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COMPATIBILITY OF QUASI-ORDERINGS AND VALUATIONS; A BAER-KRULL THEOREM FOR QUASI-ORDERED RINGS

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Abstract. In [9], we introduced the class of quasi-ordered commutative rings and proved that each such ring $(R, \preceq)$ is either an ordered ring or a valued ring. Here we take a further step in our investigation of this class. We develop the notion of $\preceq$-compatible valuations, leading to a definition of the rank of $(R, \preceq)$. We exploit it to establish a Baer-Krull Theorem; more precisely, fixing a valuation v on R, we describe all v-compatible quasi-orders on R.
In case where the quasi-order is an order, this yields a generalization of the classical Baer-Krull Theorem for ordered fields. Else, if we restrict attention to quasi-orders that come from valuations, our results give rise to a complete characterization of all the coarsenings, respectively all the refinements, of a given valuation v on R.

1. Introduction

Often, orders and valuations are treated as two different subjects. However, in his note [3], S.M. Fakhruddin introduced the notion of quasi-ordered fields $(K, \preceq)$ and proved the dichotomy, that each such field is either an ordered field or else there is a valuation v on K such that $x \preceq y$ if and only if $v(y) \leq v(x)$. Thus, Fakhruddin found a uniform approach to these notions (see [3, Theorem 2.1]). By exploiting his result, the second author of this paper was able to generalize the said dichotomy from fields to commutative rings with 1 (see [9, Theorem 4.6]).

The aim of the present paper is to continue our study of quasi-ordered rings, that way unifying the theories of orders and valuations. For this purpose we consider important results from real algebra, which are also meaningful if the order is replaced with a binary relation $\preceq$ defined by a valuation as above. More precisely, we will establish the two theorems from the title, which are both stated for ordered fields in the literature, and that both deal with the interaction of orders and valuations, for quasi-ordered rings. The main focus of attention of our work is to demonstrate that via quasi-orders, ordered and valued commutative rings can be treated simultaneously - not only when formulating these theorems, but also throughout the proofs. Furthermore, we generalize the two theorems from fields to commutative rings in case where the quasi-order is an order, whereas we obtain new statements if the quasi-order comes from a valuation. The paper is organized as follows:

In section 2 we briefly recall ordered and valued rings, and give our definition of quasi-ordered rings (see Definition 2.6). Moreover, we quote the two theorems that we want to establish for this class (see Theorems 2.10 and 2.11).

Section 3 deals with the notion of compatibility between quasi-orders and valuations. Given a quasi-ordered ring $(R, \preceq)$, we first give a characterization of all valuations v on R that are compatible with $\preceq$ (see Theorem 3.12). In case where v is Manis (i.e. surjective) and $\preceq$ comes from a Manis valuation w, we will show that v is compatible with $\preceq$ if and only if v is a coarsening of w (see Lemma 3.15), leading to a characterization of all the coarsenings v of w (see Theorem 3.16). Afterwards, we are in the position to define the rank of a quasi-ordered ring (see Definition 3.19).

In the fourth and final section we establish a Baer-Krull Theorem for quasi-ordered rings (see Theorem 4.10, respectively Corollary 4.11). Once this is proven, we can not only generalize the classical Baer-Krull Theorem to ordered rings (see Corollary 4.13), but also characterize all Manis refinements w of a valued ring (R, v), given that v is also Manis (see Corollary 4.16).

2. Preliminaries

Here we briefly introduce some basic results concerning valued, ordered and quasi-ordered rings. Moreover, we introduce the theorems, which we aim to establish for quasi-ordered rings (see Theorems 2.10 and 2.11). Throughout this section let R always denote a commutative ring with 1.

Definition 2.1. (see [1, VI. 3.1]) Let $(\Gamma, +, \leq)$ be an ordered abelian group and ∞ a symbol such that $\gamma < \infty$ and $\infty = \infty + \infty = \gamma + \infty = \infty + \gamma$ for all $\gamma \in \Gamma$. A map $v : R \to \Gamma \cup \{\infty\}$ is called a **valuation** on R if $\forall x, y \in R:$

(V1) $v(0) = \infty$,
(V2) $v(1) = 0$,
(V3) $v(xy) = v(x) + v(y)$,
(V4) $v(x + y) \geq \min\{v(x), v(y)\}$.

We always assume that Γ is the group generated by $\{v(x) : x \in v^{-1}(\Gamma)\}$ and call it the **value group** of R. We also denote it by Γ_v. The set $\mathfrak{q}_v := \text{supp}(v) := v^{-1}(\infty)$ is called the **support** of v.

Facts 2.2.

(1) An easy consequence of the axioms (V1) - (V4) is that $\mathfrak{q}_v$ is a prime ideal. Thus, when valuations are defined on fields, (V1) is usually replaced with $v(x) = \infty \Leftrightarrow x = 0$, and (V2) is omitted.
(2) Generally, v is not surjective, as its image is not necessarily closed under additive inverses. However, if R is a field and $x \in R$ with $v(x) = \gamma \in \Gamma_v$, then $v(x^{-1}) = -\gamma$. Thus, field valuations are always surjective.
(3) $R_v := \{x \in R : v(x) \geq 0\}$ is a subring of R, called the **valuation ring** of v. The set $I_v := \{x \in R : v(x) > 0\}$ is a prime ideal of R, which we will call **valuation ideal**. If R is a field, then R_v is a local ring and I_v its maximal ideal.

We conclude our introduction of valuations with a simple but very helpful lemma.

Lemma 2.3. *Let (R, v) be a valued ring and $x, y \in R$ such that $v(x) \neq v(y)$. Then $v(x + y) = \min\{v(x), v(y)\}$.*

Proof. Completely analogue as in the field case, see for instance [2, (1.3.4)]. □

Let us now turn towards the notion of orders on rings. For the sake of convenience, they are usually identified with positive cones $P \subset R$, where $x \in P$ expresses that x is non-negative.

Definition 2.4. (see [8, p.29]) A **positive cone** of R is a subset $P \subset R$ such that the following conditions are satisfied:

(P1) $P \cup -P = R$,
(P2) $\mathfrak{p} := P \cap -P$ is a prime ideal of R, called the **support** of R,
(P3) $P \cdot P \subseteq P$,
(P4) $P + P \subseteq P$.

Definition 2.5. (see [9, Definition 2.3]) Let $\leq$ be a binary, reflexive, transitive and total relation on R. Then $(R, \leq)$ is called an **ordered ring** if $\forall x, y, z \in R:$

(O1) $0 < 1$,

(O2) $xy \leq 0 \Rightarrow x \leq 0 \vee y \leq 0$,
(O3) $x \leq y,\ 0 \leq z \Rightarrow xz \leq yz$,
(O4) $x \leq y \Rightarrow x + z \leq y + z$.

The set of all orders of R is in $1:1$ correspondence with the set of all positives cones of R via $x \leq y \Leftrightarrow y - x \in P$. Note that if R is a field, then (P2) yields that $\mathfrak{p} = \{0\}$, which precisely means that the corresponding order $\leq$ is anti-symmetric (and vice versa).

Recall from the introduction that some quasi-orders $\preceq$ of R are induced by a valuation v via $x \preceq y$ if and only if $v(y) \leq v(x)$. In this case all elements are non-negative. Hence, positive cones are inappropriate to deal with quasi-orders. So in order to compare ordered and quasi-ordered rings, it is necessary to stick to Definition 2.5.

Let us now have a closer look at quasi-ordered rings. As mentioned above, Fakhruddin developped a notion of quasi-ordered fields $(K, \preceq)$ and was able to show that quasi-ordered fields are either ordered fields or else $\preceq$ comes from a valuation as above (see [3, Theorem 2.1]). In [9], the second author of this paper generalized this result to commutative rings with 1, leading to the following result:

Definition 2.6. (see [9, Definition 3.2]) Let R be a commutative ring with 1 and $\preceq$ a binary, reflexive, transitive and total relation on R. If $x, y \in R$, we write $x \sim y$ if $x \preceq y$ and $y \preceq x$, and we write $x \prec y$ if $x \preceq y$ but $y \npreceq x$.
The pair $(R, \preceq)$ is called a **quasi-ordered ring** if $\forall x, y, z \in R:$

(QR1) $0 \prec 1$,
(QR2) $xy \preceq 0 \Rightarrow x \preceq 0 \vee y \preceq 0$,
(QR3) $x \preceq y,\ 0 \preceq z \Rightarrow xz \preceq yz$,
(QR4) $x \preceq y,\ z \nsim y \Rightarrow x + z \preceq y + z$,
(QR5) If $0 \prec z$, then $xz \preceq yz \Rightarrow x \preceq y$.

The equivalence class of x (w.r.t. $\sim$) is denoted by E_x, and E_0 is called the **support** of $\preceq$.

Theorem 2.7. (*see* [9, Theorem 4.6]) *A quasi-ordered ring $(R, \preceq)$ is either an ordered ring or else a valued ring (R, v) such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$.*
Moreover, the support of the quasi-order coincides with the support of the order, respectively with the support of the valuation.

Remark 2.8.

(1) If $(R, \preceq)$ is a quasi-ordered ring with $x \sim 0$ and $y \nsim 0$, then $x + y \sim y$ (see [9, Lemma 3.6]). This result will be useful later.
(2) As indicated in the previous theorem, the support E_0 is a prime ideal of R (see [9, Proposition 3.8]).
(3) The “new” axiom (QR5) is crucial for the dichotomy, see [9, Proposition 3.1]. Moreover, note that it easily implies (QR2). Indeed, if $xy \preceq 0$ and $0 \prec x$, then (QR2) yields $y \preceq 0$. We decided to keep axiom (QR2) in order to preserve the analogy between ordered and quasi-ordered rings.
(4) If R is a field, then the axioms (QR1) and (QR2) can be replaced with the axiom $x \sim 0 \Rightarrow x = 0$, while (QR5) becomes unnecessary. As a matter of fact, this is precisely how Fakhruddin introduced quasi-ordered fields in the first place (see [3, 2]).

Later on, we will also use the following variant of axiom (QR5).

Lemma 2.9. *Let $(R, \preceq)$ be a quasi-ordered ring and $x, y, z \in R$. If $z \nsim 0$, then $xz \sim yz \Rightarrow x \sim y$.*

Proof. This was already proven in [9, Lemma 3.7]. □

We conclude this introductive section by recalling the Theorems 2.10 and 2.11 below, which we will establish for quasi-ordered rings in this paper. So let $(K, \leq)$ be an ordered field. Recall that a valuation v on K is said to be compatible with $\leq$, if $0 \leq x \leq y$ implies $v(y) \leq v(x)$ (see for instance [6, Definition 2.4]). A subset $S \subseteq K$ is convex (w.r.t. $\leq$), if from $x \leq y \leq z$ and $x, z \in S$ follows $y \in S$.

