Relative categoricity in abelian groups II

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30 April 2008

Abstract

Abstract. We consider structures A consisting of an abelian group with a subgroup A^P distinguished by a 1-ary relation symbol P, and complete theories T of such structures. Such a theory T is (κ, λ)-categorical if T has models A of cardinality λ with $|A^P| = \kappa$, and given any two such models A, B with $A^P = B^P$, there is an isomorphism from A to B which is the identity on A^P. We classify all complete theories of such structures A in terms of the cardinal pairs (κ, λ) in which they are categorical. We classify algebraically the A of finite order λ with A^P of order κ which are (κ, λ)-categorical.

AMS Subject Classification: 03C35, 20K01, 20K35.

The paper falls into four parts. Part I introduces the definition of (κ, λ)-categoricity and sets up the needed machinery. Part II shows that (κ, λ)-categorical theories of pairs of abelian groups must satisfy certain conditions which depend on κ and λ. In Part III we show that the conditions derived in Part II are not only necessary but also sufficient for (κ, λ)-categoricity, and we draw some corollaries. The main results here are in section 14, which describes the possible relative categoricity spectra of theories of pairs of abelian groups, and for each spectrum gives a structural description of the abelian group pairs involved. Part IV gives a complete classification of the finite relatively categorical p-group pairs where p is an odd prime, and also when p is 2 under the further assumption that A^P is a characteristic subgroup of A.

The case of infinite groups is due to the first author; he thanks Ian Hodkinson and the second author for helpful discussions in the early stages. His preliminary report [6] gave only partial information about the case where $\kappa < \lambda$. The classification of the finite relatively categorical p-group pairs is due to the second author. Both authors express their thanks to the organisers of the meeting 'Methods of Logic in Mathematics II' at the Euler International Mathematical Institute, St Petersburg, in July 2005, and the editors of the Proceedings; and to the referee for a very helpful report.

Part I
Introductory

1 Relative categoricity

A complete theory is a consistent first-order theory whose models are all elementarily equivalent. We write $A \equiv B$ when the structure A is elementarily equivalent to B, and $A \preccurlyeq B$ when A is an elementary substructure of B. The complete first-order theory of A, Th(A), is the set of all first-order sentences that are true in A.

Except where we say otherwise, T is a complete theory in a countable first-order language $L(P)$, one of whose symbols is a 1-ary relation symbol P; L is the language got by dropping P from $L(P)$; and for every model A of T, the set P_A of elements of A which satisfy the formula $P(x)$ is the domain of a substructure A^P of the reduct $A \restriction L$. We call A^P the P-*part* of A. Since a sentence is true in A^P if and only if its relativisation to P is true in A, the complete theory Th(A^P) is determined by T; we write it as T^P.

We write κ, λ etc. for cardinals. The cardinality of a structure A is $|A|$. By a (κ, λ)-*structure* (or a (κ, λ)-*model* if we are talking about models of T) we mean an $L(P)$-structure A with $|A^P| = \kappa$ and $|A| = \lambda$.

We say that T is (κ, λ)-*categorical* if T has (κ, λ)-models, and whenever A, B are any two such models with $A^P = B^P$, there is an isomorphism from A to B over A^P (i.e. which is the identity on A^P).

We say that T is *relatively categorical* if whenever A, B are any two models of T with $A^P = B^P$, there is an isomorphism from A to B over A^P.

Lemma 1.1 *If T is (κ, λ)-categorical and A is a (κ, λ)-model of T, then every automorphism of A^P extends to an automorphism of A.*

Proof. Let α be an automorphism of A^P. Construct a structure B and an isomorphism $\gamma : A \to B$ so that $A^P = B^P$ and γ extends α^{-1}. By assumption there is an isomorphism $\beta : A \to B$ over A^P. Then $\gamma^{-1}\beta$ is an automorphism of A extending α. □

We will sometimes describe a structure A as having a property when its complete first-order theory Th(A) has the property; for example 'A is relatively categorical' means 'Th(A) is relatively categorical'.

