

On dp-minimal ordered structures

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Abstract

We show basic facts about dp-minimal ordered structures. The main results are : dp-minimal groups are abelian-by-finite-exponent, in a divisible ordered dp-minimal group, any infinite set has non-empty interior, and any theory of pure tree is dp-minimal.

Introduction

One of the latest topic of interest in abstract model theory is the study of dependent, or NIP, theories. The abstract general study, was initiated by Shelah in [Sh715], and pursued by him in [Sh783], [Sh863] and [Sh900]. One of the questions he addresses is the definition of *super-dependent* as an analog of superstable for stable theories. Although, as he writes, he has not completely succeeded, the notion he defines of strong-dependence seems promising. In [Sh863] it is studied in details and in particular, ranks are defined. Those so-called dp-ranks are used to prove existence of an indiscernible sub-sequence in any long enough sequence. Roughly speaking, a theory is strongly dependent if no type can fork infinitely many times, each forking being independent from the previous one. (Stated this way, it is naturally a definition of “strong-NIP₂”). Also defined in that paper are notions of minimality, corresponding to the ranks being equal to 1 on 1-types. In [OnsUsv], Onshuus and Usvyatsov extract from this material the notion of dp-minimality which seems to be the relevant one. A dp-minimal theory is a theory where there cannot be two independent witnesses of forking for a 1-type. It is shown in that paper that a stable theory is dp-minimal if and only if every 1-type has weight 1. In general, unstable, theories, one can link dp-minimality to *burden* as defined by H. Adler ([Adl]).

Dp-minimality on ordered structures can be viewed as a generalization of weak-o-minimality. In that context, there are two main questions to address : what do definable sets in dimension 1 look like, (*i.e.* how far is the theory from being o-minimal), and what theorems about o-minimality go through. J. Goodrick has started to study those questions in [Goo], focussing on groups. He proves that definable functions are piecewise locally monotonous extending a similar result from weak-o-minimality.

In the first section of this paper, we recall the definitions and give equivalent formulations. In the second section, we make a few observations on general linearly ordered

inp-minimal theories showing in particular that, in dimension 1, forking is controlled by the ordering. The lack of a cell-decomposition theorem makes it unclear how to generalize results to higher dimensions.

In section 3, we study dp-minimal groups and show that they are abelian-by-finite-exponent. The linearly ordered ones are abelian. We prove also that an infinite definable set in a dp-minimal ordered divisible group has non-empty interior, solving a conjecture of Alf Dolich.

Finally, in section 4, we give examples of dp-minimal theories. We prove that colored linear orders, orders of finite width and trees are dp-minimal.

1 Preliminaries on dp-minimality

Definition 1.1. (Shelah) An independence (or inp-) pattern of length κ is a sequence of pairs $(\phi^\alpha(x, y), k^\alpha)_{\alpha < \kappa}$ of formulas such that there exists an array $\langle a_i^\alpha : \alpha < \kappa, i < \lambda \rangle$ for some $\lambda \geq \omega$ such that :

- Rows are k^α -inconsistent : for each $\alpha < \kappa$, the set $\{\phi^\alpha(x, a_i^\alpha) : i < \lambda\}$ is k^α -inconsistent,
- paths are consistent : for all $\eta \in \lambda^\kappa$, the set $\{\phi^\alpha(x, a_{\eta(\alpha)}^\alpha) : \alpha < \kappa\}$ is consistent.

Definition 1.2. • (Goodrick) A theory is inp-minimal if there is no inp-pattern of length two in a single free variable x .

- (Onshuus and Usvyatsov) A theory is dp-minimal if it is NIP and inp-minimal.

A theory is NTP_2 if there is no inp-pattern of size ω for which the formulas $\phi^\alpha(x, y)$ in the definition above are all equal to some $\phi(x, y)$. It is proven in [Che] that a theory is NTP_2 if this holds for formulas $\phi(x, y)$ where x is a single variable. As a consequence, any inp-minimal theory is NTP_2 .

We now give equivalent definitions (all the ideas are from [Sh863], we merely adapt the proofs there from the general NIP context to the dp-minimal one).

Definition 1.3. Two sequences $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ are *mutually indiscernible* if each one is indiscernible over the other.

Lemma 1.4. *Consider the following statements :*

1. T is inp-minimal.
2. For any two mutually indiscernible sequences $A = (a_i : i < \omega)$, $B = (b_j : j < \omega)$ and any point c , one of the sequences $(\text{tp}(a_i/c) : i < \omega)$, $(\text{tp}(b_i/c) : i < \omega)$ is constant.
3. Same as above, but change the conclusion to : one the sequences A or B stays indiscernible over c .
4. For any indiscernible sequence $A = (a_i : i \in I)$ indexed by a dense linear order I , and any point c , there is i_0 in the completion of I such that the two sequences $(\text{tp}(a_i/c) : i < i_0)$ and $(\text{tp}(a_i/c) : i > i_0)$ are constant.

5. Same as above, but change the conclusion to : the two sequences $(\mathbf{a}_i : i < i_0)$ and $(\mathbf{a}_i : i > i_0)$ are indiscernible over \mathbf{c} .

6. \mathbb{T} is *dp-minimal*.

Then for any theory \mathbb{T} , (2), (3), (4), (5), (6) are equivalent and imply (1). If \mathbb{T} is NIP, then they are all equivalent.

Proof. (2) \Rightarrow (1) : In the definition of independence pattern, one may assume that the rows are mutually indiscernible. This is enough.

