Quantifier elimination, valuation property & preparation theorem in subanalytic geometry via transformation to normal crossings

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Abstract. This paper investigates the geometry of the expansion $\mathcal{R}_Q$ of the real field $\mathbb{R}$ by restricted quasianalytic functions. The main purpose is to establish quantifier elimination, description of definable functions by terms, the valuation property and preparation theorem (in the sense of Parusiński–Lion–Rolin). To this end, we study non-standard models $\mathcal{R}$ of the universal diagram T of $\mathcal{R}_Q$ in the language $\mathcal{L}$ augmented by the names of rational powers.

The approach we present herein makes no appeal to the Weierstrass preparation theorem, unavailable in the general quasianalytic geometry. The basic tools applied are transformation to normal crossings and decomposition into special cubes. The latter method, introduced in our article [30], combines modifications by blowing up along smooth centers with a suitable partitioning. All but one reasonings are valid in the general quasianalytic settings. That one delicate point, being evident in the classical analytic case, can be related to an affirmative answer to a natural yet open problem posed in this article, which concerns ultradifferentiable functions (to the effect that polynomials are dense in a certain Hilbert space associated with a quasianalytic Denjoy–Carleman class).

Research partially supported by KBN Grant 1P03A00527.

AMS Classification: Primary 03C64, 03C10, 14P15; Secondary 26E10, 32B20, 32S45.

Key words: quasianalytic functions, special cubes, special modifications, analytically independent infinitesimals, active and non-active infinitesimals, valuation property, quantifier elimination, preparation theorem.

Via an analysis of $\mathcal{L}$-terms and infinitesimals, we prove the valuation property for functions given by $\mathcal{L}$-terms, and next the exchange property for substructures of a given model $\mathcal{R}$. Our proofs are based on the concepts of analytically independent as well as active and non-active infinitesimals, introduced in this article. Further, through model-theoretic compactness, quantifier elimination for T is established. Hence the universal theory T is complete and o-minimal, and $\mathcal{R}_Q$ is its prime model. Under the circumstances, every definable function is given piecewise by $\mathcal{L}$-terms, and therefore the previous results concerning $\mathcal{L}$-terms generalize immediately to definable functions. Thus in particular, we obtain in this fashion the valuation property and preparation theorem for subanalytic functions. Finally, a subanalytic version of Puiseux's theorem with parameter is demonstrated.

1. Introduction. As in our previous paper [30], we begin — following Bierstone–Milman [5] — by fixing a family $\mathcal{Q} = (\mathcal{Q}_m)_{m\in\mathbb{N}}$ of sheaves of local $\mathbb{R}$-algebras of smooth functions on $\mathbb{R}^m$. For each open subset $U \subset \mathbb{R}^m$, $\mathcal{Q}(U) = \mathcal{Q}_m(U)$ is thus a subalgebra of the algebra $\mathcal{C}^\infty_m(U)$ of real smooth functions on U. By a Q-function we mean any function $f \in \mathcal{Q}(U)$. Similarly,

$$f = (f_1, \dots, f_k) : U \longrightarrow \mathbb{R}^k$$

is called a Q-mapping if so are its components $f_1, \dots, f_k$. We impose on this family of sheaves the following six conditions:

1. each algebra $\mathcal{Q}(U)$ contains the restrictions of polynomials;
2. $\mathcal{Q}$ is closed under composition, i.e. the composition of Q-mappings is a Q-mapping (whenever it is well defined);
3. $\mathcal{Q}$ is closed under inverse, i.e. if $\varphi : U \longrightarrow V$ is a Q-mapping between open subsets $U, V \subset \mathbb{R}^m$, $a \in U$, $b \in V$ and if $\partial\varphi/\partial x(a) \neq 0$, then there are neighbourhoods U_a and V_b of a and b, respectively, and a Q-diffeomorphism $\psi : V_b \longrightarrow U_a$ such that $\varphi \circ \psi$ is the identity mapping on V_b;
4. $\mathcal{Q}$ is closed under differentiation;
5. $\mathcal{Q}$ is closed under division by a coordinate, i.e. if $f \in \mathcal{Q}(U)$ and $f(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m) = 0$ as a function in the variables x_j, $j \neq i$, then $f(x) = (x_i - a_i)g(x)$ with some $g \in \mathcal{Q}(U)$;

6. $\mathcal{Q}$ is quasianalytic, i.e. if $f \in \mathcal{Q}(U)$ and the Taylor series $\widehat{f}_a$ of f at a point $a \in U$ vanishes, then f vanishes in the vicinity of a.

By means of Q-mappings, one can build, in the ordinary manner, the category Q of Q-manifolds and Q-mappings, which is a subcategory of that of smooth manifolds and smooth mappings. Similarly, Q-semianalytic and Q-subanalytic sets can be defined. Consider now the the expansion $\mathcal{R}_Q$ of the real field $\mathbb{R}$ by restricted Q-functions, i.e. functions of the form

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [-1,1]^m \\ 0, & \text{otherwise} \end{cases}$$

where $f(x)$ is a Q-function in the vicinity of the compact cube $[-1,1]^m$. The structure $\mathcal{R}_Q$ is model complete and o-minimal (cf. [37, 36, 30]). The definable subsets in $\mathcal{R}_Q$ coincide with those subsets in $\mathbb{R}^m$ that are Q-subanalytic in a semialgebraic compactification of $\mathbb{R}^m$

In order to investigate $\mathcal{L}$-terms of the structure $\mathcal{R}_Q$, we shall consider the universal diagram T of the structure $\mathcal{R}_Q$ in the language $\mathcal{L}$ of restricted quasianalytic functions augmented by the names of rational powers (i.e. the set of all universal $\mathcal{L}$-sentences that are true in $\mathcal{R}_Q$). We impose the ordinary postulates on the reciprocal function $1/x$ and roots, namely

$$x \cdot 1/x = 1 \quad \text{if} \quad x \neq 0 \quad \text{and} \quad 1/x = 0 \quad \text{if} \quad x = 0,$$

$$(\sqrt[n]{x})^n = x \quad \text{if} \quad x \geq 0 \quad \text{and} \quad \sqrt[n]{x} = 0 \quad \text{if} \quad x < 0.$$

The role of the function symbols attached to the language of restricted quasianalytic functions can be explained as follows. The reciprocal function $1/x$ is indispensable when inverting transformation by blowing up, and roots are indispensable when inverting substitution of powers.

Our analysis of $\mathcal{L}$-terms and infinitesimals in non-standard models of T is based on transformation to normal crossings and decomposition into special cubes. The latter method, introduced in our article [30] for relatively compact Q-semianalytic sets, combines modifications by blowing up along smooth centers with a suitable partitioning. It carries over, as shown in Section 2, to any sets described by $\mathcal{L}$-terms, both in the structure $\mathcal{R}_Q$ and in non-standard models of T. Generally, in our non-standard analysis, we are interested in finding suitable special modifications which take into account a tuple of infinitesimals. A counterpart of this problem in the classical subanalytic geometry is to find a finite (or locally finite) family of suitable modifications, whose targets cover a space under consideration. The major part of Sections 2 and 3 will be concerned with such problems.

In Section 2 the notion of analytically independent infinitesimals is introduced. There we deal, inter alia, with the desingularization of $\mathcal{L}$-terms by special modifications and a modification of a Q-function to a regular one with respect to one distinguished variable. Sections 3 and 4 are devoted to the study of our concept of an active and a non-active infinitesimal, which is crucial for the whole work. An infinitesimal μ is active over a finite set λ of infinitesimals if, for certain $\mathcal{L}$-term $t(x, y)$ which is linear with respect to the variable y, the valuation of $t(\lambda, \mu)$ is not in the valuation group of the structure $\langle \lambda \rangle$ generated by the set λ.

In the third section we treat the case of a non-active infinitesimal. We consider certain modifications, which are linear with respect to the distinguished variable that corresponds to a non-active infinitesimal. Most of the theorems from this section ensure that a Q-function or an $\mathcal{L}$-term in question can be improved after applying such modifications; these are for instance: the theorem on behaviour of an $\mathcal{L}$-term at a non-active infinitesimal or the exchange property for a non-active infinitesimal. The latter amounts to solving, with respect to one distinguished variable, say y, an equation given by an $\mathcal{L}$-term. It is worth pointing out that we have reduced the problem of solving such an equation to that of solving a linear equation and to the implicit function theorem.

Let us mention that a linearization of an analytic equation with respect to one distinguished variable y can be achieved in the classical analytic geometry by means of the Weierstrass preparation theorem and the Abhyankar–Jung theorem (cf. [1, 19, 34]). Whereas the former reduces, after blowing up the remainder variables x, a given equation to a polynomial one with respect to y, the latter allows one, after modifying the remainder variables x by transformation to normal crossings and substitution of powers, to decompose the polynomial into a finite product of linear factors of the form $y - a_i(x)$ with some analytic functions $a_i(x)$ (see e.g. [35, 38, 32, 34]).

We wish to emphasize the linear character of the very definition of a non-active infinitesimal as well as of modifications with respect to a distinguished variable, on which our theory has been built. Not only does it enable us to avoid the Weierstrass preparation theorem, but also plays a vital role in the proof of the valuation property. Besides, transformation to normal crossings, taking into account the coordinate functions, propagates linearly through the valuation group.

An active infinitesimal cannot be handled in a similar way as a non-active one. Section 4 is devoted to the study of an active infinitesimal. We prove, inter alia, the theorem on behaviour of an $\mathcal{L}$-term at a regular sequence of infinitesimals, exchange property for a regular sequence, valuation property for $\mathcal{L}$-terms and, eventually, the general exchange property for substructures of a given model $\mathcal{R}$ of the universal theory T. It means that span operation on the family of all subsets of the model $\mathcal{R}$ is a pregeometry on $\mathcal{R}$. This enables us to introduce a general notion of independence for subsets of $\mathcal{R}$ as well as — by analogy with the dimension of vector spaces or with the transcendence degree of field extensions — the notion of rank and relative rank for substructures of $\mathcal{R}$. Following Zariski–Samuel [43], Chap. II, § 12, we express the former as the notion of a free set, which has proven to coincide, for the case of a set of infinitesimals, with our notion of analytical independence.

Our proof of the exchange property for a regular sequence includes a reasoning wherein a diagonal formal series is involved. It is evident in the classical case of analytic functions, but very delicate in the quasianalytic settings. This subtle point is a special case of the following problem:

Let f be a Q-function at $0 \in \mathbb{R}^k$ with Taylor series $\hat{f}$. Split the set $\mathbb{N}^k$ of exponents into two disjoint subsets A and B, $\mathbb{N}^k = A \cup B$, and decompose the formal series $\hat{f}$ into the sum of two formal series G and H, supported by A and B, respectively. Do there exist two Q-functions g and h at $0 \in \mathbb{R}^k$ with Taylor series G and H, respectively?

The above can be related to an affirmative answer to a natural yet open problem posed in this article (see also our article [31]), which concerns ultradifferentiable functions, namely whether polynomials are dense in a certain Hilbert space associated with a quasianalytic Denjoy–Carleman class. This Hilbert space, introduced recently by Thilliez [39], is an analogue of Sobolev spaces of infinite order of type l_2, which allows one to handle simultaneously an infinite number of derivatives.

In Section 5 we apply the foregoing results along with model-theoretic compactness to the problem of inversion of general special modifications. It will turn out that the requirement for the inverse mapping ψ of a special modification φ we impose in Section 2 is no constraint on special cubes at all, because it is fulfilled by every special modification. We need this inversion theorem and Gabrielov's closure theorem in order to establish quantifier elimination for the theory T. In fact, we shall prove that if a set E is described by $\mathcal{L}$-terms, so is its projection. Our proof makes use of model-theoretic compactness again. Consequently, the theory T is complete and o-minimal, and the standard model $\mathcal{R}_Q$ is its prime model.

Quantifier elimination and an elementary universal axiomatization for the expansion of the real field by restricted analytic functions were established by van den Dries–Macintyre–Marker in the language augmented by the names of the reciprocal function $1/x$ and roots (see [11], and also [9, 26, 27]). Recently, Rambaud [36] has proved a theorem of this kind for the quasianalytic setting. The author investigates families of so-called stable infinitesimals, which play a key role in his proof. He makes use of certain desingularization algorithms and an embedding of a model under consideration into an ultrapower of the real field.

