EVERY DEFINABLE C^∞ MANIFOLD IS AFFINE

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Abstract. Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, e^x, \dots)$ be an exponential o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers with C^∞ cell decomposition. We prove that every n-dimensional definable C^∞ manifold is definably C^∞ imbeddable into $\mathbb{R}^{2n+1}$.

1. Introduction

M. Shiota proved that if $0 < r < \infty$, then every C^r Nash manifold is affine [9]. Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of $\mathbb{R}$. Note that if $\mathcal{M} = \mathcal{R}$, then a definable C^r manifold is a C^r Nash manifold. Definable C^r categories based on $\mathcal{M}$ are generalizations of the C^r Nash category.

General references on o-minimal structures are [1], [2], see also [10]. The term "definable" means "definable with parameters in $\mathcal{M}$".

If r is a non-negative integer, then every definable C^r manifold is affine [7]. We have the following theorem as a generalization of this result.

Theorem 1.1. *Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, e^x, \dots)$ be an exponential o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers with C^∞ cell decomposition. Then every n-dimensional definable C^∞ manifold X is definably C^∞ imbeddable into $\mathbb{R}^{2n+1}$.*

The above theorem is the definable version of Whitney's imbedding theorem (e.g. 2.14 [3]). Even in the Nash category (i.e. $\mathcal{M} = \mathcal{R}$), we cannot drop the assumption that $\mathcal{M}$ is exponential by Theorem 1.1 [9].

Theorem 1.2 ([5])**.** *If $0 \leq s < \infty$ and $\mathcal{M}$ is an exponential o-minimal expansion of $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ with C^∞ cell decomposition, then every definable C^s map between definable C^∞ manifolds is approximated in the definable C^s topology by definable C^∞ maps.*

Using Theorem 1.2 and by a way similar to the proof of Theorem 1.1 and 1.3 [4], we have the following theorem.

Theorem 1.3. *Let $1 \leqq s < r \leqq \infty$, then every definable C^s manifold admits a unique definable C^r manifold structure up to definable C^r diffeomorphism.*

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2. Proof of our result

Theorem 2.1. *Let X be an affine definable C^∞ manifold and V a definable subset closed in X. Then there exists a non-negative definable C^∞ function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = V$.*

Proof. By definition of affineness and 3.2 [8], X is definably C^∞ diffeomorphic to a definable C^∞ submanifold of some $\mathbb{R}^l$ which is closed in $\mathbb{R}^l$. We identify X with its image. Thus V is closed in $\mathbb{R}^l$. Since $\mathcal{M}$ admits C^∞ cell decomposition, there exists a C^∞ cell decomposition $\mathcal{D}$ partitioning V. For every cell $C \in \mathcal{D}$, the closure $\overline{C}$ of C in X lies in V. Thus if $V = C_1 \cup \cdots \cup C_m$, then $V = \overline{C_1} \cup \cdots \cup \overline{C_m}$. If C_i is bounded and k-dimensional, then $\overline{C_i}$ is definably C^∞ diffeomorphic to $[-1, 1]^k$. Hence $\overline{C_i}$ is the zeros of a definable C^∞ function. Thus the case where V is compact is proved.

Let $\overline{C_i}$ be unbounded. Replacing $\mathbb{R}^l$ by $\mathbb{R}^{l+1}$, we may assume that $0 \notin \overline{C_i}$. Let $i : \mathbb{R}^{l+1} - \{0\} \to \mathbb{R}^{l+1} - \{0\}, i(x) = \frac{x}{||x||^2}$, where $||x||$ denotes the norm of x. Then $C'_i = i(\overline{C_i}) \cup \{0\}$ is the one point compactification of $\overline{C_i}$. Thus there exists a definable C^∞ function $\psi : \mathbb{R}^{l+1} \to \mathbb{R}$ with $C'_i = \psi^{-1}(0)$. Hence $\overline{C_i}$ is definably C^∞ diffeomorphic to the set $C''_i = \{(x, y) \in \mathbb{R}^{l+1} \times \mathbb{R} | \psi(x) = 0, ||x||^2 y = 1\}$. Therefore $\overline{C_i}$ is the zeros of a definable C^∞ function. Since $V = \overline{C_1} \cup \cdots \cup \overline{C_m}$, V is the zeros of a definable C^∞ function ϕ. Thus $f := \phi^2 : X \to \mathbb{R}$ is the required function. □

The following is a definable C^∞ partition of unity.