(2) \Rightarrow (3) : Assume $\mathbf{A} = \langle \mathbf{a}_i : i < \omega \rangle$, $\mathbf{B} = \langle \mathbf{b}_i : i < \omega \rangle$ and \mathbf{c} are a witness to $\neg(3)$. Then there are two tuples $(i_1 < \dots < i_n)$, $(j_1 < \dots < j_n)$ and a formula $\phi(\mathbf{x}; \mathbf{y}_1, \dots, \mathbf{y}_n)$ such that $\models \phi(\mathbf{c}; \mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}) \wedge \neg \phi(\mathbf{c}; \mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_n})$. Take an $\alpha < \omega$ greater than all the i_k and the j_k . Then, exchanging the i_k and j_k if necessary, we may assume that $\models \phi(\mathbf{c}; \mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}) \wedge \neg \phi(\mathbf{c}; \mathbf{a}_{n,\alpha}, \dots, \mathbf{a}_{n,\alpha+n-1})$. Define $\mathbf{A}' = \langle (\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}) \rangle^\wedge \langle (\mathbf{a}_{n,k}, \dots, \mathbf{a}_{n,k+n-1}) : k \geq \alpha \rangle$. Construct the same way a sequence \mathbf{B}' . Then \mathbf{A}' , \mathbf{B}' , \mathbf{c} give a witness of $\neg(2)$.

(3) \Rightarrow (2) : Obvious.

(3) \Rightarrow (5) : Let $\mathbf{A} = \langle \mathbf{a}_i : i \in I \rangle$ be indiscernible and let \mathbf{c} be a point. Then assuming (3) holds, for every i_0 in the completion of I , one of the two sequences $\mathbf{A}_{<i_0} = \langle \mathbf{a}_i : i < i_0 \rangle$ and $\mathbf{A}_{>i_0} = \langle \mathbf{a}_i : i > i_0 \rangle$ must be indiscernible over \mathbf{c} . Take any such i_0 such that both sequences are infinite, and assume for example that $\mathbf{A}_{>i_0}$ is indiscernible over \mathbf{c} . Let $j_0 = \inf\{i \leq i_0 : \mathbf{A}_{>i} \text{ is indiscernible over } \mathbf{c}\}$. Then $\mathbf{A}_{>j_0}$ is indiscernible over \mathbf{c} . If there are no elements in I smaller than j_0 , we are done. Otherwise, if $\mathbf{A}_{<j_0}$ is not indiscernible over \mathbf{c} , then one can find $j_1 < j_0$ such that again $\mathbf{A}_{<j_1}$ is not indiscernible over \mathbf{c} . By definition of j_0 , $\mathbf{A}_{>j_1}$ is not indiscernible over \mathbf{c} either. This contradicts (3).

(5) \Rightarrow (4) : Obvious.

(4) \Rightarrow (2) : Assume $\neg(2)$. Then one can find a witness of it consisting of two indiscernible sequences $\mathbf{A} = \langle \mathbf{a}_i : i \in I \rangle$, $\mathbf{B} = \langle \mathbf{b}_i : i \in I \rangle$ indexed by a dense linear order I and a point \mathbf{c} .

Now, we can find an i_0 in the completion of I such that for any $i_1 < i_0 < i_2$ in I , there are i, i' , $i_1 < i < i_0 < i' < i_2$ such that $\text{tp}(\mathbf{a}_i/\mathbf{c}) \neq \text{tp}(\mathbf{a}_{i'}/\mathbf{c})$. Find a similar point j_0 for the sequence \mathbf{B} . Renumbering the sequences if necessary, we may assume that $i_0 \neq j_0$. Then the indiscernible sequence of pairs $\langle (\mathbf{a}_i, \mathbf{b}_i) : i \in I \rangle$ gives a witness of $\neg(4)$.

(6) \Rightarrow (2) : Let \mathbf{A} , \mathbf{B} , \mathbf{c} be a witness of $\neg(2)$. Assume for example that there is $\phi(\mathbf{x}, \mathbf{y})$ such that $\models \phi(\mathbf{c}, \mathbf{a}_0) \wedge \neg \phi(\mathbf{c}, \mathbf{a}_1)$. Then set $\mathbf{A}' = \langle (\mathbf{a}_{2k}, \mathbf{a}_{2k+1}) : k < \omega \rangle$ and $\phi'(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) = \phi(\mathbf{x}; \mathbf{y}_1) \wedge \neg \phi(\mathbf{x}; \mathbf{y}_2)$. Then by NIP, the set $\{\phi'(\mathbf{x}, \bar{\mathbf{y}}) : \bar{\mathbf{y}} \in \mathbf{A}'\}$ is k -inconsistent for some k . Doing the same construction with \mathbf{B} we see that we get an independence pattern of length 2.

(5) \Rightarrow (6) : Statement (5) clearly implies NIP (because IP is always witnessed by a formula $\phi(x, y)$ with x a single variable). We have already seen that it implies inp-minimality. \square

Standard examples of dp-minimal theories include :

- O-minimal or weakly o-minimal theories (recall that a theory is weakly-o-minimal if every definable set in dimension 1 is a finite union of convex sets),
- C-minimal theories,
- $\text{Th}(\mathbf{Z}, +, \leq)$.

The reader may check this as an exercise or see [Goo].

More examples are given in section 4 of this paper.

2 Inp-minimal ordered structures

Little study has been made yet on general dp-minimal ordered structures. We believe however that there are results to be found already at that general level. In fact, we prove here a few lemmas that turn out to be useful for the study of groups.

We show that, in some sense, forking in dimension 1 is controlled by the order.

We consider $(M, <)$ an inp-minimal linearly ordered structure with no first nor last element. We denote by T its theory, and let \mathbb{M} be a monster model of T .

Lemma 2.1. *Let $X = X_{\bar{a}}$ be a definable subset of \mathbb{M} , cofinal in \mathbb{M} . Then X is non-forking (over \emptyset).*

Proof. If $X_{\bar{a}}$ divides over \emptyset , there exists an indiscernible sequence $(\bar{a}_i)_{i < \omega}$, $\bar{a}_0 = \bar{a}$, witnessing this. Every $X_{\bar{a}_i}$ is cofinal in \mathbb{M} . Now pick by induction intervals I_k , $k < \omega$, with $I_k < I_{k+1}$ containing a point in each $X_{\bar{a}_i}$. We obtain an inp-pattern of length 2 by considering $x \in X_{\bar{a}_i}$ and $x \in I_k$.