Theorem 2.10. (*see* [6, Theorem 2.3 and Proposition 2.9] *or* [2, Proposition 2.2.4]) *Let $(K, \leq)$ be an ordered field and let v be a valuation on K. The following are equivalent:*

(1) *v is compatible with $\leq$.*
(2) *The valuation ring K_v is convex.*
(3) *The maximal ideal I_v is convex.*
(4) *$I_v < 1$.*
(5) *$\leq$ induces canonically via the residue map $\varphi_v : K_v \to Kv := K_v/I_v$, $x \mapsto x + I_v$ an order $\leq'$ on the residue field Kv.*

The fifth condition of the previous result is crucial for the second theorem, the so called Baer-Krull Theorem (see [2, p.37]). Let K again be a field and v a valuation on K with value group Γ_v. Note that $\overline{\Gamma_v} = \Gamma_v/2\Gamma_v$ is in a canonical way an $\mathbb{F}_2$-vector space. Hence, we find a subset $\{\pi_i : i \in I\} \subset K$, such that $\{\overline{v(\pi_i)} : i \in I\}$ is a $\mathbb{F}_2$-basis of $\overline{\Gamma_v}$.

Theorem 2.11. (*Baer-Krull Theorem for ordered fields; see* [2, Theorem 2.2.5]) *Let K be a field and v a valuation on K. Moreover, let $\mathcal{X}(K)$ and $\mathcal{X}(Kv)$ denote the set of all orderings on K, respectively Kv. There exists a bijective map*

$$\psi : \{\leq\, \in \mathcal{X}(K) : \ \leq \ \textit{is } v\textit{-compatible}\} \to \{-1, 1\}^I \times \mathcal{X}(Kv),$$

described as follows: given an ordering $\leq$ in the domain of ψ, let $\eta_\leq : I \to \{-1, 1\}$, where $\eta_\leq(i) = 1 \Leftrightarrow 0 \leq \pi_i$. Then the map $\leq \mapsto (\eta_\leq, \leq')$ is the above bijection, where $\leq'$ denotes the order on Kv from Theorem 2.10(5).

3. Compatibility between quasi-orders and valuations

The aim of this section is to prove an analogue of Theorem 2.10 for quasi-ordered rings. First we convince ourselves that for this end, we have to restrict our attention to surjective valuations (see Example 3.2), also called Manis valuations. Then we establish that the conditions (1) - (3) and (5) from the said theorem are equivalent for quasi-ordered rings, if v is Manis (see Theorem 3.12). This gives rise to a characterization of all Manis valuations w on R, which are coarser than v (see Theorem 3.16), leading to a definition of the rank of a quasi-ordered ring (see Definition 3.19). Afterwards, we prove that $I_v < 1$ is no equivalent condition anymore, no matter of which of the two kinds the quasi-order is (see Examples 3.21 and 3.22). We conclude this section by showing that Theorem 2.10 holds to the full extend, if we additionally demand that v is local (see Lemma 3.25).

Notation 3.1. We use the following notation for the rest of this section:

(1) Let R always denote a commutative ring with 1. If a quasi-order $\preceq$ on R (see Definition 2.6) is induced by some valuation v on R (see Theorem 2.7), we also write $\preceq_v$ instead of $\preceq$ and call it a **proper quasi-order** (p.q.o).
(2) The symbol $\leq$ is reserved for orders.
(3) If v is a valuation on R, we write $\mathfrak{q}_v := \mathrm{supp}(v) := v^{-1}(\infty)$ for its **support** and Γ_v for its **value group** (see Definition 2.1). Moreover, we denote by $R_v := \{x \in R : v(x) \geq 0\}$ the **valuation ring** of v, by $I_v := \{x \in R : v(x) > 0\}$ the prime ideal of R_v induced by v, and by $U_v := R_v \backslash I_v := \{x \in R : v(x) = 0\}$. Finally, $Rv := R_v/I_v$ denotes the **residue ring** of v and $\varphi_v : R_v \to Rv,\ x \mapsto x + I_v$ the **residue map**.

In general, we cannot expect that Theorem 2.10 holds even for ordered rings, as the following basic examples show:

Example 3.2.

(1) Consider the map $v : \mathbb{Z}[X] \to \mathbb{Z} \cup \{\infty\}$, $f = \sum_{i \in \mathbb{N}} a_i X^i \mapsto -\deg f$. It is easy to verify that v is a valuation on R. We can extend the unique order on $\mathbb{Z}$ to $\mathbb{Z}[X]$ by declaring $f \leq g :\Leftrightarrow f(0) \leq g(0)$. Note that $R_v = \mathbb{Z}$ and $I_v = \{0\}$, so obviously the conditions (4) and (5) of Theorem 2.10 are satisfied. However, the inequalities $0 \leq X \leq 0$ yield that neither I_v nor R_v is convex with respect to $\leq$. Moreover, we have $0 \leq X + 1 \leq 1$, but $v(X+1) = -1 < 0 = v(1)$, so (1) is also not satisfied.

(2) Let p be a prime number and v the p-adic valuation on the integers $\mathbb{Z}$, i.e. if $x = p^r a_1 \ldots a_n$ in the unique prime factorization, then $v(x) = r$ (see [2, (1.3.1)]. Moreover, let $\leq$ denote the unique order on $\mathbb{Z}$. Then $R_v = \mathbb{Z}$ is convex, so (2) holds. However, it is easy to see that all the other conditions of Theorem 2.10 are not satisfied.

These counterexamples can be prevented by demanding surjectivity of the valuation. Recall that valuations on fields are automatically surjective. Contrary, in the ring case, $v(R \backslash \mathfrak{q}_v)$ is not necessarily closed under additive inverses, as R is not necessarily closed under multiplicative inverses (see Facts 2.2).

Definition 3.3. (see [7, p.193]) Let v be a valuation on R. Then v is said to be a **Manis valuation**, if v is surjective.

Surjectivity will be frequently used later on, as it mitigates the lack of multiplicative inverses. We now turn towards the proof of Theorem 2.10 for quasi-ordered rings. This requires some preliminaries. First of all, we define the notions of compatibility and convexity for quasi-ordered rings in their usual sense.

Definition 3.4. Let $(R, \preceq)$ be a quasi-ordered ring. A valuation v on R is said to be **compatible** with $\preceq$ (or $\preceq$-compatible), if for all $y, z \in R :$

$$0 \preceq y \preceq z \Rightarrow v(z) \leq v(y).$$

Definition 3.5. Let $(R, \preceq)$ be a quasi-ordered ring and $S \subseteq R$ a subset of R. Then S is said to be **convex**, if $x \preceq y \preceq z$ and $x, z \in S$ implies $y \in S$.

The following lemma simplifies convexity in a usual manner and holds particularly for the valuation ring R_v and its prime ideal I_v, as $v(x) = v(-x)$ for all $x \in R$.

Lemma 3.6. *Let $(R, \preceq)$ be a quasi-ordered ring. A subset $S \subseteq R$ with $0 \in S$ and $S = -S$ is convex, if and only if $0 \preceq y \preceq z$ and $z \in S$ implies $y \in S$.*

Proof. The implication $\Rightarrow$ is trivial. So suppose that the right hand side holds and let $x \preceq y \preceq z$ with $x, z \in S$. If $0 \preceq y$, it follows immediately by assumption that $y \in S$. So suppose that $y \prec 0$. Then $x \preceq y \prec 0$. We will show $0 \prec -y \preceq -x$. Note that $-x \in S$ because $S = -S$. Hence, we obtain $-y \in S$, but then also $y \in S$.
Clearly $0 \prec -x, -y$ by axiom (QR4) and the fact that E_0 is an ideal (see Remark 2.8(2)). It remains to show that $-y \preceq -x$. Assume for a contradiction $-y \not\preceq -x$, so $-x \prec -y$. Note that $y \prec 0 \prec -x, -y$, therefore $-x \not\sim y$ and $y \not\sim -y$. Via (QR4), it follows from $x \preceq y$ that $0 \preceq y - x$ and from $-x \preceq -y$ that $y - x \preceq 0$. Thus, $y - x \in E_0$. This implies $-y \sim -x$ (see Remark 2.8(2)), a contradiction. $\square$

The most difficult part of the proof will be to show that if v is a $\preceq$-compatible valuation on R, then $\preceq$ induces a quasi-order on the residue ring Rv. For that purpose we want to exploit convexity of I_v.

Lemma 3.7. *Let $(R, \preceq)$ be a quasi-ordered ring, v a valuation on R such that I_v is convex, and $u \in U_v$.*

(1) If $c \in I_v$, then $c \not\sim u$.
(2) If $0 \prec u$, then $0 \prec u + c$ for all $c \in I_v$.
(3) If $u \prec 0$, then $u + c \prec 0$ for all $c \in I_v$.

Proof.

(1) Assume $c \sim u$. Then $c \preceq u \preceq c$ and convexity of I_v yields $u \in I_v$, a contradiction.
(2) Assume that $0 \prec u$ but $0 \not\prec u+c$ for some $c \in I_v$. Then $0 \prec u$ and $u+c \preceq 0$. Note that this implies $c \notin E_0$, as otherwise $u \sim u + c$ (see Remark 2.8(1)). Hence, we obtain $u \preceq -c$. So it holds $0 \prec u \preceq -c$. Convexity of I_v yields $u \in I_v$, a contradiction.
(3) Assume $0 \preceq u + c$ for some $c \in I_v$. Then $u \prec 0 \preceq u + c$. It remains to show that $-u \not\sim u + c$. Then $0 \prec -u \preceq c$ and one may conclude by convexity of I_v. So assume for a contradiction that $-u \preceq u + c$. From Lemma 2.3 follows $u + c \in U_v$, so (1) yields that $-c \not\sim u + c$. Thus, one obtains $-u - c \preceq u$. Now note that $0 \prec -u \in U_v$. So (2) yields $0 \prec -u - c$. Therefore $0 \prec -u - c \preceq u \prec 0$, a contradiction. This finishes the proof.

□

Finally, we require a couple of results that Fakhruddin established in the more specific setting of quasi-ordered fields (see [3]).