The fact that a universal theory T admits quantifier elimination has weighty model-theoretic and geometric consequences, implying in particular, that every definable function is piecewise given by terms (a theorem of Herbrand [17]). Therefore all the results we have previously proved for $\mathcal{L}$-terms remain valid for definable functions. Section 6 provides a brief exposition of several applications. First, the valuation property for definable functions is stated. Hence, through model-theoretic compactness and definable choice (which is available for o-minimal structures), one can derive the preparation theorem in the sense of Parusiński–Lion–Rolin (see [14, 28]). Finally, we demonstrate a subanalytic version of Puiseux's theorem with parameter as well as its immediate consequence, piecewise uniform asymptotics.

We conclude this section with some useful remarks.

Remarks. 1) Let Φ be an arbitrary semialgebraic diffeomorphism of $\mathbb{R}^m$ onto $(-1,1)^m$. In view of Gabrielov's complement theorem, $E \subset \mathbb{R}^m$ is a definable subset of the structure $\mathcal{R}_Q$ iff $\Phi(E)$ is a (bounded) Q-subanalytic subset in $\mathbb{R}^m$.

2) Condition 4. imposed on the family of quasianalytic functions is a direct consequence of condition 5. We must show that if $\widetilde{f}(x)$ is a restricted Q-function, so is each partial derivative $\partial f/\partial x_i(x)$. We check it for $i = m$.

Since the function $f(x)$ is Q-analytic in the vicinity of $[-1,1]^m$, the function

$$g(x,y) := f(x_1,\dots,x_{m-1},x_m+y) - f(x)$$

is Q-analytic in the vicinity of $[-1,1]^m \times [-\delta,\delta] \subset \mathbb{R}^{m+1}$ with some $\delta > 0$. It follows from condition 5 that $g(x,y) = yh(x,y)$ for a function $h(x,y)$ Q-analytic in the vicinity of $[-1,1]^m \times [-\delta,\delta]$. Hence

$$\partial f/\partial x_m(x) = \partial g/\partial y(x,0) = h(x,0)$$

for x in the vicinity of $[-1,1]^m$, which is the desired result.

3) Under condition 6 of quasi-analyticity, condition 5 for convex subsets U is equivalent to the following one: if the Taylor series of $f \in \mathcal{Q}(U)$ at a point $a \in U$ is divisible by $x_i - a_i$, then $f(x) = (x_i - a_i)g(x)$ with some $g \in \mathcal{Q}(U)$.

4) Although it is well-known that every model $\mathcal{R}$ of the theory T is a real closed field (see e.g. [25]), we shall not use this fact in our approach.

5) The interpretation $f^{\mathcal{R}}$ of each restricted Q-function $\widetilde{f}$ in any model $\mathcal{R}$ of the theory T is an infinitely differentiable function and we have

$$\partial f^{\mathcal{R}}/\partial x_i = (\partial f/\partial x_i)^{\mathcal{R}}.$$

Indeed, if $f(x)$ is Q-analytic in the vicinity of $[-1,1]^m$, then (as in Remark 2) we have

$$f(x_1,\dots,x_{m-1},x_m+y) - f(x) = yh(x,y) = y[h(x,0) + yk(x,y)]$$

for certain functions $h(x,y)$ and $k(x,y)$ which are Q-analytic in the vicinity of $[-1,1]^m \times [-\delta,\delta]$ with some $\delta > 0$. Hence

$$\partial f^{\mathcal{R}}/\partial x_m(x) = h^{\mathcal{R}}(x,0) = (\partial f/\partial x_i)^{\mathcal{R}}(x),$$

as asserted.

6) Similarly, making use of Taylor formula, we have

$$f(x_1,\dots,x_{m-1},x_m+y) - \sum_{j=0}^{n} 1/j! \cdot \partial^j f/\partial x_m^j(x) \cdot y^j = y^{n+1} \cdot h(x,y)$$

for a function $h(x,y)$ Q-analytic in the vicinity of $[-1,1]^m \times [-\delta,\delta]$ with some $\delta > 0$.

2. Special cubes and analytically independent infinitesimals. We proved in [30] the following

Theorem on Decomposition into Special Cubes. *Every bounded Q-semianalytic subset E in $\mathbb{R}^m$ is a finite union of special cubes S_j, i.e. subsets in $\mathbb{R}^m$ of the form*

$$S_j = \varphi_j((-1,1)^{d_j}),$$

where $\varphi_j(x)$ is a Q-mapping from the vicinity of $[-1,1]^{d_j}$ into $\mathbb{R}^m$ such that the restriction of φ_j to $(-1,1)^{d_j}$ is a diffeomorphism onto S_j.

Furthermore, each of those special cubes S_j and the inverse mappings

$$\psi_j : S_j \longrightarrow (-1,1)^{d_j}$$

to the associated Q-diffeomorphisms φ_j are given piecewise by terms in the language of restricted Q-analytic functions augmented by the name of the reciprocal function $1/x$. $\diamondsuit$

Remark 1. The inverse mappings ψ_j to the diffeomorphisms φ_j are given piecewise by terms in the language of restricted Q-functions augmented by the name of the reciprocal function $1/x$, because — roughly speaking — the mappings ψ_j have been locally built in the process of blowing up as a successive superposition of restricted Q-functions and of the reciprocal function $1/x$ off the zero argument.

Corollary 1. *Let $F \subset \mathbb{R}^m$ be a bounded subset described by $\mathcal{L}$-terms and $t(x) = t(x_1, \ldots, x_m)$ be an $\mathcal{L}$-term. Then the part of the graph of $t(x)$ lying over F is a finite union of special cubes S_i in $\mathbb{R}^m \times \mathbb{R}$ of the form*

$$S_i = \varphi_i((-1,1)^{d_i}),$$

where $\varphi_i(x)$ is a Q-mapping from the vicinity of $[-1,1]^{d_i}$ into $\mathbb{R}^m \times \mathbb{R}$ such that the restriction of φ_i to $(-1,1)^{d_i}$ is a Q-diffeomorphism onto S_i.

Furthermore, each of those special cubes S_i and the inverse mappings $\psi_i : S_i \longrightarrow (-1,1)^{d_i}$ to φ_i are given piecewise by $\mathcal{L}$-terms. $\diamondsuit$

Remark 2. Decomposition into special cubes yields the above corollary according to the following observation. After adding new variables (one for each occurrence of a function symbol involved in a given $\mathcal{L}$-term), the graph of this term and the sets described by a finite number of $\mathcal{L}$-terms are bijective projections of certain Q-semianalytic subsets, and the inverse mapping to those projections are given by $\mathcal{L}$-terms. Note that if a subset contained in the domain of such a projection is described by $\mathcal{L}$-terms, so is its image under this projection. In this fashion, techniques related to Q-semianalytic sets can be adapted to the sets described by $\mathcal{L}$-terms.

Remark 3. Given an $\mathcal{L}$-term $t(x) = t(x_1, \ldots, x_m)$, there exists a partitioning of $\mathbb{R}^m$ into finitely many Q-submanifolds described by $\mathcal{L}$-terms, such that the restriction of the function given by $t(x)$ to each of these Q-submanifolds is smooth (i.e. C^∞).

Unless otherwise stated we shall deal only with special cubes (S, φ) such that

- φ is a Q-mapping in the vicinity of $[-1,1]^d$ (or sometimes $[0,1]^d$) which is a diffeomorphism of $(-1,1)^d$ onto S;
- the inverse mapping ψ to this diffeomorphism is given piecewise by $\mathcal{L}$-terms.

It will turn out that the above requirement for the inverse mapping ψ is no constraint on special cubes at all, because it is fulfilled by every special cube (Proposition 2 from Section 5). We shall also consider special cubes described above, but which are the diffeomorphic images of arbitrary cubes in $\mathbb{R}^d$, and especialy of the cubes $(0,1)^d$.

We may regard a special cube $\varphi : (0,1)^d \longrightarrow S$ as a kind of modification of the set S. When we look at the special cube (S, φ) in this manner, we shall call φ a special modification.

We now fix a model $\mathcal{R}$ of the universal theory T in the language $\mathcal{L}$. Every $\mathcal{L}$-substructure of $\mathcal{R}$ is a model of T. We always regard the standard model $\mathcal{R}_Q$ as a substructure of $\mathcal{R}$.

Since the decompositions into special cubes we deal with are described by $\mathcal{L}$-terms (both a special modification φ and its inverse ψ), they are preserved by passage to any model $\mathcal{R}$ of T:

$$E^{\mathcal{R}} = \bigcup_j S_j^{\mathcal{R}} \quad \text{and} \quad (\operatorname{graph} t(x) \cap (F \times \mathbb{R}))^{\mathcal{R}} = \bigcup_j S_j^{\mathcal{R}}.$$

For simplicity of notation, we shall usually omit the superscript $^{\mathcal{R}}$ referring to the interpretations in a model $\mathcal{R}$, which will not lead to confusion.

We now turn to an analysis of infinitesimals of the model $\mathcal{R}$. We say that infinitesimals $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{R}$ are analytically dependent, if λ lie in a special cube $S = \varphi((0,1)^d)$ with $d < m$. We call infinitesimals λ analytically independent if they are not analytically dependent. Analytical independence is preserved, of course, under permutation of infinitesimals. We say that a subset A in $\mathcal{R}$ is analytically independent if every finite subset A in $\mathcal{R}$ consists of analytically independent infinitesimals. If $A \subset B$ and the set B is analytically independent, so is A.

For any subset $A \subset \mathcal{R}$, $\langle A \rangle$ denotes the substructure of $\mathcal{R}$ generated by A. Every finitely generated model of T has, of course, a finite, analytically independent set of generators.

The convex hull of $\mathbb{R}$ in $\mathcal{R}$ is a valuation ring V of bounded (with respect to $\mathbb{R}$) elements in $\mathcal{R}$; its maximal ideal $\mathfrak{m}$ consists of all infinitesimals in $\mathcal{R}$. The valuation v induced by V is called the standard valuation on the field $\mathcal{R}$; its value group $\Gamma_{\mathcal{R}}$ is a $\mathbb{Q}$-vector space. In order to investigate the valuation v, we shall need several results about Q-functions, stated and proved in this and the next section.

Now we state yet another corollary to theorem on decomposition into special cubes, which is a direct consequence of Corollary 1, applied to the graph of a given term $t(x)$.

Corollary 2. *Desingularization of an $\mathcal{L}$-term:*
Consider an $\mathcal{L}$-term $t(x)$ and positive analytically independent infinitesimals $\lambda = (\lambda_1, \ldots, \lambda_m)$. If $t(\lambda)$ is bounded, then there exist a special cube $S = \varphi((0,1)^m) \subset \mathbb{R}^m$, a Q-function $f(x')$ in the vicinity of $[0,1]^m$, positive infinitesimals $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ such that $\lambda = \varphi(\lambda') \in S$ and

$$t^{\varphi}(x') := t(\varphi(x')) = f(x') \quad \textit{for all} \;\; x' \in (0,1)^m.$$

$\diamondsuit$

We can rephrase the conclusion of Corollary 2 as follows.

One can find a special modification

$$\varphi : (0,1)^m \longrightarrow \mathbb{R}^m \quad \textit{with} \quad \lambda = \varphi(\lambda') \;\; \textit{for some} \;\; \lambda' \in (0,1)^m$$

such that the superposition $f := t \circ \varphi$ extends to a Q-function in the vicinity of $[0,1]^m$; in particular we have $t(\lambda) = f(\lambda')$. $\diamondsuit$

The next theorem will be crucial for investigation of y-regular Q-functions. Its proof makes use of the noetherianity of the rings of formal power series.

Proposition. *Let $f_n(x) : [-1,1]^m \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, be restricted Q-functions, not all of which vanish, and let $\lambda = (\lambda_1, \dots, \lambda_m)$ be positive analytically independent infinitesimals. Then one can find a special cube*

$$S = \varphi((0,1)^m) \subset \mathbb{R}^m,$$

with special modification $\varphi(x)$ being Q-analytic in the vicinity of the cube $[0,1]^m$ and a composite of successive blowings-up, and positive infinitesimals $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ such that $\lambda = \varphi(\lambda') \in S$ and

$$f_n^{\varphi}(x') := (f_n \circ \varphi)(x') = x'^{\alpha}\, u_n(x'), \quad n \in \mathbb{N},$$

in the vicinity of $[0,1]^m$, where $\alpha \in \mathbb{N}^m$, the functions $u_n(x')$ are Q-analytic in the vicinity of $[0,1]^m$ and $u_k(0) \neq 0$ for a certain $k \in \mathbb{N}$.