If $X_{\bar{a}}$ forks over \emptyset , it implies a disjunction of formulas that divide, but one of these formulas must be cofinal : a contradiction. \square

A few variations are possible here. For example, we assumed that X was cofinal in the whole structure \mathbb{M} , but the proofs also works if X is cofinal in a \emptyset -definable set Y , or even contains an \emptyset -definable point in its closure. This leads to the following results.

For X a definable set, let $\text{Conv}(X)$ denote the convex hull of X . It is again a definable set.

Porism 2.2. *Let X be a definable set of \mathbb{M} (in dimension 1). Assume $\text{Conv}(X)$ is A definable. Then X is non-forking over A .*

Porism 2.3. *Let $M \prec N$ and let p be a complete 1-type over N . If the cut of p over N is of the form $+\infty, -\infty, a^+$ or a^- for $a \in M$, then p is non-forking over M .*

Proposition 2.5 generalizes this.

Lemma 2.4. *Let X be an A -definable subset of \mathbb{M} . Assume that X divides over some model M , then :*

1. *We cannot find $(\mathbf{a}_i)_{i < \omega}$ in M and points $(x_i)_{i < \omega}$ in $X(M)$ such that $\mathbf{a}_0 < x_0 < \mathbf{a}_1 < x_1 < \mathbf{a}_2 < \dots$.*
2. *The set X can be written as a finite disjoint union $X = \bigcup X_i$ where the X_i are definable over $M \cup A$, and each $\text{Conv}(X_i)$ contains no M -point.*

Proof. Easy ; (2) follows from (1). □

Proposition 2.5. *Let $A \subset M$, with M , $|A|^+$ -saturated, and let $p \in S_1(M)$. The following are equivalent :*

1. *The type p forks over A ,*
2. *There exist $\mathbf{a}, \mathbf{b} \in M$ such that $p \vdash \mathbf{a} < x < \mathbf{b}$, and \mathbf{a} and \mathbf{b} have the same type over A ,*
3. *There exist $\mathbf{a}, \mathbf{b} \in M$ such that $p \vdash \mathbf{a} < x < \mathbf{b}$, and the interval $I_{\mathbf{a}, \mathbf{b}} = \{x : \mathbf{a} < x < \mathbf{b}\}$ divides over A .*

Proof. (3) \Rightarrow (1) is trivial.

For (2) \Rightarrow (3), it is enough to show that if $\mathbf{a} \equiv_A \mathbf{b}$, then $I_{\mathbf{a}, \mathbf{b}}$ divides over A . Let σ be an A -automorphism sending \mathbf{a} to \mathbf{b} . Then the tuple $(\mathbf{b} = \sigma(\mathbf{a}), \sigma(\mathbf{b}))$ has the same type as (\mathbf{a}, \mathbf{b}) , and $\mathbf{a} < \mathbf{b} < \sigma(\mathbf{b})$. By iterating, we obtain a sequence $\mathbf{a}_1 < \mathbf{a}_2 < \dots$ such that $(\mathbf{a}_k, \mathbf{a}_{k+1})$ has the same type over A as (\mathbf{a}, \mathbf{b}) . Now the sets $I_{\mathbf{a}_{2k}, \mathbf{a}_{2k+1}}$ are pairwise disjoint and all have the same type over A . Therefore each of them divides over M .

We now prove (1) \Rightarrow (2)

Assume that (2) fails for p . Let $X_{\bar{\mathbf{a}}}$ be an M -definable set such that $p \vdash X_{\bar{\mathbf{a}}}$. Let $\bar{\mathbf{a}}_0 = \mathbf{a}, \bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \dots$ be an A -indiscernible sequence. Note that the cut of p is invariant under all A -automorphisms. Therefore each of the $X_{\bar{\mathbf{a}}_i}$ contains a type with the same cut over M as p . Now do a similar reasoning as in Lemma 2.1. □

Corollary 2.6. *Forking equals dividing : for any $A \subset B$, any $p \in S(B)$, p forks over A if and only if p divides over A .*

Proof. By results of Chernikov and Kaplan ([CheKap]), it is enough to prove that no type forks over its base. And it suffices to prove this for one-types (because of the general fact that if $\text{tp}(\mathbf{a}/B)$ does not fork over A and $\text{tp}(\mathbf{b}/B\mathbf{a})$ does not fork over $A\mathbf{a}$, then $\text{tp}(\mathbf{a}, \mathbf{b}/B)$ does not fork over A).

Assume $p \in S_1(A)$ forks over A . Then by the previous proposition, p implies a finite disjunction of intervals $\bigcup_{i < n} (\mathbf{a}_i, \mathbf{b}_i)$ with $\mathbf{a}_i \equiv_A \mathbf{b}_i$. Assume n is minimal. Without loss, assume $\mathbf{a}_0 < \mathbf{a}_1 < \dots$. Now, as $\mathbf{a}_0 \equiv_A \mathbf{b}_0$ we can find points $\mathbf{a}'_i, \mathbf{b}'_i$, with $(\mathbf{a}_i, \mathbf{b}_i) \equiv_A (\mathbf{a}'_i, \mathbf{b}'_i)$ and $\mathbf{a}'_0 = \mathbf{b}_0$.

Then p proves $\bigcup_{i < n} (\mathbf{a}'_i, \mathbf{b}'_i)$. But the interval $(\mathbf{a}_0, \mathbf{b}_0)$ is disjoint from that union, so p proves $\bigcup_{0 < i < n} (\mathbf{a}_i, \mathbf{b}_i)$, contradicting the minimality of n . □

Note that this does not hold without the assumption that the structure is linearly ordered. In fact the standard example of the circle with a predicate $C(x, y, z)$ saying that y is between x and z (see for example [Wag], 2.2.4.) is dp-minimal.