The *relative categoricity spectrum* of a theory T in $L(P)$ is the class of pairs of cardinals (κ, λ) such that T is (κ, λ)-categorical; we write it as RCspec(T). Section 14 will identify the possible relative categoricity spectra of theories of abelian group pairs. The analogous question about ordinary (non-relative) categoricity for countable first-order theories was the content of Łoś's conjecture, which Michael Morley proved in [9]. In Morley's work this question was the key to unlocking the structure of uncountably categorical theories. For us, relative categoricity spectra will perform a similar service in the study of abelian group pairs.

2 Abelian groups

Henceforth L is the first-order language of abelian groups, with function symbols $+$, $-$ and constant symbol 0.

We use standard abelian group notation, as for example in Fuchs [3]. We write 0 for the trivial group; if B and C are subgroups of the abelian group A, we say that B and C are *disjoint* when $B \cap C = 0$. We write $A[n]$ for the subgroup of A consisting of the elements a such that $na = 0$. An element a of A is *m-divisible* if $a = mb$ for some element b of A. If $b_1, \ldots, b_n$ are elements of a group B, then $\langle b_1, \ldots, b_n \rangle$ means the subgroup of B generated by $b_1, \ldots, b_n$; when B and C are subgroups of a group A, we write $B + C$ for $\langle B \cup C \rangle$, the smallest subgroup of A containing B and C. A group B is *bounded* if for some finite n, $nB = 0$; the least such n is the *exponent* of B. If B is a torsion group, then B is the direct sum of its p-components, $B = \bigoplus_{p \text{ prime}} B_p$, and this decomposition is unique. We write $\mathbb{Q}$ for the additive group of rationals, $\mathbb{J}_p$ for the additive group of p-adic integers, $\mathbb{Z}(p^k)$ for the cyclic group of order p^k, $\mathbb{Z}(p^\infty)$ for the Prüfer p-group and $A^{(\mu)}$ for the direct sum of μ copies of the group A.

We use [5] Appendices A and B for facts on the first-order theories of abelian groups. A key result is that every ω_1-saturated abelian group is pure-injective ([5] Theorem 10.7.3).

The following results are now classical. See Macintyre [8] for (a) and (b), while (c) is immediate from the Ryll-Nardzewski theorem.

Theorem 2.1 *Let T be a complete theory of infinite abelian groups.*

(a) T is ω-stable if and only if every model of T is the direct sum of a divisible group and a bounded group.

(b) T is uncountably categorical if and only if one of the following holds:

(i) Every model of T is a direct sum of a finite group and an infinite homocyclic group $\mathbb{Z}(p^k)^{(\mu)}$;

(ii) every model of T is a direct sum of a finite group, a divisible torsion-free group (possibly trivial), and divisible p-groups of finite rank for each prime p.

(c) T is ω-categorical if and only if every model of T is bounded.

In any first-order language, a formula is said to be *positive primitive*, or more briefly *p.p.*, if it has the form $\exists \bar{x} \bigwedge_{i \in I} \phi_i$ where each ϕ_i is atomic. A subgroup B of an abelian group A is *pure* if and only if for every tuple $\bar{b}$ of elements of B and every p.p. formula $\phi(\bar{x})$, $A \models \phi(\bar{b})$ implies $B \models \phi(\bar{b})$ (cf. Hodges [5] p. 56).

Let A be an abelian group and p a prime. If k is a natural number or ∞, we define $p^k A$ by induction on k:

$$pA = \{pa : a \in A\}; \;\; p^0 A = A; \;\; p^{k+1} A = p(p^k A); \;\; p^\infty A = \bigcap_{k<\omega} p^k A.$$

The *p-height* of an element a of A, $\mathrm{ht}^p_A(a)$, is the $k < \omega$ such that $a \in p^k A \setminus p^{k+1} A$, or ∞ if there is no such k. We put $\infty + 1 = \infty$. (This follows Eklof and Fisher [1]. Fuchs [3] continues the definitions of p-heights into the transfinite ordinals.) We write $p^n A[p]$ for $(p^n A)[p]$.

We will say that an abelian group A is *divisible-plus-bounded* if A is the direct sum of a divisible group and a bounded group. Divisible-plus-bounded groups appeared in Theorem 2.1 and they will play a central role in this paper.

The following lemma gives some group-theoretic characterisations of divisible-plus-bounded groups.

Lemma 2.2 *Let A be an abelian group. The following are equivalent:*

(a) A is divisible-plus-bounded.

