# Asymptotic classes of finite Moufang polygons

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**Abstract**

In this paper we study the model theory of classes of finite Moufang polygons. We show that each family of finite Moufang polygons forms an 'asymptotic class'. As a result, since every non-principal ultraproduct of an asymptotic class is 'measurable', and therefore supersimple of finite rank, we obtain examples of (infinite) supersimple Moufang polygons of finite rank.

In a forthcoming paper, [8], we will show that all supersimple Moufang polygons of finite rank arise over supersimple fields and belong to exactly those families which also have finite members.

This body of work will give a description of groups with supersimple finite rank theory which have a definable spherical Moufang BN-pair of rank at least two.

## 1 Introduction

There is a well-known model-theoretic conjecture, often referred to as the 'Algebraicity Conjecture', which states that simple groups over an algebraically closed field are exactly the simple groups of finite Morley rank (a model-theoretic notion of dimension which generalises that of an algebraic variety). This has been answered in the 'even type' and 'mixed type' cases, see [1], while there still seems to be a lack of methods to tackle the 'odd type' and 'degenerate type' cases. In [1], a very important tool is the classification of simple groups of finite Morley rank with a spherical Moufang BN-pair of Tits rank $\geq 2$, which was achieved in [11]. The latter makes use of the classification of Moufang polygons of finite Morley rank (also given by [11]).

The 'finite Morley rank' condition is very strong, and eliminates many interesting Moufang polygons. For example, non-principal ultraproducts of finite Moufang polygons have a very nice model theory - they are supersimple finite rank - but do not have finite Morley rank. This inspired the work carried out in [7], which was intended to generalise the results of [11] from the superstable context (with the stronger assumption of finite Morley rank) to the supersimple one.

The work in this paper is extracted from [7]. The main results can be stated as follows (more detailed statements are, respectively, Theorems 7.2 and 8.2). First, we recall that there are, up to duality, only seven families of finite Moufang polygons, i.e., those whose members are either projective planes, symplectic quadrangles, Hermitian quadrangles in projective space of dimension 3 or 4, split Cayley and twisted triality hexagons, or Ree-Tits octagons, with the latter arising over (finite) *difference* fields (see Definition 5.12). According to Definition 3.6, every (possibly infinite) Moufang polygon which belongs to one of those families listed above, is said to be *good*.

**Theorem 1.1** Let $\mathcal{C}$ be any of the families of finite Moufang polygons. Then $\mathcal{C}$ forms an asymptotic class.

**Theorem 1.2** Let $\Gamma$ be a good Moufang polygon, and let $\Sigma$ be its associated little projective group. Then $\Gamma$ and $\Sigma$ are parameter bi-interpretable. In particular, $\Gamma$ is supersimple finite rank if and only if $\Sigma$ is supersimple finite rank.

We also show the following result (which is Theorem 7.1), which does not seem to appear in the literature, although, to some extent, it is implicitly present in [16]; it is a fairly small extension of the results from [16].

**Theorem 1.3** For any fixed family $\mathcal{G}$ of either finite Chevalley groups or finite twisted groups of fixed Lie type and Lie rank, there exists an $L_{\text{group}}$-formula $\sigma$ such that for any fixed finite group $G$, we have $G \in \mathcal{G}$ if and only if $G \models \sigma$.

Theorem 1.1 says, essentially, that the class of definable sets in any family of finite Moufang polygons satisfies the Lang-Weil asymptotic behaviour of the rational points of varieties in finite fields. The remaining work done in [7] deals with those (infinite) Moufang polygons which are not good, showing that the latter are not supersimple finite rank. This work will also be extracted and presented in a forthcoming paper, [8]; it rests on the classification of Tits and Weiss [18].

This paper is organized as follows. Sections 2 and 3 give some background on Moufang polygons (in particular, Section 2 gives examples of good Moufang polygons), while Section 5 introduces the model-theoretic notions that we will use throughout this paper; in particular, the notion for a class of finite structures to be an asymptotic class. Also, Section 4 deals with the key points regarding the interpretation of the little projective group in the polygon; this is done almost exactly as in Section 1 of [11]. Sections 6 and 7 prove Theorem 1.1. More precisely, Section 6 shows a uniform bi-interpretation (using parameters) between a given family of finite Moufang polygons and its corresponding class of finite little projective groups; since the classes of finite little projective groups are well-known (they are either classes of finite Chevalley groups or finite twisted groups of fixed Lie type and Lie rank), and they are shown to be 'asymptotic classes' by [16], this uniform bi-interpretation procedure allows us to 'transfer' the asymptotic behaviour to the classes of finite Moufang polygons. This is proved in Theorem 6.3. However, there is an issue regarding the use of parameters. This, in a similar context (see Chapter 5 of [16]), led Ryten to introduce the notion of a *strong* uniform bi-interpretation between classes of finite structures. This is treated in Section 7. Indeed, Theorem 7.2(i) proves that the bi-interpretation shown in the proof of Theorem 6.3 is strong. This gives Theorem 1.1. Since by [12] non-principal ultraproducts of asymptotic classes are 'measurable' and thus supersimple finite rank, Theorem 1.1 provides examples of supersimple finite rank

Moufang polygons arising over (difference) pseudofinite fields.

Finally, Section 8 deals with Theorem 1.2, namely Theorem 8.2. One direction, the interpretation of the little projective group in the associated good Moufang polygon, requires just a result on the existence of a bound for the number of 'root groups' generating the little projective group, which is known to be true in the literature (see [3], for instance). For the other direction, to interpret the polygon from the group, we basically interpret the points and lines of the polygon as the coset space in the little projective group of, respectively, the pointwise stabilizers of a fixed point and a fixed line (where the latter are incident and play the role of a 'fundamental flag').

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## 2 Generalized polygons

In this section we introduce Moufang polygons, which are the basic objects of this paper. In view of Theorem 1.1, we will concentrate on those families of Moufang polygons which include infinitely many finite Moufang polygons.

Moufang polygons have been classified by Tits and Weiss, and their book [18] gives full details of this classification; also, [19] gives further details on generalized polygons, including polygons without the Moufang assumption. We use both references.

Let $L_{\text{inc}} = (P, L, I)$ be a language with 2 disjoint unary relations $P$ and $L$ and a binary relation $I$, where $I \subseteq P \times L \cup L \times P$ is symmetric and stands for *incidence*. An $L_{\text{inc}}$-structure is called an *incidence structure*. Usually, the elements $a$ satisfying $P$ are called *points*, those satisfying $L$ are called *lines*, and pairs $(a, l)$, or $(l, a)$, satisfying $I$ are called *flags*.

A sequence $(x_0, x_1, ..., x_k)$ of elements $x_i \in P \cup L$ such that $x_i$ is incident with $x_{i-1}$ for $i = 1, 2, ..., k$ is called a *k-chain*; if $x, y \in P \cup L$, and $k$ is least such that there is a $k$-chain $(x_0, x_1, ..., x_k)$ with $x_0 = x$ and $x_k = y$, we write $\mathrm{d}(x_0, x_k) = k$. For $x \in P \cup L$, we define $B_k(x) := \{y \in P \cup L : 1 \leq \mathrm{d}(x, y) \leq k\}$. If $a$ is a point, $B_1(a)$ is called a *line pencil*; if $l$ is a line, $B_1(l)$ is called a *point row*.

**Definition 2.1** A *generalized n-polygon*, or *generalized n-gon*, is an incidence structure $\Gamma = (P, L, I)$ satisfying the following three axioms:

(i) every element $x \in P \cup L$ is incident with at least three other elements;

(ii) for all elements $x, y \in P \cup L$ we have $\mathrm{d}(x, y) \leq n$;

(iii) if $\mathrm{d}(x, y) = k < n$, there is a unique $k$-chain $(x_0, x_1, ..., x_k)$ with $x_0 = x$ and $x_k = y$.
A *subpolygon* $\Gamma'$ of $\Gamma$ is an incidence substructure $\Gamma' = (P', L', I') \subseteq \Gamma$, i.e., $P' \subseteq P$, $L' \subseteq L$ and $I' = I \cap (P' \times L')$, satisfying the axioms (i)-(iii) above.

Generalized $n$-gons are often called *thick* generalized $n$-gons; this is because sometimes the definition above is given with 'two' in place of 'three' in (i), and if so by dropping the assumption (i) and replacing it by:

(i)$'$ "every element $x \in P \cup L$ is incident with exactly two other elements",
we obtain *thin* generalized $n$-gons, namely *ordinary polygons*.

If confusion does not arise, we will often refer to generalized $n$-gons as *n-gons*, for short. Generalized polygons are objects interesting in their own right, but have particular significance because of the following result; for backgroung on buildings, see any of: [2],[3],[15],[17] and [18].

**Proposition 2.2** (Proposition 3.2 of [15]) Let $(\Delta, \mathcal{A})$ be an irreducible, spherical building of Tits rank $\geq 3$, with associated Coxeter matrix $M = (m_{ij})_{i,j \in I}$. Then, every residue of rank 2 is a generalized $m_{ij}$-gon for some $i, j \in I$.

**Remark 2.3** By the work of Tits (see, for instance, Proposition 40.5 of [18]), these rank 2 residues have the 'Moufang' property (see Definition 2.6). In general, generalized $n$-polygons seem too wild to classify, but under the Moufang assumption they are classified in [18]; in particular, by Theorem 17.1 of [18], $n \in \{3, 4, 6, 8\}$.

For any $n$-gon $\Gamma = (P, L, I)$, the cardinality of a line pencil $B_1(a)$, for some $a \in P$, and the cardinality of a point row $B_1(l)$, for some $l \in L$, do not depend, respectively, on $a$ and $l$; therefore, if $s = |B_1(a)|$ and $t = |B_1(l)|$, for some $a \in P$ and $l \in L$, where $s$ and $t$ can be either finite or infinite cardinals, then we define $(s, t)$ to be the *order* of $\Gamma$. We denote by $\Gamma^{\text{dual}} = (L, P, I)$ the dual of $\Gamma$; here, $\Gamma^{\text{dual}}$ is obtained by interchanging points and lines of $\Gamma$.