Lemma 2.7. *Let E be a definable equivalence relation on M , we consider the imaginary sort $S = M/E$. Then there is on S a definable equivalence relation \sim with finite classes such that there is a definable linear order on S/\sim .*

Proof. Define a partial order on S by $a/E \prec b/E$ if $\inf(\{x : xEa\}) < \inf(\{x : xEb\})$. Let \sim be the equivalence relation on S defined by $x \sim y$ if $\neg(x \prec y \vee y \prec x)$. Then \prec defines a linear order on S/\sim . The proof that \sim has finite classes is another variation on the proof of 2.1. \square

From now until the end of this section, we also assume NIP.

Lemma 2.8. (NIP). *Let $p \in S_1(\mathbb{M})$ be a type inducing an M -definable cut, then p is definable over M .*

Proof. We know that p does not fork over M , so by NIP, p is M -invariant. Let M_1 be an $|M|^+$ -saturated model containing M . Then the restriction of p to M_1 has a unique global extension inducing the same cut as p . In particular p has a unique heir. Being M -invariant, p is definable over M . \square

The next lemma states that members of a uniformly definable family of sets define only finitely many “germs at $+\infty$ ”.

Lemma 2.9. (NIP). *Let $\phi(x, y)$ be a formula with parameters in some model M_0 , x a single variable. Then there are b_1, \dots, b_n such that for every b , there is $\alpha \in \mathbb{M}$ and k such that the sets $\phi(x, b) \wedge x > \alpha$ and $\phi(x, b_k) \wedge x > \alpha$ are equal.*

Proof. Let E be the equivalence relation defined on tuples by bEb' iff $(\exists \alpha)(x > \alpha \rightarrow (\phi(x, b) \leftrightarrow \phi(x, b')))$. Let b, b' having the same type over M_0 . By NIP, the formula $\phi(x, b) \Delta \phi(x, b')$ forks over M_0 . By Lemma 2.1, this formula cannot be cofinal, so b and b' are E -equivalent. This proves that E has finitely many classes. \square

If the order is dense, then this analysis can be done also locally around a point a with the same proof :

Lemma 2.10. (NIP + dense order). *Let $\phi(x, y)$ be a formula with parameters in some model M_0 , x a single variable. Then there exists n such that : For any point a , there are b_1, \dots, b_n such that for all b , there is $\alpha < a < \beta$ and k such that the sets $\phi(x, b) \wedge \alpha < x < \beta$ and $\phi(x, b_k) \wedge \alpha < x < \beta$ are equal.*

3 Dp-minimal groups

We study inp-minimal groups. Note that by an example of Simonetta, ([Sim]), not all such groups are abelian-by-finite. It is proven in [MacSte] that C-minimal groups are abelian-by-torsion. We generalize the statement here to all inp-minimal theories.

Proposition 3.1. *Let G be an inp-minimal group. Then there is a definable normal abelian subgroup H such that G/H is of finite exponent.*

Proof. Let A, B be two definable subgroups of G . If $a \in A$ and $b \in B$, then there is $n > 0$ such that either $a^n \in B$ or $b^n \in A$. To see this, assume $a^n \notin B$ and $b^n \notin A$ for all $n > 0$. Then, for $n \neq m$, the cosets $a^m B$ and $a^n B$ are distinct, as are $A.b^m$ and $A.b^n$. Now we obtain an independence pattern of length two by considering the sequences of formulas $\phi_k(x) = "x \in a^k B"$ and $\psi_k(x) = "x \in A.b^k"$.

For $x \in G$, let $C(x)$ be the centralizer of x . By compactness, there is k such that for $x, y \in G$, for some $k' \leq k$, either $x^{k'} \in C(y)$ or $y^{k'} \in C(x)$. In particular, letting $n = k!$, x^n and y^n commute.

Let $H = C(C(G^n))$, the bicommutant of the n th powers of G . It is an abelian definable subgroup of G and for all $x \in G$, $x^n \in H$. Finally, if H contains all n powers then it is also the case of all conjugates of H , so replacing H by the intersection of its conjugates, we obtain what we want. \square

Now we work with ordered groups.

Lemma 3.2. *Let G be an inp-minimal ordered group. Let H be a definable sub-group of G and let C be the convex hull of H . Then H is of finite index in C .*

Proof. We may assume that H and C are \emptyset -definable. So without loss, assume $C = G$.

If H is not of finite index, there is a coset of H that forks over \emptyset . All cosets of H are cofinal in G . This contradicts Lemma 2.1. \square

Proposition 3.3. *Let G be an inp-minimal ordered group, then G is abelian.*

Proof. Note that if $a, b \in G$ are such that $a^n = b^n$, then $a = b$, for if for example $0 < a < b$, then $a^n < a^{n-1}b < a^{n-2}b^2 < \dots < b^n$.

For $x \in G$, let $C(x)$ be the centralizer of x . We let also $D(x)$ be the convex hull of $C(x)$. By 3.2, $C(x)$ is of finite index in $D(x)$. Now take $x \in G$ and $y \in D(x)$. Then xy is in $D(x)$, so there is n such that $(xy)^n \in C(x)$. Therefore $(yx)^n = x^{-1}(xy)^n x = (xy)^n$. So $xy = yx$ and $y \in C(x)$. Thus $C(x) = D(x)$ is convex.

Now if $0 < x < y \in G$, then $C(y)$ is a convex subgroup containing y , so it contains x , and x and y commute. \square

This answers a question of Goodrick ([Goo] 1.1).

Now, we assume NIP, so G is a dp-minimal ordered group. We denote by G^+ the set of positive elements of G .

Let $\phi(x)$ be a definable set (with parameters). For $\alpha \in G$, define $X_\alpha = \{g \in G^+ : (\forall x > \alpha)(\phi(x) \leftrightarrow \phi(x+g))\}$. Let H_α be equal to $X_\alpha \cup -X_\alpha \cup \{0\}$. Then H_α is a definable subgroup of G and if $\alpha < \beta$, H_α is contained in H_β . Finally, let H be the union of the H_α for $\alpha \in G$, it is the subgroup of *eventual periods* of $\phi(x)$.