(b) For some positive integer m, mA is divisible.

(c) There is a positive integer m such that $mA = mnA$ for all positive integers n.

(d) The number of pairs (p, n), such that p is prime and n is a positive integer such that $|p^n A/p^{n+1}A| > 1$, is finite.

(e) There is a finite k such that for each prime p, the p-heights of elements of A are all either ∞ or $\leqslant k$; and for all but finitely many primes p the p-component of A is divisible.

Proof. (a) $\Rightarrow$ (b) is by taking for m the exponent of the bounded part of A. Then (b) $\Rightarrow$ (c) is immediate.

(c) $\Rightarrow$ (d): If (d) fails for one prime p and infinitely many n, then for every positive integer m we have $mA \neq mpA$, since multiplication by a number relatively prime to p makes no difference to p-heights. If (d) fails for infinitely many primes, then for every positive integer m we have $mA \neq qmA$ for some q relatively prime to m, for the same reason.

(e) is a paraphrase of (d).

(e) $\Rightarrow$ (a): If (e) holds with an integer k, then take m divisible by p^{k+1} for the finitely many exceptional primes p. If $a \in mA$ then a has infinite p-height for every prime p, and by Euclid it follows that a is divisible by every positive integer. In particular for every prime p there is b such that $mpb = a$, and so a is pc for an element $c = mb$ of mA. Therefore mA is divisible, proving (b). To derive (a), choose a subgroup B of A which is maximal disjoint from mA. Then $A = mA \oplus B$, and $mB \subseteq mA$ so that $mB = 0$ and B is bounded. □

If an abelian group satisfies (c) or (d) in Lemma 2.2, then clearly so does every group elementarily equivalent to A. So the class of divisible-plus-bounded groups is closed under elementary equivalence. Also Th(A) distinguishes between (a) and (b) in the following corollary.

Corollary 2.3 *Suppose the abelian group A is ω_1-saturated and not divisible-plus-bounded. Then one of the following holds:*

(a) There is a prime p such that $|p^k A/p^{k+1}A| > 1$ for all $k < \omega$; and A has a direct summand of the form $\mathbb{J}_p$.

(b) *For each prime p there is a finite k_p such that $p^{k_p}A = p^{k_p+1}A$; and there are a strictly increasing sequence of primes $(q_i : i < \omega)$ and a sequence $(m_i : i < \omega)$ of positive integers such that A has a direct summand which is the pure-injective hull of the direct sum $\bigoplus_{i<\omega} \mathbb{Z}(q_i^{m_i})$.*

Proof. If there is a prime p such that all $p^k A/p^{k+1}A$ are nontrivial, then ω_1-saturation guarantees the existence of an element $a \in A$ which has infinite order but is not divisible by p. Comparison with the structure theory of pure-injective groups ([3] §40) shows that $\mathbb{J}_p$ must be a direct summand of A. (Since $\mathbb{J}_p \oplus \mathbb{Q}^{(\omega_1)}$ is ω_1-saturated but not divisible-plus-bounded, this case does occur.)

If there is no such p, then for each prime p there is a finite k_p such that $p^{k_p}A = p^{k_p+1}A$. But then by (d) of the lemma, there must be infinitely many primes q such that for some finite m, $q^m A \neq q^{m+1}A = q^{m+2}A$. Again comparison with the structure theory of pure-injective groups completes the case. □

The next lemma lists some properties of the complete theories of divisible-plus-bounded abelian groups.

Lemma 2.4 *Let T be a complete first-order theory of abelian groups. The following are equivalent:*

(a) *Some (or all) models of T are divisible-plus-bounded.*

(b) *T has finite models or is ω-stable.*

(c) *Every model of T is pure-injective.*

(d) *For every model B of T, $\mathrm{Ext}(\mathbb{Q}, B) = 0$.*

Proof. If (a) holds, then we can verify (b). Also both bounded and divisible abelian groups are pure-injective, and a direct sum of two pure-injectives is pure-injective, so that (c) holds too, and hence also (d) by [3] Proposition 54.1.