**Definition 2.4** Given two incidence structures $\Gamma_1 = (P_1, L_1, I_1)$, and $\Gamma_2 = (P_2, L_2, I_2)$, an *isomorphism* of $\Gamma_1$ onto $\Gamma_2$ is a pair of bijections $\alpha : P_1 \longrightarrow P_2$ and $\beta : L_1 \longrightarrow L_2$ preserving incidence and non-incidence; a *duality* of $\Gamma_1$ onto $\Gamma_2$ is an isomorphism of $\Gamma_1$ onto $\Gamma_2^{\text{dual}}$.

**Definition 2.5** Let $\Gamma = (P, L, I)$ be an $n$-gon. Suppose that $x, y \in P \cup L$ and $\mathrm{d}(x, y) = k < n$. By axiom (iii) of Definition 2.1, there is a unique element $z \in B_{k-1}(x) \cap B_1(y)$, which is denoted by $z = \mathrm{proj}_k(x, y)$. In particular, if $\mathrm{d}(x, y)$ is exactly $n$, then there is a bijection $[y, x] : B_1(x) \longrightarrow B_1(y)$, given by $z \longmapsto \mathrm{proj}_{n-1}(z, y)$, with inverse $[x, y]$. We call the map $[y, x]$ a *perspectivity* between $x$ and $y$; a composition of perspectivities is called a *projectivity*, and we put $[x_3, x_2][x_2, x_1] = [x_3, x_2, x_1]$, and so on.

**Definition 2.6** A *root* of an $n$-gon $\Gamma$ is an $n$-chain $\alpha = (x_0, x_1, ..., x_n)$ with $x_{i-1} \neq x_i$ for $i = 1, 2, ..., n$. Given such a root $\alpha$, consider the set $X = \cup_{i=1}^{n-1} B_1(x_i)$. We define the *root group* $U_\alpha$ to be the group of all automorphisms of $\Gamma$ that fix $X$ elementwise. Since $U_\alpha$ fixes $x_0$ and $x_n$, the root group $U_\alpha$ acts on both sets $B = B_1(x_0)\backslash\{x_1\}$ and $B' = B_1(x_n)\backslash\{x_{n-1}\}$. The group $\Sigma := \langle U_\alpha : \alpha \text{ root } \rangle$ is called the *little projective group* of the polygon $\Gamma$.

A root $\alpha$ is called *Moufang* if the group $U_\alpha$ acts transitively on the set $B$ and, symmetrically, on the set $B'$; or, equivalently, on the set of all ordinary $n$-polygons containing $\alpha$. Then $\Gamma$ is called Moufang if every root $\alpha$ is Moufang.

There are basically two ways of coordinatizing a generalized polygon. We follow a purely geometric approach as in [11] and [19], while the Tits and Weiss classification follows a more algebraic path.

**Definition 2.7** Let $u, v$ be a flag of an $n$-gon $\Gamma$. Then, for some $k < n$, we define $B_k(u,v) = B_k(v) \backslash B_{k-1}(u)$ to be a *Schubert cell* of $\Gamma$. In particular, since $P = B_0(l,a) \cup B_1(a,l) \cup B_2(l,a) \cup ...$, the set of points $P$ is partitioned into $n$ Schubert cells. Likewise for the set of lines $L$.

**Definition 2.8** Consider an element $x \in B_k(x_{2n-1}, x_0)$, for some $k < n$, and let $(x_{2n-1}, x_0, x'_1, x'_2, ..., x'_k = x)$ denote the corresponding $(k+1)$-chain. Note that $\mathrm{d}(x'_i, x_{n+i}) = n$, for $i = 1, 2, ..., k$, so we may put $t_i(x) = \mathrm{proj}_{n-1}(x'_i, x_{n+i-1}) \in T_i$, where $T_i = B_1(x_{n+i-1}) \backslash \{x_{n+i}\}$ are the *parameter sets*. We have therefore attached *coordinates* $(t_1(x), t_2(x), ..., t_k(x)) \in T_1 \times T_2 \times ... \times T_k$ to the element $x$.

Above, we considered only elements at distance $k$ from $x_0$ which are not at distance $k-1$ from $x_{2n-1}$. Thus, we can attach coordinates to the remaining elements treating them as elements of the Schubert cells $B_k(x_0, x_{2n-1})$; for example, if $x \in B_k(x_0, x_{2n-1})$, for $k \leq n-1$, then the first element $x'_1$ of the $(k+1)$-chain joining $x$ with the flag $(x_0, x_{2n-1})$ is now *opposite* to (i.e., at distance $n$) $x_{n-2}$, and not to $x_n$ as in the previous case; thus, the coordinates of $x$ with respect to the Schubert cell $B_k(x_0, x_{2n-1})$ are $t_i(x) = \mathrm{proj}_{n-1}(x'_i, x_{n-i}) \in T_{n-1+i}$ for $i = 1, 2, ..., k$, where $T_{n-1+i} = B_1(x_{n-i}) \backslash \{x_{n-i-1}\}$.

It follows that the coordinatization uses $2n-2$ *parameter sets*, namely the sets $T_1, T_2, ..., T_{n-1}$ for the Schubert cells $B_k(x_{2n-1}, x_0)$ with $k = 1, 2, ..., n-1$, and the sets $T_n, T_{n+1}, ..., T_{2n-2}$ for the Schubert cells $B_k(x_0, x_{2n-1})$ with $k = n, n+1, ..., 2n-2$.

**Remark 2.9** Let $\Gamma = (P, L, I)$ be a generalized $n$-gon, and let $A = (x_0, x_1, ..., x_{2n-1})$ be an ordinary polygon in $\Gamma$. We call the set $X = \cup_{i=0,...,2n-1} B_1(x_i)$ the *hat-rack* of $\Gamma$. Since from the coordinatization every element $x \in P \cup L$ has coordinates from the parameter sets $T_i$, it follows that, model theoretically, $\mathrm{dcl}(X) = \Gamma$ (see second paragraph of the beginning of Section 5).

**Remark 2.10** Typically, there is an algebraic structure $S$ (i.e., an alternative division ring, a vector space over a field, a Jordan division algebra, and so on), two subsets $S_1$ and $S_2$ of $S$, and functions from $S_1 \times S_1$, $S_1 \times S_2$ and/or $S_2 \times S_1$ to $S_1$ and/or $S_2$ (e.g. a bilinear form, a quadratic form, a norm map, and so on), which 'determine' (up to 'duality') the associated generalized polygon, and vice versa. For instance, sometimes $S_1$ has the structure of a field, $S_2$ that of a vector space over $S_1$, and the map $S_2 \longrightarrow S_1$ is a quadratic form (this is the case of an orthogonal quadrangle - see Example 3.2).

In this paper, we will sometimes use the following informal meaning of coordinatization: given a generalized polygon $\Gamma$, we say that $\Gamma$ is *coordinatized by*, or *coordinatized over*, the structure $S$, if $S$ is the algebraic structure associated to $\Gamma$ as in Remark 2.10 (see Part II, Sections 9-16, and Part III, Section 30, of [18], for all the details about these algebraic structures). This is not used in a precise model-theoretic sense.

# 3 Some examples of Moufang polygons

We now give an introduction to certain families of (Moufang) generalized polygons from the Tits and Weiss classification which, up to some restriction, also arise in the finite case. In the following, by a *skew-field* we mean a non-commutative division ring; in our context this is not very relevant (there are no finite or commutative skew-fields), but will be relevant for [8].

**Example 3.1** Triangles ($n = 3$): Generalized 3-gons are precisely projective planes. By Theorem 17.2 of [18], Moufang projective planes are coordinatized by alternative division rings; for the latter, see Construction 2.2.4 of [19]. Alternative division rings are either associative (fields or skew-fields) or non-associative (Cayley-Dickson algebras, see Definition 9.11 of [18]).

We denote by $\mathrm{PG}_2(A)$ the Moufang projective plane coordinatized by an alternative division ring $A$. Any finite Moufang projective plane is Desarguesian over a finite field, and we denote it by $\mathrm{PG}_2(q)$ for some finite field $\mathbb{F}_q$ with $q$ a prime power.

**Example 3.2** Orthogonal and Hermitian quadrangles ($n = 4$): Let $V$ be a vector space over some (possibly skew) field $K$, and let $\sigma$ be a field anti-automorphism of order at most 2, i.e., $(ab)^\sigma = b^\sigma a^\sigma$, for all $a, b \in K$. Put $K_\sigma = \{t^\sigma - t : t \in K\}$. Then consider a *$\sigma$-quadratic form* $q : V \longrightarrow K/K_\sigma$ such that $q(x) = g(x, x) + K_\sigma$, for all $x \in V$, where $g$ is the '$(\sigma, 1)$-linear form' associated to $q$; see Section 2.3 of [19] for the details.

We say that $q$ has *Witt index* $l$, for some $l \in \mathbb{N}$, if $q^{-1}(0)$ contains $l$-dimensional subspaces but no higher dimensional ones. For a non-degenerate $\sigma$-quadratic form $q$ on $K$ with With index 2, we define the following geometry $\Gamma = Q(V, q)$: the points are the 1-spaces in $q^{-1}(0)$, the lines are the 2-spaces in $q^{-1}(0)$ and incidence is symmetrized inclusion. By Corollary 2.3.6 of [19], $\Gamma$ is a generalized quadrangle if and only if $V$ has dimension $\geq 5$ or $\sigma \neq id_{\mathrm{K}}$ (and dim $V \geq 4$). All such quadrangles with $\sigma$ being the $id_{\mathrm{K}}$ are called *orthogonal quadrangles*. We denote them by $Q(l, K)$, for $l := \dim(V) \geq 5$. The remaining ones, where $\sigma \neq id_{\mathrm{K}}$, give rise to *Hermitian quadrangles*, which are constructed over vector spaces of dimension $l \geq 4$; we denote them by $HQ(l, K)$.

Over finite fields, orthogonal quadrangles arise only over a vector space of dimension 5 or 6 (see Section 2.3.12 of [19]), and we denote them by, respectively, $Q(5, q)$ and $Q(6, q)$, for some finite field $\mathbb{F}_q$ with $q$ a prime power. Likewise, over some finite field $\mathbb{F}_q$, there are only two examples (up to duality) of Hermitian quadrangles, and we denote them by $HQ(4, q)$ and $HQ(5, q)$ (see again Section 2.3.12 of [19]).