Now apply Lemma 2.9 to the formula $\psi(x, y) = \phi(x-y)$. It gives n points b_1, \dots, b_n such that for all $b \in G$, there is k such that $b - b_k$ is in H . This implies that H has finite index in G .

If furthermore G is densely ordered, then we can do the same analysis locally. This yields a proof of a conjecture of Alf Dolich : in a dp-minimal divisible ordered group, any infinite set has non empty interior. As a consequence, a dp-minimal divisible definably complete ordered group is o-minimal.

As before, $I_{a,b}$ denotes the open interval (a, b) , and τ_b is the translation by $-b$. We will make use of two lemmas from [Goo] that we recall here for convenience.

Lemma 3.4 ([Goo], 3.3). *Let G be a divisible ordered inp-minimal group, then any infinite definable set is dense in some non trivial interval.*

In the following lemma, \overline{M} stands for the completion of M . By a definable function f into \overline{M} , we mean a function of the form $a \mapsto \inf \phi(a; M)$ where $\phi(x; y)$ is a definable function. So one can view \overline{M} as a collection of imaginary sorts (in which case it naturally contains only *definable* cuts of M), or understand $f : M \rightarrow \overline{M}$ simply as a notation.

Lemma 3.5 ([Goo], 3.19). *Let $f : M \rightarrow \overline{M}$ be a definable partial function such that $f(x) > 0$ for all x in the domain of f . Then for every interval I , there is a sub-interval $J \subseteq I$ and $\epsilon > 0$ such that for $x \in J \cap \text{dom}(f)$, $|f(x)| \geq \epsilon$.*

Theorem 3.6. *Let G be a divisible ordered dp-minimal group. Let X be an infinite definable set, then X has non-empty interior.*

Proof. Let $\phi(x)$ be a formula defining X .

By Lemma 3.4, there is an interval I such that X is dense in I . By Lemma 2.10 applied to $\psi(x; y) = \phi(y + x)$ at 0, there are $b_1, \dots, b_n \in M$ such that for all $b \in M$, there is $\alpha > 0$ and k such that $|x| < \alpha \rightarrow (\phi(b + x) \leftrightarrow \phi(b_k + x))$.

Taking a smaller I and X , if necessary, assume that for all $b \in I \cap X$, we may take $k = 1$.

Define $f : x \mapsto \sup\{y : I_{-y,y} \cap \tau_{b_1}X = I_{-y,y} \cap \tau_x X\}$, it is a function into \overline{M} , the completion of M . By Lemma 3.5, there is $J \subset I$ such that, for all $b \in J$, we have $|f(b)| \geq \epsilon$.

Fix $\nu < \frac{\epsilon}{2}$ and $b \in J$ such that $I_{b-2\epsilon, b+2\epsilon} \subseteq J$ (taking smaller ϵ if necessary). Set $L = I_{b-\nu, b+\nu}$ and $Z = L \cap X$. Assume for simplicity $b = 0$. Easily, if $g_1, g_2 \in Z$, then $g_1 + g_2 \in Z \cup L^c$ and $-g_1 \in Z$ (because any two points of Z have isomorphic neighborhoods of size ϵ). So Z is a group interval : it is the intersection with $I_{b-\nu, b+\nu}$ of some subgroup H of G . Now if $x, y \in L$ satisfy that there is $\alpha > 0$ such that $I_{-\alpha, \alpha} \cap \tau_x X = I_{-\alpha, \alpha} \cap \tau_y X$, then $x \equiv y$ modulo H . It follows that points of L lie in finitely many cosets modulo H . Assume Z is not convex, and take $g \in L \setminus Z$. Then for each $n \in \mathbf{N}$, the point g/n is in L and the points g/n define infinitely many different cosets; a contradiction.

Therefore Z is convex and X contains a non trivial interval. □

Corollary 3.7. *Let G be a dp-minimal ordered group. Assume G is divisible and definably complete, then G is o-minimal.*

Proof. Let X be a definable subset of G . By 3.6, the (topological) border Y of X is finite.

Let $a \in X$, then the largest convex set in X containing a is definable. By definable completeness, it is an interval and its end-points must lie in Y .

This shows that G is o-minimal. □

4 Examples of dp-minimal theories

We give examples of dp-minimal theories, namely : linear orders, order of finite width and trees.

We first look at linear orders. We consider structures of the form (M, \leq, C_i, R_j) where \leq defines a linear order on M , the C_i are unary predicates (“colors”), the R_j are binary monotone relations (that is $x_1 \leq xR_jy \leq y_1$ implies $x_1R_jy_1$).

The following is a (weak) generalization of Rubin’s theorem on linear orders (see [Poi]).

Proposition 4.1. *Let (M, \leq, C_i, R_j) be a colored linear order with monotone relations. Assume that all \emptyset -definable sets in dimension 1 are coded by a color and all monotone \emptyset -definable binary relations are represented by one of the R_j . Then the structure eliminates quantifiers.*

Proof. The result is obvious if M is finite, so we may assume (for convenience) that this is not the case.

We prove the theorem by back-and-forth. Assume that M is ω -saturated and take two tuples $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$ from M having the same quantifier free type.

Take $x_0 \in M$; we look for a corresponding y_0 . Notice that \leq is itself a monotone relation, a finite boolean combinations of colors is again a color, a positive combination of monotone relations is again a monotone relation, and if xRy is monotone $\phi(x, y) = \neg yRx$ is monotone. By compactness, it is enough to find a y_0 satisfying some finite part of the quantifier-free type of x_0 ; that is, we are given

- One color C such that $M \models C(x_0)$,
- For each k , monotone relations R_k and S_k such that $M \models x_0R_kx_k \wedge x_kS_kx_0$.