Lemma 3.8. *Let $(R, \preceq)$ be a quasi-ordered ring and $x \in R$. Then $x \sim -x$ if and only if $0 \preceq x, -x$.*

Proof. Just as in the case of quasi-ordered fields, see [3, Lemma 3.1]. □

Lemma 3.9. *Let $(R, \preceq)$ be a quasi-ordered ring and $x, y \in R$. If $x \sim y$, then $x \sim -y$ or $0 \sim x - y$.*

Proof. If $x, y \sim 0$, then $x \sim -y$, as E_0 is an ideal. So suppose that $x, y \nsim 0$. Assume further that $x \nsim -y$. We show $0 \sim x - y$. Note that $y \preceq x \nsim -y$. Therefore $0 \preceq x - y$. Moreover, $x \preceq y \sim x \nsim -y$, so $y \nsim -y$, and therewith $x - y \preceq 0$. Thus, $0 \sim x - y$. □

Corollary 3.10. *Let $(R, \preceq)$ be a quasi-ordered ring. Then $\sim$ is preserved under multiplication, i.e. if $x, y, a \in R$ such that $x \sim y$, then $ax \sim ay$.*

Proof. The cases $0 \preceq a$ (axiom (QR3)) and x, y in E_0 (E_0 is an ideal) are both trivial. So suppose that $0 \not\preceq a$ and $x, y \nsim 0$. Then $0 \preceq -a$. The previous lemma gives rise to a case distinction. First suppose $0 \sim x - y$. Since $-x \nsim 0$ it holds $-x \nsim x - y$. Hence, $0 \sim x - y$ yields $-x \sim -y$. Since $0 \preceq -a$, axiom (QR3) yields $ax \sim ay$. Now suppose that $0 \nsim x - y$. Then also $0 \nsim y - x$. The previous lemma implies $x \sim -y$ and $y \sim -x$. Therefore $-y \sim x \sim y \sim -x$. Since $0 \preceq -a$, we obtain $ay = (-a)(-y) \sim (-a)(-x) = ax$. □

Lemma 3.11. *Let $(R, \preceq)$ be a quasi-ordered ring such that $0 \prec -1$. Then it holds $x + y \preceq \max\{x, y\}$ for all $x, y \in R$.*

Proof. Basically as in the field case, see [3, Lemma 4.1]. Suppose $x \preceq y$. Assume for a contradiction that $y \prec x + y$. Note that $0 \preceq 1$ by axiom (QR1). Lemma 3.8 implies $-1 \sim 1$, so the previous corollary yields $-r \sim r$ for all $r \in R$. It follows $-x \sim x \preceq y \prec x + y$. Particularly, $-y \nsim x + y$, since $y \nsim x + y$. So, by applying (QR4), we obtain $x + y \sim -x - y \preceq x \preceq y$, a contradiction. □

Finally, we can prove the main theorem of this section:

Theorem 3.12. *Let $(R, \preceq)$ be a quasi-ordered ring and let v be a Manis valuation on R. The following are equivalent:*

(1) v is compatible with $\preceq$.
(2) R_v is convex.
(3) I_v is convex.
(4) $\preceq$ induces canonically via the residue map $x \mapsto x + I_v$ a quasi-order $\preceq'$ with support $\{0\}$ on the residue ring Rv.

Proof. We first prove that (1) implies (2). So let $0 \preceq y \preceq z$ with $z \in R_v$. (1) yields that $v(z) \leq v(y)$. Since $v(z) \geq 0$, this shows that $v(y) \geq 0$, i.e. $y \in R_v$.

Next we show that (2) implies (3). So let $0 \preceq y \preceq z$ with $z \in I_v$. Assume for a contradiction that $y \notin I_v$, so $v(y) \leq 0$. Since $z \in I_v$, it holds $\gamma := v(z) > 0$. As v is Manis, there exists some $a \in R$ such that $v(a) = -\gamma < 0$. Moreover, one can choose a such that $0 \preceq a$ (if $a \prec 0$, then $0 \preceq -a$ and $v(-a) = v(a)$). Axiom (QR3) yields $0 \preceq ay \preceq az$. As 0 and az lie in R_v, it follows by convexity that $ay \in R_v$, i.e. $v(ay) \geq 0$. However, $v(ay) = v(a) + v(y) < 0$, a contradiction.

Now we prove that (3) implies (1). Suppose that I_v is convex. We claim that v is compatible with $\preceq$. Assume for a contradiction there exist $0 \preceq y \preceq z$ such that $v(y) < v(z)$. Choose $a \succ 0$ such that $v(a) = -v(y)$. Via axiom (QR3) one obtains $0 \preceq ay \preceq az$ with $v(ay) = 0$ and $v(az) = v(z) - v(y) > 0$, so $az \in I_v$ but $ay \notin I_v$. This contradicts the convexity of I_v.

We conclude by showing that (3) and (4) are equivalent. First suppose that (4) holds and let $0 \preceq y \preceq z$ with $z \in I_v$. Assume for a contradiction $y \notin I_v$. Choose $a \in R$ with $0 \preceq a$ and $v(a) = -v(y)$. Then $0 \preceq ay \preceq az$ with $v(ay) = 0$ and $v(az) > 0$. Taking residues, it follows $0 \preceq' \overline{ay} \preceq' 0$. Since the support of $\preceq'$ is trivial, this yields that $\overline{ay} = 0$, contradicting $v(ay) = 0$, i.e. $ay \notin I_v$. Therefore, $y \in I_v$.

Now suppose that (3) holds. The quasi-order induced by the residue map is given by

$$\overline{x} \preceq' \overline{y} :\Leftrightarrow \exists c_1, c_2 \in I_v : x + c_1 \preceq y + c_2.$$

First of all we verify that $\preceq'$ is well-defined, so we have to show that $\overline{x} \preceq' \overline{y}$ implies $\overline{x_1} \preceq' \overline{y_1}$ for $\overline{x} = \overline{x_1}$ and $\overline{y} = \overline{y_1}$, say $x = x_1 + c_1$ and $y = y_1 + c_2$ for some $c_1, c_2 \in I_v$. So let $\overline{x} \preceq' \overline{y}$. Then there exist $c_3, c_4 \in I_v$ such that $x + c_3 \preceq y + c_4$. But then $x_1 + (c_1 + c_3) \preceq y_1 + (c_2 + c_4)$, thus, $\overline{x_1} \preceq' \overline{y_1}$.

Evidently, $\preceq'$ is reflexive and total. Next we show transitivity, i.e. that $\overline{x} \preceq' \overline{y}$ and $\overline{y} \preceq' \overline{z}$ yields $\overline{x} \preceq' \overline{z}$. Thus, assume that $x + c_1 \preceq y + c_2$ and $y + d_1 \preceq z + d_2$ for some $c_1, c_2, d_1, d_2 \in I_v$. We argue by case distinction.

First suppose that $y \in U_v$. Assume for a contradiction that $x + e_1 \succ z + e_2$ for all $e_1, e_2 \in I_v$. In particular, $x + c_1 + d_1 - c_2 \succ z + d_2$. Note that $y + c_2 \in U_v$ and $d_1 - c_2 \in I_v$, so Lemma 3.7(1) yields $y + c_2 \not\sim d_1 - c_2$. So from the inequality $x + c_1 \preceq y + c_2$ follows

$$x + c_1 + d_1 - c_2 \preceq y + c_2 + d_1 - c_2 = y + d_1 \preceq z + d_2,$$

a contradiction.

If $y \in I_v$, then $y + c_2$ and $y + d_1 \in I_v$. By convexity and Lemma 3.7(2) and (3), this yields that x is either a negative unit or in I_v, and that z is either a positive unit or in I_v. We only have to consider the case where both elements are in the ideal. But then $x + (z - x) \preceq z + 0$, and thus $\overline{x} \preceq' \overline{z}$.

Now we show that the support of $\preceq'$ equals $\{0\}$. So let $\overline{x} \sim 0$ and assume for a contradiction that $x \in R_v \backslash I_v = U_v$. Then there exist $c_1, c_2 \in I_v$ such that $x + c_1 \preceq c_2$ and there exist $d_1, d_2 \in I_v$ such that $d_1 \preceq x + d_2$.

If $0 \prec x$, then $0 \prec x + c_1 \preceq c_2$ by Lemma 3.7(2), and we have $x + c_1 \in I_v$ by convexity, a contradiction. Likewise, if $x \prec 0$, then $d_1 \preceq x + d_2 \prec 0$ by Lemma 3.7(3), again contradicting the convexity.
It remains to check the axioms (QR1) and (QR3) - (QR5) (axiom (QR2) is omitted because of Remark 2.8(3)).

(QR1) Assume for a contradiction $\overline{1} \preceq' \overline{0}$. Then there exist $c_1, c_2 \in I_v$ such that $0 \prec 1 + c_1 \preceq c_2$ (Lemma 3.7(2)). Convexity of I_v yields $1 + c_1 \in I_v$, and therefore $1 \in I_v$, a contradiction. Thus, $\overline{0} \prec' \overline{1}$.

(QR3) We have to verify that $0 \preceq' \overline{x}$ and $\overline{y} \preceq' \overline{z}$ implies $\overline{xy} \preceq' \overline{xz}$. For $\overline{x} = 0$, there is nothing to show, so assume without loss of generality $x \notin I_v$. From $0 \preceq' \overline{x}$ follows that there are some $c_1, c_2 \in I_v$ such that $c_1 \preceq x+c_2$. Applying Lemma 3.7(1) yields that $c_1 - c_2 \preceq x$. So convexity of I_v gives us $0 \preceq x$. Moreover, $\overline{y} \preceq' \overline{z}$ means $y+d_1 \preceq z+d_2$ for some $d_1, d_2 \in I_v$. (QR3) implies $xy + xd_1 \preceq xz + xd_2$ and therefore $\overline{xy} \preceq \overline{xz}$.

(QR4) We have to prove that $\overline{x} \preceq' \overline{y}$ and $\overline{y} \not\sim \overline{z}$ yields $\overline{x+z} \preceq' \overline{y+z}$. Let $c_1, c_2 \in I_v$ such that $x + c_1 \preceq y + c_2$. We have to show: $x + z + d_1 \preceq y + z + d_2$ for some $d_1, d_2 \in I_v$. Note that $\overline{y} \not\sim \overline{z}$ implies either $\forall e_1, e_2 : y + e_1 \prec z + e_2$ or $\forall e_1, e_2 : y+e_1 \succ z+e_2$. Either way, $z \not\sim y+c_2$. But then $x+z+c_1 \preceq y+z+c_2$ by (QR4), i.e. $\overline{x+z} \preceq' \overline{y+z}$.

(QR5) We have to show that if $0 \prec' \overline{a}$, then $\overline{ax} \preceq' \overline{ay}$ implies $\overline{x} \preceq' \overline{y}$. Note that if $ax \preceq ay$, then $x \preceq y$ by axiom (QR5), hence $\overline{x} \preceq' \overline{y}$. So from now on assume that $ay \prec ax$. First we show that one may also assume that $x, y \in U_v$. Indeed, suppose that $\overline{x} = 0$ and assume for a contradiction that $\overline{y} \prec' 0$. Then $\overline{ay} \preceq' 0$ by axiom (QR3). But $=$ cannot hold because neither $a \in I_v$, nor $y \in I_v$. Thus, $\overline{ay} \prec' 0 = \overline{ax}$, contradicting the assumption. Now suppose that $\overline{y} = 0$ and assume for a contradiction that $0 \prec' \overline{x}$. Then $\overline{ay} = 0 \prec \overline{ax}$, again a contradiction. Hence, one may assume that both x and y lie in U_v. So from $\overline{ax} \preceq' \overline{ay}$ follows that there exists some $c \in I_v$ such that $ax \preceq ay + c$. Thus, it holds $ay \prec ax \preceq ay + c$. The rest of the proof is done by case distinction.
If $0 \prec -1$, then $0 \preceq -r$ for all $r \in R$ with $0 \preceq r$ by (QR3). Lemma 3.8 yields that all elements are non-negative. Particularly, since ay is a unit and I_v is convex, it holds $c \prec ay$ (otherwise $0 \prec ay \preceq c \in I_v$). From Lemma 3.11 follows $ay \prec ay + c \preceq \max\{ay + c\} = ay$, a contradiction.
Finally suppose $-1 \prec 0$. Consider the inequalities $ay \prec ax \preceq ay + c$. By Lemma 3.7(2) and (3), ay and $ay + c$ have the same sign, and so ax has also the same sign, which is contary to the sign of $-ay$. Particularly, we may add $-ay$ to these two inequalities and obtain $0 \preceq a(x - y) \preceq c$. By convexity of I_v follows $a(y - x) \in I_v$ and since I_v is a prime ideal with $a \notin I_v$, one obtains $\overline{x} = \overline{y}$. Particularly, $\overline{x} \preceq' \overline{y}$, as desired.