**Remark 3.3** The orthogonal quadrangle $Q(5, K)$ is sometimes seen, under the Klein correspondence, as a quadrangle isomorphic to the dual (see Definition 2.4) of the symplectic quadrangle $W(K)$, say, over the same field (see Proposition 3.4.13 of [19] for the details). The *symplectic quadrangle* $W(K)$ consists of the totally isotropic 1 and 2-subspaces of the 4-dimensional vector space equipped with a nondegenerate symplectic form ($(x, x) = 0$, for all $x \in V$).

There are also two examples of anti-isomorphisms between orthogonal and Hermitian quadrangles: $HQ(4, L)$ is isomorphic to the dual of the orthogonal quadrangle $Q(6, K)$, for some quadratic Galois extension $L$ of the field $K$ equipped with a non-trivial element $\sigma \in \mathrm{Gal}(L/K)$ (see Proposition 3.4.9 of [19]); $HQ(5, L)$ is dual to the orthogonal quadrangle $Q(8, K)$, where $L$ is a skew field which is a quaternion algebra (see Definition 9.3 of [18]) of dimension 4 over its centre $K$ (see Proposition 3.4.11 of [19]).

**Example 3.4** Split Cayley and twisted triality hexagons ($n = 6$): Let $V$ be an 8-dimensional vector space over a field $K$, and let $\mathcal{Q}_7(K)$ be the nondegenerate *quadric hypersurface* of Witt index 4 living in the associated 7-dimensional projective space $P(V)$ of $V$; see Section 2.4 of [18] for the details. The hexagons we are interested in arise from the quadric $\mathcal{Q}_7(K)$. By the Witt index 4 assumption, $\mathcal{Q}_7(K)$ contains 3-dimensional projective subspaces of $P(V)$.

With regards to the quadric $q$, there exists a certain 'trilinear form' $T : V \times V \times V \longrightarrow K$ (see Section 2.4.6 of [19]) such that, for some fixed $v \in V \backslash \{0\}$, the set of all $w \in V$ for which $T(v, w, x)$ vanishes in $x$ is a projective 3-space of $\mathcal{Q}_7(K)$; moreover, the vanishing of $T(v, w, x)$ provides an incidence structure whose points are such projective 3-spaces, in a way that it also allows us to represent these points as points of $P(V)$. Then this arising point-line incidence structure (where the lines are just the lines of $P(V)$) turns out to be a generalized hexagon; see Theorem 2.4.8 of [19]. There are two kinds of hexagons, and they both depend on a certain automorphism $\sigma$ of $K$ of order 1 or 3. If $\sigma = \mathrm{id}_\mathrm{K}$, we call the associated hexagon a *split Cayley hexagon*, and denote it by $H(K)$, and if $\sigma \neq id_K$, we call it a *twisted triality hexagon*, and denote it by $T(K, K^\sigma)$.

In the finite case, the field automorphism $\sigma$ is determined by the field $\mathbb{F}_{q^3}$, where $q$ is a prime power, and therefore the finite twisted triality hexagon is unique, namely $T(q^3, q)$.

**Example 3.5** Ree-Tits octagons ($n = 8$): These are associated to a certain *metasymplectic space* $M$, which is a building arising from a Dynkin diagram of type $F_4$; in particular (see Theorem 2.5.2 of [19]), associated to $M$ there is a field $K$ of characteristic 2 which admits a Tits endomorphism $\sigma$ (see Definition 5.13). The generalized octagons associated to the pair $(K, \sigma)$ are called *Ree-Tits octagons*, and denoted by $O(K, \sigma)$.

Moufang octagons do arise over finite fields $\mathbb{F}_{2^{2k+1}}$, and in this case the Tits endomorphism is always the automorphism $x \longrightarrow x^{2^k}$ (see Lemma 7.6.1 of [19]). Thus, we denote a finite Ree-Tits octagon by $O(2^{2k+1}, x \longrightarrow x^{2^k})$.

**Definition 3.6** With the notation of Examples 3.1, 3.2, 3.4 and 3.5, we call *good polygons* all the following generalized polygons, assumed to satisfy the Moufang condition: $\mathrm{PG}_2(A)$ over a Desarguesian division ring $A$, $Q(l, K)$ for $l = 5, 6$, $HQ(l, K)$ for $l = 4, 5$, $H(K)$, $T(K, K^\sigma, \sigma)$ and $O(K, \sigma)$, with $K$ a perfect field.

## 4 Definability of the root groups

With the notation of Section 2, let us fix a (Moufang) generalized $n$-polygon $\Gamma$, an ordinary subpolygon $A = (x_0, x_1, ..., x_{2n-1})$ in $\Gamma$, a root $\alpha = (x_0, x_1, ..., x_n) \subseteq A$, and the root group $U_\alpha$ associated to $\alpha$. We now discuss the procedure which allows us to define $U_\alpha$ in the language $L_{\mathrm{inc}}$; this is extracted from [11]. For the model-theoretic concept of definability and, in particular, the proof of Lemma 4.1, as well the notation used in the lemma, consult the beginning of Section 5.

Put $B = B_1(x_{2n-1}, x_0) = B_1(x_0)\backslash\{x_{2n-1}\}$ and $0 = x_1$. Next, choose an element $a \in B_1(x_{2n-1})\backslash\{x_0, x_{2n-2}\}$. For $y \in B$, put $a_y = \mathrm{proj}_{n-1}(a, [x_n, x_0](y))$ and consider the projectivity $\pi_y = [x_0, a_y, x_{2n-2}, a_0, x_0]$ (see Definition 2.5). This projectivity fixes $x_{2n-1}$, induces a permutation on $B$, and maps 0 to $y$; also, $\pi_y$ is parameter definable from the coordinatization (with parameters in $A \cup \{a\}$), since it is a composition of perspectivities. Hence, we have definable maps $\pm : B \times B \longrightarrow B$ by putting $x + y = \pi_y(x)$ and $x -$

$y = \pi_y^{-1}(x)$. The structure $(B,+)$ is a *right* (or *left*) *loop*, i.e., satisfies the following: $(x+y)-y = (x-y)+y = x$, $0+x = x+0 = x-0 = x$.

Let now $g$ be an element of $U_\alpha$, and put $c := g(0)$, say. Then, since $g$ is an automorphism, $g(a_0) = g(a_c)$. By Lemma 1.13 of [11], it follows that $g(x) = [x_0, a_c, x_{2n-2}, a_0, x_0](x) = x+c$ for all $x \in B$ and, similarly, that $g^{-1}(x) = x-c$. In particular, the lemma tells us that given any element $g \in U_\alpha$, the restriction of $g$ to $B$, denoted $g|_B$, is definable with parameters in $A \cup \{a\}$. Also, from Lemmas 1.15 and 1.16 of [11], it follows, respectively, that the action of $U_\alpha$ on $B$ is *semi-regular*, i.e., $1_{U_\alpha}$ is the only element in $U_\alpha$ fixing any element of $B$, and $U_\alpha$ is embedded into $(B,+)$ via the map $g \longrightarrow g^{-1}(0)$.

Assuming now the Moufang condition, $U_\alpha$ acts transitively (and thus regularly, by the above semi-regularity) on the set $B$; therefore, we can definably identify the root group $U_\alpha$ with the additive loop $(B,+)$. In order to obtain a definable action of $U_\alpha$ on $\Gamma$, we need to definably extend the action of $(B,+)$ to the whole of $\Gamma$. This can be done using the coordinatization as in Definition 2.8.

**Lemma 4.1** The action of $U_\alpha$ on $\Gamma$ is parameter definable in $L_{\text{inc}}$.

*Proof*: Let $\Gamma$, $A$, $\alpha$, and so on, as in the above setting. We have shown how to definably identify (using parameters from $A \cup \{a\}$) the root group $U_\alpha$ with the right loop $(B,+)$. Hence, for the assertion, we need to show that the action of every element $g \in U_\alpha$ definably extends to the whole of $\Gamma$.

Let $g$ be any element of $U_\alpha$, and consider $g|_B$. Let $X = B \cup B_1(x_1) \cup B_1(x_2) \cup ... \cup B_1(x_{n-1})$. Then $g|_X$ has a unique extension to an automorphism of $\Gamma$; indeed, the action of $g|_X$ on the ordinary polygon $A$ is uniquely determined, using perspectivities the action on the corresponding hatrack is determined, and hence, by Remark 2.9, the action of $g$ on $\Gamma$ is uniquely determined.

Consider now the language $L = (P, L, I, c_0, c_1, ..., c_n, Q, g|_Q)$, where the $c_i$ are constant symbols, and $Q$ and $g|_Q$ are relation symbols of arity 1 and 2, respectively. Let $T$ be the first-order $L$-theory describing the above structure with the constant symbols $c_i$

interpreted as the elements $x_i$ of $\alpha$, and $Q$ interpreted as $B \cup B_1(x_1) \cup B_1(x_2) \cup ... \cup B_1(x_{n-1})$. Let also $L^+ = L \cup \{g\}$, where $g$ is a binary relation symbol. Let $T^+$ be the $L^+$-theory extending $T$ which asserts that $g$ is the graph of an automorphism of $\Gamma = (P, L, I)$ extending $g|_X$. By the last paragraph, any model of $T$ has a unique extension to a model of $T^+$. Hence, by Beth's Definability Theorem (see, for instance, Proposition 0.1 of [13]), $g$ is uniformly definable in models of $T$. □

## 5 Asymptotic classes of finite structures

We assume that the reader is familiar with the basic notions of model theory; in particular, the concepts of a first order language $L$ and an $L$-structure $M$, that is, a structure interpreting $L$. In general $T$ will denote a complete theory in the language $L$. For a first order $L$-formula $\sigma$, with parameters in $\bar{M}$ (a sufficiently saturated extension of $M$), by the expression $\models \sigma$ we mean that $\sigma$ is true in $\bar{M}$. Given an $L$-structure $M$, we often refer to $\mathrm{Th}(M)$ as the theory of $M$, i.e., the theory consisting of those first order sentences true in $M$. Usually, $A$, $B$, etc., will denote subsets of $M$, and $x, y$, etc., will denote elements of $M$. Unless it is clear from the context, $\bar{x}$ will denote a tuple $(x_1, x_2, ..., x_n) \in M^n$, for some integer $n$. If $\bar{b} = (b_1, b_2, ..., b_n) \in B^n$, we often abuse notation by writing $\bar{b} \subseteq B$.