Proposition 2.2. *Let $\{U_i\}_{i=1}^k$ be a definable open covering of a definable C^∞ manifold X. Then there exist definable C^∞ functions $\lambda_i : X \to \mathbb{R}$ $(1 \le i \le k)$ such that $0 \le \lambda_i \le 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^k \lambda_i = 1$.*

If X is affine, then the definable C^r version of Proposition 2.2 is known in 4.8 [6].

Proof. We now prove that there exists a definable open covering $\{V_i\}_{i=1}^k$ of X such that $\overline{V_i} \subset U_i$, $(1 \le i \le k)$, where $\overline{V_i}$ denotes the closure of V_i in X.

We proceed by induction on k. If $k = 1$, then there is nothing to prove. Assume that there exists a definable open covering $\{V_i\}_{i=1}^{k-1} \cup \{U_k\}$ of X such that $\overline{V_i} \subset U_i$, $(1 \le i \le k-1)$.

Let $X_{k-1} := \cup_{i=1}^{k-1} V_i$. By the inductive hypothesis, there exists a definable open covering $\{W_i\}_{i=1}^{k-1}$ of X_{k-1} such that $cl\ W_i \subset V_i$, where $cl\ W_i$ means the closure of W_i in X_{k-1}.

We may assume that U_k is affine. Let $Z_k := U_k \cap \cup_{i=1}^{k-1} V_i$ and $Cl\ Z_k$ denote the closure of Z_k in U_k. By Theorem 2.1, there exists a non-negative definable C^∞ function $\phi_k : U_k \to \mathbb{R}$ such that $\phi_k^{-1}(0) = Cl\ Z_k$. Since $cl\ W_1 \subset V_1$, ϕ_k is extensible to a non-negative definable C^∞ function $\phi_k^1 : U_k \cup W_1 \to \mathbb{R}$ such that ${\phi_k^1}^{-1}(0) = Cl\ Z_k \cup W_1$. Inductively, we have a non-negative definable C^∞ function $\phi : X \to \mathbb{R}$ such that $\phi^{-1}(0) = Cl\ Z_k \cup W_1 \cdots \cup W_{k-1}$. Let $V_k := \{x \in U_k | \phi(x) > 0\}$. Then $V_k = \{x \in X | \phi(x) > 0\}$, $\overline{V_k} \subset U_k$ and $\{V_i\}_{i=1}^k$ is the required definable open covering of X.

By Theorem 2.1, we have a non-negative definable C^∞ function $\mu_i : U_i \to \mathbb{R}$ such that $\mu_i^{-1}(0) = U_i - V_i$. Thus μ_i is extensible to a non-negative definable C^∞ function $\mu'_i : X \to \mathbb{R}$ such that ${\mu'_i}^{-1}(0) = X - V_i$. Therefore $\lambda_i := \mu'_i / \sum_{i=1}^k \mu'_i$ is the required definable C^r partition of unity. □

Proof of Theorem 1.1. Let $\{\phi_i : U_i \to \mathbb{R}^n\}_{i=1}^k$ be a definable C^r atlas of X. By Proposition 2.2, we have definable C^∞ functions $\lambda_i : X \to \mathbb{R}$, $(1 \le i \le k)$ such that $0 \le \lambda_i \le 1$, *supp* $\lambda_i \subset U$ and $\sum_{i=1}^k \lambda_i = 1$. Thus the map $F : X \to \mathbb{R}^{nk} \times \mathbb{R}^k$ defined by $F(x) = (\lambda_1(x)\phi_1(x), \dots \lambda_k(x)\phi_k(x), \lambda_1(x), \dots, \lambda_k(x))$ is a definable C^∞ imbedding. Hence X is affine. Thus it is either compact or compactifiable by 1.2 [6]. Hence we may assume that X is affine and compact at the beginning. A similar argument of the proof of 1.4 [11], every definable C^∞ map $f : X \to \mathbb{R}^{2n+1}$ can be approximated in the C^r topology by an injective definable C^∞ immersion $h : X \to \mathbb{R}^{2n+1}$. Since X is compact, h is the required definable C^∞ imbedding. □

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