A model theoretic application of Gelfond-Schneider theorem

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ABSTRACT

We prove that weakly o-minimal expansions of the ordered field of all real algebraic numbers are polynomially bounded. Apart of this we make a couple of observations concerning weakly o-minimal expansions of ordered fields of finite transcendence degree over the rationals. We show for instance that if Schanuel's conjecture is true and $K \subseteq \mathbb{R}$ is a field of finite transcendence degree over the rationals, then weakly o-minimal expansions of $(K, \leq, +, \cdot)$ are polynomially bounded.

0 Introduction

G. Faber in [Fa] (see also [Ma], Chapter II, §36) gives a construction of an entire transcendental function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with rational coefficients a_n such that all its values along with all derivatives are algebraic numbers at all algebraic arguments. Thus, one cannot detect the transcendence of f by examining the values assumed by f or its derivatives at algebraic arguments. An extreme form of such a behavior in the real case was discovered in 1995 by A. Wilkie (see [Wi]). He described a construction of an everywhere analytic transcendental function $g : \mathbb{R} \longrightarrow \mathbb{R}$, whose transcendence cannot be detected by any first order methods. In other words, the ordered field of all real algebraic numbers $\mathcal{R}_{\mathrm{alg}} := (\mathbb{R}_{\mathrm{alg}}, \leq, +, \cdot)$ when expanded by $g \restriction \mathbb{R}_{\mathrm{alg}}$ is an elementary substructure of $(\mathbb{R}, \leq, +, \cdot, g)$. The Wilkie's construction, thanks to results of [DD], provides an example of a proper o-minimal expansion of $\mathcal{R}_{\mathrm{alg}}$, answering positively a question asked in [LS] (see the last paragraph of §5).

As Wilkie's o-minimal expansion of $\mathcal{R}_{\mathrm{alg}}$ turns out to be polynomially bounded, and there are several known o-minimal expansions of $\mathcal{R} := (\mathbb{R}, \leq, +, \cdot)$ which are not polynomially bounded, it is natural to ask about the existence of o-minimal expansions of $\mathcal{R}_{\mathrm{alg}}$ which are not polynomially bounded. This paper gives a negative answer to that question, even if one relaxes the hypothesis of o-minimality to that of weak o-minimality.

The paper is organized as follows. In §1 we recall some basic notation and terminology concerning o-minimality and weak o-minimality, paying special attention to so called weakly o-minimal non-valuational expansions of ordered groups. We skip the most complicated inductive definitions referring the reader to [We07], where the basic model theory of weakly o-minimal non-valuational expansions of ordered groups is developed.

In §2 we prove the main result of the paper (Theorem 2.2). It says that all weakly o-minimal structures expanding $\mathcal{R}_{\mathrm{alg}}$ are polynomially bounded. The proof combines the author's work on weakly o-minimal non-valuational expansions of ordered groups with Baizhanov's results concerning expansions of models of weakly o-minimal theories by families of convex predicates, Miller's dichotomy for o-minimal expansions of $\mathcal{R}$, and the Gelfond-Schneider theorem.

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It was suggested by A. Macintyre, M. Tressl and A. Wilkie that the assertion of Theorem 2.2 might hold in a more general context, namely for ordered fields $K \subseteq \mathbb{R}$ of finite transcendence degree over $\mathbb{Q}$. In §3 we show that such a generalization holds provided that the Schanuel's conjecture is true (see Theorem 3.2). At the present moment the use of Schanuel's conjecture could hardly be avoided, since our proof relies on the fact that logarithms of certain algebraic numbers are algebraically independent over $\mathbb{Q}$. Although quite a lot is known about linear independence of logarithms of algebraic numbers (see for example [Ba]), the existence of two algebraic numbers a, b with $\log a, \log b$ algebraically independent over $\mathbb{Q}$ still remains an open problem in number theory.

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1 Notation and preliminaries

Let $(M, \leq)$ be a dense linear ordering without endpoints. An expansion $\mathcal{M}$ of $(M, \leq)$ is called o-minimal [weakly o-minimal] iff every subset of M definable in $\mathcal{M}$ is a finite union of intervals [respectively: convex sets]. Weak o-minimality, unlike o-minimality, in general is not preserved under elementary equivalence. We say that a first order theory is weakly o-minimal if all its models are weakly o-minimal structures. Weakly o-minimal theories do not have the independence property (see [MMS], Proposition 7.3). As shown in [Bz], any expansion of a model of a weakly o-minimal theory by a family of convex predicates has a weakly o-minimal theory. A weaker version of this theorem appears in [BP]. S. Shelah in [Sh783] proves a quantifier-elimination result for theories without the independence property, from which Baizhanov's theorem easily follows.