A set $D \subseteq M^n$ is said to be *definable*, over $B \subseteq M$, if there is some $L$-formula $\phi(\bar{x}, \bar{b})$, with parameters $\bar{b} \subseteq B$, such that $\phi(\bar{x}, \bar{b})$ is satisfied exactly by elements of $D$. Sometimes we will denote it by $D = \phi(M^n, \bar{b})$, for ease. When we define a set over the *empty set*, we talk about a 0-definable set. If $D$ is a finite $B$-definable set $\{a_1, a_2, ..., a_n\}$, the elements $a_i$ are said to be *algebraic* over $B$; in particular, if $A$ is a singleton $\{a\}$, then $a$ is said to be in the *definable closure* of $B$, which is denote by $\mathrm{dcl}(B)$. An *interpretable set* is a set of the form $A/E$, where $A \subseteq M^n$ is a definable set and $E$ a definable $n$-ary equivalence relation on $A$. A partial type over $A$ is a set of formulas with parameters from $A$, which is realized in $\bar{M}$; while for a *complete type over* $A$, denoted by $\mathrm{tp}(\bar{x}/A)$, we mean a partial type which contains either $\sigma$ or $\neg\sigma$ for every first order $L$-formula $\sigma$ over $A$.

An important model-theoretic concept in this paper is that of supersimplicity, or more strictly that of measurability. Supersimple theories represent a subclass of simple theories equipped with a rank on types. A nice account on supersimple theories can be found in [20]. As examples of supersimple structures, we mention pseudofinite fields and also smoothly approximable structures (see [6]).

We now focus on a model-theoretic generalization of results on finite fields in [5], stemming ultimately from Lang-Weil. It was introduced (in dimension 1) in [12], and extended by arbitrary finite dimension by Elwes in [9].

**Definition 5.1** (Definition 2.1 of [9]) Let $L$ be a countable first order language, $N \in \omega$, and $\mathcal{C}$ a class of finite $L$-structures. Then we say that $\mathcal{C}$ is an *N-dimensional asymptotic class* if for every $L$-formula $\phi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there is a finite set of pairs $D \subseteq (\{0, 1, ..., Nn\} \times \mathbb{R}^+) \cup \{(0, 0)\}$ and for each $(d, \mu) \in D$ a collection $\Phi_{(d,\mu)}$ of elements of the form $(M, \bar{a})$, where $M \in \mathcal{C}$ and $\bar{a} \in M^m$, so that $\{\Phi_{(d,\mu)} : (d, \mu) \in D\}$ is a partition of $\{\{M\} \times M^m : M \in \mathcal{C}\}$, and $|\phi(M, \bar{a})| - \mu|M|^{d/N} = o(|M|^{d/N})$, as $|M| \longrightarrow \infty$ and $(M, \bar{a}) \in \Phi_{(d,\mu)}$. Moreover, each $\Phi_{(d,\mu)}$ is required to be definable, that is to say $\{\bar{a} \in M^m : (M, \bar{a}) \in \Phi_{(d,\mu)}\}$ is uniformly 0-definable across $\mathcal{C}$.

**Remark 5.2** $\mathcal{C}$ In order to check that $\mathcal{C}$ is an $N$-dimensional asymptotic class, it suffices to verify the above conditions for all formulas $\phi(\bar{x}, \bar{y})$ where $l(\bar{x}) = 1$; see Lemma 2.2 of [9].

**Definition 5.3** Let $\mathcal{C}$ be a class of finite $L$-structures, and let $N$ be a positive integer. We say that $\mathcal{C}$ is a *weak N-dimensional asymptotic class* if it satisfies the asymptotic behaviour as in Definition 5.1 but without the assumption that $\Phi_{(d,\mu)}$ is definable. Also, we say that $\mathcal{C}$ is a *semiweak N-dimensional asymptotic class* if $\Phi_{(d,\mu)}$ is uniformly definable but not necessarily 0-definable.

As examples of 1-dimensional asymptotic classes we mention: finite fields; for every finite $d \geq 2$, the class of all finite vertex transitive graphs of valency $d$; finite extraspecial groups of exponent a fixed odd prime number $p$; finite cyclic groups. See [12] for the details about these examples. By [6], any smoothly approximable structure is approximated by a sequence of 'envelopes'; a carefully chosen class of finite envelopes forms an $N$-dimensional asymptotic class. As an example, we mention, over a fixed finite field $\mathbb{F}_q$, the class of all finite dimensional vector spaces equipped with a non-degenerate alternating bilinear form.

Elwes proved that there is a strong connection between asymptotic classes of finite structures and the infinite ultraproducts arising from the members of the classes, in the following sense (which is Lemma 4.1 of [12] generalised to $N$-dimensional asymptotic classes; see also Corollary 2.8 of [9]); there are various notions of rank suitable for supersimle theories, and the $S_1$-rank is one of these (see, for instance, Section 5.1 of [20]).

**Proposition 5.4** Let $\mathcal{C}$ be an $N$-dimensional asymptotic class and let $M$ be an infinite ultraproduct of members of $\mathcal{C}$. Then $\mathrm{Th}(M)$ is supersimple and the $S_1$-rank of $M$ is at most $N$.

The proof of Proposition 5.4 shows that ultraproducts of members of asymptotic classes are in fact 'measurable structures' (see Definition 5.1 of [12]), which are supersimple structures of finite rank (with extra conditions). There are corresponding notions of weakly measurability and semiweak measurability, as in Definition 5.3.

We now introduce the concept of bi-interpretation, i.e., the interpretation of a structure into another, and vice versa, which plays an important role in this paper. Bi-interpretation can be formulated as a concept between *classes* of finite, or infinite, structures.

**Definition 5.5** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of structures in first order languages $L_1$ and $L_2$, respectively. We say that $\mathcal{C}_1$ is *uniformly parameter interpretable*, UPI, in $\mathcal{C}_2$ if there exists an injection $i : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ so that for each $M \in \mathcal{C}_1$, the $L_1$-structure $M$ is (parameter) interpreted in $i(M)$, uniformly across $\mathcal{C}_1$, i.e., there exists an $L_2$-formula $\phi(\bar{u}, \bar{z})$ such that for every $M \in \mathcal{C}_1$ there are $r \in \omega$, a definable set $X = \phi(\bar{u}, \bar{a}) \subset i(M)^r$ for some tuple $\bar{a}$ of $i(M)$ of the same length as $\bar{z}$, an $L_2$-definable equivalence relation $E(\bar{u}_1, \bar{u}_2)$ (defined over $\bar{a}$) on $X$ with $l(\bar{u}_1) = l(\bar{u}_2) = l(\bar{u})$, a map $f_{\mathcal{C}_1} : M \longrightarrow X/E$, and $L_2$-definable subsets (defined over $\bar{a}$) of the Cartesian powers of $X/E$ which interpret the constant, relation, and function symbols of $L_1$ in such a way that $f_{\mathcal{C}_1}$ is an $L_1$-isomorphism. We call $M^\star$ the interpretation of $M$ in $i(M)$, and denote by $f : M \longrightarrow M^\star$ the associated $L_1$-isomorphism. If $\bar{a}_z$, say, is the tuple of $i(M)$, or an 'imaginary' tuple of $X/E$, that is used as parameters to interpret $M^\star$, then we call $\bar{a}_z$ the *witness* to the UPI in $\mathcal{C}_2$.

Suppose now that the map $i$ is a bijection, and that $\mathcal{C}_2$ is also UPI in $\mathcal{C}_1$ (i.e., there exists an $L_1$-formula $\psi(\bar{x}, \bar{y})$ such that for every $N \in \mathcal{C}_2$ there are $s$, $Y = \psi(\bar{x}, \bar{a}_y) \subset i^{-1}(N)^s$, $E'$, $g_{\mathcal{C}_2} : N \longrightarrow Y/E'$, etc., as before). Thus, denote by $g : N \longrightarrow N^\star$ the $L_2$-isomorphism associated with the interpretation of $N$ in $i^{-1}(N)$, for every $N \in \mathcal{C}_2$. Then $g$ induces an $L_1$-isomorphism $g^\star : M^\star \longrightarrow M^{\star\star}$, where $M^{\star\star}$ is the interpretation of $M^\star$ in $i^{-1}(M^\star)$; likewise, we have an induced $L_2$-isomorphism $f^\star : N^\star \longrightarrow N^{\star\star}$. With these assumptions, we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are UPI *bi-interpretable* if the isomorphisms $g^\star f$ and $f^\star g$ are defined uniformly in the members of $\mathcal{C}_1$, and in the members of $\mathcal{C}_2$, respectively. When we say that $\bar{a}_y$ and $\bar{a}_z$ are *witnesses* to this UPI bi-interpretation, we mean, in addition to the above, that the isomorphism $g^\star f$ is $\bar{a}_y$-definable, and the isomorphism $f^\star g$ is $\bar{a}_z$-definable.

In [16], Ryten considered a slightly more constrained notion, which he called a uniformly parameter definable (UPD) bi-interpretation. In the UPD case, no quotient is involved: for each $M \in \mathcal{C}$, $M$ is parameter bi-definable with $i(M)$.

**Remark 5.6** When we have a class of finite structures $\mathcal{C}_1$ and a UPI bi-interpretation of the class $\mathcal{C}_1$ with an asymptotic class $\mathcal{C}_2$, then the asymptotic behaviour of $\mathcal{C}_2$ can be 'transferred', to the class $\mathcal{C}_1$; similarly, if we have an infinite structure $M$ parameter bi-interpretable with a measurable structure $N$, then (semiweak) measurability can be 'transferred' from $N$ to $M$. If no parameters are involved in the bi-interpretation, then it preserves the property of being an asymptotic class (or being measurable). These results are due to Elwes and Ryten (see below).

Notice that, given an asymptotic class of finite structures $\mathcal{C}$ in a language $L$, and a sublanguage $L_1 \subset L$, the class of reducts $\{M_{L_1} : M \in \mathcal{C}\}$ may not be an asymptotic class anymore; the problem is with the definability assumption required in the last clause of Definition 5.1. However, the set of reducts is a *weak* asymptotic class. On the other hand, trivially, expanding the language $L$ by constants preserves the property of being an asymptotic class.