□

Remark 3.13.

(1) $I_v \prec 1$ (compare Theorem 2.10) is an easy consequence of these conditions. It follows for instance immediately from the convexity I_v.
(2) In the setting of the previous theorem, if $\preceq$ is an order (respectively a proper quasi-order), then $\preceq'$ is also an order (respectively a proper quasi-order).

Proof. First suppose that $\preceq$ is an order. Comparing the definitions of ordered rings (Definition 2.5) and quasi-ordered rings (Definition 2.6), we only have to show that $\overline{x} \preceq' \overline{y}$ implies $\overline{x+z} \preceq' \overline{y+z}$ for all $\overline{z} \in Rv$. From $\overline{x} \preceq' \overline{y}$ follows $x + c_1 \preceq y + c_2$ for some $c_1, c_2 \in I_v$. Since $\preceq$ is an order, we get $x + z + c_1 \preceq y + z + c_2$, thus, $\overline{x+z} \preceq' \overline{y+z}$. So $\preceq'$ is indeed an order.

Finally, suppose that $\preceq = \preceq_w$ for some valuation w on R. We consider the map $w/v : Rv \to \Gamma_{w/v} \cup \{\infty\}$ given by

$$w/v(a + I_v) := \begin{cases} \infty & \text{if } a \in I_v \\ w(a) & \text{else} \end{cases}.$$

(compare [2, p.45] for the field case). We prove that w/v is well-defined. For $a \in I_v$ this is clear by definition. So suppose that $a \in U_v$ and $c \in I_v$. We have to show that $w(a) = w(a + c)$. From condition (1) of the previous theorem we obtain for all $x, y \in R$ that if $w(x) \leq w(y)$, then $v(x) \leq v(y)$. Hence, it follows from $v(a) = 0 < v(c)$ that also $w(a) < w(c)$. Lemma 2.3 yields $w(a + c) = \min\{w(a), w(c)\} = w(a)$.
It is easy to see that w/v satisfies the axioms (V1) and (V2) from Definition 2.1. For (V3) note that $ab \in I_v$ if and only if $a \in I_v$ or $b \in I_v$, since I_v is prime, so $w/v(ab + I_v) = \infty$ if and only if $w/v(a + I_v) + w/v(b + I_v) = \infty$. From this observation (V3) is easily deduced. The prove of (V4) is done by a similar case distinction. Hence, w/v defines a valuation on Rv. Its support is $\{0\}$, as $\mathfrak{q}_w \subseteq \mathfrak{q}_v \subseteq I_v$, which again follows from Theorem 3.12(1). Moreover, for $x, y \in U_v$ (i.e. $\overline{x}, \overline{y} \neq 0$) it holds

$$\begin{aligned} \overline{x} \preceq'_w \overline{y} &\Leftrightarrow x + c_1 \preceq_w y + c_2 \text{ for some } c_1, c_2 \in I_v \\ &\Leftrightarrow w(y + c_2) \leq w(x + c_1) \text{ for some } c_1, c_2 \in I_v \\ &\Leftrightarrow w(y) \leq w(x) \\ &\Leftrightarrow w/v(\overline{y}) \leq w/v(\overline{x}), \end{aligned}$$

where the third equivalence follows precisely as in the proof of the well-definedness of w/v, while the last equivalence is just the definition of w/v. This proves that $\preceq'_w = \preceq_{w/v}$. □

If $\preceq$ is an order, then Theorem 3.12 generalizes Theorem 2.10 from ordered fields to ordered rings. The next lemma implies that if $\preceq = \preceq_w$ for some Manis valuation w, then Theorem 3.12 precisely characterizes the Manis valuations v on R that are coarser than w.

Definition 3.14. (see [10, p.264]) Let v, w be valuations on R. Then v is said to be a **coarsening** of w (or w a **refinement** of v), in short, $v \leq w$, if $R_w \subseteq R_v$ and $I_v \subseteq I_w$. The valuation v is called **strict coarsening**, in short, $v < w$, if one of these inclusions is strict.

Lemma 3.15. *Let v and w be Manis valuations on R. The following are equivalent:*

1. *v is $\preceq_w$-compatible (i.e. $w(z) \leq w(y) \Rightarrow v(y) \leq v(z)$).*
2. *v is a coarsening of w.*

Proof. We first show that (1) implies (2). Let $x \in R_w$. Then $0 = w(1) \leq w(x)$, so also $0 = v(1) \leq v(x)$, thus $x \in R_v$. Likewise, if $x \notin I_w$, then $w(x) \leq w(1) = 0$, which yields that $v(x) \leq v(1) = 0$. Therefore $x \notin I_v$.
Conversely, assume that (2) holds and suppose that $w(x) \leq w(y)$. By [10, Proposition 3.1] it holds $\mathfrak{q}_w = \mathfrak{q}_v$, so we may without loss of generality assume that x is not in the support of these valuations. Moreover note that $U_w \subseteq U_v$: indeed, if $u \in U_w$, then $u \in R_w$ and $u \notin I_w$, thus $u \in R_v$ and $u \notin I_v$. Therefore, $u \in U_v$. As $w(x) \in \Gamma_w$ and w is Manis, there exists some $a \in R$ such that $w(a) = -w(x)$. It follows $ax \in U_w$ and $ay \in R_w$. Therefore, $ax \in U_v$ and $ay \in R_v$. It is easy to see that this implies $v(x) \leq v(y)$. □

Hence, we obtain as a special case of Theorem 3.12 the following characterization of coarsenings:

Theorem 3.16. *Let v, w be Manis valuations on R. Then v is a coarsening of w, if and only if one of the following equivalent conditions is satisfied for all $x, y, z \in R$:*

(1) $w(x) \leq w(y) \Rightarrow v(x) \leq v(y)$,
(2) $w(x) \leq w(y), 0 \leq v(x) \Rightarrow 0 \leq v(y)$,
(3) $w(x) \leq w(y), 0 < v(x) \Rightarrow 0 < v(y)$,
(4) $w/v : Rv \to \Gamma_{w/v} \cup \{\infty\}, x + I_v \mapsto \begin{cases} \infty & \text{if } x \in I_v \\ w(x) & \text{else} \end{cases}$ *defines a valuation with support* $\{0\}$.

Proof. This is precisely Theorem 3.12 in the special case where the quasi-order $\preceq$ comes from a Manis valuation w, and Lemma 3.15. Moreover, we simplified the convexity of R_v and I_v (in (2) and (3)) according to Lemma 3.6. □

Theorem 3.12 leads to a definition of the rank of a quasi-ordered field (see Definition 3.19).

Definition 3.17. (compare [5, Ch. I, Definition 2]) Two valuations v, w on R are said to be **equivalent**, in short, $v \sim w$, if $v(x) \leq v(y) \Leftrightarrow w(x) \leq w(y)$ for all $x, y \in R$.

Let v, w be two Manis valuations on R. From Definition 3.14 and Theorem 3.16(1) follows

$$v \sim w \Leftrightarrow v \leq w \text{ and } w \leq v \Leftrightarrow R_v = R_w \text{ and } I_v = I_w.$$

By abuse of language, we identify equivalent valuations (this is quite common in the literature, see for instance [5, p.11] or [10, p.256]).

Lemma 3.18. *Let $(R, \preceq)$ be a quasi-ordered ring. The set*

$$\mathcal{R} := \{w : w \textit{ is a } \preceq\textit{-compatible Manis valuation on } R\}$$

is totally ordered by $\leq$ ("coarser").

Proof. Let $v, w \in \mathcal{R}$. By Theorem 3.12(2), the valuation rings R_v and R_w are convex subrings of R, say without loss of generality $R_w \subseteq R_v$. If equality holds, it follows again by convexity (Theorem 3.12(3)) that, without loss of generality, $I_w \subseteq I_v$. So either $I_w = I_v$, and then $v = w$, or $I_w \subsetneq I_v$, and then $w < v$. So suppose from now on that $R_w \subsetneq R_v$. We first show that then also $U_w \subsetneq U_v$. Choose some $0 \prec z$ with $z \in R_v \backslash R_w$. Then $w(z) < 0 = w(1)$. From Theorem 3.12(1) follows $1 \preceq z$. This implies $z \notin I_v$. So $z \in R_v \backslash I_v = U_v$, whereas $z \notin R_w \supseteq U_w$.
Finally, let $y \in I_v$. From Remark 3.13(1) follows $y \prec 1$. Since R_w is convex, it holds $z \in R_w = I_w \sqcup U_w$. If $z \in U_w$, then $z \in U_v$, a contradiction. Therefore, $z \in I_w$. This shows $I_v \subseteq I_w$. Thus, $v < w$. □

Definition 3.19. Let $(R, \preceq)$ be a quasi-ordered ring. The order type of the set $\mathcal{R}$ (see Lemma 3.18) is called the **rank** of $(R, \preceq)$.

Next we show that $I_v \prec 1$ is not equivalent to the other conditions of Theorem 3.12, regardless of whether $\preceq$ is a proper quasi-order (Example 3.21) or an order (Example 3.22).

Theorem 3.20. (see [2, Theorem 2.2.1]) *Let K be a field, $\Gamma \subseteq \Gamma'$ ordered abelian groups, $u : K \to \Gamma \cup \{\infty\}$ a valuation on K, and $\gamma \in \Gamma'$. For $f = \sum_{i=0}^{n} a_i X^i \in K[X]$ define*

$$v(f) = \begin{cases} \infty & \text{if } f = 0 \\ \min\limits_{0 \leq i \leq n} \{u(a_i) + i\gamma\} & \text{otherwise} . \end{cases}$$

For $f, g \in K[X] \backslash \{0\}$ declare $v(f/g) = v(f) - v(g)$. Then $v : K(X) \to \Gamma' \cup \{\infty\}$ is a valuation that extends u.

Example 3.21. Let $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ denote the p-adic valuation for some prime number $p \in \mathbb{N}$ (see [2, p.18]), i.e. if $0 \neq x = \frac{p^r a_i}{p^s b_i} \in \mathbb{Q}$ (using the unique prime factorization in $\mathbb{Z}$), then $v(x) = r - s$. Apply the previous theorem with $\gamma = 1$ to extend v_p to a valuation $v : \mathbb{Q}(X) \to \mathbb{Z} \cup \{\infty\}$. The restriction of v to $\mathbb{Q}[X]$, which we will denote by v as well, is easily seen to be a Manis valuation on $\mathbb{Q}[X]$. This is because v_p is surjective, as it is a field valuation, and $\Gamma = \Gamma'$. We do the same procedure with w instead of v, except that this time $\gamma = 0$.
Note that $v = w$ on $\mathbb{Q}$ and $v(f) = w(f) + i$ for some $i > 0$ if $f \in \mathbb{Q}[X] \backslash \mathbb{Q}$. This implies $I_v \prec_w 1$. Indeed, $f \in I_v$ means $v(f) > 0$. But then also $w(f) > 0 = w(1)$, and therefore $f \prec_w 1$. However, v is not compatible with $\preceq_w$. For instance we have

$$w(X^2) = 0 < w(p) = 1 < w(0) = \infty,$$

i.e. $0 \prec_w p \prec_w X^2$, but $v(p) = 1 < v(X^2) = 2$.