Remark 4. Note that the inverse mapping $\psi : S \longrightarrow (0,1)^m$ to the special modification φ is given piecewise by terms in the language of restricted Q-functions augmented by the name of the reciprocal function $1/x$. Such a special modification φ can be achieved by transformation to normal crossings by blowing up combined with a suitable partitioning on each successive stage of the process of blowing up (according to our method of decomposition into special cubes, presented in [30]).

For the proof, consider the ideal $\mathcal{I} \subset \mathbb{R}[[x]]$ generated by the Taylor series at zero $\widehat{f_n}(x)$ of the functions $f_n(x)$ and take generators $\widehat{f_1}, \dots, \widehat{f_N}$ of $\mathcal{I}$. One can simultaneously transform by blowing up the functions

$$f_1(x), \dots, f_N(x), x_1, \dots, x_m$$

to normal crossings, so that

$$f_n^{\varphi}(x') = x'^{\beta_n}\, v_n(x'),\ v_n(0) \neq 0 \quad \text{for} \quad n = 1, \dots, N,$$

and the exponents $\beta_1, \dots, \beta_N$ are totally ordered with respect to the induced partial ordering from $\mathbb{N}^m$ (i.e. $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$ for all $i = 1, \dots, m$) — see e.g. [2, 5]. Putting

$$\alpha := \beta_k = \min\{\beta_1, \dots, \beta_N\},$$

we see that all the Taylor series at zero $\widehat{f_n^\varphi}(x')$ are divisible by x'^α, whence so are the functions $f_n^\varphi(x')$ (by Condition 5 imposed on our family of Q-functions, which asserts that the quasianalytic family of functions is closed under division by a coordinate; see also Remark 3 from Section 1). The conclusion can thus be achieved by our method of decomposition into special cubes [30], when one takes into account the following two observations:

- when transforming to normal crossings by blowing up, the successive inverse images of the infinitesimals λ lie on no center of the successive blowings-up, because they continue to be analytically independent;
- the final inverse image under the transformation of each orthant is a union of orthants, so that one may assume that the inverse image λ' of λ lies in the first orthant. $\diamondsuit$

By the y-order $\operatorname{ord} f(x,y)$ of a Q-function $f(x,y)$ at zero we mean the smallest non-negative integer $n \in \mathbb{N}$ for which $\partial^n f/\partial y^r(0,0) \neq 0$, if such integers exist, or ∞ otherwise. We say that the function $f(x,y)$ is y-regular at zero, if $\operatorname{ord} f(x,y) < \infty$, i.e. $f(0,y) \not\equiv 0$. A useful fact, which is an immediate consequence of postulate 5 imposed on the family of Q-functions, will be stated below.

Lemma. *If $f(x,y) : [-1,1]^m \times [-1,1] \longrightarrow \mathbb{R}$ is a restricted Q-function such that each partial derivative*

$$\partial^n f/\partial y^n(x,0) \quad \text{for} \ \ n = 0,1,2,\dots$$

is divisible by x^α, then so is the function $f(x,y)$. $\diamondsuit$

Hence and by the foregoing proposition applied to the sequence of Q-functions $f_n(x) := \partial^n f/\partial y^n(x,0)$ for $n = 0,1,2,\dots$, we obtain

Corollary 1. *Modification of a Q-function to y-regular one:*
Let $f(x,y) : [-1,1]^m \times [-1,1] \longrightarrow \mathbb{R}$ be a restricted Q-function, $f \not\equiv 0$, and $\lambda = (\lambda_1,\dots,\lambda_m)$ be positive analytically independent infinitesimals. Then one can find a special cube $S = \varphi((0,1)^m) \subset \mathbb{R}^m$ with φ being a composite of successive blowings-up, and positive infinitesimals $\lambda' = (\lambda'_1,\dots,\lambda'_m)$ such that $\lambda = \varphi(\lambda') \in S$ and

$$f^\varphi(x',y) := f(\varphi(x'),y) = x'^\alpha g(x',y)$$

in the vicinity of $[0,1]^{m+1}$, where $\alpha \in \mathbb{N}^m$ and $g(x',y)$ is a Q-function y-regular at $(0,0) \in \mathbb{R}^m \times \mathbb{R}$. $\diamondsuit$

Corollary 2. *Consider infinitesimals* $\lambda = (\lambda_1, \dots, \lambda_m)$ *and* μ. *If* $v(\mu) \notin \Gamma_{\langle\lambda\rangle}$, *then for any Q-function* $f(x, y)$ *at* $(0, 0) \in \mathbb{R}^m_x \times \mathbb{R}_y$ *we have*

$$v(f(\lambda, \mu)) \in \Gamma_{\langle\lambda\rangle} \oplus \mathbb{Q} \cdot v(\mu).$$

Indeed, we may assume that the infinitesimals λ are analytically independent. In view of Corollary 1, we can assume that the function $f(x, y)$ is y-regular at zero, say, of y-order n. Then, in the vicinity of zero, we have

$$f(x, y) = f_0(x) + f_1(x)y + \dots f_{n-1}(x)y^{n-1} + f_n(x, y)y^n,$$

where $f_0(x), \dots, f_{n-1}(x), f_n(x, y)$ are Q-functions at zero and $f_n(0, 0) \neq 0$. By the assumption of the corollary, the values

$$v(f_0(\lambda)),\ v(f_1(\lambda)\mu),\ \dots,\ v(f_{n-1}(\lambda)\mu^{n-1}),\ v(f_n(\lambda, \mu)\mu^n) = v(\mu^n)$$

are pairwise distinct, and thus the assertion follows. ♢

3. Active and non-active infinitesimals. We say that an infinitesimal μ is non-active over infinitesimals $\lambda = (\lambda_1, \dots, \lambda_m)$ if for each $\mathcal{L}$-term $t(x)$ we have

$$v(\mu - t(\lambda)) \in \Gamma_{\langle\lambda\rangle}.$$

Otherwise, the infinitesimal μ is called active over λ. It is clear that if μ is non-active over λ, so is the infinitesimal $\mu' = s(\lambda)\mu + t(\lambda)$ that is the value at (λ, μ) of any y-linear $\mathcal{L}$-term.

Theorem on lowering y-order. *Let* $f(x, y)$ *be a Q-function,* y*-regular at* $(0, 0) \in \mathbb{R}^m_x \times \mathbb{R}_y$ *of* y*-order* $n > 0$, $\lambda = (\lambda_1, \dots, \lambda_m)$ *be positive analytically independent infinitesimals and* μ *a positive infinitesimal, non-active over* λ. *Then there exist a special cube* $S = \varphi((0, 1)^m) \subset \mathbb{R}^m$, *a Q-function* $\omega(x', y')$ *in the vicinity of* $[0, 1]^m \times \mathbb{R}$ *being linear with respect to the last variable with* $\omega(0, 0) = 0$, *a Q-function* $g(x', y')$ *in the vicinity of* $[0, 1]^{m+1}$ *of* y'*-order* $< n$, *positive infinitesimals* $\lambda' = (\lambda'_1, \dots, \lambda'_m)$, *a positive infinitesimal* μ', *non-active over* λ', *and* $\alpha \in \mathbb{N}^m$ *such that* $\lambda = \varphi(\lambda') \in S$, $\mu = \omega(\lambda', \mu')$ *and*

$$f^\sigma(x', y') := f(\varphi(x'), \omega(x', y')) = x'^\alpha g(x', y')$$

in the vicinity of $[0, 1]^{m+1}$; *here* $\sigma(x', y') := (\varphi(x'), \omega(x', y'))$.

Our proof starts with the observation that the implicit function theorem yields a Q-function $\chi(x)$ at $0 \in \mathbb{R}^m$ such that

$$\partial^{n-1} f / \partial y^{n-1}(x, \chi(x)) = 0 \quad \text{and} \quad \chi(0) = 0.$$

Making the y-linear change of variables $y' = y - \chi(x)$, the infinitesimal $\mu' := \mu - \chi(\lambda)$ remains non-active over λ. Therefore, one may assume that

$$\partial^{n-1} f / \partial y^{n-1}(x, 0) \equiv 0.$$

Since μ is non-active over λ, $v(\mu) = v(t(\lambda))$ for some $\mathcal{L}$-term $t(x)$. Via decomposition into special cubes, there exist a special cube $S = \varphi((0,1)^m) \subset \mathbb{R}^m$ such as described in the theorem, a Q-function $\xi(x)$ in the vicinity of $[0,1]^m$ and positive infinitesimals $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ such that

$$\lambda = \varphi(\lambda') \quad \text{and} \quad t(\lambda) = \xi(\lambda').$$

Through transformation to normal crossings by blowing up, one can also assume that $\xi(x)$ is normal crossings at zero, whence

$$v(\mu) = v(t(\lambda)) = v((\lambda')^{\alpha})$$

for some multi-index $\alpha \in \mathbb{N}^m$. We are thus reduced to the case where $v(\mu) = v(\lambda^{\alpha})$; then $\mu = (c + \epsilon)\lambda^{\alpha}$ with some $c \in \mathbb{R}$ and an infinitesimal ϵ.

Consider now the y-linear change of variables $y' = y - cx^{\alpha}$ and put

$$\mu' := \mu - c\lambda^{\alpha} = \epsilon\lambda^{\alpha};$$

obviously, $v(\mu') > v(\mu)$. Then

$$f(x, y) = f(x, y' + cx^{\alpha}) =: g(x, y')$$

and

$$\partial^{n-1} g / \partial (y')^{n-1}(x, y') = \partial^{n-1} f / \partial y^{n-1}(x, y' + cx^{\alpha}).$$

Note that

$$\partial^{n-1} g / \partial (y')^{n-1}(x, 0) = \partial^{n-1} f / \partial y^{n-1}(x, cx^{\alpha}) \not\equiv 0,$$

because, by the initial reduction, $y = 0$ is a unique solution near zero of the equation $\partial^{n-1} f / \partial y^{n-1}(x, y) = 0$. Again, through transformation to normal crossings by blowing up, one can assume that the function $\partial^{n-1}g/\partial(y')^{n-1}(x,0)$ is normal crossings at $0 \in \mathbb{R}^m$, i.e.

$$\partial^{n-1}g/\partial(y')^{n-1}(x,0) = u(x)x^{\beta}$$

for some $\beta \in \mathbb{N}^m$ and a Q-function $u(x)$ at $0 \in \mathbb{R}^m$ with $u(0) \neq 0$.

We now show that $\mu'' := \mu'/\lambda^{\beta}$ is an infinitesimal too. Indeed, we have

$$v(\lambda^{\beta}) = v(\partial^{n-1}g/\partial(y')^{n-1}(\lambda,0)) = v(\partial^{n-1}f/\partial y^{n-1}(\lambda, c\lambda^{\alpha})).$$

Since $\partial^n f/\partial y^n(0,0) \neq 0$, we get

$$\partial^{n-1}f/\partial y^{n-1}(\lambda,\mu) = \partial^{n-1}f/\partial y^{n-1}(\lambda,0) + \text{unit}\cdot\mu = \text{unit}\cdot\mu$$

and

$$\partial^{n-1}f/\partial y^{n-1}(\lambda,\mu) = \partial^{n-1}f/\partial y^{n-1}(\lambda,c\lambda^{\alpha}) + \text{unit}\cdot\mu'.$$

Hence

$$\partial^{n-1}f/\partial y^{n-1}(\lambda,c\lambda^{\alpha}) = \text{unit}\cdot\mu,$$

and thus $v(\lambda^{\beta}) = v(\mu) < v(\mu')$, as desired.

The above allows one to introduce yet another y-linear change of variables, namely $y'' = y'/x^{\beta}$. Then

$$f(x,y) = g(x,y') = g(x, y''\cdot x^{\beta}) =: h(x,y'')$$

whence we get

$$\partial^k h/\partial(y'')^k(x,y'') = x^{k\cdot\beta}\cdot\partial^k g/\partial(y')^k(x,y''\cdot x^{\beta}) \quad \text{for all} \quad k \in \mathbb{N}.$$

We have, in particular, the equalities:

$$\partial^k h/\partial(y'')^k(x,0) = x^{k\cdot\beta}\cdot\partial^k g/\partial y^k(x,0) \quad \text{for all} \quad k \in \mathbb{N}.$$

But for $k = n-1$ we get

$$\partial^{n-1}h/\partial(y'')^{n-1}(x,0) = x^{(n-1)\cdot\beta}\cdot\partial^{n-1}g/\partial y^{n-1}(x,0) = x^{n\cdot\beta}\cdot u(x)$$

with $u(0) \neq 0$. Every partial derivative $\partial^k h/\partial(y'')^k(x,0)$, $k \geq n-1$, is thus divisible by $x^{n\cdot\beta}$. Since the quotient for $k = n-1$ is just $u(x)$ with $u(0) \neq 0$, we are able to lower the y-order of the function $f(x,y)$ by means of the proposition and lemma from Section 2 applied to the functions $x^{n\cdot\beta}$ and $\partial^k h/\partial(y'')^k(x,0)$, $k = 0,1,\dots,n-2$. This completes the proof. $\diamondsuit$

Repeated application of the above theorem enables us to draw the following two conclusions, which will play a crucial role in the sequel. We keep the foregoing assumptions.