In order to show how to 'transfer' the asymptotics of a class to another class of finite structures, we need the following result of Elwes, for which he needed to suitably extend the language by finitely many constants; however (see Definition 5.8 and Proposition 5.10 below), if we introduce a 'strong' condition on the UPI bi-interpretation, then the asymptotics transfer without need of extending the language.

**Proposition 5.7** (Corollary 3.8 of [9]) If $\mathcal{C}_1$ and $\mathcal{C}_2$, in the finite languages $L_1$ and $L_2$ respectively, are UPI bi-interpretable, and $\mathcal{C}_2$ is an asymptotic class, then there is an extension $L_1'$ of $L_1$ by finitely many constants, and for each $M \in C_1$ an expansion $M'$ to $L_1'$ so that $\mathcal{C}_1' := \{M' : M \in C_1\}$ is an asymptotic class.

**Definition 5.8** Let $\mathcal{C}$ and $\mathcal{D}$ be two classes of finite structures, respectively, in the finite languages $L_{\mathcal{C}}$ and $L_{\mathcal{D}}$, and suppose that $\mathcal{C}$ and $\mathcal{D}$ are UPI bi-interpretable. Then they are *strongly* UPI bi-interpretable if additionally there is a 0-definable $L_{\mathcal{C}}$-formula $\gamma(\bar{y}, \bar{t})$, such that if $C \in \mathcal{C}$ and $D = i(C)$ then for any $\bar{a}_y$, $\bar{a}_t \in C$, we have $C \models \gamma(\bar{a}_y, \bar{a}_t)$ if and only if $\bar{a}_y$ and $\bar{a}_z$ are witnesses to the UPI bi-interpretation between $\mathcal{C}$ and $\mathcal{D}$ (as in Definition 5.5) and $g(\bar{a}_z) = \bar{a}_t$.

**Remark 5.9** Notice that the strongness condition is not in general symmetric; however, it is clear from the definition which direction we are taking, since the formula $\gamma(\bar{y},\bar{t})$ is an $L$-formula with $L$ either $L_{\mathcal{C}}$ or $L_{\mathcal{D}}$. Thus, if $\gamma(\bar{y},\bar{t})$ is an $L_{\mathcal{C}}$-formula as in the definition above, then we say that the UPI bi-interpretation is *strong on the $\mathcal{C}$-side*.

**Proposition 5.10** Suppose that $\mathcal{D}$ is an asymptotic class, and $\mathcal{C}$ is strongly UPI bi-interpretable on the $\mathcal{C}$-side with $\mathcal{D}$. Then $\mathcal{C}$ is an asymptotic class.

*Proof*: By Proposition 5.7, $\mathcal{C}$ is a weak asymptotic class. We must show that $\mathcal{C}$ satisfies the definability assumption required in the last clause of Definition 5.1, i.e., we have to show that parameters are not needed to define dimension and measure in $\mathcal{C}$. This is done as in Proposition 4.2.10(1) of [16]. □

In Section 7 we will need the following, which is essentially Lemma 4.2.11 of [16]. The statement is adjusted here to allow UPI rather than UPD bi-interpretation.

**Lemma 5.11** Suppose $\mathcal{C}$ and $\mathcal{D}$ are UPI bi-interpretable classes of finite structures, as above. For each $C \in \mathcal{C}$, let $\bar{a}_y$, $\bar{a}_z$ be witnesses to the bi-interpretation of $C$ and $i(C) = D$. Suppose in addition:

(i) there is an $L_{\mathcal{D}}$-formula $\zeta(\bar{z})$ such that $\zeta(\bar{a}_z)$ holds, and if $\bar{a}'_z \in D$ with $\zeta(\bar{a}'_z)$, then the $L_{\mathcal{C}}$-structure whose interpretation in $L_{\mathcal{D}}$ is witnessed by $\bar{a}'_z$ is isomorphic to $C^\star$;

(ii) there is an $L_{\mathcal{C}}$-formula $\eta(\bar{y})$ such that $\eta(\bar{a}_y)$ holds, and if $\bar{a}'_y \in C$ with $\eta(\bar{a}'_y)$, then the $L_{\mathcal{D}}$-structure whose interpretation in $L_{\mathcal{C}}$ is witnessed by $\bar{a}'_y$ is isomorphic to $D^\star$.

Then $\mathcal{C}$ and $\mathcal{D}$ are strongly UPI bi-interpretable, on the $\mathcal{C}$-side.

*Proof*: This is virtually identical to the proof of Lemma 4.2.11 of [16], except that now the interpretations allow quotients; these are handled by Proposition 5.7. We omit the details. □

We conclude this section with a further example of an asymptotic class which plays an important role in the content of this paper, for the Ree-Tits octagons.

**Definition 5.12** Let $L_{\text{diff}}$ be the language $L_{\text{ring}}$ augmented by a unary function symbol $\sigma$. A *difference field* is a pair $(K, \sigma)$ consisting of a field $K$ and an automorphism $\sigma$ of $K$.

**Definition 5.13** Let $K$ be a field of characteristic $p$, for some prime $p$. A *Frobenius endomorphism* $\sigma$ is the map which sends $x$ to $x^p$, for every $x \in K$, and we denote it by Frob. Also, a *Tits endomorphism* of $K$ is a square root of the Frobenius endomorphism, i.e., the endomorphism $\sigma : K \longrightarrow K$ such that $x^{\sigma^2} = \text{Frob}(x)$ for all $x \in K$.

**Remark 5.14** We refer to [4] for a survey on difference fields. In [16], Ryten developed a particular theory of pseudofinite difference fields, denoted by $\text{PSF}(m, n, p)$ (see Section 3.3.2 of [16] for the axiomatization of $\text{PSF}(m, n, p)$). He shows that the class of finite difference fields $\mathcal{C}_{(m,n,p)} := \{(\mathbb{F}_{p^{nk+m}}, \sigma^k) : k \in \omega\}$, where $m, n \in \mathbb{N}$ with $m \geq 1, n > 1$ and $(m, n) = 1$, and $\sigma$ is the Frobenius automorphism, forms a 1-dimensional asymptotic class; moreover, he shows that every non-principal ultraproduct of members of $\mathcal{C}_{(m,n,p)}$ is a model of $\text{PSF}(m, n, p)$, and, vice versa, every model of $\text{PSF}(m, n, p)$ is elementarily equivalent to a non-principal ultraproduct of members of $\mathcal{C}_{(m,n,p)}$ (see Theorem 3.3.15 of [16]).

# 6 The UPI bi-interpretation

In this section we prove Theorem 6.3, which together with Theorem 7.2 will yield Theorem 1.1. With the notation of Examples 3.1, 3.2, 3.4 and 3.5, and according to the classification of Tits and Weiss (see Section 34 of [18]), the finite Moufang generalized polygons are, up to duality, $\text{PG}_2(q)$, $W(q)$, $HQ(4, q)$, $HQ(5, q)$, $H(q)$, $T(q^3, q)$ and $O(2^{2k+1}, x \mapsto x^{2^k})$.

We give the list (up to duality) of finite Moufang polygons in the following table, where we associate to each polygon $\Gamma$ the corresponding little projective group $\Sigma$; see, for instance, Section 8.3 of [19].

| $\Gamma$ | $\Sigma$ |
|---|---|
| $\mathrm{PG}_2(q)$ | $\mathrm{PSL}_3(q)$ |
| $W(q)$ | $\mathrm{PS}_{p_4}(q)$ |
| $HQ(4,q)$ | $\mathrm{PSU}_4(q)$ |
| $HQ(5,q)$ | $\mathrm{PSU}_5(q)$ |
| $H(q)$ | $G_2(q)$ |
| $T(q^3,q)$ | ${}^3D_4(q)$ |
| $O(2^{2k+1}, x \mapsto x^{2^k})$ | ${}^2F_4(q)$ |

*Table 6.1*

Let $\mathcal{C}$ be one of the above classes of finite Moufang polygons. To prove Theorem 1.1, we must show that $\mathcal{C}$ forms an asymptotic class (see Definition 5.1). By Proposition 5.10 (see also Proposition 5.7), in order to prove that $\mathcal{C}$ is an asymptotic class we firstly need to show that $\mathcal{C}$ is UPI bi-interpretable (see Definition 5.5) with a class $\mathcal{G}$, say, which is already known to be an asymptotic class. The 'natural candidate' in this setting is the class of corresponding finite little projective groups.

By Table 6.1 above, we also know that $\mathcal{G}$ is either a class of finite Chevalley groups or a class of finite twisted groups of fixed Lie type and Lie rank. These classes of finite groups all belong to classes already analyzed by Ryten in [16]; there he showed that $\mathcal{G}$ forms an asymptotic class by proving that it is strongly UPI bi-interpretable with a class of finite (difference) fields. We can summarize the main results from [16] in the following.

**Theorem 6.1** (Ryten, [16]) Let $\mathcal{G}$ be any family of finite simple Lie groups of fixed Lie rank.

(i) $\mathcal{G}$ is strongly UPD bi-interpretable with either the class of finite fields $\mathcal{F}$ or one of the classes $\mathcal{C}_{(m,n,p)}$;

(ii) $\mathcal{G}$ is an asymptotic class.

**Remark 6.2** There are just finitely many exceptions consisting of finite Moufang polygons whose corresponding little projective groups are not simple (see Lemma 5.8.1 of [19]), but we can disregard the cases in our context as Ryten does in his thesis (Remark 5.2.9 of [16]): finitely many exceptional cases can be ruled out in the bi-interpretation, by describing the elementary diagram of the corresponding models.

Therefore, for the remainder of this section we will be exhibiting a UPI bi-interpretation between any class of finite Moufang polygons $\mathcal{C}$ and the corresponding asymptotic class of finite little projective groups $\mathcal{G}$.

**Theorem 6.3** *The classes $\mathcal{C}$ and $\mathcal{G}$ are UPI bi-interpretable.*

The proof of the theorem is given throughout the rest of the section. The map $i : \mathcal{C} \longrightarrow \mathcal{G}$ is defined to take each $\Gamma \in \mathcal{C}$ to its little projective group. Since both are parametrized by a (difference) field, $i$ is injective; in fact, by Remark 6.2 (since $\mathcal{G} \backslash i(\mathcal{C})$ is finite) we can assume, without loss of generality, that $i$ is a bijection. We first show that $\mathcal{C}$ is UPI in $\mathcal{G}$.