Example 3.22. Order $\mathbb{Q}[X]$ by declaring $f \leq g :\Leftrightarrow f(0) \leq g(0)$. Consider the trivial valuation $v : \mathbb{Q}[X] \to \{0, \infty\}$, given by $v(0) = \infty$ and $v(x) = 0$ for $x \neq 0$. Then v is Manis and $I_v = \{0\} \prec 1$, but I_v is not convex since $0 \sim X$ and $X \notin I_v$.

Remark 3.23. In the case of ordered fields $(K, \leq)$, the condition $I_v < 1$ is often times replaced with the equivalent condition $1 + I_v \geq 0$ (see for instance [6, Definition 2.4] or [2, Proposition 2.2.4]). Note, however, that for ordered rings they are also equivalent and that the latter is inappropriate for quasi-ordered rings of the form $(R, \preceq_w)$, as then all elements are non-negative, which means that $1 + I_v \succeq 0$ is trivially satisfied.

We conclude this section by imposing a suitable extra condition on v, such that $I_v \prec 1$ becomes an equivalent condition again.

Definition 3.24. (see [5, Ch. I, Definition 5]) A valuation v on R is called **local**, if the pair (R_v, I_v) is local, i.e. if I_v is the unique maximal ideal of R_v.

The maximal ideal of a local ring consists precisely of all non-units of the said ring. A characterization of local valuations is given in [5, Ch. I, Proposition 1.3] and [4, Proposition 5], respectively. Note that if R is a field, then v is always a local Manis valuation.

Lemma 3.25. *Let $(R, \preceq)$ be a quasi-ordered ring and v a local Manis valuation on R. The following are equivalent:*

(1) v is compatible with $\preceq$.
(2) $I_v \prec 1$.

Proof. (1) implies (2) is clear, see Remark 3.13(1). Now suppose that (2) holds, and assume for a contradiction that there are some $y, z \in R$ such that $0 \preceq y \preceq z$, but $v(y) < v(z)$. The latter implies $y \notin \mathfrak{q}_v$. Since v is Manis, we find some $0 \preceq a$ such that $v(a) = -v(y)$. We obtain $0 = v(ay) < v(az)$, so $ay \in U_v$ and $az \in I_v$. As v is local and $ay \notin I_v$, ay is a unit. It follows

$$0 < v(az) - v(ay) = v\left(\frac{az}{ay}\right),$$

i.e. $\frac{az}{ay} \in I_v$. Hence, (2) yields $\frac{az}{ay} \prec 1$. This implies $az \prec ay$ (for if $az \sim ay$, then $\frac{az}{ay} \sim 1$ by Corollary 3.10, a contradiction). On the other hand, it follows from $y \preceq z$ and $0 \preceq a$ that $ay \preceq az$, a contraction.

□

Corollary 3.26. *Let v, w be Manis valuations on R such that v is local. Then v is coarser than w if and only if $I_v \subseteq I_w$.*

Proof. This is an immediate consequence of Lemma 3.15 and Lemma 3.25 in the case where $\preceq = \preceq_w$. □

4. The Baer-Krull Theorem

In the previous section we fixed a quasi-ordered ring $(R, \preceq)$ and characterized all the (local) Manis valuations on R, that are compatible with $\preceq$ (see Theorem 3.12 and Lemma 3.25). It is natural to ask what happens the other way round, i.e. if we fix a valued ring (R, v) with v Manis, can we describe all the quasi-orders on R that are compatible with v? A positive answer is given by the Baer-Krull Theorem (see Theorem 4.10, respectively Corollary 4.11 or Corollary 4.12). Recall that if $\preceq$ is a v-compatible quasi-order on R, then it gives rise to a quasi-order $\preceq'$ on the residue ring $Rv := R_v/I_v$. The said theorem establishes a connection between the v-compatible quasi-orders on R with support $\mathrm{supp}(v)$, and the quasi-orders on Rv with support $\{0\}$. After establishing the Baer-Krull Theorem for quasi-ordered rings, we deduce a version for ordered, respectively proper quasi-ordered, rings (see Corollary 4.13, respectively Corollary 4.16). The first one yields a generalization of the classical Baer-Krull Theorem (see Theorem 2.11), while the latter gives rise to a characterization of all Manis valuations on R that are finer than v.

When one deals with quasi-ordered rings, this theorem becomes more complicated than in the ordered field case (see Theorem 2.11). Note that the map η there is completely determined by the signs of the elements π_i. If the quasi-order is an order, then all $\eta \in \{-1, 1\}^I$ are realizeable and one gets a bijective correspondence as in Theorem 2.11. However, if the quasi-order is induced by some valuation, then all elements are non-negative, so the only η possible is the constant map $\eta = 1$. Therefore, when we consider quasi-ordered rings $(R, \preceq)$, the best we can hope for is that ψ is an injective map such that the image of ψ contains all possible tuples $(\eta_\preceq, \preceq')$ as just described. Establishing such a result will be the aim for the rest of this paper.

Notation 4.1. We use the following notation for the rest of this section:

(1) Let R be a commutative ring with 1 and $v\colon R \to \Gamma_v \cup \{\infty\}$ a Manis valuation on R with support $\mathfrak{q}_v$, valuation ring R_v, prime ideal I_v and residue ring $Rv := R_v/I_v$, just as in Notation 3.1. Moreover, we define $\tilde{R} := R \backslash \mathfrak{q}_v = v^{-1}(\Gamma_v)$.

(2) We fix some $\mathbb{F}_2$-basis $\{\overline{\gamma_i} : i \in I\}$ of $\overline{\Gamma_v} = \Gamma_v/2\Gamma_v$, and let $\{\pi_i : i \in I\} \subseteq \tilde{R}$ be such that $v(\pi_i) = \gamma_i$.

(3) Given a v-compatible quasi-order on R, we denote by $\preceq'$ the induced quasi-order on Rv (see Theorem 3.12). By $\eta_\preceq$ we denote the map $I \to \{-1, 1\}$ defined by $\eta_\preceq(i) = 1$ if and only if $0 \preceq \pi_i$.

Now we fix some tuple $(\eta^*, \preceq^*)$ from the disjoint union

$$\{-1, 1\}^I \times \{\text{orders on } Rv \text{ with support } \{0\}\} \sqcup \{1\}^I \{\text{p.q.o on } Rv \text{ with support } \{0\}\}$$

The main part of the proof of the Baer-Krull Theorem is to construct a quasi-order on R that is mapped to $(\eta^*, \preceq^*)$ under the analogue of the map ψ from Theorem 2.11. For that purpose we define a binary relation $\preceq$ on R as a function of $\preceq^*$ and η^* as follows: If $x, y \in \mathfrak{q}_v$, declare $x \preceq y$. Otherwise, if $x \in \tilde{R}$ or $y \in \tilde{R}$, consider

$$\gamma := \gamma_{x,y} := \max\{-v(x), -v(y)\} \in \Gamma_v.$$

Write $\overline{\gamma} = \sum_i \overline{\gamma_i}$. Then $\gamma = \sum_i \gamma_i + 2v(a)$ for some $a \in \tilde{R}$, which is uniquely determined up to its value (i.e. instead of a one may have chosen any other element $b \in R$ with $v(b) = v(a)$). Consider $x \prod_i \pi_i a^2$ and $y \prod_i \pi_i a^2$.

Lemma 4.2. *Let $x, y \in \tilde{R}$ and I, π_i, a as above. Then $x\prod_i \pi_i a^2$, $y\prod_i \pi_i a^2 \in R_v$. Moreover, $\overline{x\prod_i \pi_i a^2} = 0$ if and only if $v(x) > v(y)$.*

Proof. Note that

$$v\left(x\prod_i \pi_i a^2\right) = v(x) + \sum_i v(\pi_i) + 2v(a) = v(x) + \gamma$$
$$= v(x) + \max\{-v(x), -v(y)\} \geq 0,$$

and likewise for $y\prod_i \pi_i a^2$, so both are in R_v. Moreover,

$$\overline{x\prod_i \pi_i a^2} = 0 \Leftrightarrow v(x) + \max\{-v(x), -v(y)\} > 0 \Leftrightarrow v(x) > v(y).$$

□

Particularly, we can take residues of both $x\prod_i \pi_i a^2$ and $y\prod_i \pi_i a^2$. The moreover part of the statement will be of great importance in the proof of Main Lemma 4.5. For the latter, we also require the following two lemmas, which extend the statements from axiom (QR3), respectively (QR5).

Lemma 4.3. *Let $(R, \preceq)$ be a quasi-ordered ring. If $x \preceq y$ and $z \preceq 0$, then $yz \preceq xz$.*

Proof. As E_0 is an ideal, we may without loss of generality assume that $z \nsim 0$. Moreover, note that if $y \sim 0$, then $x, z \preceq 0$, thus $0 \preceq -x, -z$. It follows via (QR3) that $yz \sim 0 \preceq xz$. So we may also assume that $y \notin E_0$. From $x \preceq y$ and $z \preceq 0$ follows $-xz \preceq -yz$. We claim that $yz \nsim -yz$. Once this is shown, it follows from $-xz \preceq -yz$ that $yz - xz \preceq 0$. The latter implies $yz \preceq xz$. Indeed, either $x \nsim 0$ and therefore $xz \nsim 0$ (E_0 is a prime ideal), so that we can apply (QR4); or $x \sim 0$, i.e. $xz \sim 0$, and therefore $yz - xz \sim yz \preceq 0 \sim xz$ (see Remark 2.8(1)). So assume for a contradiction that $yz \sim -yz$. Lemma 3.8 yields $0 \preceq yz, -yz$. As $y \notin E_0$, either $0 \prec y$ or $0 \prec -y$. So via (QR5) it follows either from $0 \preceq yz$ (if $0 \prec y$) or from $0 \preceq -yz$ (if $0 \prec -y$) that $0 \preceq z$. Hence $z \sim 0$, a contradiction. □

Lemma 4.4. *Let $(R, \preceq)$ be a quasi-ordered ring and $x, y, z \in R$. If $xz \preceq yz$ and $z \prec 0$, then $y \preceq x$.*

Proof. Assume for a contradiction $x \prec y$. Multiplying with z yields $yz \preceq xz$. Hence $xz \sim yz$. Lemma 2.9 yields $x \sim y$, a contradiction. □

Main Lemma 4.5. *With the notation from above, define for $x \in \tilde{R}$ or $y \in \tilde{R}$ that*

$$x \preceq y :\Leftrightarrow \begin{cases} \text{Either} \quad \overline{x\prod_i \pi_i a^2} \preceq^* \overline{y\prod_i \pi_i a^2} \quad \text{and} \quad \prod_i \eta^*(i) = 1 \\ \\ \text{or} \quad \overline{y\prod_i \pi_i a^2} \preceq^* \overline{x\prod_i \pi_i a^2} \quad \text{and} \quad \prod_i \eta^*(i) = -1. \end{cases}$$

Moreover, declare $x \preceq y$ for $x, y \in \mathfrak{q}_v$. Then $\preceq$ defines a quasi-order on R with support $E_0 = \mathfrak{q}_v$.