Proposition 1. *Behaviour of a Q-function at non-active infinitesimals: We can find a special cube* $S = \varphi((0,1)^m) \subset \mathbb{R}^m$*, a Q-function* $\omega(x', y')$ *in the vicinity of* $[0,1]^m \times \mathbb{R}$ *being linear with respect to the last variable with* $\omega(0,0) = 0$*, a Q-function* $g(x', y')$ *in the vicinity of* $[0,1]^{m+1}$*, positive infinitesimals* $\lambda' = (\lambda'_1, \dots, \lambda'_m)$*, an infinitesimal* μ' *and* $\alpha \in \mathbb{N}^m$ *such that* $\lambda = \varphi(\lambda') \in S$*,* $\mu = \omega(\lambda', \mu')$*,* $g(0,0) \neq 0$ *and*

$$f^\sigma(x', y') := f(\varphi(x'), \omega(x', y')) = x'^\alpha g(x', y')$$

in the vicinity of $[0,1]^{m+1}$*. In particular,* $v(f(\lambda,\mu)) = v(\lambda'^\alpha) \in \Gamma_{\langle\lambda\rangle}$*.* $\diamondsuit$

Proposition 2. *Exchange property for a non-active infinitesimal: We can find a special cube* $S = \varphi((0,1)^m) \subset \mathbb{R}^m$*, a Q-function* $\omega(x', y')$ *in the vicinity of* $[0,1]^m \times \mathbb{R}$ *being linear with respect to the last variable with* $\omega(0,0) = 0$*, a Q-function* $g(x', y')$ *in the vicinity of* $[0,1]^{m+1}$*, a Q-function* $h(x')$ *in the vicinity of* $[0,1]^m$*, positive infinitesimals* $\lambda' = (\lambda'_1, \dots, \lambda'_m)$*, an infinitesimal* μ' *and* $\alpha \in \mathbb{N}^m$ *such that* $\lambda = \varphi(\lambda') \in S$*,* $\mu = \omega(\lambda', \mu')$*,* $\partial g/\partial y(0,0) \neq 0$ *and*

$$f^\sigma(x', y') - h(x') = f(\varphi(x'), \omega(x', y')) - h(x') = x'^\alpha g(x', y')$$

in the vicinity of $[0,1]^{m+1}$*. Consequently, the non-active infinitesimal* μ *is the value of an* $\mathcal{L}$*-term* $\tau(x,y)$ *at the infinitesimals* λ *and* $\nu := f(\lambda,\mu)$*:*

$$\mu = \tau(\lambda_1, \dots, \lambda_m, \nu) \quad \text{or equivalently} \quad \mu \in \langle \lambda, \nu \rangle.$$

$\diamondsuit$

We shall now derive some consequences of Proposition 1.

Corollary. *Given a finite number of infinitesimals* $\lambda = (\lambda_1, \dots, \lambda_m)$*, the value group* $\Gamma_{\langle\lambda\rangle}$ *is a vector space over* $\mathbb{Q}$ *of dimension* $\leq m$*.*

We lead the proof by induction on the number m of generators. We may, of course, assume that the infinitesimals λ are analytically independent. It suffices to show that the vector space spanned over the set

$$\{v(f(\lambda)): \ f \ \text{ is a Q-functon at } \ 0 \in \mathbb{R}^m\}$$

is of dimension $d(\lambda) \leq m$. For, supposing the vectors $v(t_1(\lambda)), \ldots, v(t_{m+1}(\lambda))$ are linearly independent over $\mathbb{Q}$, we would find, by applying $m+1$ times corollary 2 to the theorem on decomposition into special cubes, infinitesimals $\lambda' = (\lambda'_1, \ldots, \lambda'_m) \in \langle\lambda\rangle$ and Q-functions $f_1(x), \ldots, f_{m+1}(x)$ at $0 \in \mathbb{R}^m$ such that

$$t_i(\lambda) = f_i(\lambda') \quad \text{for} \ \ i = 1, \ldots, m+1.$$

Hence we would get $d(\lambda') > m$, which contradicts our hypothesis.

Suppose now that the assertion holds for m and take $m+1$ infinitesimals $\lambda = (\lambda_1, \ldots, \lambda_m), \mu$. We must show that $d(\lambda, \mu) \leq m+1$. If μ is non-active over λ, it follows from Proposition 1 and the induction hypothesis that

$$d(\lambda, \mu) = d(\lambda) \leq \dim \Gamma_{\langle\lambda\rangle} \leq m.$$

In the other case, $v(\mu - t(\lambda)) \notin \Gamma_{\langle\lambda\rangle}$ for an $\mathcal{L}$-term $t(x)$. By the desingularization of $\mathcal{L}$-terms (Corollary 2 to the theorem on decomposition into special cubes from Section 2), one can find a special modification φ and infinitesimals λ' such that $\lambda = \varphi(\lambda')$ and $f := t \circ \varphi$ is a Q-function at $0 \in \mathbb{R}^m$. Then $t(\lambda) = f(\lambda')$ and $v(\mu - f(\lambda')) \notin \Gamma_{\langle\lambda\rangle}$. Consequently, we get from Corollary 2 to the proposition from Section 2 and the induction hypothesis the following inequalities

$$d(\lambda, \mu) \leq d(\lambda', \mu) = d(\lambda') + 1 \leq \dim \Gamma_{\langle\lambda'\rangle} + 1 \leq m+1,$$

which is the desired result. $\diamondsuit$

Proposition 3. *Behaviour of an $\mathcal{L}$-term at non-active infinitesimals: For each $\mathcal{L}$-term $t(x, y)$, there exist a special modification $\varphi : (0,1)^m \longrightarrow \mathbb{R}^m$, a Q-function $\omega(x', y')$ in the vicinity of $[0,1]^m \times \mathbb{R}$, linear with respect to the last variable, $\omega(0) = 0$, a Q-function $f(x', y')$ in the vicinity of $[0,1]^{m+1}$, positive infinitesimals $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ and a positive infinitesimal μ' such that $\lambda = \varphi(\lambda') \in S$, $\mu = \omega(\lambda', \mu')$ and for all $(x', y') \in (0,1)^{m+1}$ we have*

$$t^\sigma(x', y') = f(x', y') \quad \text{or} \quad t^\sigma(x', y') = \frac{1}{f(x', y')},$$

according as $t(\lambda, \mu)$ is bounded or not; here $t^\sigma(x', y') := t(\varphi(x'), \omega(x', y'))$.

In particular, we have the dichotomy:
- *either an infinitesimal* μ *is active over* λ
- *or* $\Gamma_{\langle\lambda,\mu\rangle} = \Gamma_{\langle\lambda\rangle}$.

The proof is by induction with respect to the complexity of the term $t(x,y)$ and consists in repeated application of Proposition 1 and transformation to normal crossings of the functions in the non-distinguished variables x', which occur in the process. In the case of a product of two $\mathcal{L}$-terms, one should simultaneously transform to normal crossings the two functions in the non-distinguished variables x', so that the two exponents obtained are totally ordered (as in the proof of the proposition from Section 2). In the case of a root of an $\mathcal{L}$-term, after transformation to normal crossings, one should substitute suitable power functions. The detailed verification is straightforward, and we leave it to the reader. Note, however, that the equality in question holds only for $(x',y') \in (0,1)^{m+1}$, and not in the vicinity of $[0,1]^{m+1}$. $\diamondsuit$

Propositions 2 and 3 both yield immediately the following

Corollary. *Exchange property for a non-active infinitesimal:*
We can find a special modification $\varphi : (0,1)^m \longrightarrow \mathbb{R}^m$, *a Q-function* $\omega(x',y')$ *in the vicinity of* $[0,1]^m \times \mathbb{R}$, *being linear with respect to the last variable with* $\omega(0,0) = 0$, *a Q-function* $g(x',y')$ *in the vicinity of* $[0,1]^{m+1}$, *a Q-function* $h(x')$ *in the vicinity of* $[0,1]^m$, *positive infinitesimals* $\lambda' = (\lambda'_1,\dots,\lambda'_m)$, *an infinitesimal* μ' *and* $\alpha \in \mathbb{N}^m$ *such that* $\lambda = \varphi(\lambda') \in S$, $\mu = \omega(\lambda',\mu')$, $\partial g/\partial y(0,0) \neq 0$ *and*

$$t^{\sigma}(x',y') - h(x') = t(\varphi(x'),\omega(x',y')) - h(x') = x'^{\alpha} g(x',y')$$

in the vicinity of $[0,1]^{m+1}$. *Consequently, the non-active infinitesimal* μ *is the value of an* $\mathcal{L}$*-term* $\tau(x,y)$ *at the infinitesimals* λ *and* $\nu := t(\lambda,\mu)$:

$$\mu = \tau(\lambda_1,\dots,\lambda_m,\nu) \quad \text{or equivalently} \quad \mu \in \langle\lambda,\nu\rangle.$$

$\diamondsuit$

4. Valuation property for $\mathcal{L}$-terms. A sequence $\lambda = (\lambda_1,\dots,\lambda_k)$ of infinitesimals shall be called regular if their valuations

$$v(\lambda_1),\dots,v(\lambda_k) \in \Gamma_{\langle\lambda\rangle}$$

are linearly independent over $\mathbb{Q}$. We begin this section with the following

Proposition. *Behaviour of an $\mathcal{L}$-term at a regular sequence: Consider a regular sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive infinitesimals. Then, for each term $t(x_1, \ldots, x_k)$ with $t(\lambda)$ bounded, there are a Q-function f at $0 \in \mathbb{R}^k$ and multi-indices $\beta_i = (\beta_{i1}, \ldots, \beta_{ik}) \in \mathbb{Q}^k$, $i = 1, \ldots, k$, linearly independent over $\mathbb{Q}$, such that $\lambda^{\beta_1}, \ldots, \lambda^{\beta_m}$ are positive infinitesimals and*

$$t(\lambda) = f(\lambda^{\beta_1}, \ldots, \lambda^{\beta_k}).$$

Our proof starts with the observation that, due to the desingularization of $\mathcal{L}$-terms (Corollary 2 to the theorem on decomposition into special cubes from Section 2), there exist a special cube $S = \varphi((0,1)^k) \subset \mathbb{R}^k$, a Q-function $g(x')$ in the vicinity of $[0,1]^m$, positive infinitesimals $\lambda' = (\lambda'_1, \ldots, \lambda'_k)$ such that $\lambda = \varphi(\lambda') \in S$ and

$$t^{\varphi}(x') := t(\varphi(x')) = g(x') \quad \text{for all} \quad (x') \in (0,1)^k.$$

Via transformation to normal crossings by blowing up, applied to the functions $\varphi_1, \ldots, \varphi_k$ and the coordinate functions $x_1, \ldots, x_k$, one can assume that the infinitesimals λ' are a regular sequence too, and that

$$\varphi_i(x') = u_i(x')(x')^{\alpha_i} \quad \text{with} \quad \alpha_i = (\alpha_{i1}, \ldots, \alpha_{ik}) \in \mathbb{N}^k, \; i = 1, \ldots, k,$$

where the Q-functions $u_i(x')$ are units at zero and the multi-indices α_i, $i = 1 \ldots, k$, are linearly independent over $\mathbb{Q}$. Let A be the matrix whose rows are just the multi-indices α_i, and B the inverse matrix with rows $\beta_l = (\beta_{l1}, \ldots, \beta_{lk}) \in \mathbb{Q}^k$. Consider the mapping

$$\psi : (0,\infty)^k \longrightarrow (0,\infty)^k, \; \psi(x) := (x^{\beta_1}, \ldots, x^{\beta_k}).$$

Then the superposition $\chi = (\chi_1, \ldots, \chi_k) := \psi \circ \varphi$ is of the form $\chi_i(x') = x'_i v_i(x')$, $i = 1, \ldots, k$, where Q-functions $v_i(x')$ are units at zero. Hence

$$\lambda^{\beta_1} = \chi_1(\lambda'), \; \ldots \; , \lambda^{\beta_k} = \chi_k(\lambda')$$

are infinitesimals, and the mapping χ is invertible; put $\omega := \chi^{-1}$. Then $x' = \omega(x^{\beta_1}, \ldots, x^{\beta_k})$, and thus we get

$$t(\lambda) = g(\lambda') = (g \circ \omega)(\lambda^{\beta_1}, \ldots, \lambda^{\beta_k}).$$