Fix a dense linear ordering $(M, \leq)$ without endpoints. An ordered pair $\langle C, D\rangle$ of non-empty convex subsets of M is called a cut in $(M, \leq)$ if $C < D$ and $C \cup D = M$. A cut $\langle C, D\rangle$ is said to be definable in $\mathcal{M}$ iff the sets C, D are definable. Given a weakly o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$, we denote by $\overline{M}^{\mathcal{M}}$ the set of all definable (in $\mathcal{M}$) cuts $\langle C, D\rangle$ such that D does not have the lowest element. In a natural way we can equip $\overline{M}^{\mathcal{M}}$ with a dense linear ordering without endpoints extending that of $(M, \leq)$, i.e. we identify an element $a \in M$ with the cut $\langle(-\infty, a], (a, +\infty)\rangle$. If $X \subseteq M^m$ is a non-empty set definable in $\mathcal{M}$, then a function $f : X \longrightarrow \overline{M}^{\mathcal{M}}$ is said to be definable in $\mathcal{M}$ if the set $\{\langle\overline{x}, y\rangle \in X \times M : y < f(\overline{x})\}$ is definable in $\mathcal{M}$.

An ordered group admitting a weakly o-minimal expansion is abelian and divisible (see [MMS], Theorem 5.1), and could be regarded as an extension of $(\mathbb{Q}, \leq, +)$. An ordered field which admits a weakly o-minimal expansion is real closed (see [MMS], Theorem 5.3) and could be regarded as an extension of $\mathcal{R}_{\mathrm{alg}}$.

A weakly o-minimal expansion $\mathcal{M} = (M, \leq, +, \ldots)$ of an ordered group $(M, \leq, +)$ is said to be non-valuational (or of non-valuational type) if for any cut $\langle C, D\rangle$ definable in $\mathcal{M}$, we have that $\inf\{y - x : x \in C, y \in D\} = 0$. Note that if $(M, \leq, +)$ has a dense ordered subgroup isomorphic to $(\mathbb{Q}, \leq, +)$, then $(M, \leq, +)$ is archimedean and every weakly o-minimal expansion of $(M, \leq, +)$ is of non-valuational type. In particular, every weakly o-minimal expansion of $\mathcal{R}_{\mathrm{alg}}$ is of non-valuational type.

As shown in [We07], weakly o-minimal non-valuational expansions of ordered groups enjoy a property called strong cell decomposition (see [We07, Theorem 2.15]). It implies that every weakly o-minimal non-valuational expansion of an ordered group has weakly o-minimal theory. Moreover, if $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$, then there is a canonical o-minimal extension $\overline{\mathcal{M}}$ of $\mathcal{M}$, expanding the abelian divisible group $(\overline{M}^{\mathcal{M}}, \leq, +)$ and closely related to $\mathcal{M}$. The construction of $\mathcal{M}$ guarantees that if $X \subseteq (\overline{M}^{\mathcal{M}})^m$ is definable in $\overline{\mathcal{M}}$, then the trace $X \cap M^m$ is definable in $\mathcal{M}$ (see [We07, §3]). Similarly, if $\mathcal{M} = (R, \leq, +, \cdot, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered (so real closed) field $(R, \leq, +, \cdot)$, then there is a canonical o-minimal extension $\overline{\mathcal{M}}$ of $\mathcal{M}$ expanding the real closed field $(\overline{R}, \leq, +, \cdot)$.

A weakly o-minimal expansion $\mathcal{M} = (R, \leq, +, \cdot, \ldots)$ of a real closed subfield of the reals is said to be polynomially bounded iff for every definable function $f : (a, +\infty) \longrightarrow \overline{R}^{\mathcal{M}}$, where $a \in R$, there is a positive integer n such that for sufficiently large $x \in R$ we have that $|f(x)| \leq x^n$. Ch. Miller in [Mi1] proved that any o-minimal expansion of $\mathcal{R}$ is either polynomially bounded or defines the exponential function exp. This result was later generalized in [Mi2], where it was shown that every o-minimal expansion a real closed field is either power bounded or defines a non-zero unary function equal to its derivative.