If (a) fails, then by (c) of the previous lemma there is an infinite increasing sequence $(n_0, n_1, \ldots)$ of positive integers such that for each $i < \omega$, $n_{i+1}A$ is a proper subgroup of n_iA. Thus the cosets of the n_iA $(i \in \omega)$ form an infinite branching tree. There are continuum many branches, so that in a countable model B of T not all the branches are realised by elements. Hence (b) fails, and we infer (a) ⇔ (b).

It remains to complete the cycle (a) ⇒ (c) ⇒ (d) ⇒ (a) by proving (d) ⇒ (a). Suppose (a) fails and B is a countable Szmielew model of T (cf. [5]

p. 663f). Then B has either infinitely many pairwise non-isomorphic cyclic direct summands, or for some prime p a direct summand of the form $\mathbb{Z}_{(p)}$ (the localisation of $\mathbb{Z}$ at p, which is not a pure-injective group). In either case (d) fails (by [3] Corollaries 54.4,5). □

An abelian group B satisfying the condition of (d) of the lemma is said to be *cotorsion* (Fuchs [3] p. 232).

The next lemma describes how divisible-plus-bounded groups behave in short exact sequences.

Lemma 2.5 *Suppose A is an abelian group and B a subgroup of A.*

(a) If A is divisible-plus-bounded then A/B is divisible-plus-bounded.

(b) Suppose A is divisible-plus-bounded. If there is a positive integer k such that for all positive n, every element of B that is kn-divisible in A is n-divisible in B, then B is divisible-plus-bounded. In particular B is divisible-plus-bounded if either it is a pure subgroup of A, or A/B is bounded.

(c) If B and A/B are divisible-plus-bounded then A is divisible-plus-bounded.

Proof. (a) Suppose mA is divisible. Then $m(A/B) = (mA + B)/B \cong mA/(mA \cap B)$, which is divisible since it is a homomorphic image of a divisible group.

(b) Assume there is k as stated, and let m be a positive integer such that mA is divisible. We show that mB is divisible. Let b be in B and n a positive integer. Then $mb \in mA \cap B$. By divisibility there is $a \in A$ such that $mb = knma$. So by assumption there is $c \in B$ such that $mb = nmc$, as required.

If B is pure in A then the assumption holds with $k = 1$. If $k(A/B) = 0$ and $b = kna$ with $b \in B$ and $a \in A$, then $ka \in B$ and $b = n(ka)$, so that again the assumption holds.

(c) Suppose kB and $m(A/B)$ are divisible. We show that kmA is divisible. Taking any $a \in A$, consider a positive integer n. By assumption on A/B there is $c \in A$ such that $ma + B = nmc + B$, hence $ma = nmc + b$ for some $b \in B$. Then $kma = knmc + kb$. By assumption on B there is $d \in B$ such that $kb = nmkd$, and so $kma = nkm(c + d)$ as required. □

We give two easy counterexamples to strengthenings of the previous two lemmas. They motivate several of the constructions in this paper.

Example 2.6 (a) A divisible with subgroup B that is not divisible-plus-bounded. (This is impossible when B is pure in A or A/B is bounded, by Lemma 2.5(b).)

A is $\mathbb{Q}$ and B is $\mathbb{Z}$.

(b) A nonsplit short exact sequence $B \longrightarrow A \longrightarrow \mathbb{Z}(p^\infty)$ with B bounded. (This is impossible with $\mathbb{Q}$ in place of $\mathbb{Z}(p^\infty)$, by Lemma 2.4(d).)

A is $\mathbb{Z}(p^\infty)$ and B is $A[p]$.

3 Group pairs

Definition 3.1 (a) By a *group pair* we mean an $L(P)$-structure A which is an abelian group with A^P a subgroup.

(b) A homomorphism $h : A \to B$ of group pairs is a homomorphism of abelian groups such that $hA^P \subseteq B^P$. (Then isomorphisms, short exact sequences etc. are defined in the obvious way.)

(c) We say that a group pair A is *bounded over* P if the group A/A^P is bounded. (And likewise for other notions where the meaning is clear.)

(d) If A is a group pair and T is Th(A), we write T^P for Th(A^P) and T/T^P for Th(A/A^P). Since A^P and A/A^P are both interpretable in A, they depend only on T.