Putting $f := g \circ \omega$ finishes the proof. $\diamondsuit$

Combining the above with Proposition 3 from Section 3 (on behaviour of an $\mathcal{L}$-term at non-active infinitesimals), we immediately obtain the

Corollary. *Suppose that* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *are positive analytically independent infinitesimals such that* $\lambda_1, \ldots, \lambda_k$ *is a regular sequence with* $k = \dim \Gamma_{\langle\lambda\rangle}$, $1 \leq k \leq m-1$. *Every infinitesimal* λ_j, $j = k+1, \ldots, m$, *is, of course, non-active over the preceding infinitesimals. Then, for each term* $t(x_1, \ldots, x_m)$ *with* $t(\lambda)$ *bounded, there are a Q-function* f *at* $0 \in \mathbb{R}^m$ *and multi-indices* $\beta_i = (\beta_{i1}, \ldots, \beta_{ik}) \in \mathbb{Q}^k$, $i = 1, \ldots, k$, *linearly independent over* $\mathbb{Q}$, *such that* $(\lambda_1, \ldots, \lambda_k)^{\beta_1}, \ldots, (\lambda_1, \ldots, \lambda_k)^{\beta_m}$ *are positive infinitesimals and*

$$t(\lambda) = f((\lambda_1, \ldots, \lambda_k)^{\beta_1}, \ldots, (\lambda_1, \ldots, \lambda_k)^{\beta_k}, \lambda_{k+1}, \ldots, \lambda_m).$$

$\diamondsuit$

Having disposed of these preparations, we can now return to the exchange property.

Theorem 1. *Exchange property for a regular sequence:*
Let $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_{k-1}, \mu)$ *be a regular sequence of infinitesimals and* ν *an infinitesimal. If* $\nu \in \langle \lambda_1, \ldots, \lambda_{k-1}, \mu \rangle$ *and* $\mu \notin \langle \lambda_1, \ldots, \lambda_{k-1} \rangle$, *then the infinitesimal* ν *is active over* $\lambda_1, \ldots, \lambda_{k-1}$. *Consequently, we have*

$$\langle \lambda_1, \ldots, \lambda_{k-1}, \mu \rangle = \langle \lambda_1, \ldots, \lambda_{k-1}, \nu \rangle.$$

The assumption that $\nu \in \langle \lambda_1, \ldots, \lambda_{k-1}, \mu \rangle$ means that $\nu = t(\lambda, \mu)$ for an $\mathcal{L}$-term $t(x, y)$. We begin the proof by reducing the form taken by this term at the infinitesimals $\lambda_1, \ldots, \lambda_{k-1}, \mu$. We may, of course, assume that all the infinitesimals under consideration are positive. By the above proposition, we have

$$\nu = t(\lambda, \mu) = f((\lambda, \mu)^{\beta_1}, \ldots, (\lambda, \mu)^{\beta_k})$$

for some Q-function f at $0 \in \mathbb{R}^k$ and multi-indices $\beta_i = (\beta_{i1}, \ldots, \beta_{ik}) \in \mathbb{Q}^k$, $i = 1, \ldots, k$, linearly independent over $\mathbb{Q}$, such that $\lambda^{\beta_1}, \ldots, \lambda^{\beta_m}$ are positive infinitesimals. Without loss of generality, the problem can be reduced to the case where $\beta_i = (\beta_{i1}, \ldots, \beta_{ik}) \in \mathbb{Z}^k$, $i = 1, \ldots, k$, and next to the case

$$\nu = f(\lambda^{\alpha_1} \mu^{\epsilon_1}, \ldots, \lambda^{\alpha_k} \mu^{\epsilon_k})$$

with $\epsilon_i \in \{-1, 0, 1\}$, $i = 1, \ldots, k$.

But one can always replace a Q-function $f(u, v)$ by $g(u, v/u)$, where the Q-function $g(u', v')$ is given by the formula $g(u', v') := f(u', u'v')$. Since the valuations of the infinitesimals $\lambda^{\alpha_1}\mu^{\epsilon_1}, \ldots, \lambda^{\alpha_k}\mu^{\epsilon_k}$ are pairwise distinct, we can thus reduce the situation to the case where

$$\nu = f(\lambda^{\alpha_1}\mu, \lambda^{\alpha_2}/\mu, \lambda^{\alpha_3}, \ldots, \lambda^{\alpha_k}),$$

and next to the case

$$\nu = f(\mu, \lambda^{\alpha_2}/\mu, \lambda^{\alpha_3}, \ldots, \lambda^{\alpha_k}).$$

Since the multi-indices $\alpha_2, \ldots, \alpha_k \in \mathbb{Z}^k$ are obviously linearly independent over $\mathbb{Q}$, we can eventually assume, without loss of generality, that

$$\nu = f(\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1}/\mu, \mu),$$

where f is a Q-function at $0 \in \mathbb{R}^k$.

Let

$$\widehat{f}(x_1, \ldots, x_{k-2}, u, v) = \sum_{\gamma \in \mathbb{N}^{k-2}} \sum_{i,j \in \mathbb{N}} a_{\gamma,i,j} x^\gamma u^i v^j, \quad x = (x_1, \ldots, x_{k-2}),$$

be the Taylor series at zero of the function $f(x, u, v)$. We wish to obtain a Q-function $g(x, w)$ at zero such that the Taylor series at zero of the Q-function $g(x, uv)$ is of the form

$$\widehat{g}(x_1, \ldots, x_{k-2}, uv) = \sum_{\gamma \in \mathbb{N}^{k-2}} \sum_{i \in \mathbb{N}} a_{\gamma,i,i} x^\gamma u^i v^i.$$

The existence of a Q-function $g(x, w)$ with such a "diagonal" Taylor series is evident provided that one confines oneself to the classical case of analytic functions.

Remark 1. In the case where $Q = Q_M$ is the quasianalytic Denjoy–Carleman class corresponding to an increasing sequence $M = (M_n)$ of real numbers, the existence of a Q-function $g(x, w)$ described above can be related to an affirmative answer to a natural yet open problem stated below; see also the introduction to this paper (Section 1) and our article [31].

Consider the Hilbert space $\mathcal{H}_M$ of ultradifferentiable functions, introduced by Thilliez [39] in Section 2. This Hilbert space resembles Sobolev spaces of infinite order of l_2-type. Are polynomial mappings dense in $\mathcal{H}_M$?

Now we can readily prove that the infinitesimal ν is active over $\lambda = (\lambda_1, \ldots, \lambda_{k-1})$, and more precisely:

$$v(\nu - g(\lambda)) = v(f(\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1}/\mu, \mu) - g(\lambda)) \notin \Gamma_{\langle\lambda\rangle}.$$

First observe that the coefficients of the monomials $x^\gamma u^i v^i$ ($\gamma \in \mathbb{N}^{k-2}$, $i \in \mathbb{N}$) of the Taylor series of the function

$$h(x, u, v) := f(x, u, v) - g(x, uv)$$

vanish. Not all of the remaining coefficients $a_{\gamma,i,j}$ ($\gamma \in \mathbb{N}^{k-2}$, $i, j \in \mathbb{N}$, $i \neq j$) vanish. Otherwise $f(x, u, v) = g(x, uv)$, whence

$$\nu = f(\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1}/\mu, \mu) = g(\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1}/\mu \cdot \mu) = g(\lambda) \in \langle\lambda\rangle,$$

which would contradict our assumption.

Similarly as in Corollary 1 to the proposition from Section 2, we can obtain an analogous modification of the Q-function $h(x, u, v)$ stated below.

Claim. *One can find a special cube $S = \varphi((0, 1)^{k-2}) \subset \mathbb{R}^{k-2}$ with φ being a composite of successive blowings-up, and positive infinitesimals $\lambda' = (\lambda'_1, \ldots, \lambda'_{k-2})$ such that $(\lambda_1, \ldots, \lambda_{k-2}) = \varphi(\lambda'_1, \ldots, \lambda'_{k-2}) \in S$ and*

$$h^\varphi(x', u, v) := h(\varphi(x'), u, v) = x'^\alpha \eta(x', u, v) \quad \text{near} \quad (0, 0, 0) \in \mathbb{R}^m_{x'} \times \mathbb{R}_u \times \mathbb{R}_v,$$

where $\alpha \in \mathbb{N}^m$ and $\eta(x', u, v)$ is a Q-function at zero with

$$\partial^{p+q}\eta/\partial u^p \partial v^q(0, 0, 0) \neq 0$$

for some $p, q \in \mathbb{N}$. $\diamondsuit$

For simplicity we shall drop the sign of apostrophe over x. Our assumption about the vanishing of certain coefficients of Taylor series remains valid for the function $\eta(x, u, v)$. This assumption yields $p \neq q$. Further, we have

$$\eta(x, u, v) - \sum_{j=0}^{q-1} \partial^j \eta/\partial v^j(x, u, 0) \cdot v^j/j! = v^q \cdot \zeta(x, u, v)$$

for some Q-function $\zeta(x, u, v)$ at zero, and

$$\zeta(x, u, v) - \sum_{i=0}^{p-1} \partial^i \zeta/\partial u^i(x, 0, v) \cdot u^i/i! = u^p \cdot \omega(x, u, v)$$

for some Q-function $\omega(x,u,v)$ at zero.

Again, through repeating successive modifications of the variables x by means of special cubes, as in Corollary 1 to the proposition from Section 2, one can assume that

$$1/j!\,\partial^j\eta/\partial v^j(x,u,0) = x^{\gamma_j}\eta_j(x,u),$$

where $\eta_j(x,u)$, $j = 0,\dots,q-1$, are u-regular Q-functions at zero, say, of order r_j; similarly,

$$1/i!\,\partial^i\zeta/\partial u^i(x,0,v) = x^{\delta_i}\zeta_i(x,v),$$

where $\zeta_i(x,v)$, $i = 0,\dots,p-1$, are v-regular Q-functions at zero, say, of order s_i. Therefore we are able to write down these functions in the following form (cf. Corollary 2 to the proposition from Section 2):

$$\eta_j(x,u) = \sum_{r=0}^{r_j-1} \eta_{j,r}(x)u^r + \eta_{j,r_j}(x,u)u^{r_j} \quad \text{with} \quad \eta_{j,r_j}(0,0) \neq 0$$

and

$$\zeta_i(x,v) = \sum_{s=0}^{s_i-1} \zeta_{i,s}(x)v^s + \zeta_{i,s_i}(x,v)v^{s_i} \quad \text{with} \quad \zeta_{i,s_i}(0,0) \neq 0;$$

all functions which occur above are Q-analytic at zero.

Moreover, making use of transformation to normal crossings by blowing up, one can also assume that

$$\eta_{j,r}(x) = x^{\alpha_{j,r}} \cdot \mathrm{unit}(x) \quad \text{and} \quad \zeta_{i,s}(x) = x^{\beta_{i,s}} \cdot \mathrm{unit}(x),$$

with some $\alpha_{j,r}, \beta_{i,s} \in \mathbb{N}^{k-2}$.

Then

$$\eta(x,u,v) = \sum_{j=0}^{q-1} v^j \left[\sum_{r=0}^{r_j-1} u^r x^{\gamma_j}\eta_{j,r}(x) + u^{r_j}x^{\gamma_j}\cdot \mathrm{unit}(x,u)\right] +$$

$$+\sum_{i=0}^{p-1} u^i v^q \left[\sum_{s=0}^{s_i-1} v^s x^{\delta_i}\zeta_{i,s}(x) + v^{s_i}x^{\delta_i}\cdot \mathrm{unit}(x,v)\right] + u^p v^q \omega(x,u,v).$$

Since $\partial^{p+q}\eta/\partial u^p\partial v^q(0,0,0) \neq 0$, we get $\omega(0,0,0) \neq 0$. Substituting

$$\lambda_1,\dots,\lambda_{k-1},\lambda_k/\mu,\mu \quad \text{for} \quad x_1,\dots,x_{k-1},u,v,$$

we deduce that the valuation of the summands in the above formula are pairwise distinct. Consequently, $v(\eta(\lambda_1,\dots,\lambda_{k-1},\lambda_k/\mu,\mu))$ coincides with the valuation of one summand in the above formula.