Proof. The proof of the Main Lemma is extensive, however, the methods are widely the same. Notably, the moreover part from Lemma 4.2 is frequently exploited. We always use the notation from above. For the sake of convenience and uniformity, we treat $\preceq^*$ and η^* as an arbitrary quasi-order on Rv with support $\{0\}$, respectively an arbitrary map from I to $\{-1, 1\}$, for as long as possible. In fact, the distinction whether $\preceq^*$ is an order or induced by a valuation (in which case the map η^* is trivial) is only necessary at some points when we verify axiom (QR4).
First of all we show that $\preceq$ is well-defined. Recall that a was only determined up to its value. So let $b \in \tilde{R}$ with $v(a) = v(b)$ and suppose that $\overline{x\prod_i \pi_i a^2} \preceq^* \overline{y\prod_i \pi_i a^2}$. We have to show that also $\overline{x\prod_i \pi_i b^2} \preceq^* \overline{y\prod_i \pi_i b^2}$. As v is Manis, there exists some

$z \in \tilde{R}$ with $v(z) = -v(b)$, so $v(bz) = 0$, i.e. $\overline{bz} \neq 0$. Particularly, $0 \prec^* \overline{bz}^2$. It follows via (QR3), after rearranging, that $\overline{x \prod_i \pi_i b^2}\ \overline{az}^2 \preceq^* \overline{y \prod_i \pi_i b^2}\ \overline{az}^2$. We conclude by eliminating $0 \prec^* \overline{az}^2$ via axiom (QR5).
Clearly $\preceq$ is reflexive and total. At next we prove transitivity. So let $x \preceq y$ and $y \preceq z$. We have to show that $x \preceq z$. First of all note that the case $x, z \in \mathfrak{q}_v$ is trivial. So we assume without loss of generality that not both are in the support. In the following denote by I the index set to compare x and y, by J the one to compare y and z, and by L the one to compare x and z, with corresponding squares a^2, b^2 and c^2, respectively. For the proof we distinguish four cases.
First of all assume that $v(p) = v(q) \leq v(r)$ with $p, q, r \in \{x, y, z\}$ pairwise distinct. Then $\gamma_{x,y} = \gamma_{x,z} = \gamma_{y,z} \in \Gamma_v$ all coincide, so $I = J = L$ and $a = b = c$. Hence, transitivity of $\preceq$ follows immediately by transitivity of $\preceq^*$. It remains to verify the cases where there is a unique smallest element among $v(x), v(y)$ and $v(z)$. First suppose that $v(x) < v(y), v(z)$. Then $\gamma_{x,y} = -v(x) = \gamma_{x,z}$, i.e. $I = L$ and $a = c$. We do the case $\prod_i \eta^*(i) = -1$, the case $\prod_i \eta^*(i) = 1$ is proven analogously. From $x \preceq y$ and $v(x) < v(y)$ then follows $\overline{y \prod_i \pi_i a^2} = 0 \preceq^* \overline{x \prod_i \pi_i a^2}$ (Lemma 4.2). Now $v(x) < v(z)$ and Lemma 4.2 imply that $\overline{z \prod_i \pi_i a^2} = 0$. Therefore, $x \preceq z$. Next, suppose that $v(y) < v(x), v(z)$. Then $\gamma_{x,y} = -v(y) = \gamma_{y,z}$, i.e. $I = J$ and $a = b$. Again, we only do the case $\prod_i \eta^*(i) = -1$. From $v(y) < v(x)$ and $x \preceq y$ follows $\overline{y \prod_i \pi_i a^2} \preceq^* \overline{x \prod_i \pi_i a^2} = 0$. Likewise, $v(y) < v(z)$ and $y \preceq z$ implies $0 \preceq^* \overline{y \prod_i \pi_i a^2}$. Since the support of $\preceq^*$ is trivial, it follows $\overline{y \prod_i \pi_i a^2} = 0$. On the other hand, $v(y) < v(x), v(z)$ yields via Lemma 4.2 that $\overline{y \prod_i \pi_i a^2} \neq 0$, a contradiction. The case $v(z) < v(x), v(y)$ is proven as the case where $v(x)$ is the unique smallest value.
Now we establish that the support of $\preceq$ is $\mathfrak{q}_v$. Assume there is some $x \in E_0$ with $x \notin \mathfrak{q}_v$. Then $\overline{x \prod_i \pi_i a^2} \sim 0$. As the support of $\preceq^*$ is $\{0\}$, this yields $x \prod_i \pi_i a^2 \in I_v$. However, as $v(x) < v(0) = \infty$, this contradicts Lemma 4.2. We obtain that $E_0 \subseteq \mathfrak{q}_v$. The other implication follows immeediately from the definition of $\preceq$.
It remains to verify the axioms (QR1) - (QR5) and compatibility with v. For the proof of **(QR1)** assume for a contradiction that $1 \preceq 0$. Note that $\gamma_{0,1} = 0$, so $I = \emptyset$ and $\prod_i \eta^*(i) = 1$. It follows from $1 \preceq 0$ that $\overline{a}^2 \preceq^* 0$ for some $a \in R$ with $v(a) = 0$. This contradicts the facts that squares are non-negative and that the support of $\preceq^*$ is trivial.
For **(QR2)** is nothing to show by Remark 2.8(3). Next, we verify **(QR3)**, i.e. we show that $x \preceq y$ and $0 \preceq z$ implies $xz \preceq yz$. By the definition of $\preceq$ and the fact that $\mathfrak{q}_v$ is an ideal, we may without loss of generality assume that $z \notin \mathfrak{q}_v$, and that not both x and y are in $\mathfrak{q}_v$. Now note that

$$\begin{aligned}\gamma_{xz,yz} &= \max\{-v(xz), -v(yz)\} = \max\{-v(z), -v(0)\} + \max\{-v(x), -v(y)\} \\ &= \gamma_{0,z} + \gamma_{x,y} \in \Gamma_v.\end{aligned}$$

Write $\gamma_{x,y} = \sum_i \gamma_i + 2v(a)$ and $\gamma_{0,z} = \sum_j \gamma_j + 2v(b)$. Set $L = I \sqcup J$, the (wlog) disjoint union of the index sets I and J. Then

$$\gamma_{xz,yz} = \gamma_{y,z} + \gamma_{0,z} = \sum_l \gamma_l + 2v(ab).$$

So to compare xz and yz with respect to $\preceq$, one has to consider $xz \prod_l \pi_l a^2 b^2$ and $yz \prod_l \pi_l a^2 b^2$. Further note that $\prod_l \eta^*(l) = \prod_i \eta^*(i) \cdot \prod_j \eta^*(j)$. First consider the case $\prod_j \eta^*(j) = 1$. This yields $0 \preceq^* \overline{z \prod_j \pi_j b^2}$. Suppose $\prod_i \eta^*(i) = -1 = \prod_l \eta^*(l)$. From $x \preceq y$ then follows $\overline{y \prod_i \pi_i a^2} \preceq^* \overline{x \prod_i \pi_i a^2}$. Applying (QR3) yields $\overline{yz \prod_l \pi_l a^2 b^2} \preceq^* \overline{xz \prod_l \pi_l a^2 b^2}$. Therefore $xz \preceq yz$. The case $\prod_i \eta^*(i) = -1 = \prod_l \eta^*(l)$ is analogue. The proof for $\prod_j \eta^*(j) = -1$ is also almost the same; we just apply Lemma 4.3 instead of axiom (QR3).

The proof of axiom **(QR4)** is divided into five subcases. Let I, J, L and a, b, c as in the verification of transitivity. First suppose that $v(x) < v(z)$ or $v(y) < v(z)$. Either way, $\gamma_{x,y} = \gamma_{x+z,y+z}$. Moreover, in both cases $\overline{z \prod_i \pi_i a^2} = 0$. From this observation, the claim follows immediately. Further note that if $\preceq^*$ is an order and $x \prec y$, we obtain $x + z \prec y + z$, because orders preserve strict inequalities under addition. We will need this fact below.
If $v(z) < v(x), v(y)$, then $\gamma_{y,z} = \gamma_{x+z,y+z}$. Note that $\overline{x \prod_j \pi_j b^2} = 0 = \overline{y \prod_j \pi_j b^2}$ by Lemma 4.2. It is easy to see that then $x + z \preceq y + z$. Suppose now that $v(x) = v(z) < v(y)$. Then $\gamma_{x,y} = \gamma_{y,z} = \gamma_{x+z,y+z}$. From $v(x) < v(y)$ follows $\overline{y \prod_i \pi_i a^2} = 0$. Distinguish two subcases. If $\prod_i \eta^*(i) = 1$, then $x \preceq y$ yields $\overline{x \prod_i \pi_i a^2} \preceq^* \overline{y \prod_i \pi_i a^2} = 0$. As $z \nsim y$, one may add $\overline{z \prod_i \pi_i a^2}$ on both sides. This concludes that case. On the other hand, if $\prod_i \eta^*(i) = -1$, then $\eta^* \neq 1$, so $\preceq^*$ must be an order and one may add $\overline{z \prod_i \pi_i a^2}$ on both sides as well. Next, suppose that $v(y) = v(z) < v(x)$. Then $\gamma_{x,y} = \gamma_{x+z,y+z}$. It holds $\overline{x \prod_i \pi_i a^2} = 0$. If $\prod_i \eta^*(i) = 1$, then $x \preceq y$ yields $0 \preceq^* \overline{y \prod_i \pi_i a^2}$ and one may conclude by adding $\overline{z \prod_i \pi_i a^2}$ on both sides. Analogously, if $\prod_i \eta^*(i) = -1$, one may also add $\overline{z \prod_i \pi_i a^2}$ on both sides, as the support of $\preceq^*$ is zero. Finally, suppose that $v(x) = v(y) = v(z)$. If they are all in $\mathfrak{q}_v$, then also $x+z, y+z \in \mathfrak{q}_v$, and therefore $x+z \preceq y+z$. So suppose that $v(x) = v(y) = v(z) \in \Gamma_v$. It holds