2 The main result

The theorem presented here was proved shortly after the author had noticed the failure of weak o-minimality for the structure $(\mathcal{R}_{\mathrm{alg}}, P)$, where P denotes a binary predicate defined as

$$P = \{\langle x, y\rangle \in \mathbb{R}^2_{\mathrm{alg}} : y < \exp(x)\}.$$

It became clear that if one finds an "elementary-like" way of embedding weakly o-minimal expansions of $\mathcal{R}_{\mathrm{alg}}$ into o-minimal structures expanding $\mathcal{R}$, then it might be possible to explore the effects of Miller's dichotomy for the larger structure on the original one. Actually, in [LS], Ch. Laskowski and Ch. Steinhorn proved that every o-minimal expansion of an ordered archimedean group could be elementarily and densely embedded into an o-miniamal expansion of $\mathcal{R}$. This result is sufficient to show that all o-minimal expansions of $\mathcal{R}_{\mathrm{alg}}$ are polynomially bounded. Indeed, consider an o-minimal expansion $\mathcal{M}$ of $\mathcal{R}_{\mathrm{alg}}$ which is not polynomially bounded. The expansion $\mathcal{N}$ of $\mathcal{R}$ into which $\mathcal{M}$ elementarily embeds is not polynomially bounded either. By Miller's dichotomy, the exponential function must be definable (in fact 0-definable) in $\mathcal{N}$. Hence the set of pairs $\langle a, b\rangle \in \mathbb{R}_{\mathrm{alg}}$ with $b < \exp(a)$ is 0-definable in $\mathcal{M}$, which by Lindemann's theorem contradicts the o-minimality of $\mathcal{M}$.

Before starting the proof of Theorem 2.2, we will state a classical number-theoretic result which provided a solution to Hilbert's seventh problem and was proved independently by A.O. Gelfond and T. Schneider in 1934. An excellent exposition of its proof can be found in Chapter X of [Ni].

Theorem 2.1 (Gelfond-Schneider theorem) *If α and β are algebraic numbers with $\alpha \neq 0$, $\alpha \neq 1$, and if β is not a real rational number, then any value of α^β is transcendental.*

At this point we are in a position to proceed towards the principal result of the paper.

Theorem 2.2 *Any weakly o-minimal expansion of $(\mathbb{R}_{\mathrm{alg}}, \leq, +, \cdot)$ is polynomially bounded.*

Proof. Suppose that $\mathcal{M}$ is a weakly o-minimal and not polynomially bounded expansion of $\mathcal{R}_{\mathrm{alg}}$. As $\mathcal{M}$ is of non-valuational type, by Corollary 2.16 from [We07], $Th(\mathcal{M})$ is weakly o-minimal. For every real transcendental number α, denote by P_α the set of all real algebraic numbers smaller than α. By [Bz], the structure $\mathcal{N} := (\mathcal{M}, P_\alpha : \alpha \in \mathbb{R} \setminus \mathbb{R}_{\mathrm{alg}})$ has weakly o-minimal theory. Obviously, $\mathcal{N}$ is not polynomially bounded and is of non-valuational type. The canonical o-minimal extension $\overline{\mathcal{N}}$ of $\mathcal{N}$ (as constructed in §3 of [We07]) is isomorphic to an o-minimal and not polynomially bounded expansion of the ordered field of reals. Therefore, without loss of generality we can assume that the universe of $\overline{\mathcal{N}}$ is $\mathbb{R}$. By [Mi1], the exponential function exp is definable in $\overline{\mathcal{N}}$, so the set $\{\langle a,b\rangle \in \mathbb{R}^2 : b < \exp(a)\}$ is definable in $\overline{\mathcal{N}}$. By results of [We07, §3], its intersection with $\mathbb{R}^2_{\rm alg}$ is definable in $\mathcal{N}$. So the set $S := \{\langle a,b\rangle \in \mathbb{R}^2_{\rm alg} : b < \exp(a)\}$ is definable in $\mathcal{N}$.

Using the fact that at non-zero algebraic arguments the exponential function assumes only transcendental values, we can easily see that for every $\langle a,b\rangle \in \mathbb{R}^2_{\rm alg}$, the following conditions are equivalent:

- $a > 0 \wedge b < a^{\sqrt{a}}$;
- $a > 0 \wedge (\exists x \neq 0)(\langle x,b\rangle \in S \wedge \langle \frac{x}{\sqrt{a}}, a\rangle \notin S)$.