Define $U_k(x) = \{t : tR_kx_k\}$ and $V_k(x) = \{t : xS_k t\}$. The $U_k(x)$ are initial segments of M and the $V_k(x)$ final segments. For each k, k' , either $U_k(x_k) \subseteq U_{k'}(x_{k'})$ or $U_{k'}(x_{k'}) \subseteq U_k(x_k)$. Assume for example $U_k(x_k) \subseteq U_{k'}(x_{k'})$, then this translates into a relation $\phi(x_k, x_{k'})$, where $\phi(x, y) = (\forall t)(tR_kx \rightarrow tR_{k'}y)$. Now $\phi(x, y)$ is a monotone relation itself. The assumptions on \bar{x} and \bar{y} therefore imply that also $U_k(y_k) \subseteq U_{k'}(y_{k'})$.

The same remarks hold for the final segments V_k .

Now, we may assume that $U_1(x_1)$ is minimal in the $U_k(x_k)$ and $V_1(x_1)$ is minimal in the $V_k(x_k)$. We only need to find a point y_0 satisfying $C(x)$ in the intersection $U_1(y_1) \cap V_1(y_1)$.

Let $\psi(x, y)$ be the relation $(\exists t)(C(t) \wedge tR_1y \wedge xR_1t)$. This is a monotone relation. As it holds for (x_0, x_1) , it must also hold for (y_0, y_1) , and we are done. \square

The following result was suggested, in the case of pure linear orders, by John Goodrick.

Proposition 4.2. *Let $\mathcal{M} = (M, \leq, C_i, R_j)$ be a linearly ordered infinite structure with colors and monotone relations. Then $\text{Th}(\mathcal{M})$ is dp-minimal.*

Proof. By the previous result, we may assume that $T = \text{Th}(\mathcal{M})$ eliminates quantifiers. Let $(x_i)_{i \in I}, (y_i)_{i \in I}$ be mutually indiscernible sequences of n -tuples, and let $\alpha \in M$ be a point. We want to show that one of the following holds :

- For all $i, i' \in I$, x_i and $x_{i'}$ have the same type over α , or
- for all $i, i' \in I$, y_i and $y_{i'}$ have the same type over α .

Assume that I is dense without end points.

By quantifier elimination, we may assume that $n = 1$, that is the x_i and y_i are points of M . Without loss, the (x_i) and (y_i) form increasing sequences. Assume there exists $i < j \in I$ and R a monotone definable relation such that $M \models \neg \alpha R x_i \wedge \alpha R x_j$. By monotonicity of R , there is a point i_R of the completion of I such that $i < i_R \rightarrow \neg \alpha R x_i$ and $i > i_R \rightarrow \alpha R x_i$.

Assume there is also a monotone relation S and an i_S such that $i < i_S \rightarrow \neg \alpha S y_i$ and $i > i_S \rightarrow \alpha S y_i$.

For points x, y define $I(x, y)$ as the set of $t \in M$ such that $M \models \neg t R x \wedge t R y$. This is an interval of M . Furthermore, if $i_1 < i_2 < i_3 < i_4$ are in I , then the intervals $I(x_{i_1}, x_{i_2})$ and $I(x_{i_3}, x_{i_4})$ are disjoint. Define $J(x, y)$ the same way using S instead of R .

Take $i_0 < i_R < i_1 < i_2 < \dots$ and $j_0 < i_S < j_1 < j_2 < \dots$. For $k < \omega$, define $I_k = I(x_{i_{2k}}, x_{i_{2k+1}})$ and $J_k = J(y_{j_{2k}}, y_{j_{2k+1}})$. The two sequences (I_k) and (J_k) are mutually indiscernible sequences of disjoint intervals. Furthermore, we have $\alpha \in I_0 \wedge J_0$. By mutual indiscernibility, $I_i \wedge J_j \neq \emptyset$ for all indices i and j , which is impossible.

We treated the case when α was to the left of the increasing relations R and S . The other cases are similar. \square

An ordered set (M, \leq) is of *finite width*, if there is n such that M has no antichain of size n .

Corollary 4.3. *Let $\mathcal{M} = (M, \leq)$ be an infinite ordered set of finite width, then $\text{Th}(\mathcal{M})$ is dp-minimal.*

Proof. We can define such a structure in a linear order with monotone relations : see [Shm]. More precisely, there exists a structure $\mathcal{P} = (P, \prec, R_j)$ in which \prec is a linear order and the R_j are monotone relations. There is a definable relation $O(x, y)$ such that the structure (\mathcal{P}, O) is isomorphic to (M, \leq) .

The result therefore follows from the previous one. \square

We now move to trees. A tree is a structure (T, \leq) such that \leq defines a partial order on T , and for all $x \in T$, the set of points smaller than x is linearly ordered by \leq . We will also assume that given $x, y \in T$, the set of points smaller than x and y has a maximal element $x \wedge y$ (and set $x \wedge x = x$). This is not actually a restriction, since we could always work in an imaginary sort to ensure this.

Given $a, b \in T$, we define the open ball $B(a; b)$ of center a containing b as the set $\{x \in T : x \wedge b > a\}$, and the closed ball of center a as $\{x \in T : x \geq a\}$.

Notice that two balls are either disjoint or one is included in the other.

Lemma 4.4. *Let (T, \leq) be a tree, $a \in T$, and let D denote the closed ball of center a . Let $\bar{x} = (x^1, \dots, x^n) \in (T \setminus D)^n$ and $\bar{y} = (y^1, \dots, y^m) \in D^m$. Then $\text{tp}(\bar{x}/a) \cup \text{tp}(\bar{y}/a) \vdash \text{tp}(\bar{x} \cup \bar{y}/a)$.*

Proof. A straightforward back-and-forth, noticing that $\text{tp}(\bar{x}/\mathbf{a}) \cup \text{tp}(\bar{y}/\mathbf{a}) \vdash \text{tp}_{\text{qf}}(\bar{x} \cup \bar{y}/\mathbf{a})$ (quantifier-free type). \square

We now work in the language $\{\leq, \wedge\}$, so a sub-structure is a subset closed under \wedge .

Proposition 4.5. *Let $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$, $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ be two sub-structures from \mathbb{T} . Assume :*

1. *\mathbf{A} and \mathbf{B} are isomorphic as sub-structures,*
2. *for all i, j such that $\mathbf{a}_i \geq \mathbf{a}_j$, $\text{tp}(\mathbf{a}_i, \mathbf{a}_j) = \text{tp}(\mathbf{b}_i, \mathbf{b}_j)$.*

Then $\text{tp}(\mathbf{A}) = \text{tp}(\mathbf{B})$.