But our assumption about the vanishing of certain coefficients of Taylor series of the function $\eta(x,u,v)$ implies that no summands with factor of the form $u^i v^i$ occur in the above summation. It follows immediately that

$$v(\eta(\lambda_1,\dots,\lambda_{k-1},\lambda_k/\mu,\mu)) \in v((\lambda_{k-1}/\mu)^i\mu^j) + \Gamma_{\langle\lambda_1,\dots,\lambda_{k-1}\rangle} = v(\mu^{j-i}) + \Gamma_{\langle\lambda\rangle}$$

for some $i,j \in \mathbb{N}$, $i \neq j$. Hence $v(\eta(\lambda_1,\dots,\lambda_{k-1},\lambda_k/\mu,\mu)) \notin \Gamma_{\langle\lambda\rangle}$, and thus

$$v(\nu - g(\lambda)) = v(f(\lambda_1,\dots,\lambda_{k-2},\lambda_{k-1}/\mu,\mu) - g(\lambda)) \notin \Gamma_{\langle\lambda\rangle}.$$

This means that the infinitesimal ν is active over λ, as asserted.

In order to complete the proof, consider the equation $\nu - t(\lambda,\mu) = 0$. Since μ is non-active over the infinitesimals (λ,ν), the exchange property for a non-active infinitesimal (cf. Proposition 2 and the corollary to Proposition 3 from Section 3) may be applied. Therefore this equation can be solved by means of the implicit function theorem with respect to μ, i.e. $\mu = \tau(\lambda,\nu) \in \langle\lambda,\nu\rangle$ for an $\mathcal{L}$-term $\tau(x,y)$, which is the desired conclusion. $\diamondsuit$

A slight change in the above proof, which consists in application of the corollary to Proposition 1 instead of Proposition 1 itself, actually leads us to the following strengthening of Theorem 1.

Theorem 1'. *Let μ and $\lambda = (\lambda_1,\dots,\lambda_m)$ be together analytically independent infinitesimals such that $\mu,\lambda_1,\dots,\lambda_k$ is a regular sequence with $\dim\Gamma_{\langle\mu,\lambda\rangle} = k+1$, for some $k \in \{1,\dots,m\}$. For any $\mathcal{L}$-term $t(y,x)$, $x = (x_1\dots,x_m)$, if the infinitesimal $\nu := t(\mu,\lambda) \notin \langle\lambda\rangle$, then ν is active over the infinitesimals λ.* $\diamondsuit$

Remark 2. In the proof of Theorem 1' one should take into account that the infinitesimals $\lambda_{k+1},\dots,\lambda_m$ do not affect the valuation group.

Now we wish to draw some conclusions from the foregoing analysis of infinitesimals.

Theorem 2. *Consider a regular sequence $\mu,\lambda_1,\dots,\lambda_k$ of infinitesimals and infinitesimals $\lambda_{k+1},\dots,\lambda_{k+l}$; put $\lambda = (\lambda_1,\dots,\lambda_{k+l})$. If $\dim\Gamma_{\langle\lambda\rangle} = k$, then $\dim\Gamma_{\langle\mu,\lambda\rangle} = k+1$.*

We may, of course, assume that the infinitesimals λ are analytically independent. The proof is by induction with respect to l. The assertion is trivial for $l = 0$. For the induction step, suppose the formula holds for l, i.e. $\dim \Gamma_{\langle \mu, \lambda_1, \ldots, \lambda_{k+l} \rangle} = k + 1$. We must show that $\dim \Gamma_{\langle \mu, \lambda_1, \ldots, \lambda_{k+l+1} \rangle} = k + 1$.

Were $\dim \Gamma_{\langle \mu, \lambda_1 \ldots, \lambda_{k+l+1} \rangle} > k + 1$, the infinitesimal λ_{k+l+1} would be active over the preceding infinitesimals, i.e.

$$v(\lambda_{k+l+1} - t(\mu, \lambda_1, \ldots, \lambda_{k+l})) \not\in \Gamma_{\langle \mu, \lambda_1 \ldots, \lambda_{k+l} \rangle} = \Gamma_{\langle \mu, \lambda_1 \ldots, \lambda_k \rangle}$$

for an $\mathcal{L}$-term t. Since λ_{k+l+1} is non-active over $\lambda_1, \ldots, \lambda_{k+l}$, we have

$$t(\mu, \lambda_1, \ldots, \lambda_{k+l}) \not\in \langle \lambda_1, \ldots, \lambda_{k+l} \rangle.$$

Otherwise we would get a contradiction

$$v(\lambda_{k+l+1} - t(\mu, \lambda_1, \ldots, \lambda_{k+l})) \in \Gamma_{\langle \lambda_1 \ldots, \lambda_{k+l+1} \rangle} = \Gamma_{\langle \lambda_1 \ldots, \lambda_k \rangle}.$$

Further, the induction hypothesis means that the infinitesimals

$$\mu, \lambda_1, \ldots, \lambda_k, \lambda_{k+1} \ldots, \lambda_{k+l}$$

satisfy the assumptions of Theorem 1'. This implies that the infinitesimal $\nu := t(\mu, \lambda_1, \ldots, \lambda_{k+l})$ would be active over $\lambda_1, \ldots, \lambda_{k+l}$, and thus

$$t(\mu, \lambda_1, \ldots, \lambda_{k+l}) - \tau(\lambda_1, \ldots, \lambda_{k+l}) \not\in \Gamma_{\langle \lambda_1, \ldots, \lambda_{k+l} \rangle} = \Gamma_{\langle \lambda_1, \ldots, \lambda_k \rangle}$$

for an $\mathcal{L}$-term τ. Then

$$v(\lambda_{k+l+1} - t(\mu, \lambda_1, \ldots, \lambda_{k+l})) =$$

$$= v((\lambda_{k+l+1} - \tau(\lambda_1, \ldots, \lambda_{k+l})) - (t(\mu, \lambda_1, \ldots, \lambda_{k+l}) - \tau(\lambda_1, \ldots, \lambda_{k+l}))) =$$

$$= \min \{ v(\lambda_{k+l+1} - \tau(\lambda_1, \ldots, \lambda_{k+l})), v(t(\mu, \lambda_1, \ldots, \lambda_{k+l}) - \tau(\lambda_1, \ldots, \lambda_{k+l})) \}.$$

Again, since λ_{k+l+1} is non-active over $\lambda_1, \ldots, \lambda_{k+l}$, we have

$$v(\lambda_{k+j+1} - \tau(\lambda_1, \ldots, \lambda_{k+j})) \in \Gamma_{\langle \lambda_1 \ldots, \lambda_{k+j} \rangle} = \Gamma_{\langle \lambda_1 \ldots, \lambda_k \rangle}.$$

Consequently, both the valuations in the above minimum are distinct, and thus we would get

$$v(\lambda_{k+l+1} - t(\mu, \lambda_1, \ldots, \lambda_{k+l})) \in \Gamma_{\langle \mu, \lambda_1 \ldots, \lambda_{k+l} \rangle} = \Gamma_{\langle \mu, \lambda_1 \ldots, \lambda_k \rangle},$$

because the other valuation $v(t(\mu, \lambda_1, \dots, \lambda_{k+l}) - \tau(\lambda_1, \dots, \lambda_{k+l}))$ lies in the group $\Gamma_{\langle \mu, \lambda_1 \dots, \lambda_{k+l} \rangle} = \Gamma_{\langle \mu, \lambda_1 \dots, \lambda_k \rangle}$ too. This contradiction completes the proof of Theorem 2. $\diamondsuit$

Corollary 1. *Valuation property for $\mathcal{L}$-terms:*
If $\lambda = (\lambda_1, \dots, \lambda_m)$ and μ are infinitesimals, we have the following dichotomy:

- *either μ is non-active over λ, and then $\Gamma_{\langle \lambda, \mu \rangle} = \Gamma_{\langle \lambda \rangle}$*
- *or μ is active over λ, and then $\dim \Gamma_{\langle \lambda, \mu \rangle} = \dim \Gamma_{\langle \lambda \rangle} + 1$.*

In the latter case, one can find an $\mathcal{L}$-term $t(x)$ such that

$$v(\mu - t(\lambda)) \notin \Gamma_{\langle \lambda \rangle} \quad \textit{and} \quad \Gamma_{\langle \lambda, \mu \rangle} = \Gamma_{\langle \lambda \rangle} \oplus \mathbb{Q} \cdot v(\mu - t(\lambda)).$$

Indeed, we may assume that $\dim \Gamma_{\langle \lambda \rangle} = k$ with $1 \leq k \leq m$. Then we can find a regular sequence $\lambda' = (\lambda'_1, \dots, \lambda'_k)$ with $\lambda'_1, \dots, \lambda'_k \in \langle \lambda \rangle$. The former case, where μ is non-active over λ, is clear according to Proposition 3 from Section 3.

If μ is active over λ, then $v(\mu - t(\lambda)) \notin \Gamma_{\langle \lambda \rangle}$ for an $\mathcal{L}$-term $t(x)$, and the sequence $\mu' := \mu - t(\lambda), \lambda'_1, \dots, \lambda'_k$ is regular. Since

$$\Gamma_{\langle \mu, \lambda, \rangle} = \Gamma_{\langle \mu', \lambda, \rangle} = \Gamma_{\langle \mu', \lambda', \lambda \rangle},$$

the assertion follows immediately from Theorem 2, applied to the sequence $\mu', \lambda'_1, \dots, \lambda'_k, \lambda_1, \dots, \lambda_m$. $\diamondsuit$

Corollary 2. *Steinitz's Exchange Property:*
Consider a finite number of infinitesimals $\lambda = (\lambda_1, \dots, \lambda_m), \mu, \nu$. If $\nu \in \langle \lambda, \mu \rangle$ and $\nu \notin \langle \lambda \rangle$, then $\mu \in \langle \lambda, \nu \rangle$.

The case where μ is non-active over the infinitesimals λ has been proved as the exchange property for a non-active infinitesimal in Proposition 2 and the corollary to Proposition 3 from Section 3. Consider now the other case.

We may, of course, assume that $\dim \Gamma_{\langle \lambda \rangle} = k$, $1 \leq k \leq m$, and that the infinitesimals $\lambda_1, \dots, \lambda_k, \mu$ are a regular sequence. It follows from Theorems 1' and 2 that $\dim \Gamma_{\langle \lambda, \mu \rangle} = k + 1$ and that ν is active over the infinitesimals λ. Hence μ is non-active over the infinitesimals λ, ν. Therefore we can repeat the reasoning from the proof of the last assertion of Theorem 1, where we applied the exchange property for a non-active infinitesimal and the implicit function theorem. $\diamondsuit$

5. Description of Q-subanalytic sets by terms. We begin by drawing some conclusions from Steinitz's Exchange Property (Corollary 2 to Theorem 2 from Section 4). The span operation $s(A) := \langle A \rangle$, which assigns to each subset $A \subset \mathcal{R}$ the substructure generated by A, fulfils the following conditions:

(S1) $A \subset B \ \Rightarrow \ s(A) \subset s(B)$;

(S2) $b \in s(A) \ \Rightarrow \ b \in s(a_1, \ldots, a_m)$ for some $a_1, \ldots, a_m \in A$;

(S3) $A \subset s(A)$;

(S4) $s(s(A)) = s(A)$;

(S5) $c \in s(A, b)$ and $c \notin s(A) \ \Rightarrow \ b \in s(A, c)$.

Model-theorists call such a span operation s a pregeometry on the structure $\mathcal{R}$ (see e.g. [24]). Conditions (S1)–(S4) are fulfilled by algebraic closure in any structure. A first-order structure is called geometric if algebraic closure satisfies the exchange property (S5). Definable closure and algebraic closure coincide in a structure with linear ordering, because in any finite set one can define the least element, the next least element and so on.

Every o-minimal structure satisfies condition (S5) too, and thus is geometric. One can build — by analogy with the dimension of vector spaces or with the transcendence degree of field extensions — a general dimension theory for geometric structures. There is a general notion of independence in such structures. We say that a subset A in $\mathcal{R}$ is a free (or, an independent) set, if

$$a \notin s(A \setminus \{a\} \quad \text{for any } a \in A;$$

A is called a basis of $\mathcal{R}$ if it is both a generating system of $\mathcal{R}$ and a free set (cf. [43], Vol. I, Chap. II, § 12). It can be checked that every maximal free subset of an algebraically closed structure $\mathcal{R}$ is a basis, and that any two bases have the same size, called the rank of $\mathcal{R}$. We have also at our disposal the notion of relative rank for a pair of geometric structures $\mathcal{R} \subset \mathcal{S}$.