$$\gamma_{x+z,y+z} = \max\{-v(x+z), -v(y+z)\} \leq -v(z).$$

First suppose that equality holds. Then $\max\{-v(x+z), -v(y+z)\} = -v(x)$, i.e. all $\gamma's$ coincide and. If $\prod_i \eta^*(i) = 1$, the claim follows immediately from (QR4) and the fact that $y \nsim z$ by simply adding $\overline{z \prod_i \pi_i a^2}$ to both sides of the inequality $\overline{x \prod_i \pi_i a^2} \preceq^* \overline{y \prod_i \pi_i a^2}$. Contary, if $\prod_i \eta^*(i) = -1$, then $\preceq^*$ must be an order and we may simply add $\overline{z \prod_i \pi_i a^2}$ on both sides anyway.
Last but not least assume that $<$ holds, i.e. $\max\{-v(x+z), -v(y+z)\} < -v(z)$. Then $\overline{x \prod_i \pi_i a^2}, \overline{y \prod_i \pi_i a^2}$ and $\overline{z \prod_i \pi_i a^2}$ are all non-zero. Moreover,

$$\overline{(x+z) \prod_i \pi_i a^2} = 0 = \overline{(y+z) \prod_i \pi_i a^2}.$$

This yields $\overline{x \prod_i \pi_i a^2} = \overline{y \prod_i \pi_i a^2} = -\overline{z \prod_i \pi_i a^2}$. Particularly, we may assume that $\preceq^*$ is an order, since in the proper quasi-order case

$$\overline{y \prod_i \pi_i a^2} \sim -\overline{y \prod_i \pi_i a^2} = \overline{z \prod_i \pi_i a^2},$$

contradicting the assumption $y \nsim z$. We claim that $x + z \sim 0 \sim y + c$, which clearly implies $x + z \preceq y + z$. Assume for a contradiction that $x + z \nsim 0$. If $x + z \prec 0$, it follows from the case "$v(x) < v(z)$" above and the fact that $\preceq^*$ is an order, that $x \prec -z$, contradicting $x \sim -z$. Likewise, if $0 \prec x + z$, it follows from the case "$v(y) < v(z)$" that $-z \prec x$, again a contradiction. Therefore $x + z \sim 0$. The same arguments show that $y + z \sim 0$ as well, whereby (QR4) is established.
Now we prove axiom **(QR5)**. Suppose $xz \preceq yz$ and $0 \prec z$. We have to show $x \preceq y$. Clearly $z \in \tilde{R}$, as $0 \nsim z$. Moreover, without loss of generality $x \in \tilde{R}$ or $y \in \tilde{R}$. Note that $\gamma_{xz,yz} = \gamma_{x,y} + \gamma_{z,0}$. Let I denote the index set to compare x and y, J the one to compare z and 0, and L the one to compare xz and yz, with squares a^2, b^2 and $(ab)^2$, respectively. Note that $\prod_l \eta^*(l) = \prod_i \eta^*(i) \prod_j \eta^*(j)$.
First consider the case $\prod_j \eta^*(j) = 1$. Then $0 \prec^* \overline{z \prod_j \pi_j b^2}$. If $\prod_l \eta^*(l) = -1$, also $\prod_i \eta^*(i) = -1$. It holds $\overline{yz \prod_l \pi_l a^2 b^2} \preceq^* \overline{xz \prod_l \pi_l a^2 b^2}$. Eliminating $\overline{z \prod_j \pi_j b^2}$ via (QR5) yields $\overline{y \prod_i \pi_i a^2} \preceq^* \overline{x \prod_i \pi_i a^2}$, and therefore $x \preceq y$. On the other hand, if $\prod_l \eta^*(l) = 1$, then also $\prod_i \eta^*(i) = 1$, and the prove is analogoue.

If $\prod_j \eta^*(j) = -1$, then $\overline{z\prod_j \pi_j b^2} \prec^* 0$. If $\prod_l \eta^*(l) = -1$, then $\prod_i \eta^*(i) = 1$. It follows $\overline{yz\prod_l \pi_l a^2 b^2} \preceq^* \overline{xz\prod_l \pi_l a^2 b^2}$. Applying Lemma 4.4 yields $\overline{x\prod_i \pi_i a^2} \preceq^* \overline{y\prod_i \pi_i a^2}$. Therefore $x \preceq y$. The case $\prod_l \eta^*(l) = 1$ is analogue.
We conclude by showing that $\preceq$ is compatible with v, i.e. by showing that $0 \preceq x \preceq y$ yields $v(y) \leq v(x)$. So assume for a contradiction that $v(x) < v(y)$. Note that $\gamma_{0,x} = -v(x) = \gamma_{x,y}$. If $\prod_i \eta^*(i) = -1$, then $0 \preceq x$ yields $\overline{x\prod_i \pi_i a^2} \prec^* 0 = \overline{y\prod_i \pi_i a^2}$, i.e. $y \prec x$, a contradiction. The same argument works in the case where $\prod_i \eta^*(i) = 1$. This finishes the proof of the main lemma. □

Remark 4.6. The quasi-order $\preceq$ from the Main Lemma becomes very simple in the case where $x \in U_v$ and $y \in R_v$ (or vice versa). Note that then $\gamma_{x,y} = 0$. This implies $I = \emptyset$. Hence, $\prod_i \eta^*(i) = 1$. Moreover, the element a satisfies $v(a) = 0$, so by well-definedness of $\preceq$ we may simply choose $a = 1$. Therefore $x \preceq y \Leftrightarrow \overline{x} \preceq^* \overline{y}$.

For the proof of the Baer-Krull Theorem we require two more lemmas. They will be used to compare the "size" of two quasi-orders on R.

Lemma 4.7. *Let $(R, \preceq)$ be a quasi-ordered ring. Then $E_0 + \{x\} \subseteq E_x$ for all $x \in R$.*

Proof. For $x \in E_0$ there is nothing to show. So let $y \in R \backslash E_0$ such that $y = c + x$ for some $c \in E_0$. Remark 2.8(1) yields $c + x \sim x$, so $y \in E_x$. □

Lemma 4.8. *Let $(R, \preceq)$ be a quasi-ordered ring and $x \in R$. If $E_0 + \{x\} \subsetneq E_x$, then $E_x = -E_x$.*

Proof. Let $z \in E_x$ be arbitrary and $y \in E_x \backslash (E_0 + \{x\})$. We will show that $-y \in E_x$. From $z \sim x \sim y$ and Corollary 3.10 then follows $-z \sim -y \sim x$, i.e. also $-z \in E_x$, what proves that $E_x = -E_x$.
The proof that $-y \in E_x$ is like in [3, p.208]. Assume for a contradiction that $-y \notin E_x$. Then $y \preceq x \not\sim -y$, thus $0 \preceq x - y$. Likewise, it follows from $x \preceq y \not\sim -y$ that $x - y \preceq 0$. Therefore, $x - y \in E_0$, i.e. $y \in E_0 + \{x\}$, a contradiction. Hence, $-y \in E_x$, i.e. $E_x = -E_x$. □

Notation 4.9. For a prime ideal $\mathfrak{p}$ of R denote by $\mathcal{X}_{\mathfrak{p}}(R)$ the set of all quasi-orders on R with support $\mathfrak{p}$. Analogously, denote by $\mathcal{X}_{o,\mathfrak{p}}(R)$ (respectively $\mathcal{X}_{p,\mathfrak{p}}(R)$) the set of all orders (respectively proper quasi-orders) on R with support $\mathfrak{p}$.

Theorem 4.10. *(Baer-Krull Theorem for quasi-ordered Rings I)*
Let R be a commutative ring with 1 and v a Manis valuation on R. Then

$$\psi\colon \{\preceq \in \mathcal{X}_{\mathfrak{q}_v}(R)\colon \preceq \text{ is } v\text{-compatible}\} \to \{-1,1\}^I \times \mathcal{X}_{\{0\}}(Rv),$$
$$\preceq \mapsto (\eta_{\preceq}, \preceq')$$

is a well-defined map such that $\psi \restriction \psi^{-1}(\mathcal{A}) : \psi^{-1}(\mathcal{A}) \to \mathcal{A}$ is a bijection, where $\mathcal{A} := \{-1,1\}^I \times \mathcal{X}_{o,\{0\}}(Rv) \sqcup \{1\}^I \times \mathcal{X}_{p,\{0\}}(Rv)$.

Proof. By Theorem 3.12 we know that $\preceq'$ is indeed a quasi-order on Rv with support $\{0\}$. Therefore, the map ψ is well-defined. Next, let $(\eta^*, \preceq^*) \in \mathcal{A}$ be arbitrary. We prove that ψ maps the quasi-order $\preceq$ constructed in the Main Lemma (depending on η^* and $\preceq^*$) to the tuple $(\eta^*, \preceq^*)$. First verify that $\eta_{\preceq} = \eta^*$. To compare π_i and 0 w.r.t. $\preceq$, let $\gamma := \max\{-v(\pi_i), -v(0)\} = -\gamma_i$, i.e. $\gamma = v(\pi_i a^2)$ for some $a \in \tilde{R}$. Hence, we have to consider 0 and $\pi_i \pi_i a^2 = (\pi_i a)^2$. Note that $0 \prec^* \overline{\pi_i a}^2$, as it is a square and $\preceq^*$ has trivial support. From this observation we obtain

$$\eta_{\preceq}(i) = 1 \Leftrightarrow 0 \preceq \pi_i \Leftrightarrow \eta^*(i) = 1,$$

and therefore $\eta_{\preceq} = \eta^*$.
Now we want to prove that $\preceq' = \preceq^*$, i.e. that $\overline{x} \preceq' \overline{y}$ if and only if $\overline{x} \preceq^* \overline{y}$ for all

$x, y \in R_v$. Assume without loss of generality that not both $x, y \in I_v$. Then also $x + c$ and $y + d$ are not both in I_v for all $c, d \in I_v$. It follows from Remark 4.6 that $x + c \preceq y + d \Leftrightarrow \overline{x + c} \preceq^* \overline{y + d}$. Thus,

$$\begin{aligned}\overline{x} \preceq' \overline{y} &\Leftrightarrow \exists c_1, c_2 \in I_v : x + c_1 \preceq y + c_2 \\ &\Leftrightarrow \exists c_1, c_2 \in I_v : \overline{x + c_1} \preceq^* \overline{y + c_2} \\ &\Leftrightarrow \overline{x} \preceq^* \overline{y},\end{aligned}$$

where the first equivalence is simply the definition of $\preceq'$.
We conclude by showing that $\psi \restriction \psi^{-1}(\mathcal{A})$ is injective. Let $\preceq_1 \in \psi^{-1}(\mathcal{A})$ be arbitrary, and denote by $\preceq_2$ the quasi-order on R defined by $\eta_{\preceq_1}$ and $\preceq_1'$ (see Main Lemma). We prove that $\preceq_1 = \preceq_2$. First of all we claim that $\preceq_1 \subseteq \preceq_2$. So let $x, y \in R$. Since $\preceq_1$ and $\preceq_2$ have both support $\mathfrak{q}_v$, we may without loss of generality assume that $x \notin \mathfrak{q}_v$ or $y \notin \mathfrak{q}_v$. Let I, π_i and a be as in the definition of the quasi-order $\preceq_2$. First suppose that $\prod_i \eta_{\preceq_1}(i) = -1$, i.e. $\prod_i \pi_i a^2 \prec_1 0$. With Lemma 4.3 and Lemma 4.4, we obtain

$$\begin{aligned}x \preceq_1 y &\Leftrightarrow y \prod_i \pi_i a^2 \preceq_1 x \prod_i \pi_i a^2 \\ &\Rightarrow \overline{y \prod_i \pi_i a^2} \preceq_1' \overline{x \prod_i \pi_i a^2} \\ &\Leftrightarrow x \preceq_2 y.\end{aligned}$$