As noted earlier, we say that a complete theory has a property when all its models have the property. Thus for example every complete theory of a divisible-plus-bounded group is pure-injective. It's clear that if A is a group pair that is bounded over P, then every group pair elementarily equivalent to A is also bounded over P; so in this case we will say that Th(A) is bounded over P, or that T/T^P is bounded (where T is Th(A)).

Also if A is a group pair, we describe A as divisible (torsion, etc.) if its reduct $A \restriction L$ is a divisible (torsion, etc.) abelian group.

A good deal of the model theory of abelian groups carries over immediately to group pairs in the language $L(P)$. For example if $\phi(x_0, \ldots, x_{n-1})$ is a p.p. formula of $L(P)$ with no parameters and A is a group pair, then $\{\bar{a} : A \models \phi(\bar{a})\}$ is a subgroup $\phi(A^n)$ of A^n; subgroups of this form are called *p.p. subgroups* of A. By the *Baur-Monk invariant* $\phi(\bar{x})/\psi(\bar{x})$ of A we mean the cardinality of the quotient group

$$\phi(A^n)/(\phi(A^n) \cap \psi(A^n)),$$

counted as either finite or ∞. Note that by replacing ψ by a p.p. formula logically equivalent to the conjunction $\phi \wedge \psi$, we can assume that every Baur-Monk invariant is the cardinality of a quotient group $\phi(A^n)/\psi(A^n)$ where ϕ and ψ are p.p. formulas such that ψ entails ϕ. An *invariant sentence* is a sentence of $L(P)$ expressing that a certain Baur-Monk invariant has value $\leqslant k$, where k is a positive integer.

The same proof as for modules (e.g. [5] section A1) gives:

Theorem 3.2 *For every formula $\phi(\bar{x})$ of $L(P)$ there is a boolean combination $\phi'(\bar{x})$ of invariant sentences and p.p. formulas of $L(P)$, which is equivalent to $\phi(\bar{x})$ in all group pairs. Every complete theory of group pairs is stable.* □

Corollary 3.3 *Let A be a group pair. If Th(A) has a (κ, λ)-model with $\omega \leq \kappa < \lambda$, then Th$(A)$ has a (κ', λ')-model whenever $\omega \leq \kappa' \leq \lambda'$.*

Proof. This follows from Shelah [11] Conclusion V.6.14(2) (noting the assumption on his p. 223 that T is stable). □

In the next corollary and henceforth, 'pure' in group pairs is defined as after Theorem 2.1, but with group pairs and $L(P)$ instead of abelian groups and their first-order language L.

Corollary 3.4 *Suppose $A \subseteq B \subseteq C$ are group pairs. If $A \equiv C$, A is pure in B and B is pure in C, then the inclusions are elementary embeddings. In particular if $A \equiv C$, $A \subseteq C$ and the inclusion is pure, then $A \preccurlyeq C$.*

Proof. Since $A \equiv C$, A and C have the same Baur-Monk invariants. Suppose $\bar{a}$ is a tuple of elements of A, and $A \models \chi(\bar{a})$. By Theorem 3.2, χ is equivalent in all group pairs to a boolean combination of invariant sentences and p.p. formulas. Since A is pure in B and B is pure in C, $\bar{a}$ satisfies the same p.p. formulas in B and C as it does in A. So $C \models \chi(\bar{a})$, showing that $A \preccurlyeq C$.

To complete the proof it suffices to show that B has the same Baur-Monk invariants as A and C, since then $A \equiv B \equiv C$ by Theorem 3.2 again. Consider the quotient group

$$\phi(B^n)/\psi(B^n)$$

where ϕ, ψ are p.p. formulas such that ψ entails ϕ. Since A is pure in B, $\phi(A^n) = \phi(B^n) \cap A^n$ and $\psi(A^n) = \psi(B^n) \cap A^n$. So by the Second Isomorphism Theorem

$$\begin{array}{rcl} \phi(A^n)/\psi(A^n) & = & (\phi(B^n) \cap A^n)/(\phi(B^n) \cap A^n \cap \psi(B^n)) \\ & \cong & ((\phi(B^n) \cap A^n) + \psi(B^n))/\psi(B^n) \subseteq \phi(B^n)/\psi(B^n). \end{array}$$

It follows that each Baur-Monk invariant of A is $\leqslant$ the corresponding invariant of B. The same argument shows the same for B and C. Since A and C have equal invariants, we can replace the $\leqslant$ by $=$. □

We close this section with four examples of group pairs. The relative categoricity spectra of the complete theories of these group pairs are easy to calculate directly. It will turn out in section 14 below that they illustrate all the possible nonempty relative categoricity spectra for theories of group pairs where the cardinals involved are infinite. (But not all examples are as straightforward as these.)