Proof. We do a back-and-forth. Assume \mathcal{J} is ω -saturated and \mathbf{A}, \mathbf{B} satisfy the hypothesis. We want to add a point \mathbf{a} to \mathbf{A} . We may assume that $\mathbf{A} \cup \{\mathbf{a}\}$ forms a sub-structure (otherwise, if some $\mathbf{a}_i \wedge \mathbf{a}$ is not in $\mathbf{A} \cup \{\mathbf{a}\}$, add first this element).

We consider different cases :

1. The point \mathbf{a} is below all points of \mathbf{A} . Without loss \mathbf{a}_0 is the minimal element of \mathbf{A} (which exists because \mathbf{A} is closed under \wedge). Then find a \mathbf{b} such that $\text{tp}(\mathbf{a}_0, \mathbf{a}) = \text{tp}(\mathbf{b}_0, \mathbf{b})$. For any index i , we have : $\text{tp}(\mathbf{a}_i, \mathbf{a}_0) = \text{tp}(\mathbf{b}_i, \mathbf{b}_0)$ and $\text{tp}(\mathbf{a}, \mathbf{a}_0) = \text{tp}(\mathbf{b}, \mathbf{b}_0)$. By Lemma 4.4, $\text{tp}(\mathbf{a}_i, \mathbf{a}) = \text{tp}(\mathbf{b}_i, \mathbf{b})$.
2. The point \mathbf{a} is greater than some point in \mathbf{A} , say \mathbf{a}_1 , and the open ball $\mathbf{a} := \mathbf{B}(\mathbf{a}_1; \mathbf{a})$ contains no point of \mathbf{A} .

Let \mathcal{A} be the set of all open balls $\mathbf{B}(\mathbf{a}_i; \mathbf{a}_i)$ for $\mathbf{a}_i > \mathbf{a}_1$. Let n be the number of balls in \mathcal{A} that have the same type p as \mathbf{a} . Then $\text{tp}(\mathbf{a}_1)$ proves that there are at least $n + 1$ open balls of type p of center \mathbf{a}_1 . Therefore, $\text{tp}(\mathbf{b}_1)$ proves the same thing. We can therefore find an open ball \mathbf{b} of center \mathbf{b}_1 of type p that contains no point from \mathbf{B} . That ball contains a point \mathbf{b} such that $\text{tp}(\mathbf{b}_1, \mathbf{b}) = \text{tp}(\mathbf{a}_1, \mathbf{a})$. Now, if \mathbf{a}_i is smaller than \mathbf{a}_1 , we have $\text{tp}(\mathbf{a}_i, \mathbf{a}_1) = \text{tp}(\mathbf{b}_i, \mathbf{b}_1)$ and $\text{tp}(\mathbf{a}_1, \mathbf{a}) = \text{tp}(\mathbf{b}_1, \mathbf{b})$, therefore by Lemma 4.4, $\text{tp}(\mathbf{a}, \mathbf{a}_i) = \text{tp}(\mathbf{b}, \mathbf{b}_i)$.

The fact that we have taken \mathbf{b} in a new open ball of center \mathbf{b}_1 ensures that $\mathbf{B} \cup \{\mathbf{b}\}$ is again a sub-structure and that the two structures $\mathbf{A} \cup \{\mathbf{a}\}$ and $\mathbf{B} \cup \{\mathbf{b}\}$ are isomorphic.

3. The point \mathbf{a} is between two points of \mathbf{A} , say \mathbf{a}_0 and \mathbf{a}_1 ($\mathbf{a}_0 < \mathbf{a}_1$), and there are no points of \mathbf{A} between \mathbf{a}_0 and \mathbf{a}_1 .

Find a point \mathbf{b} such that $\text{tp}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}) = \text{tp}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b})$. Then if i is such that $\mathbf{a}_i > \mathbf{a}$, we have $\mathbf{a}_i \geq \mathbf{a}_1$ and again by Lemma 4.4, $\text{tp}(\mathbf{a}_i, \mathbf{a}) = \text{tp}(\mathbf{b}_i, \mathbf{b})$. And same if $\mathbf{a}_i < \mathbf{a}$.

\square

Corollary 4.6. *Let $\mathbf{A} \subset \mathbb{T}$ be any subset. Then $\bigcup_{(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathcal{A}^3} \text{tp}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vdash \text{tp}(\mathbf{A})$.*

Proof. Let \mathbf{A}_0 be the substructure generated by \mathbf{A} . By the previous theorem the following set of formulas implies the type of \mathbf{A}_0 :

- the quantifier-free type of \mathbf{A}_0 ,

- the set of 2-types $\text{tp}(\mathbf{a}, \mathbf{b})$ for $(\mathbf{a}, \mathbf{b}) \in A_0^2$, $\mathbf{a} < \mathbf{b}$.

We need to show that those formulas are implied by the set of 3-types of elements of A . We may assume A is finite.

First, the knowledge of all the 3-types is enough to construct the structure A_0 . To see this, start of example with a point $\mathbf{a} \in A$ maximal. Knowing the 3-types, one knows in what order the $\mathbf{b} \wedge \mathbf{a}$, $\mathbf{b} \in A$ are placed. Doing this for all such \mathbf{a} , enables one to reconstruct the tree A_0 .

Now take $\mathbf{m}_1 = \mathbf{a} \wedge \mathbf{b}$, $\mathbf{m}_2 = \mathbf{c} \wedge \mathbf{d}$ for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in A$ such that $\mathbf{m}_1 \leq \mathbf{m}_2$. The points \mathbf{m}_1 and \mathbf{m}_2 are both definable using only 3 of the points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, say $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Then $\text{tp}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vdash \text{tp}(\mathbf{m}_1, \mathbf{m}_2)$. \square

The previous results are also true, with the same proofs, for colored trees.

