1\}) = f_1^{n-1}(G\sigma(g,s)) \subset Z_{n-1}$. Then there exists a definable G extension of F_s', $F_s : G/He \times \Delta^n \times [0,1]_R \to Z$ such that $F_s(G/He \times \Delta^n \times \{1\}) \subset Z_n$. F_s induces a definable G map of $Ge \times [0,1]_R$ into Z which is an extension of f_t^{n-1}, therefore we have a definable G homotopy $f_t^n : X^n \to Z$ rel. $X^n \cap Y$ such that $f_t|X^{n-1} = f_t^{n-1}, f_0^n = f|X^{n-1}$ and $f_1^n(X^n) \subset Z_n$. By the induction on n, we have f_t^n for any n. The map defined by $f_t : X \to Z$ by $f_t|X^n = f_t^n$ is the required definable G homotopy. □

Lemma 3.3. *Let $Z \supset C$ be a definable G set pair and H a definable subgroup of G. If C^H is nonempty and $\pi_n(Z^H, C^H)$ vanishes, then any definable G map $Gf : (G/H \times \Delta^n, G/H \times \partial\Delta^n \to (Z,C)$ is definably G homotopic rel. $G/H \times \partial\Delta^n$ to a definable G map $G/H \times \Delta^n \to C$.*

Proof. Restricting Gf to $H/H \times \Delta^n$, we have a non-equivariant definable map $f : (\Delta^n, \partial\Delta^n) \to (Z^H, C^H)$. This map is definably homotopic rel. $\partial\Delta$ to a definable map $f_1 : \Delta^n \to C^H$. Let $f_t : \Delta^n \to Z^H$ be this homotopy. Define $Gf_t : G/H \times \Delta^n \to Z$ by $Gf_t(g,s) = gf_t(s)$. Since $f_t(s) \in Z^H$, this is well-defined. Thus $Gf_0 = Gf$ and Gf_t is a definable G homotopy rel. $G/H \times \partial\Delta^n$ of Gf_0 to $Gf_1 : G/H \times \Delta^n \to C$. □

The above two lemma proves the following proposition which is a generalization of Proposition 3.1.

Proposition 3.4. *Let $Z \supset C$ be a definable G set pair and $Y \subset X$ a definable G CW complex pair such that the dimensions of whose cells do not exceed N. If for each definable subgroup H of G which appears as an isotropy subgroup of a X, C^H is nonempty and $\pi_n(Z^H, C^H)$ vanishes for each $n < N+1$, then any definable G map $(X,Y) \to (Z,C)$ is definably G homotopic rel. Y to a definable G map $X \to C$.*

Proposition 3.5. *Let $f : X \to Y$ be a definable map between definable sets. Then* $\dim f(X) \le \dim X$.

We now consider the G cellular approximation theorem. Non-equivariant case of it is studied in [9].

Lemma 3.6. *Let $f : (\Delta^k, \partial\Delta^k) \to (\Delta^n, \partial\Delta^n)$ be a definable map and $k < n$. Then f is definably homotopic rel. $f^{-1}(\partial\Delta)$ to a definable map Δ^k to Δ^n.*

Proof. By Proposition 3.5, f is not surjective, $(\Delta^k, \partial\Delta^k) \to (\Delta^n, \partial\Delta^n)$ which transforms f' to a definable map which is definably homotopic to f. □

Lemma 3.7. *Let $Z = G/H' \times \Delta^n$ and $C = G/H' \times \partial\Delta$. Then any definable map $f : (\Delta^k, \partial\Delta^k) \to (Z^H, C^H)$ is homotopic rel. $f^{-1}(C^H)$ to a definable map of Δ^k into C^H for $k < n$ and any definable subgroup H of G.*

Proof. Composite f with the projection $Z^H = (G/H')^H \times \Delta^n \to \Delta^n$. Then we have a definable map $f' : (\Delta^k, \partial\Delta^k) \to (\Delta^n, \partial\Delta^n)$ which is definably homotopic rel. $(f')^{-1}(\partial\Delta^n)$ to a definable map from Δ^k to $\partial\Delta^n$ by Lemma 3.6. This gives a definable homotopy rel. $f^{-1}(C^H)$ of f to a definable map from Δ^k to C^H. □