In our case, span operation consists just in generating the substructure for subsets of a model $\mathcal{R}$ of the universal theory T. Clearly, any analytically independent set of infinitesimals in $\mathcal{R}$ is a free set. Therefore the assertion below is a special case of the one for free sets.

Proposition 1. *Inversion of analytically independent infinitesimals: Consider two analytically independent sets*

$$\lambda = (\lambda_1, \ldots, \lambda_m) \quad \textit{and} \quad \mu = (\mu_1, \ldots, \mu_m)$$

of infinitesimals and m $\mathcal{L}$-terms $t(x) = (t_1(x), \ldots, t_m(x))$, $x = (x_1, \ldots, x_m)$. If $\mu = t(\lambda)$, then there are m $\mathcal{L}$-terms $\tau_1(y), \ldots, \tau_m(y)$, $y = (y_1, \ldots, y_m)$, such that $\lambda = \tau(\mu)$. In other words, $\langle\lambda\rangle = \langle\mu\rangle$. $\diamondsuit$

As a direct consequence, we obtain

Corollary 1. *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be an analytically independent set of infinitesimals and μ an infinitesimal. Then the set $(\lambda_1, \ldots, \lambda_m, \mu)$ is analytically independent iff $\mu \notin \langle\lambda\rangle$.* $\diamondsuit$

Corollary 2. *A set $\lambda = (\lambda_1, \ldots, \lambda_m)$ of infinitesimals is analytically independent iff it is a free set.*

This can be easily checked by induction with respect to the number m of infinitesimals. $\diamondsuit$

We now state a theorem concerning the inversion of general special modifications, which asserts that the requirement for the inverse mapping ψ of a special modification φ we impose in Section 2 is no constraint on special cubes.

Proposition 2. *Let $\varphi : (0,1)^d \longrightarrow S \subset \mathbb{R}^m$ be a general special modification, i.e. φ is a diffeomorphism of $(0,1)^d$ onto S which extends to a Q-mapping in the vicinity of $[0,1]^d$. If S is described by $\mathcal{L}$-terms, then the inverse mapping $\varphi^{-1} : S \longrightarrow (0,1)^d$ is given piecewise by $\mathcal{L}$-terms.*

For the proof, we shall show that there exists a family $(t_\iota(y))_{\iota\in I}$ of $\mathcal{L}$-terms, $t_\iota(y) = (t_{\iota,1}(y), \ldots, t_{\iota,d}(y))$, such that the infinite disjunction

$$\bigvee_{\iota\in I} \left[\left(b = \varphi(a) \wedge a \in (0,1)^d \right) \Rightarrow (a = t_\iota(b) \wedge b \in S) \right]$$

holds for any tuples $a \in \mathcal{R}^d$ and $b \in \mathcal{R}^m$ in an arbitrary model $\mathcal{R}$ of the theory T. Then, through model-theoretic compactness, one can find a finite set $\iota_1, \ldots, \iota_n \in I$ of indices for which the finite disjunction

$$\bigvee_{k=1,\ldots,n} \left[\left(b = \varphi(a) \wedge a \in (0,1)^d \right) \Rightarrow (a = t_{\iota_k}(b) \wedge b \in S) \right]$$

holds for any such tuples a and b in an arbitrary model $\mathcal{R}$ of the theory T. Hence

$$\left(b = \varphi(a) \wedge a \in (0,1)^d\right) \Rightarrow \left[(a = t_{\iota_1}(b) \vee \ldots \vee a = t_{\iota_n}(b)) \wedge b \in S\right],$$

and thus the inverse mapping φ^{-1} is given piecewise by $\mathcal{L}$-terms, which is the desired conclusion.

So take any elements $a \in (0,1)^d$ and $b \in S^{\mathcal{R}}$ for which $b = \varphi(a)$. We may, of course, confine our analysis to the case where $a = \lambda$ and $b = \mu$ are infinitesimals. Observe that $\mathrm{rk}\,\langle\lambda\rangle \leq \mathrm{rk}\,\langle\mu\rangle$, for otherwise $\mu \in T^{\mathcal{R}}$ for a special cube T of dimension $< \mathrm{rk}\,\langle\lambda\rangle$, whence $\lambda \in (\varphi^{-1}(T))^{\mathcal{R}}$ and $\dim \varphi^{-1}(T) = \dim T < \mathrm{rk}\,\langle\lambda\rangle$, which is impossible.

Consequently, we have

$$\langle\mu\rangle \subset \langle\lambda\rangle \quad \text{and} \quad \mathrm{rk}\,\langle\lambda\rangle \leq \mathrm{rk}\,\langle\mu\rangle,$$

and thus $\langle\mu\rangle = \langle\lambda\rangle$. Therefore our auxiliary assertion follows and the proof is complete. $\diamondsuit$

Before turning to quantifier elimination for the theory T, we state the quasianalytic version of Gabrielov's theorem [15] on the closure and frontier of a semianalytic set, which will be needed in the proof.

Gabrielov's Closure Theorem. *If $E \subset \mathbb{R}^m$ is a Q-semianalytic set, so are the closure $\overline{E}$ and the frontier ∂E. Moreover, if E is of the form*

$$E = \{x \in [-1,1]^m : f_1(x) = \ldots = f_k(x) = 0, \quad g_1(x) > 0, \ldots, g_l(x) > 0\},$$

where f_i's and g_j's are Q-analytic functions in the vicinity of the cube $[-1,1]^m$, then $\overline{E}$ and ∂E are described by Q-analytic functions which are polynomials in the variables x, in the functions f_i's, g_j's and in their (finitely many) partial derivatives. $\diamondsuit$

Remark. Gabrielov's proof used a method of truncating Taylor series, which allows one to reduce the problem to sets described by polynomials where the Tarski–Seidenberg theorem applies. This method does not involve the Weierstrass preparation theorem, but relies on the Łojasiewicz inequality instead. Consequently, it can be transferred almost verbatim to the quasi-analytic settings.

Corollary. *If $E \subset \mathbb{R}^m$ is a set described by $\mathcal{L}$-terms, so are the closure $\overline{E}$ and the frontier ∂E.*

It is sufficient, of course, to consider the case of closure. The proof consists then in adding new variables, one for each occurrence of a function symbol involved in a given $\mathcal{L}$-term (as explained in Remark 2 from Section 2). $\diamondsuit$

We can now turn to quantifier elimination for the theory T.

Theorem on Quantifier Elimination. *Let $\pi : \mathbb{R}^m_x \times \mathbb{R}^n_y \longrightarrow \mathbb{R}^m_x$ be the canonical projection. If a set $E \subset \mathbb{R}^m_x \times \mathbb{R}^n_y$ is defined by a quantifier-free $\mathcal{L}$-formula $\phi(x, y)$ (i.e. E is described by a finite number of $\mathcal{L}$-terms involved in ϕ), so is its projection*

$$F = \pi(E) = \{x \in \mathbb{R}^m : \; \exists\, y_n \ldots \exists\, y_1 \;\; \phi(x, y)\}.$$

Accordingly, the theory T admits quantifier elimination.

The proof is by induction with respect to the dimension $\dim E =: d$. It suffices, of course, to consider the case $n = 1$. The case $\dim E = 0$ is trivial; take $d \geq 1$. Assuming the assertion to hold for $0, 1, \ldots, d-1$, we shall prove it for d.

For this purpose, we shall show that there exist a family of quantifier-free formulae $(\phi_\iota(x))_{\iota \in I}$ such that

$$\bigwedge_{\iota \in I} \mathcal{R} \models \forall\, x \;\; [\phi_\iota(x) \Rightarrow \exists\, y \;\; \phi(x, y)]$$

and the infinite disjunction

$$\bigvee_{\iota \in I} \;[(\exists\, y \;\; \phi(a, y)) \Rightarrow \phi_\iota(a)]$$

holds for any fixed tuple $a \in \mathcal{R}^m$ in an arbitrary model $\mathcal{R}$ of the theory T. Then, through model-theoretic compactness, we get

$$F = \{x \in \mathbb{R}^m : \phi_{\iota_1}(x)\} \cup \ldots \cup \{x \in \mathbb{R}^m : \phi_{\iota_l}(x)\}$$

for some $\iota_1, \ldots, \iota_p \in I$, which is the desired result.

Obviously, we may assume that the set E is bounded. Take any element $(a, b) = (a_1, \ldots, a_m, b) \in E^{\mathcal{R}} \in \mathcal{R}^m$. We can, of course, confine our analysis to the case where $(a, b) = (\lambda, \mu) = (\lambda_1, \ldots, \lambda_m, \mu)$ are infinitesimals. Note that in non-standard models $\mathcal{R}$ we shall work only with the interpretations $E^{\mathcal{R}}$ of the set E, because the set F is a priori not described by $\mathcal{L}$-terms, and thus we are not able to analyse its interpretations yet.

We have two possibilities: either $\mu \in \langle \lambda \rangle$ or $\mu \notin \langle \lambda \rangle$. The former is easy; it yields $\mu = t(\lambda)$ for an $\mathcal{L}$-term, and thus one should attach to our family of formulae one that describes the set $(x_1, \dots, x_m, t(x))^{-1}(E)$.

The latter needs a more careful treatment. We can reduce our problem to the case where the infinitesimals λ are analytically independent. Indeed, one can find a basis chosen from among λ's, say $\lambda_1, \dots, \lambda_r$ with $r := \operatorname{rk} \langle \lambda \rangle$. Clearly,

$$\lambda_j = \tau_j(\lambda_1, \dots, \lambda_r), \quad j = r+1, \dots, m,$$

for some $\mathcal{L}$-terms $\tau_{r+1}, \dots, \tau_m$. Let $\rho : \mathbb{R}^m \longrightarrow \mathbb{R}^r$ be the canonical projection onto the first r coordinates. Then our analysis of the projection π of the set E can be replaced by that of the projection $\rho \circ \pi$ of the set

$$\tilde{E} := E \cap \{x_{r+1} = \tau_{r+1}(x_1, \dots, x_r)\} \cap \ldots \cap \{x_m = \tau_m(x_1, \dots, x_r)\},$$

which is equivalent to that of the projection onto the first r coordinates of the following set also described by $\mathcal{L}$-terms:

$$(x_1, \dots, x_r, y)(\tilde{E}) \subset \mathbb{R}^r \times \mathbb{R}_y.$$

Hence the reduction goes. The replacement described above is a non-standard counterpart of fiber cutting in the classical subanalytic geometry.

So suppose the infinitesimals λ are analytically independent. According to Corollary 1 to Proposition 1, the infinitesimals $(\lambda, \mu) \in E^{\mathcal{R}}$ are analytically independent too, and thus we can assume that E is a special cube of dimension $d = m + 1$. Further, due to Gabrielov's closure theorem, the frontier $V := \partial E$ is a set described by $\mathcal{L}$-terms of dimension $< d$.

Observe that we can replace the set E by its part lying over the complement of any closed subset $Z \subset \mathbb{R}^m$ described by $\mathcal{L}$-terms of dimension $< m$, because the infinitesimals λ are analytically independent. Therefore it is sufficient to investigate the parts of the sets E and V over such a complement. Consequently, we can assume that the set V over such a complement is a finite union of special cubes $S_1, \dots, S_l$ such that each projection

$$\pi : S_i \longrightarrow \mathbb{R}^m, \quad i = 1, \dots, l,$$

is a local Q-diffeomorphism. Indeed, if V is a finite union of special cubes, we should remove the special cubes whose projections onto $\mathbb{R}^m$ are of dimension

$< m$ and cut out from the remaining special cubes the sets of ramification points, which are described by $\mathcal{L}$-terms and of dimension $< m$.

Further, we can replace the special cubes S_i by the sets $S_i \setminus \pi^{-1}(\pi(\partial S_i))$, $i = 1, \dots, l$. These sets are also described by $\mathcal{L}$-terms due to Gabrielov's closure theorem and the induction hypothesis. Each projection

$$\pi : S_i \setminus \pi^{-1}(\pi(\partial S_i)) \longrightarrow \pi(S_i) \setminus \pi(\partial S_i)$$

is thus a proper mapping whence a topological covering.

After decomposition of the sets $\pi(S_i) \setminus \pi(\partial S_i)$ into special cubes and removing those of dimension $< m$, the part of the set V under study is now a finite union of topological coverings V_C over simply connected open special cubes $C \subset \mathbb{R}^m$.