It is proven in [Par] that theories of trees are NIP. We give a more precise result.

Proposition 4.7. *Let $\mathcal{T} = (\mathbb{T}, \leq, C_i)$ be a colored tree. Then $\text{Th}(\mathcal{T})$ is dp-minimal.*

Proof. We will use criterium (5) of 1.4 : if $(\mathbf{a}_i)_{i \in I}$ and $(\mathbf{b}_j)_{j \in J}$ are mutually indiscernible sequences and $\alpha \in \mathbb{T}$ is a point, then one of the sequences (\mathbf{a}_i) and (\mathbf{b}_j) is indiscernible over α .

We will always assume that the index sets (I and J) are dense linear orders without end points.

1) We start by showing the result assuming the \mathbf{a}_i and \mathbf{b}_j are points (not tuples).

We classify the indiscernible sequence (\mathbf{a}_i) in 4 classes depending on its quantifier-free type.

- I** The sequence (\mathbf{a}_i) is monotonous (increasing or decreasing).
- II** The \mathbf{a}_i are pairwise incomparable and $\mathbf{a}_i \wedge \mathbf{a}_j$ is constant equal to some point β .
- III** The \mathbf{a}_i are incomparable and $\mathbf{a}_i \wedge \mathbf{a}_j$, $i < j$ depends only on i . Then let $\mathbf{a}'_i = \mathbf{a}_i \wedge \mathbf{a}_j$ (for some $i < j$). The \mathbf{a}'_i form an increasing indiscernible sequence.
- IV** The \mathbf{a}_i are incomparable and $\mathbf{a}_i \wedge \mathbf{a}_j$, $i < j$ depends only on j . Then the $\mathbf{a}'_j = \mathbf{a}_i \wedge \mathbf{a}_j$ ($i < j$) form a decreasing indiscernible sequence.

Assume (\mathbf{a}_i) lands in case **I**. Consider the set $\{x : x < \alpha\}$. If that set contains a non-trivial subset of the sequence (\mathbf{a}_i) , we say that α *cuts* the sequence. If this is not the case, then the sequence (\mathbf{a}_i) stays indiscernible over α . To see this, assume for example that (\mathbf{a}_i) is increasing and that α is greater than all the \mathbf{a}_i . Take two sets of indices $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ and a $k \in I$ greater than all those indices. Then $\text{tp}(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}/\mathbf{a}_k) = \text{tp}(\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_n}/\mathbf{a}_k)$. Therefore by Lemma 4.4, $\text{tp}(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}/\alpha) = \text{tp}(\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_n}/\alpha)$.

In case **II**, note that if (\mathbf{a}_i) is not α -indiscernible, then there is $i \in I$ such that α lies in the open ball $B(\beta; \mathbf{a}_i)$ (we will also say that α *cuts* the sequence (\mathbf{a}_i)). This follows easily from Proposition 4.5.

In the last two cases, if (a_i) is α -indiscernible, then it is also the case for (a'_i) . Conversely, if (a'_i) is α -indiscernible, then α does not cut the sequence (a'_i) . From 4.5, it follows easily that (a_i) is also α -indiscernible. We can therefore replace the sequence (a_i) by (a'_i) which belongs to case **I**.

Going back to the initial data, we may assume that (a_i) and (b_j) are in case **I** or **II**. It is then straightforward to check that α cannot cut both sequences. For example, assume (a_i) is increasing and (b_j) is in case **II**. Then define β as $b_i \wedge b_j$ (any i, j). If α cuts (b_j) , then $\alpha > \beta$. But (a_i) is β -indiscernible. So β does not cut (a_i) . The only possibility for α to cut (a_i) is that β is smaller than all the a_i and the a_i lie in the same open ball of center β as α . But then the a_i lie in the same open ball of center β as one of the b_j . This contradicts mutual indiscernability.

2) Reduction to the previous case. We show that if $(a_i)_{i \in I}$ is an indiscernible sequence of n -tuples and $\alpha \in T$ such that (a_i) is not α -indiscernible, then there is an indiscernible sequence $(d_i)_{i \in I}$ of points of T in $\text{dcl}((a_i))$ such that (d_i) is not α -indiscernible.

First, by 4.6, we may assume that $n = 2$. Write $a_i = (b_i, c_i)$ and define $m_i = b_i \wedge c_i$. We again study different cases :

1. The m_i are all equal to some m .
As (a_i) is not α -indiscernible, necessarily, $\alpha > m$ and the ball $B(m; \alpha)$ contains one b_i (resp. c_i). Then take $d_i = b_i$ (resp. $d_i = c_i$) for all i .
2. The m_i are linearly ordered by $<$ and no b_i nor c_i is greater than all the m_i .
Then the balls $B(m_i; b_i)$ and $B(m_i; c_i)$ contain no other point from $(b_i, c_i, m_i)_{i \in I}$. Then, α must cut the sequence (m_i) and one can take $d_i = m_i$ for all i .
3. The m_i are linearly ordered by $<$ and, say, each b_i is greater than all the m_i .
Then each ball $B(m_i; a_i)$ contains no other point from $(b_i, c_i, m_i)_{i \in I}$. If α cuts the sequence m_i , then again one can take $d_i = m_i$. Otherwise, take a point γ larger than all the m_i but smaller than all the d_i . Applying 4.4 with a there replaced by γ , we see that (b_i) cannot be α -indiscernible. Then take $d_i = b_i$ for all i .
4. The m_i are pairwise incomparable.
The the sequence (m_i) lies in case **II**, **III** or **IV**. The open balls $B(m_i; b_i)$ and $B(m_i; c_i)$ cannot contain any other point from $(b_i, c_i, m_i)_{i \in I}$. Considering the different cases, one sees easily that taking $d_i = m_i$ will work.

This finishes the proof. □

Remark 4.8. If we define dp-minimal^+ analogously to strongly^+ -dependent (see [Sh863]), all theories studied in this section are dp-minimal^+ .

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