**Lemma 6.4** Let $\mathcal{G}$ be a family of finite simple groups of Lie type. Then each conjugacy class of parabolic subgroups is UPI across $\mathcal{G}$.

*Proof*: We first deal with the untwisted case. By Proposition 8.3.1(iii) of [3], for $\Sigma \in \mathcal{G}$ and a parabolic subgroup $P_J$ of $\Sigma$ corresponding to a set $J$ of fundamental roots, we have $P_J = BN_JB$, so as $N_J$ is finite, it suffices to show that Borels $B$ are uniformly definable. We also have $B = UH$; by Lemma 5.2.7 of [16], $H$ is uniformly definable. Also, $U = X_{r_1}X_{r_2}...X_{r_n}$ (see 5.5.3(ii) of [3]), for some integer $n$, where the $X_{r_i}$ are positive root groups, so it suffices to show that the $X_{r_i}$ are uniformly definable. For this see Corollary 5.2.8 of [16].

For the twisted groups, the arguments are essentially the same. We have $B^1 = U^1H^1$ (notation of Chapter 13 of [3]), and the appropriate uniform definability results can be found in Chapter 5 of [16]. □

**Lemma 6.5** There exists a uniform parameter interpretation of the Moufang polygon $\Gamma = \Gamma(\Sigma)$ in $\Sigma$, for $\Sigma$ varying through $\mathcal{G}$.

*Proof*: Let $\Sigma$ be the little projective group associated to a Moufang polygon $\Gamma = (P, L, I) \in \mathcal{C}$, and let $pIl$ be a fixed flag in $\Gamma$. Denote by $\Sigma_p$ and $\Sigma_l$, respectively, the stabilizer of $p$ in $\Sigma$ and the stabilizer of $l$ in $\Sigma$. Then $\Sigma_p$ and $\Sigma_l$ turn out to be parabolic subgroups of the BN-pair associated to $\Sigma$. Since $\Sigma$ is simple, the set of pairs $(\Sigma_p, \Sigma_l)$ such that $pIl$ is a flag, is UPI across $\mathcal{G}$ by Lemma 6.4. Hence, from the definable parabolic subgroups of $\Sigma$ we can interpret the polygon $\Gamma$ as follows: interpret the points of $\Gamma$ as the cosets $\Sigma/\Sigma_p$ and the lines of $\Gamma$ as the cosets $\Sigma/\Sigma_l$; incidence $I$ is interpreted as $g\Sigma_p I h\Sigma_l$ if and only if $g\Sigma_p \cap h\Sigma_l \neq \emptyset$. □

**Proposition 6.6** There exists a uniform parameter interpretation of the little projective group $\Sigma = \Sigma(\Gamma)$ in $\Gamma$, for $\Gamma$ varying through $\mathcal{C}$.

*Proof*: Let $\Gamma \in \mathcal{C}$, and let $\Sigma = \langle U_\alpha : \alpha \text{ is a root } \rangle$ be its little projective group. We aim to interpret $\Sigma$ in a uniform way across $\mathcal{C}$. First, fix an ordinary polygon $A = (x_0, x_1, ..., x_{2n-1})$ in $\Gamma$, a root $\alpha = (x_0, x_1, ..., x_n) \subset A$ and a line pencil $B$ centered at the point $x_0$ of $\alpha$. In Section 4 we showed how to define, with parameters, a right loop on $B$, how to definably identify it with the action of the root group $U_\alpha$ on the set $B$ and, ultimately, how to extend such an action on the whole polygon (see Lemma 4.1); put $X = \{x_0, x_1, ..., x_{2n-1}, a\}$, the set of parameters used to define $U_\alpha$ and its action on $\Gamma$. Since $\Sigma$ is generated by all its root groups, we aim to find a bound $m$ such that for some root groups $U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_m}$ (not necessarily distinct) $\Sigma = U_{\alpha_1} U_{\alpha_2} ... U_{\alpha_m}$. For we follow [3].

By the studies of Tits, we know that $\Sigma$ is either a Chevalley group or a twisted group of fixed Lie type and Lie rank. If $\Sigma$ is a Chevalley group, by the Bruhat decomposition (Corollary 8.4.4 of [3]) we need to find such bounds for $U, V, H$ and $N$, where $U$ is the subgroup of $\Sigma$ generated by the 'positive' root groups $U_1, U_2, ..., U_n$ and $V$ that generated by the 'negative' root groups, $N$ the group associated to the $BN$-pair of $\Sigma$ and $H = N \cap B$. Since the set of positive roots is finite, $U$ is definable by Theorem 5.3.3 of [3]; likewise

$V$. Also, by Chapter 6 of [3] and the assumption of finite Lie rank $r$, say, every element of $H$ is a product of $4r$ root groups; from this and the finiteness of the associated Weyl group $W = N/H$, it follows that $N$ is also generated by a product of boundedly many root groups. For the twisted case the situation is similar as the Bruhat decomposition still holds (see Proposition 13.5.3 of [3]).

Let now $\Sigma$ be any finite little projective group in $\mathcal{G}$. It follows from the above paragraph that, for some integer $m$, we can construct $\Sigma$ as a group with domain $U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_m}/\sim$, where the equivalence relation $\sim$ is defined as follows:

$(g_1, g_2, ..., g_m) \sim (h_1, h_2, ..., h_m)$ if and only if

$g_1 g_2 ... g_m(x) = h_1 h_2 ... h_m(x)$, for $x \in P \cup L$, $g_i, h_i \in U_{\alpha_i}$ and $1 \leq i \leq m$.

Denote by $[(g_1, g_2, ..., g_m)]_\sim$ the equivalence class of $(g_1, g_2, ..., g_m) \in U_{\alpha_1} \times ... \times U_{\alpha_m}$ with respect to $\sim$. Now we define the group multiplication "$\cdot$", say, as follows:

$[(g_1, g_2, ..., g_m)]_\sim \cdot [(h_1, h_2, ..., h_m)]_\sim = [(k_1, k_2, ..., k_m)]_\sim$ if and only if

$g_1 g_2 ... g_m(h_1 h_2 ... h_m(x)) = k_1 k_2 ... k_m(x)$, for all $x \in P \cup L$.

This is clearly well-defined. Without loss of generality, we may assume that $U_{\alpha_1}$ is $U_\alpha$; then, as we defined $U_\alpha$, as well its action on the whole of the associated $\Gamma \in \mathcal{C}$ (over the set of parameters $X$), we can do the same for the remaining root groups $U_{\alpha_i}$ for $i \in 2, 3, ..., m$; namely, by adding new parameters $X_i = \{x_0^{(i)}, x_1^{(i)}, ..., x_{2n-1}^{(i)}, a^{(i)}\}$, say, for some ordinary polygons $A_i = (x_0^{(i)}, x_1^{(i)}, ..., x_{2n-1}^{(i)})$ and some $a \in B_1(x_1^{(i)}) \backslash \{x_0^{(i)}, x_2^{(i)}\}$, the root groups $U_{\alpha_i}$ and their respective actions on the whole of $\Gamma$ are definable over the set of parameters $X_i$ for $i = 2, 3, ..., m$. Hence, it follows that the relation $\sim$ is definable: for any $x \in P \cup L$, $g_1 g_2 ... g_m(x) = h_1 h_2 ... h_m(x)$ if and only if the image of $x$ under the definable action of $g_1 g_2 ... g_m$ is the same under the definable action of $h_1 h_2 ... h_m$. $\square$

**Remark 6.7** This can be used to find the bound $m$ in alternative to the method used in the proof of Proposition 6.6. Consider an infinite ultraproduct $(\Sigma^\star, U_\alpha^\star) = \Pi(\Sigma, U_\alpha)/\mathcal{U}$, for some non-principal ultrafilter $\mathcal{U}$. It follows from [16] that $U_\alpha$ is uniformly definable in $\Sigma$ (in [16] the root groups are denoted by $X_r(K)$; since by Discussion 5.2.1 and, in the twisted cases, 5.3.3 and 5.4.1 of [16], the root groups $X_r(K)$ are UPD in the class $\mathcal{F}$ of the corresponding finite (difference) fields $K$, and since by Theorem 6.1(i) the classes $\mathcal{G}$ and $\mathcal{F}$ are strongly UPD bi-interpretable, it follows that each $X_r(K)$ is also UPD in $\mathcal{G}$). Hence, by Los' theorem on ultraproducts, the root group $U_\alpha^\star$, as its action on the whole of $\Pi\Gamma/\mathcal{U}$, is parameter definable in $\Sigma^\star$. Also, since the cardinality $|U_\alpha|$ grows with $\Sigma$, the root group $U_\alpha^\star$ is infinite.

By Remark 6.2 and by the main result from [14], $\Sigma^\star$ is a simple group, definable in the pseudofinite (difference) field $\Pi\mathbb{F}_q/\mathcal{U}$, where $\mathbb{F}_q$ denotes the underlying field of $\Sigma$ as well the underlying field of $\Gamma$.

Since $\Sigma^\star$ is generated by $\{(U_\alpha^\star)^g : g \in \Sigma^\star,\ \alpha \text{ root }\}$, and this set is $\Sigma^\star$-invariant, by the Zilber Indecomposibility Theorem (ZIT) in its supersimple finite rank version (see Remark 3.5 of [10]) there exists a definable subgroup $H \leq U_{\alpha_1}^\star U_{\alpha_2}^\star ... U_{\alpha_n}^\star$ in $\Sigma^\star$ which is also $\Sigma^\star$-invariant, so normal; moreover, for each $i = 1, 2, ..., n$, by ZIT we also have that $U_{\alpha_i}^\star/H$ is finite, thus $H \neq 1$. Therefore, as $\Sigma^\star$ is simple, $H = \Sigma^\star$. This argument applies to all infinite ultraproducts of the $(\Sigma, U_\alpha)$. Hence, there is a single $m$ such that in all ultraproducts $(\Sigma^\star, U_\alpha^\star)$, we have $\Sigma^\star = U_{\alpha_1} U_{\alpha_2} ... U_{\alpha_m}$. It follows that for all but finitely many finite $(\Sigma, U_\alpha)$ we have $\Sigma = U_{\alpha_1} U_{\alpha_2} ... U_{\alpha_m}$. By increasing $m$ to deal with the remaining finite $(\Sigma, U_\alpha)$, we may suppose that for all $(\Sigma, U_\alpha)$, we have that $\Sigma = U_{\alpha_1} U_{\alpha_2} ... U_{\alpha_m}$ for some $\alpha_1, \alpha_2, ..., \alpha_m$.