Hence the set $T := \{\langle a,b\rangle \in \mathbb{R}_{\rm alg} : a > 0, b < a^{\sqrt{a}}\}$ is definable in $\mathcal{N}$. Let

$$X = \{a \in \mathbb{R}_{\rm alg} : a > 0 \text{ and the set } \{y \in \mathbb{R}_{\rm alg} : \langle a,y\rangle \in T\} \text{ has supremum in } \mathbb{R}_{\rm alg}\}.$$

Certainly, X is definable in $\mathcal{N}$. Note that if $n > 0$ is a square of an integer, then $n \in X$. On the other hand, the Gelfond-Schneider theorem implies that if p is a prime, then $p \notin X$. Consequently, X cannot be a finite union of convex sets, which contradicts the weak o-minimality of $\mathcal{N}$. ■

3 Weakly o-minimal expansions of ordered fields of finite transcendence degree

Recall that Schanuel's conjecture is a statement asserting that whenever $\alpha_1, \ldots, \alpha_n$ are complex numbers linearly independent over the rationals, then the field generated by

$$\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)$$

has transcendence degree over $\mathbb{Q}$ greater than or equal to n.

Fact 3.1 *Suppose that Schanuel's conjecture is true.*

(a) If $p_1, \ldots, p_n$ are distinct primes, then $\log p_1, \ldots, \log p_n$ are algebraically independent over $\mathbb{Q}$.

(b) If $p_1, \ldots, p_n$ are distinct odd primes, then $\frac{\log p_1}{\log 2}, \ldots, \frac{\log p_n}{\log 2}$ are algebraically independent over $\mathbb{Q}$.

Proof. In order to prove (a), assume that $p_1, \ldots, p_n$ are distinct primes. It is easy to see that $\log p_1, \ldots, \log p_n$ are linearly independent over $\mathbb{Q}$. Schanuel's conjecture (applied to $\log p_1, \ldots, \log p_n$) implies that the transcendence degree over $\mathbb{Q}$ of the field generated by

$$p_1, \ldots, p_n, \log p_1, \ldots, \log p_n$$

equals n, which means that $\log p_1, \ldots, \log p_n$ are algebraically independent over $\mathbb{Q}$.

(b) is an immediate consequence of (a). ■

Theorem 3.2 *Assume that $(K, \leq, +, \cdot)$ is an ordered subfield of $(\mathbb{R}, \leq, +, \cdot)$ of finite transcendence degree over $\mathbb{Q}$. If Schanuel's conjecture is true, then every weakly o-minimal expansion of $(K, \leq, +, \cdot)$ is polynomially bounded.*

Proof. Suppose for a contradiction that there is a weakly o-minimal structure $\mathcal{M}$ expanding $(K, \leq, +, \cdot)$ which is not polynomially bounded. Let $\mathcal{N} = (\mathcal{M}, P_\alpha)_{\alpha \in \mathbb{R} \setminus K}$, where $P_\alpha = \{x \in K : x > \alpha\}$. As in the proof of Theorem 2.2, $Th(\mathcal{N})$ is weakly o-minimal and the set $S := \{\langle x, y \rangle \in K^2 : y < \exp(x)\}$ is definable in $\mathcal{N}$.

Claim. The set

$$T := \{\langle a, b \rangle \in K : a > 0, b < \log_2(a)\}$$

is definable in $\mathcal{N}$.

Proof of the Claim. Let $X_1 = \{a \in K : a < \log 2\}$, $X_2 = \{a \in K : a > \log 2\}$, and denote by S_1 the interior of $K^2 \setminus S$. Clearly, the sets X_1, X_2 and S_1 are definable in $\mathcal{N}$. Note that for $\langle a, b \rangle \in K^2$ the following conditions are equivalent.

- $a > 0 \wedge b < \log_2 a$;
- $a > 0 \wedge (\exists z)(b \cdot \log 2 < z \wedge \langle z, a \rangle \in S_1)$.

Also for $\langle b, c \rangle \in K \times K$, the following conditions are equivalent.