Clearly, each set V_C is a finite union of leaves $\Lambda_1, \dots, \Lambda_n$ that are the graphs of certain smooth functions $\xi_1(x) < \dots < \xi_n(x)$. Then the set

$$\Lambda := \{(x, y_1, \dots, y_n) : (x, y_1) \in \Lambda_1, \dots, (x, y_n) \in \Lambda_n\} =$$

$$= \{(x, y_1, \dots, y_n) : x \in C,\ (x, y_1), \dots, (x, y_n) \in V,\ y_1 < \dots < y_n\}$$

(which is an open subset of the fibre product of the leaves $\Lambda_1, \dots, \Lambda_n$ over C) is described by $\mathcal{L}$-terms of dimension m, and so are its projections $\Lambda_1, \dots, \Lambda_n$, again by the induction hypothesis. Observe now that, due to Proposition 2 and the induction hypothesis, the functions $\xi_1(x), \dots, \xi_n(x)$ are given piecewise by $\mathcal{L}$-terms too.

We must consider only such special cubes C above, say $C_1, \dots, C_s$, that $\lambda \in C^{\mathcal{R}}$. The part of the set V lying over the intersection $D := C_1 \cap \dots \cap C_s$ is then a finite union of the graphs of certain smooth functions $\xi_1(x), \dots, \xi_N(x)$ given piecewise by $\mathcal{L}$-terms. Again, we can cut out the set W of points of D at which any two distinct functions above are equal, because this set is of dimension $< m$. Take the connected component U of $D \setminus W$ (which is described by $\mathcal{L}$-terms through decomposition into special cubes) such that $\lambda \in U^{\mathcal{R}}$. It is obvious that the functions $\xi_1(x), \dots, \xi_N(x)$ are totally ordered over U, say $\xi_1(x) < \dots < \xi_N(x)$ for all $x \in U$.

But then the part E_U of the set E lying over U is a finite union of strata between some of the leaves Λ_k and Λ_{k+1}, $k \in K \subset \{1, \dots, N-1\}$:

$$E_U = \{(x, y) : \ x \in U,\ \xi_k(x) < y < \xi_{k+1}(x) \ \text{ for some } k \in K\}.$$

This can be expressed by means of the following universal $\mathcal{L}$-formula

$$(x,y) \in E_U \Leftrightarrow \left[x \in U, \bigvee_{k \in K} \xi_k(x) < y < \xi_{k+1}(x) \right],$$

which is true in every model $\mathcal{R}$ of the theory T. Therefore (λ, μ) lies in one of those strata, and hence

$$\xi(\lambda) < \mu < \zeta(\lambda) \quad \text{and} \quad \{\lambda\} \times (\xi(\lambda), \zeta(\lambda)) \subset E^{\mathcal{R}},$$

for some functions $\xi(x), \zeta(x)$ given piecewise by $\mathcal{L}$-terms. In order to complete the proof, we should attach to our family of formulae one that describes the set $(x_1, \ldots, x_m, t(x))^{-1}(E)$ with $t(x) := (\xi(x) + \zeta(x))/2$. ♢

Hence and by decomposition into special cubes, we obtain immediately

Corollary. *The theory T is complete and o-minimal, and the standard model $\mathcal{R}_Q$ is its prime model.* ♢

6. Applications. The fact that a universal theory T admits quantifier elimination has weighty model-theoretic and geometric consequences. Then every definable function is piecewise given by terms:

Proposition. (Herbrand [17]) *Consider a first-order language $\mathcal{L}$, a universal theory T in $\mathcal{L}$ and a quantifier-free $\mathcal{L}$-formula $\phi(x_1, \ldots, x_m, y)$. If*

$$T \models \quad \forall x_1 \ldots \forall x_m \exists y \quad \phi(x_1, \ldots, x_m, y),$$

then there exist a finite number of terms $t_1(x), \ldots, t_k(x)$ such that

$$T \models \quad \forall x_1 \ldots \forall x_m \quad \phi(x_1, \ldots, x_m, t_1(x)) \vee \ldots \vee \phi(x_1, \ldots, x_m, t_k(x)).$$

Corollary. *If a universal theory T admits quantifier elimination, then for each definable function $f(x_1, \ldots, x_m)$ there are finitely many $\mathcal{L}$-terms $t_1(x), \ldots, t_k(x)$ such that in every model $\mathfrak{M}$ of T we have*

$$\mathfrak{M} \models \quad \forall x_1 \ldots \forall x_m \quad f(x_1, \ldots, x_m) = t_1(x) \vee \ldots \vee f(x_1, \ldots, x_m) = t_k(x),$$

i.e. f is piecewise given by the terms $t_1(x), \ldots, t_k(x)$. ♢

Consequently, the structure $\mathcal{R}_Q$ admits smooth cell decomposition given piecewise by $\mathcal{L}$-terms (cf. [30]). Further, the operation of definable closure for subsets of a given model $\mathcal{R}$ of the theory T coincides with that of span discussed in Section 5. We can thus deduce from the valuation property for $\mathcal{L}$-terms (Corollary 1 to Theorem 2 from Section 4) the following general version (cf. [13, 14, 28]).

Valuation Property for Definable Functions. *Consider a simple (with respect to definable closure) extension $\mathcal{R} \subset \mathcal{R}\langle a\rangle$ of substructures in a fixed model of the theory T. Then we have the following dichotomy:*

$$\textit{either} \quad \dim \Gamma_{\mathcal{R}\langle a\rangle} = \dim \Gamma_{\mathcal{R}} \quad \textit{or} \quad \dim \Gamma_{\mathcal{R}\langle a\rangle} = \dim \Gamma_{\mathcal{R}} + 1.$$

In the latter case, one can find an element $r \in \mathcal{R}$ such that

$$v(a-r) \notin \Gamma_{\mathcal{R}} \quad \textit{and} \quad \Gamma_{\mathcal{R}\langle a\rangle} = \Gamma_{\mathcal{R}} \oplus \mathbb{Q} \cdot v(a-r).$$

$\diamondsuit$

The significance of the valuation property lies to a great extent in its geometric content (see e.g. [14, 28]), namely it is equivalent to the preparation theorem in the sense of Parusiński–Lion–Rolin [33, 21, 34], which says that every definable function of several variables depends piecewise on (or can be prepared with respect to) any fixed variable in a certain simple fashion. The preparation theorem, in turn, yields many geometric, differential and integral applications, as the Lipschitz structure of subanalytic sets (cf. [33, 40]) and asymptotic expansions related to integration (cf. [20, 22]).

The preparation theorem can be derived from the valuation property through model-theoretic compactness and definable choice (cf. [14, 28]). Note that definable choice is available once we know the theory T is o-minimal. We recall below a version of this theorem for our o-minimal theory T with exponent field $\mathbb{Q}$.

Preparation Theorem. *Consider a definable function $f : \mathcal{R}^{n+1} \longrightarrow \mathcal{R}$ and an $\epsilon \in \mathbb{Q}$, $\epsilon > 0$. Then there exist $\lambda_1, \dots, \lambda_k \in \mathbb{Q}$ and definable functions*

$$r_1, \dots, r_k, c_1, \dots, c_k : \mathcal{R}^n \longrightarrow \mathcal{R}, \quad u_1, \dots, u_k : \mathcal{R}^{n+1} \longrightarrow (1-\epsilon, 1+\epsilon) \subset \mathcal{R}$$

such that for each $x \in \mathcal{R}^n$ and $y \in \mathcal{R}$ we have

$$f(x,y) = |\, y - r_i(x)|^{\lambda_i} \cdot c_i(x) \cdot u_i(x,y) \quad \textit{for an} \;\; i = 1, \dots, k.$$

$\diamondsuit$

Yet another consequence of the description of definable functions by $\mathcal{L}$-terms is a subanalytic version of Puiseux's theorem with parameter (see [35] for a classical version).

Puiseux's Theorem with Parameter. *Let $E \subset \mathbb{R}^m_x$ be a definable subset and*

$$f : E \times (0, 1) \longrightarrow \mathbb{R}$$

be a definable function. Then one can find a cell decomposition of E into finitely many Q-cells $C_1, \ldots, C_s$ (i.e. Q-analytic cells; cf. [30]) for which

- *either the function $f_x(t) := f(x, t)$ vanishes near zero for all $x \in C_i$,*
- *or there exist $k \in \mathbb{N}$, $p \in \mathbb{Q}$ and a definable function $F(x, t)$, Q-analytic in a neighbourhood U_i of $C_i \times \{0\} \subset C \times \mathbb{R}_t$, such that*

$$(*) \quad f(x, t) = t^p F(x, t^{1/k}) \quad \textit{and} \quad F(x, 0) \neq 0 \quad \textit{for all} \quad (x, t) \in U_i,\ x \in C_i.$$

We may, of course, assume that the function $f(x, t)$ is given by an $\mathcal{L}$-term $\tau(x, t)$. Then the proof is by induction with respect to the complexity of the term $\tau(x, t)$. The theorem is evident if $\tau(x, t)$ is a function symbol of $\mathcal{L}$. So assume that the term $\tau(x, t)$ is composed:

$$\tau(x, t) = \varphi(\tau_1(x, t), \ldots, \tau_r(x, t)),$$

and that the terms $\tau_j(x, t)$, $j = 1, \ldots, r$, satisfy condition $(*)$. The case where φ is the multiplication function, reciprocal function or root function is easy. What remains is to check the assertion when φ is the addition function or a restricted Q-function. We shall only demonstrate how to check the assertion for a restricted Q-function of two variables, because the remaining cases are similar.

Suppose two definable functions $g(x, t)$ and $h(x, t)$ satisfy condition $(*)$. After refining Q-cell decompositions, we can assume that condition $(*)$ is satisfied in a common Q-cell decomposition for both f and g, so that

$$g(x, t) = t^p G(x, t^{1/k}),\ h(x, t) = t^q H(x, t^{1/l}) \ \text{ for all } \quad (x, t) \in U_i,$$

with obvious assumptions about the numbers p, q, k, l and the functions G, H. The problem is non-trivial only when $p, q \geq 0$, and then we are reduced to the case where

$$\tau(x, t) = \varphi(G(x, t^{1/k}), H(x, t^{1/k}))$$

and $G(x,0), H(x,0) \in [-1,1]$ for all $x \in C_i$. Put

$$K(x,t) := \varphi(G(x,t), H(x,t))$$

and

$$V_j := \{x \in C_i : 0 = K(x,0) = \partial K/\partial t(x,0) = \ldots = \partial^{j-1}K/\partial t^{j-1}(x,0)\}.$$

Then the decreasing sequence of Q-analytic sets $(V_j)_{j\in\mathbb{N}}$ gets stabilized (see e.g. [5, 29, 30]):

$$C_i = V_0 \supseteq V_1 \supseteq V_2 \supseteq \ldots \supseteq V_n = V_{n+1} = \ldots$$

For each j, the Q-functions $K_x(t) := K(x,t)$ are of constant rank j for all $x \in V_j \setminus V_{j+1}$. Therefore, again after decomposing the Q-cell C_i into finer Q-cells, we can assume that, for each finer new Q-cell C, we have

$$K(x,t) = t^j F(x,t) \quad \text{for all} \quad (x,t) \in U,$$

where $U := U_i \cap (C \times \mathbb{R})$ is a neighbourhood of

$$C \times \{0\} \subset C \times \mathbb{R}_t,$$

$F(x,t)$ is a definable Q-function in U and $F(x,0) \neq 0$ for all $x \in C$. Hence

$$\varphi(G(x,t^{1/k}), H(x,t^{1/k})) = t^{j/k} F(x,t^{1/k}),$$

which is the desired conclusion. $\diamondsuit$

As a corollary, we immediately obtain

Piecewise Uniform Asymptotics. *Under the assumptions of the above Puiseux's theorem, there exist rational numbers $q_1, \ldots, q_r \in \mathbb{Q}$ such that for each $x \in E$*

- *either the function $f_x(t) := f(x,t)$ vanishes near zero,*
- *or is asymptotic to $c\,t^{r_i}$ for some $i = 1, \ldots, r$ and $c \in \mathbb{R}$, $c \neq 0$, i.e.*

$$\lim_{t\to 0^+} \frac{f(x,t)}{c\,t^{r_i}} = 1.$$

$\diamondsuit$

We conclude this article with the following comment. The fundamental method applied in our paper is transformation to normal crossings by blowing up. This method, developed in Zariski's school of algebraic geometry as one of the most powerful tools for the resolution of singularities, culminated in the famous work of Hironaka [18]. The very concept of a normal crossing had originated from ideas of the Italian school of algebraic geometry (see e.g. [42], Chap. I, and [6]).

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