**Lemma 6.8** There exists a uniform parameter definable isomorphism between $\Gamma$ and its re-interpretation in itself.

*Proof*: Given a Moufang polygon $\Gamma$, we can re-interpret $\Gamma$ in itself by first interpreting $\Sigma$ in $\Gamma$ as in Proposition 6.6, and then by interpreting a copy of $\Gamma$ from $\Sigma$ as in Lemma 6.5. This is possible because the bi-interpretation comes equipped with isomorphisms from objects to their re-interpretations, on both sides. Namely, $\Gamma$ is uniformly parameter interpreted as the polygon $(\Sigma/\Sigma_p, \Sigma/\Sigma_l, \{(u\Sigma_p, u\Sigma_l) : u \in \Sigma\})$ from the group $U_{\alpha_1} \times ... \times U_{\alpha_m}/\sim$, which is itself uniformly parameter interpreted from $\Gamma$; here the fundamental flag $pIl$ of Lemma 6.5 is the flag $x_0 I x_{2n-1}$ fixed in Proposition 6.6. Call $\Gamma'$ this re-interpretation of $\Gamma$ in itself.

With the notation of Definition 5.5, we have an isomorphism $g^\star f : \Gamma \longrightarrow \Gamma'$. Put $g^\star f = \phi$. Then, by construction of $\sim$, the isomorphism $\phi$ is well-defined; precisely, $\phi$ sends any point $x \in \Gamma$ (or line $l \in \Gamma$) to the unique point $y = u\Sigma_p \in \Gamma'$ (or line $y = u\Sigma_l \in \Gamma'$), with $u$ the unique group element $u = [(u_1, u_2, ..., u_m)]_\sim \in U_{\alpha_1} \times ... \times U_{\alpha_m}/\sim$ such that $x = u(p)$ (or $x = u(l)$). Since by Proposition 6.6 we have a uniform parameter interpretation of the group $U_{\alpha_1} \times ... \times U_{\alpha_m}/\sim$ and its action on the whole of $\Gamma$, we can thus uniformly define (with parameters $Y = X \cup (\cup_{i=2}^r X_i) \cup \{a\}$, see Proof of Proposition 6.6) the isomorphism $\phi$ by specifying the coset $u\Sigma_p$ such that $u$ sends $p$ to $x$.

Hence, it follows that we need the definability of the set $\{(x, u\Sigma_p) : x = u(p)\}$ in $\Gamma$. However, the latter is the following:

$$\{(x, u\Sigma_p) : x = u(p)\}$$
$$= \{(x, (u_1, u_2..., u_m)/\sim \Sigma_p) : x = u_1 u_2 ... u_m(p)\}$$
$$= \{(x, u_1, u_2, ..., u_m, k_1, k_2, ..., k_m) : x = u_1 u_2 ... u_m(p), k_1 k_2 ... k_m(p) = p\}.$$

The latter is then parameter definable in $\Gamma$, using parameters from $Y$. □

**Lemma 6.9** There exists a uniform definable isomorphism between $\Sigma$ and its re-interpretation in itself.

*Proof*: We start from $\Sigma \in \mathcal{G}$ and re-interpret it in itself: we first interpret (see Lemma 6.5) $\Gamma = i^{-1}(\Sigma)$ as the coset geometry $\Gamma' := (\Sigma/\Sigma_p, \Sigma/\Sigma_l, \{(u\Sigma_p, u\Sigma_l) : u \in \Sigma\})$, and then we re-interpret (see Proposition 6.6) $\Sigma$ as $\Sigma' = U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_m}/\sim$, where $pIl$ is the fundamental flag of $\Gamma$ as in Lemma 6.5.

With the notation of Definition 5.5, we have an isomorphism $f^\star g : \Sigma \longrightarrow \Sigma'$. Put $f^\star g = \psi$. Let now $u \in \Sigma$. Then, we define $\psi(u) = u'$, where for each $s\Sigma_p$ of $\Gamma'$, we have $u'(s\Sigma_p) = us\Sigma_p$. Here, $u'$ is an element in $U'_{\alpha_1} \times ... \times U'_{\alpha_m}/\sim$, and the $U'_{\alpha_i}$, for $i = 1, 2, ..., m$, are the root groups of $\Gamma'$. Hence, we can define the set $\{(u, u') : \psi(u) = u'\}$ in $\Sigma$. □

*Proof of Theorem 6.3*: Let $\mathcal{C}$ be any class of finite Moufang polygons, and let $\mathcal{G}$ be its associated class of finite little projective groups. Then, the UPI bi-interpretability between $\mathcal{C}$ and $\mathcal{G}$ follows immediately from Lemma 6.5 and Proposition 6.6, and also Lemmas 6.8 and 6.9. □

# 7 Strongness of the UPI bi-interpretation

At this stage, using Theorems 6.1 and 6.3, we know that each class of finite Moufang polygons is a semiweak asymptotic class (see Definition 5.3); i.e., we know that dimension and measure are definable, but not yet that they are 0-definable. We address this issue in this section.

The next theorem may have independent interest, but it is essentially a small extension of results from [16]. We postpone its proof till after the proof of Theorem 7.2. It will be used to verify condition (ii) of Lemma 5.11, for the UPI bi-interpretation between a class of finite Moufang polygons and the associated class of finite little projective groups. In the following, by $L_{\text{group}}$ we mean the language of the first-order theory of groups, i.e., $L_{\text{group}} = \{\cdot, ^{-1}, c\}$, where $\cdot$, $^{-1}$ and $c$ stand for, respectively, group operation, inverse group operation and group identity symbols.

**Theorem 7.1** For any fixed family $\mathcal{G}$ of finite simple Chevalley groups, or finite twisted groups of fixed Lie type and Lie rank, there exists an $L_{\text{group}}$-sentence $\sigma$ such that for any finite group $G$, we have $G \in \mathcal{G}$ if and only if $G \models \sigma$.

**Theorem 7.2** (i) The UPI bi-interpretation between $\mathcal{C}$ and $\mathcal{G}$ of Theorem 6.3 is strong, on the $\mathcal{C}$-side.
(ii) Each family of finite Moufang polygons forms an asymptotic class.

*Proof*: First, note that (ii) follows from (i). For (i), we need to show conditions (i) and (ii) of Lemma 5.11, with $\mathcal{C}$ being a class of finite Moufang polygons and $\mathcal{D}$ the associated class $\mathcal{G}$ of finite little projective groups, as in Theorem 6.3. To see that Lemma 5.11(i) holds, note that if $\Sigma \in \mathcal{G}$ with $\Sigma = i(\Gamma)$, then in the interpretation of $\Gamma$ in $\Sigma$, the points and lines of $\Gamma$ are interpreted as cosets of certain maximal parabolic subgroups. There are two cases: $\Gamma$ is either a self-dual, i.e., dual of itself (see Definition 2.4), or a non self-dual generalized polygon. Suppose first that the class $\mathcal{C}$ has self-dual members, and let the maximal parabolic subgroups $P_1$ and $P_2$, say, be defined over $\bar{a}_z$, by the formulas $\phi_1(\bar{u}, \bar{a}_z)$ and $\phi_2(\bar{u}, \bar{a}_z)$, respectively. Then it suffices for $\zeta(\bar{a}_z)$ to say that $\phi_1(\bar{x}, \bar{a}_z)$ and $\phi_2(\bar{x}, \bar{a}_z)$ are non-conjugate maximal parabolics, and that the corresponding geometry on the cosets is a generalized polygon. Consider now the non self-dual case. Let $P_i$ and $\phi_i$, for $i = 1, 2$, as before. Then the two conjugacy classes $P_1^\Sigma$ and $P_2^\Sigma$ are definable, and invariant under $\mathrm{Aut}(\Sigma)$ (even for saturated elementary extensions of $\Sigma$); for if there was $g \in \mathrm{Aut}(\Sigma)$ interchanging $P_1^\Sigma$ and $P_2^\Sigma$, this would give an isomorphism from the corresponding polygon to its dual. Thus, e.g. by a compactness argument, $P_1^\Sigma$ and $P_2^\Sigma$ are 0-definable, i.e., there are formulas $\psi_i(\bar{x}, \bar{z})$, for $i = 1, 2$, such that:

$$H \in P_1^\Sigma \Longleftrightarrow H = \psi_1(\Sigma, \bar{b}_1) \text{ for some } \bar{b}_1 \in \Sigma^{l(\bar{z})}$$

$$H \in P_2^\Sigma \Longleftrightarrow H = \psi_2(\Sigma, \bar{b}_2) \text{ for some } \bar{b}_2 \in \Sigma^{l(\bar{z})}.$$

Then $\zeta(\bar{a}_z)$ should express that $\phi_1(\Sigma, \bar{a}_z) = \psi_1(\Sigma, \bar{b}_1)$ for some $\bar{b}_1$, $\phi_2(\Sigma, \bar{a}_z) = \psi_2(\Sigma, \bar{b}_2)$ for some $\bar{b}_2$, and that the coset geometry of $\phi_1(\Sigma, \bar{a}_z)$ and $\phi_2(\Sigma, \bar{a}_z)$ is a generalized polygon.

For Lemma 5.11(ii), let $\sigma$ be the sentence, as in Theorem 7.1, picking out (among finite groups) the members of $\mathcal{G}$; by Remark 6.2, these may be assumed simple. Then, $\eta(\bar{y})$ just says that the little projective group may be interpreted as in Proposition 6.6, and that it is simple and satisfies $\sigma$. □

*Proof of Theorem 7.1*: The proof is based on [16], where it is shown that each family $\mathcal{G}$ of finite simple groups is UPI bi-interpretable (in fact UPD bi-interpretable) with a family of finite (difference) fields $\mathcal{F}$; we already quoted this as Theorem 6.1(i).