Proposition 3.8. *Let X be a definable G CW complex and $k \le n$. Then $\pi_k(X^H, (X^n)^H) = 0$.*

Proof. Let $f : (\Delta^k, \partial\Delta^k) \to (X^H, (X^n)^H)$ be a definable map. Let $Ge_1^m, \dots, Ge_k^m$ be G m cells of the highest dimension which intersects with $f(\Delta^k)$. The we can consider f to be a definable map $(\Delta, \partial\Delta^k)$ into $(Z^H, (X^n)^H)$, where $Z = Ge_2^m, \cup \cdots \cup Ge_k^m \cup X^{m-1}$. Since the difference between Z and C is only one cell G cell Ge_1^m, by the proof of Lemma 3.7, we have a definable homotopy rel. $f^{-1}(C^H)$ of f to a definable map $f' : \Delta^k \to C^H$, provided $k < m$. Repeating this argument, we have a definable homotopy rel. $\partial\Delta^k$ of f to a definable map $f'' : \Delta^k \to (X^n)^H$. □

By Proposition 3.8, 3.5 and 3.4, we have the following theorem.

Theorem 3.9. *Let $f : X \to Y$ be a definable G map between definable G CW complexes. Then f is definably G homotopic to a definable G map $h : X \to Y$ such that $h(X^n) \subset Y^n$.*

Lemma 3.10. *Let $\phi : C \to Z$ be a definable G map between definable G sets, and $X \supset Y$ a definable G CW pair such that the dimensions of whose cells do not exceed N. If for each definable subgroup H of G which appears as an isotropy subgroup of X, C^H and Z^H are nonempty and the induced map $\phi_* : \pi(C^H) \to \pi_n(Z^H)$ is bijiective for $n < N$ and surjective for $n = N$, then any definable G map pair $g : X \to Z, f' : Y \to C$ with $g|Y = \phi \circ f'$, there exits a definable G map $f : X \to C$ such that $f|Y = f'$ and $\phi \circ f$ definably G homotopic rel. C to g.*

Proof. Let M be the definable mapping cylinder of $\phi : C \to Z$. Then M^H coincides with the mapping cylinder of $\pi^H : C^H \to Z^H$ for each definable subgroup H of G. Thus $\pi(M^H, C^H)$ vanishes for $n < N + 1$. Hence for $n \geq 1$, we can use the exact sequence in the Hurwicz homotopy theory. Therefore we may deduce this lemma from Proposition 3.4. □

Theorem 3.11. *Let $\phi : X \to Y$ be a definable G map between definable G sets. If each of X^H, Y^H is nonempty for each definable subgroup H of G, then the following conditions are equivalent.*

(1) For each definable sungroup H of G, induced map $\phi_* : \pi_n(X^H) \to \pi_n(Y^H)$ is bijiective for $1 \leq n < N$ and surjective for $n = N$.

(2) The induced map $\phi_* : [K, X]_G^{def} \to [K, Y]_G^{def}$ is bijective for $\dim K < N$ and surjective for $\dim K = N$ for any definable G CW complex K, where $[\cdot, \cdot]_G^{def}$ denotes the set of definable G homotopy classes of definable G maps.

Proof. (1) implies (2) because of Lemma 3.10. If we take $K = G/H \times (\Delta/\partial\Delta)$, (2) implies (1). □

Proof of Theorem 1.1. Put $K = B$. Then $\phi|A$ has a definable G homotopy left inverse ψ because the induced map $\phi_* : [B, A]_G^{def} \to [B, B]_G^{def}$ is an isomorphism. By the definable G homotopy extension property, we have a definable G map $\phi' : Y \to Y$ which is definably G homotopic to the identity and satisfies $\psi'|B = \psi$. Then by Lemma 3.10, we have a definable G map $\psi'' : Y \to X$ such that $\psi''|B = \psi$ and $\phi \circ \psi'' = \psi'$ is definably G homotopic to the identity of Y. That is, ψ'' is a definable G homotopy left inverse of ϕ. Moreover we have a definable G homotopy left inverse of ψ'' and by algebraic argument, (ψ'', ψ) is a definable G homotopy inverse of $(\phi, \phi|B)$. □

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