Likewise, if $\prod_i \eta_{\preceq_1}(i) = 1$, we just apply (QR3) instead of Lemma 4.3 and (QR5) instead of Lemma 4.4 to get the same result. Thus, $\preceq_1 \subseteq \preceq_2$. For the rest of the proof we distinguish the cases $-1 \nsim_{\preceq_2} 1$ and $-1 \sim_{\preceq_2} 1$.
If $-1 \nsim_{\preceq_2} 1$, then Lemma 2.9 yields $-x \nsim_{\preceq_2} x$ for all $x \in \tilde{R}$, so $E_{x,\preceq_2} \neq -E_{x,\preceq_2}$ for all such x. From Lemma 4.7 and Lemma 4.8 follows $E_{x,\preceq_2} = \mathfrak{q}_v + \{x\}$ for all $x \in R$. So Lemma 4.7 yields that $\preceq_2$ is a minimal quasi-order with support $\mathfrak{q}_v$. Therefore $\preceq_1 \subseteq \preceq_2$ implies equality, as desired. So suppose for the rest of this proof that $-1 \sim_{\preceq_2} 1$. We distinguish the subcases $v(x) \neq v(y)$ and $v(x) = v(y)$.
If $v(x) \neq v(y)$, then Lemma 4.2 states $\overline{x \prod_i \pi_i a^2} \neq 0$ and $\overline{y \prod_i \pi_i a^2} = 0$, or vice versa. By exploiting that I_v is convex with respect to $\preceq_1$ (see Theorem 3.12), it is easy to see that the $\Rightarrow$ above is also an equivalence, so $x \preceq_1 y \Leftrightarrow x \preceq_2 y$. Suppose for instance that $\overline{x \prod_i \pi_i a^2} \neq 0$ and $\overline{y \prod_i \pi_i a^2} = 0$, and assume for a contradiction that $0 = \overline{y \prod_i \pi_i a^2} \preceq_1' \overline{x \prod_i \pi_i a^2}$, but $x \prod_i \pi_i a^2 \prec_1 y \prod_i \pi_i a^2$. Then we find some $c_1, c_2 \in I_v$ such that $c_1 \preceq_1 x \prod_i \pi_i a^2 + c_2$. With Lemma 3.7(1) follows $c_1 - c_2 \preceq_1 x \prod_i \pi_i a^2 \prec_1 y \prod_i \pi_i a^2$, thus convexity of I_v yields $x \prod_i \pi_i a^2 \in I_v$, contradicting $\overline{x \prod_i \pi_i a^2} \neq 0$.
So finally suppose that $v(x) = v(y)$ and assume for a contradiction that $x \sim_{\preceq_2} y$, but $x \prec_1 y$. Choose $a \in \tilde{R}$ such that $0 \prec_1 a$ (and hence $0 \prec_2 a$) and $v(a) = -\gamma$. Note that $ax \prec_1 ay$ if and only if $x \prec_1 y$ (by (QR5) and (QR3)), and also $ax \sim_{\preceq_2} ay$ if and only if $x \sim_{\preceq_2} y$ (Lemma 2.9 and Corollary 3.10). So we may replace x and y with ax and ay. In other words, we may without loss of generality assume that $v(x) = v(y) = 0$. It holds $y \preceq_2 x$. So by definition of $\preceq_2$ and the fact that $v(x) = v(y) = 0$, we get that $\overline{y} \preceq_1' \overline{x}$ (see Remark 4.6). Thus, there exist some $c_1, c_2 \in I_v$ such that $y + c_1 \preceq_1 x + c_2$, respectively, $y \preceq_1 x + c$ for $c := c_2 - c_1$ (see Lemma 3.7(1)). Recall that $-1 \sim_{\preceq_2} 1$. But then also $-1 \sim_{\preceq_1} 1$. Otherwise $-1 \preceq_1 0$, but $-1 \npreceq_2 0$, contradicting the fact that $\preceq_1 \subseteq \preceq_2$. Therefore, Corollary 3.10 and Lemma 3.8 yield that all elements in R are non-negative with respect to $\sim_1$. Particularly, $0 \prec_1 -1$. So Lemma 3.11 implies $y \preceq_1 x + c \preceq_1 \max\{x, c\} \prec_1 y$, a contradiction (note that $y \preceq c$ would contradict the convexity of I_v, as $0 \prec_1 y$). This finishes the proof of the Baer-Krull Theorem. □

Note that for the sake of uniformity, we avoided the dichotomy in Theorem 2.7, stating that every quasi-ordered ring is either an ordered or else a valued ring, throughout the entire paper. Taking this theorem into consideration, the Baer-Krull Theorem simplifies as follows:

Corollary 4.11. *(Baer-Krull Theorem for quasi-ordered Rings II)*
Let R be a commutative ring with 1 and v a Manis valuation on R. Then the map

$$\psi\colon \{\preceq\in \mathcal{X}_{\mathfrak{q}_v}(R)\colon\ \preceq \textit{ is } v\textit{-compatible}\} \to \{-1,1\}^I \times \mathcal{X}_{\{0\}}(Rv),$$
$$\preceq\ \mapsto\ (\eta_{\preceq}, \preceq')$$

is an embedding with $\{-1,1\}^I \times \mathcal{X}_{o,\{0\}}(Rv) \sqcup \{1\}^I \times \mathcal{X}_{p,\{0\}}(Rv) \subseteq Im(\psi)$.

Proof. Theorem 2.7 and Remark 3.13(2) yield that $\psi^{-1}(\mathcal{A})$ coincides with the domain of ψ. The statement follows now immediately from the previous theorem. □

The Baer-Krull Theorem simplifies even much further in the case where the value group Γ_v is 2-divisible, because then $\overline{\Gamma_v} = \Gamma_v/2\Gamma_v$ is trivial and therefore $I = \emptyset$.

Corollary 4.12. *(Baer-Krull Theorem for quasi-ordered Rings III)*
Let R be a commutative ring with 1 and v a Manis valuation on R such that its value group Γ_v is 2-divisible. Then the map

$$\psi\colon \{\preceq\in \mathcal{X}_{\mathfrak{q}_v}(R)\colon\ \preceq \textit{ is } v\textit{-compatible}\} \to \mathcal{X}_{\{0\}}(Rv),$$
$$\preceq\mapsto\preceq'$$

is a bijection.

Proof. This follows immediately from the previous corollary and the 2-divisibility of Γ_v, see the explanation above. □

We conclude this paper by deducing Baer-Krull Theorems for ordered, respectively proper quasi-ordered, rings, from Corollary 4.11.

Corollary 4.13. *(Baer-Krull Theorem for ordered Rings)*
Let R be a commutative ring with 1 and v a Manis valuation on R. Then the map

$$\psi\colon \{\leq\in \mathcal{X}_{o,\mathfrak{q}_v}(R)\colon\ \leq \textit{ is } v\textit{-compatible}\} \to \{-1,1\}^I \times \mathcal{X}_{o,\{0\}}(Rv),$$
$$\preceq\mapsto\preceq'$$

is a bijection.

If R is a field, then this result coincides with Theorem 2.11. Further note that if Γ_v is 2-divisible, then Corollary 4.13 simplifies in the same manner as Corollary 4.12. The Baer-Krull Theorem for quasi-ordered rings also gives rise to a characterization of all Manis valuations w on R, that are finer than v. Note that the proper quasi-orders in the domain of ψ are precisely the Manis refinements of v.

Corollary 4.14. *(Baer-Krull Theorem for proper quasi-ordered Rings I)*
Let R be a commutative ring with 1 and v a Manis valuation on R. Then the map

$$\psi\colon \{\preceq_w\in \mathcal{X}_{p,\mathfrak{q}_v}(R)\colon\ \preceq_w \textit{ is } v\textit{-compatible}\} \to \mathcal{X}_{p,\{0\}}(Rv),$$
$$\preceq\mapsto\preceq'$$

is a bijection.

Now recall from Theorem 3.12 and Remark 3.13(2) that if $\preceq=\preceq_w$, then $\preceq'=\preceq_{w/v}$ (see Remark 3.13(2) for the proof and a definition of w/v). This allows us to reformulate the previous corollary more precisely (see Corollary 4.16).

Lemma 4.15. *The valuation w is Manis, if and only if w/v is Manis.*

Proof. If u is some arbitrary valuation of R, then $u(R\backslash\mathfrak{q}_u)$ is additively closed by axiom (V3) of Definition 2.1. So in order to show that u is Manis, it suffices to prove that $u(R\backslash\mathfrak{q}_u)$ is closed under additive inverses.
First suppose that w is Manis. Let $\gamma := w/v(\overline{a}) \in \Gamma_{w/v}$ for some $\overline{a} \in Rv$. Note that $w/v(\overline{a}) = w(a)$. Since w is Manis, there exists some $b \in R$ such that $w(b) = -w(a)$. This yields $w(ab) = 0 = w(1)$. By Lemma 3.16(1), respectively by the fact that $U_w \subseteq U_v$, we obtain that $v(ab) = 0$. Since $a \in U_v$, also $b \in U_v$. Therefore, it holds $w/v(\overline{b}) = w(b) = -\gamma \in \Gamma_{w/v}$.
Now assume that w/v is Manis and let $a \in R$ such that $w(a) =: \gamma \in \Gamma_w$. We show that there exists some $b \in R$ with $w(b) = -\gamma$. Note that $a \notin \mathfrak{q}_v$, since $\mathfrak{q}_v = \mathfrak{q}_w$ (see [10, Proposition 3.1]). Since v is Manis, we find some $y \in R$ such that $ay \in U_v$. So $w/v(\overline{ay}) = w(ay) =: \gamma_1$. By surjectivity of w/v, there exists some $z \in R$ such that $w/v(\overline{z}) = w(z) = -\gamma_1$. Therefore, $w(z) = -w(a) - w(y)$. This yields $w(yz) = -w(a) = -\gamma$, i.e. $b = yz$. □

Corollary 4.16. *(Baer-Krull Theorem for proper quasi-ordered Rings II)*
Let R be a commutative ring with 1 and v a Manis valuation on R. Then the map

$$\psi\colon \{w\colon w \text{ Manis and } v \leq w\} \to \{u\colon u \text{ Manis valuation on } Rv \text{ and } \mathfrak{q}_u = \{0\}\},$$
$$w \mapsto w/v$$

is a bijection.

Proof. We deduce this corollary from Corollary 4.14. As mentioned above, if $\preceq=\preceq_w$ is a proper quasi-order compatible with v, then $\preceq'=\preceq'_{w/v}$. Moreover we have shown in the previous lemma that w is Manis if and only if w/v is Manis. So we may restrict both the domain and co-domain of ψ to proper quasi-orders that come from a Manis valuation. □

Since v and w are both Manis and $\preceq_w$ is compatible with v, it follows via Lemma 3.15 that the previous corollary characterizes all Manis refinements w of v.

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