Example 3.5 A is $\mathbb{Z}_2^{(\omega)} \oplus \mathbb{Q}^{(\omega)}$ and A^P is $\mathbb{Z}_2^{(\omega)}$. The relative categoricity spectrum is the class of all pairs (κ, λ) with $\omega \leqslant \kappa < \lambda$. Note that (ω, ω) is not in the spectrum, because $\mathbb{Q} \equiv \mathbb{Q}^{(\omega)}$.

Example 3.6 A is $\mathbb{Z}_2^{(\omega)} \oplus \mathbb{Z}_3^{(\omega)}$ and A^P is $\mathbb{Z}_2^{(\omega)}$. The relative categoricity spectrum is the class of all pairs (κ, λ) where either $\omega \leqslant \kappa < \lambda$ or $\omega = \kappa = \lambda$.

Example 3.7 A is $\mathbb{Z}_2^{(\omega)} \oplus \mathbb{Z}_3^{(\omega)} \oplus \mathbb{Z}_5^{(\omega)}$ and A^P is $\mathbb{Z}_2^{(\omega)}$. The relative categoricity spectrum consists of the single pair (ω, ω).

Example 3.8 A is $\mathbb{Z}_4^{(\omega)}$ and A^P is $2A$. The relative categoricity spectrum is the class of all pairs of the form (κ, κ) with κ infinite.

Łoś's conjecture partitions the infinite cardinals into two classes, $\{\omega\}$ and the uncountable cardinals; categoricity in one of these classes is independent of categoricity in the other. The four examples above partition the pairs of infinite cardinals into three classes: $\{\omega, \omega\}$, $\{(\kappa, \kappa) : \omega < \kappa\}$ and $\{(\kappa, \lambda) : \omega \leqslant \kappa < \lambda\}$. As the examples illustrate, it will turn out that categoricity in one of these classes is not always independent of categoricity in another. In fact (κ, κ)-categoricity for some uncountable κ implies (ω, ω)-categoricity (cf. Theorem 14.1) but is incompatible with (κ', λ')-categoricity when $\kappa' < \lambda'$ (cf. Theorem 9.3).

4 Direct sums

In the class of group pairs, we can form direct sums $A = \bigoplus_{i \in I}^{P} A_i$. The definition is the same as for abelian groups, except that we also require that for any element $a = \sum_{i \in I} a_i$ with a_i in A_i,

$$a \in A^P \;\Leftrightarrow\; \text{for all } i \in I, a_i \in A_i^P.$$

For example if the A_i are sub-group-pairs of the group pair A, we can ask whether A is the group pair direct sum of the A_i. It suffices to check that A is an abelian group direct sum of its subgroups A_i, and that the equivalence above holds from left to right. (Right to left holds automatically since P picks out a subgroup.) It will be useful to have criteria which guarantee that the equivalence does hold from left to right.

Lemma 4.1 *Let A be an abelian group, and let A_i ($i \in I$) be sub-group-pairs of A such that $A = \bigoplus_{i \in I} A_i$ as abelian groups. Suppose also that there are a subset J of I and an element j_0 of J such that $A^P \subseteq \bigoplus_{i \in J} A_i$, and $A_j \subseteq A^P$ for all $j \in J \setminus \{j_0\}$. Then $A = \bigoplus_{i \in I}^{P} A_i$.*

Proof. Suppose $a = \sum_{i \in I} a_i$, $a \in A^P$. Then $a = (\sum_{j_0 \neq i \in J} a_i) + a_{j_0}$. By assumption each a_i ($j_0 \neq i \in J$) is in A^P. So a_{j_0} is in A^P too. Also when $i \notin J$, $a_i = 0 \in A^P$. □

A direct sum $A \oplus^P B$ is in fact a direct product $A \times B$, so that the Feferman-Vaught theorem applies (e.g. [5] section 9.6). Thus:

Lemma 4.2 *(a) If $B_1 \equiv C_1$ and $B_2 \equiv C_2$ then $B_1 \oplus^P B_2 \equiv C_1 \oplus^P C_2$.*

(b) If $B_1 \preccurlyeq C_1$ and $B_2 \preccurlyeq C_2$ then $B_1 \oplus^P B_2 \preccurlyeq C_1 \oplus^P C_2$.