- $b \cdot \log 2 < c$;
- $(b > 0 \wedge \frac{c}{b} \in P_2) \vee (b = 0 < c) \vee (b < 0 \wedge \frac{c}{b} \in P_1)$.

Consequently, the set T is definable in $\mathcal{N}$.

Now, let

$$X = \{a \in K : a > 0 \text{ and the set } \{y \in K : \langle a, y \rangle \in S\} \text{ has supremum in } K\}.$$

For all $n \in \mathbb{N}$, $2^n \in X$. By Fact 3.1 and Schanuel's conjecture, the field K contains only finitely many numbers of the form $\log_2 p$, where p is an odd prime. Hence only finitely many primes belong to X. Consequently, X is not a union of finitely many convex sets, a contradiction. ■

The arguments applied in the proof of Theorem 3.2 could be easily modified to give a proof of the following corollary.

Corollary 3.3 *Assume that*

- $(K, \leq, +, \cdot)$ *is an ordered subfield of* $\mathcal{R}$;
- $I \subseteq \mathbb{R}$ *is an open interval;*
- $f : I \longrightarrow \mathbb{R}$ *is a function definable in the structure* $(\mathbb{R}, \leq, +, \cdot, \exp)$;
- $(a_n)_{n \in \mathbb{N}}$ *and* $(b_n)_{n \in \mathbb{N}}$ *are sequences of elements from* $I \cap K$;
- $\lim\limits_{n \longrightarrow \infty} a_n = \lim\limits_{n \longrightarrow \infty} b_n = \sup I$;
- *the sets* $\{n \in \mathbb{N} : f(a_n) \in K\}$ *and* $\{n \in \mathbb{N} : f(b_n) \notin K\}$ *are both infinite.*

Then every weakly o-minimal expansion of $(K, \leq, +, \cdot)$ *is polynomially bounded.*

Corollary 3.4 *Assume that* $(K, \leq, +, \cdot)$ *is an ordered subfield of* $(\mathbb{R}, \leq, +, \cdot)$ *of finite transcendence degree over* $\mathbb{Q}$. *If* $e \in K$, *then every weakly o-minimal expansion of* $(K, \leq, +, \cdot)$ *is polynomially bounded.*

Proof. In Corollary 3.3 assume that $I := \mathbb{R}$, $f := \exp$, $(a_n)_{n\in\mathbb{N}}$ is a sequence of rational numbers with $\lim_{n\longrightarrow\infty} a_n = \infty$, and $(b_n)_{n\in\mathbb{N}}$ is a sequence of algebraic numbers from K linearly independent over $\mathbb{Q}$ with $\lim_{n\longrightarrow\infty} b_n = \infty$. Then $\exp(a_n) \in K$ whenever $n \in \mathbb{N}$. By Lindemann's theorem, as $\mathrm{tr}_{\mathbb{Q}}(K)$ is finite, $\exp(b_n) \notin K$ for almost all $n \in \mathbb{N}$. So by Corollary 3.3, every weakly o-minimal expansion of $(K, \leq, +, \cdot)$ is polynomially bounded. ■

We finish this paper showing an o-minimal counterpart of Theorem 3.2 for ordered fields which are not necessarily archimedean.

Fact 3.5 *If $(K, \leq, +, \cdot)$ is an ordered field of finite transcendence degree over $\mathbb{Q}$, then every o-minimal expansion of $(K, \leq, +, \cdot)$ is power bounded.*

Proof. Suppose that there is an o-minimal expansion $\mathcal{M}$ of $(K, \leq, +, \cdot)$ which is not power bounded. By [Mi2], there is a non-zero definable (in $\mathcal{M}$) function $f : K \longrightarrow K$ satisfying $f = f'$ and $f(0) = 1$. As K is real closed, we can assume that $\mathcal{R}_{\mathrm{alg}}$ is a substructure of $(K, \leq, +, \cdot)$. Let L be the subfield of K generated by $\mathbb{R}_{\mathrm{alg}} \cup f[\mathbb{R}_{\mathrm{alg}}]$. Our assumptions guarantee that L has finite transcendence degree over $\mathbb{Q}$. There is a natural embedding of $(L, \leq, +, \cdot)$ into $(\mathbb{R}, \leq, +, \cdot)$ such that $F(f(a)) = \exp(a)$ for $a \in L$. By Lindemann's theorem, the field $F[L]$ has an infinite transcendence degree over $\mathbb{Q}$, a contradiction. ■

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