Let $G = G(K)$ be a finite group from the class $\mathcal{G}$, where $K$ denotes the underlying finite (difference) field of $G$ (i.e., for $\mathrm{PSL}_n(q)$ it is $\mathbb{F}_q$, for $\mathrm{PSU}_n(q)$ - a subgroup of $\mathrm{PSL}_n(q^2)$ - it is $\mathbb{F}_q$, for ${}^2F_4$ it is $(\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k})$, and so on). We want the sentence $\sigma$ to describe the following:

(a) a uniform definition of a copy $K^\star$ of $K$ with $K^\star \subseteq G$;

(b) $\sigma$ should express that $K^\star \in \mathcal{F}$;

(c) a uniform construction of a copy $G^{\star\star}$ of $G$, living on a power of $G$, whose underlying field is exactly $K^\star$;

(d) a uniform definition of an isomorphism $G \longrightarrow G^{\star\star}$.

Since all the cases above are extensively treated in [16], we do not give any detail. Each part of (a)-(d) is dealt in [16] in two different contexts, namely the untwisted case and the twisted case; also, in the twisted case there are two sub-cases: groups with roots of the same length and groups with roots of different lengths, i.e., Suzuki and Ree groups (see, in particular, Discussion 5.4.1 of [16]). For the Suzuki and Ree groups (i.e., ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$) difference fields, rather than pure fields, are required (i.e., $(\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k})$ and $(\mathbb{F}_{3^{2k+1}}, x \mapsto x^{3^k})$).

We first do the argument excluding the case of Suzuki and Ree groups, since for the other families, all but finitely many finite fields arise. Part (a), for the uniform interpretation of $K \in \mathcal{F}$ in $G$, comes all from sections 5.2.4, 5.3.4 and 5.4 of [16]. Let $\theta(\bar{x}, \bar{y})$ be a formula, and let $\bar{a}_y$ be a tuple of $G$ such that $\theta(\bar{x}, \bar{a}_y)$ interprets $K$, as well its field structure (field addition and multiplication), and denote by $K^\star$ such interpretation of $K$ in $G$. For part (b), Ryten showed that the class $\{K^\star : K \in \mathcal{F}\}$ is cofinite in the class $\mathcal{D}$ of all finite fields. Thus, the formula $\theta(\bar{x}, \bar{y})$ can be augmented to a formula $\theta^\star = \theta^\star(\bar{x}, \bar{y})$ interpreting exactly the members of $\mathcal{F}$, i.e., ruling out members of $\mathcal{D} \backslash \mathcal{F}$ by listing their isomorphic types. We can now collect the following: there exists a formula $\eta(\bar{y})$ such that if $\bar{a}_y \in G$, for some $G \in \mathcal{G}$, then $\eta(\bar{a}_y)$ holds if and only if $\theta^\star(\bar{x}, \bar{a}_y)$ interprets a member of $\mathcal{F}$, with $\eta(\bar{y})$ being as in Lemma 5.11(ii). This gives (a) and (b). Part (c) is given by Lemmas 5.2.5 and 5.3.5, and Corollary 5.4.3(i) of [16]. Finally, for part (d), Lemma 4.3.10 of [16] tells us how to extend the uniform $L_{\text{group}}$-definable, with parameters, isomorphism between $K^\star$ and $K^{\star\star\star}$, i.e., the re-interpretation of $K^\star$ in itself, to the whole of $G$, so that we have a uniformly parameter $L_{\text{group}}$-definable isomorphism between $G$ and $G^{\star\star}$.

Let now $\tau$ be a sentence which axiomatizes the appropriate class $\mathcal{F}$ of finite fields. Also, let $\phi^\star(\bar{u}, \bar{z})$ interpret $G^{\star\star}$ in $K^\star$, as in part (c). Finally, let $\psi(\bar{x}, \bar{u}, \bar{v})$ be a formula defining an isomorphism from $G$ to $G^{\star\star}$, as in part (d). Then, $\sigma$ is a first order sentence expressing:

$$\exists \bar{y} \exists \bar{z} \exists \bar{v} (\theta^\star(G, y) \models \tau \wedge \phi^\star(\bar{u}, \bar{z}) \wedge \psi(\bar{x}, \bar{u}, \bar{v}))$$

This is first order expressible; for example, $\theta^\star \models \tau$ is expressed by relativising the quantifiers in $\tau$ to $\{\bar{x} \in G : \theta^\star(\bar{x}, \bar{y}) \text{ holds }\}$.

A small modification of this argument handles the Suzuki and Ree groups. For example, the class of finite difference fields $(\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k})$ can be characterized among all finite difference fields $(F, \sigma)$, by expressing that $\mathrm{char}(F) = 2$ and $\sigma^2 \circ$ Frob $=$ id. $\Box$

## 8 Supersimple Moufang polygons

In this section, we extend the methods used above to prove Theorem 8.2 (which yields Theorem 1.2). In the following, by $\Gamma(K)$ we mean a good polygon (see Definition 3.6) coordinatized over $K$, in the informal meaning of Remark 2.10; likewise, we denote by $\Sigma(K)$ the little projective group associated to $\Gamma(K)$. Notice that, despite Sections 6 and 7, in Theorem 8.2 below $\Sigma(K)$ is not necessarily assumed to be finite; thus, the group structures associated to good Moufang polygons are not necessarily those listed in Table 6.1. However, $\Sigma(K)$ is, essentially (up to the kernel of the action of $\Sigma(K)$ on $\Gamma(K)$), an extension of the group of $K$-rational points of a simple algebraic group of relative rank 2, a classical group of rank 2, or a group of mixed type; see, for instance, Chapter 41 of [18].

**Corollary 8.1** Let $\mathcal{C}$ be any family of finite Moufang polygons as in Theorem 6.3, and let $\mathcal{F}$ be the corresponding class of finite (difference) fields associated to $\mathcal{C}$. Then, $\mathcal{C}$ is UPI bi-interpretable with $\mathcal{F}$.

*Proof*: This follows directly from Theorems 6.1(i) and 7.2(i). □

**Theorem 8.2** Let $\Gamma(K)$ be a good Moufang generalized $n$-polygon, and let also $\Sigma(K)$ be its associated little projective group. Then:

(i) $\Gamma(K)$ and $\Sigma(K)$ are bi-interpretable (with parameters).

In particular:

(ii) $\Gamma(K)$ is supersimple finite rank if and only if $\Sigma(K)$ is supersimple finite rank.

(iii) If $\Gamma(K)$ is measurable, then $K$ is weakly measurable.

(iv) If $\Sigma(K)$ is measurable, then $\Gamma(K)$ is weakly measurable.

(v) If $\Sigma(K)$ is pseudofinite, then $\Gamma(K)$ is measurable.

**Remark 8.3** 'Weakly measurable' can probably be strengthened in (iii) and (iv) to measurable, using an analogue of Definition 5.8. The work has not been done.

*Proof of Theorem 8.2*: First, notice that (ii) and (iv) follow from (i). For (iii), we can appeal to Lemmas 3.3, 3.4 and 4.9 of [11], where it is shown how to define the field $K$ from a Moufang polygon $\Gamma(K)$, provided that some conditions on the associated little projective group $\Sigma(K)$ are satisfied; since all good Moufang polygons satisfy the assumptions required by these lemmas, part (iii) follows. Moreover, in the particular case of a Ree-Tits octagon $O(K,\sigma)$, in the end of Chapter 3 of [7] is shown that the difference field $(K,\sigma)$ is interpretable in $\Gamma(K)$; hence, in (iii), if $\Gamma(K)$ is a measurable Ree-Tits octagon, then $(K,\sigma)$ is weakly measurable.

For (v), if $\Sigma(K)$ is pseudofinite, then by the main theorem of [21] it is elementarily equivalent to a non-principal ultraproduct of a class $\mathcal{G}$ of either finite Chevalley groups of a fixed type or finite twisted groups of fixed Lie type and Lie rank. Thus, by the Los theorem, the associated good Moufang polygon $\Gamma(K)$ interpreted in $\Sigma(K)$ is also elementarily equivalent to a non-principal ultraproduct of a class $\mathcal{C}$ of finite structures; namely, $\mathcal{C}$ is a class of finite Moufang polygons. Therefore, $\mathcal{C}$ is an asymptotic class and, by Theorem 7.2 and Proposition 5.4, $\Gamma(K)$ is measurable.

To prove (i), let $\Gamma = \Gamma(K)$ be a good Moufang polygon and let $\Sigma = \Sigma(K)$ be its corresponding little projective group. For the interpretation of $\Sigma$ in $\Gamma$, it is done exactly as in the proof of Proposition 6.6, by appealing to results from [3]. To interpret $\Gamma$ in $\Sigma$, we also follow [3]. Here we also have to distinguish between the self-dual and non self-dual cases, but this is addressed exactly as in the proof of Theorem 7.2(i); thus, we omit it and refer back to Theorem 7.2 for the details about the non self-dual case. First, in $\Gamma$, let $A = (x_0, x_1, ..., x_{2n-1})$ be a fixed ordinary polygon, $\alpha = (x_0, x_1, ..., x_n)$ a fixed root in $A$, and $x_0 I x_{2n-1}$ a fixed flag in $\alpha$. Also, let $B$ be the stabilizer (in $\Sigma$) of $x_0 I x_{2n-1}$ and $N$ be the setwise stabilizer (in $\Sigma$) of $A$; then, as in 33.4 of [18], $\Sigma$ has a BN-pair. With the notation of [3], let now $\mathcal{P} := \{P_J = U_J L_J : J \subseteq I\}$ be the set of maximal parabolic subgroups of $\Sigma$ containing $B$. Then, by Section 8.5 of [3], the parabolics $P_J$ are uniformly definable. Hence, since every parabolic subgroup is an intersection of finitely many maximal parabolics, it follows that we can interpret $\Gamma$ from $\mathcal{P}$; see Section 15.5 of [3] (it deals with buildings, but by Proposition 2.2 the Tits rank 2 case gives exactly the construction of generalized polygons). Finally, for the definability of the isomorphisms $g^\star f$ and $f^\star g$ (with the notation of Definition 5.5) we can essentially proceed as done in Lemmas 6.8 and 6.9 for the finite case; we omit the details.
□

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