(c) If X is a set of elements of B and a, b are elements of B, then a, b realise distinct types over X in $B \oplus^P C$ if and only if they realise distinct types over X in B.

(d) Suppose B and C both have the property that over every countable set of parameters there are at most countably many (1-)types realised. Then the same holds for $B \oplus^P C$.

(e) Suppose $\phi(\bar{x})$ is a formula of $L(P)$ and T is a complete theory in $L(P)$. Then there is a formula $\theta(\bar{x})$ such that if A, B are $L(P)$-structures and B is a model of T, then for every $\bar{a}$ in A,

$$A \oplus^P B \models \phi[\bar{a}] \;\Leftrightarrow\; A \models \theta[\bar{a}].$$

Proof. (a), (b) and left to right in (c) are straightforward from the Feferman-Vaught theorem. For right to left in (c), suppose a and b satisfy different types over X in B. By quantifier elimination (Theorem 3.2) there is some p.p. formula $\phi(x)$ which is satisfied in B by a and not by b (say). Since ϕ is existential, a satisfies it also in $B \oplus^P C$. But b doesn't satisfy it in $B \oplus^P C$, since there is a projection from $B \oplus^P C$ to B that fixes B pointwise.

For (d), here is a more direct argument which works in our case. Suppose to the contrary that X is a countable set of elements of $B \oplus^P C$ over which the elements $b_i + c_i$ ($i < \omega_1$) realise distinct types. Without loss we can assume that $X = Y + Z$ where Y, Z are respectively subgroups of B, C. By assumption at most countably many types are realised by the b_i over Y; so we can assume that all the b_i realise the same type over Y. Since $b_0 + c_0$ and $b_1 + c_1$ realise different types over $Y + Z$, quantifier elimination gives us a p.p. formula ψ and elements $\bar{d}, \bar{e}$ of Y, Z respectively, such that

$$B \oplus^P C \models \psi(b_0 + c_0, \bar{d}, \bar{e}) \wedge \neg\psi(b_1 + c_1, \bar{d}, \bar{e})$$

(or vice versa). Since ψ is p.p., we infer

$$B \models \psi(b_0, \bar{d}, \bar{0})$$

and hence

$$B \models \psi(b_1, \bar{d}, \bar{0}).$$

These two conditions hold also with $B \oplus^P C$ in place of B, since ψ is existential. So by subtraction

$$B \oplus^P C \models \psi(c_0, \bar{0}, \bar{e}) \wedge \neg\psi(c_1, \bar{0}, \bar{e}),$$

whence

$$C \models \psi(c_0, \bar{0}, \bar{e}) \wedge \neg\psi(c_1, \bar{0}, \bar{e}).$$

This argument shows that the c_i realise uncountably many different types over Z in C, contradicting the assumption on C.

To prove (e), assume $\phi(\bar{x})$ is given, and use the Feferman-Vaught theorem as at [5] Theorem 9.6.1 to find $\theta_0(\bar{x}), \ldots, \theta_{k-1}(\bar{x})$ in $L(P)$ such that for any A, B and any $\bar{a}$ in A, the truth of $\phi(\bar{a})$ in $A \oplus^P B$ is determined by which of the $\theta_i(\bar{a})$ are true in A and which of the $\theta_i(\bar{0})$ are true in B. If B is a model of T then it is determined which $\theta_i(\bar{0})$ are true in B, and so the truth of $\phi(\bar{a})$ in $A \oplus^P B$ is determined by whether $\theta(\bar{a})$ is true in A, for some boolean combination θ of the θ_i. □

5 Pushouts

Some of our results will need a construction which is one step more complicated than direct sums, namely pushouts or fibred sums.

Let B be a group and let A_i ($i \in I$) be groups which have B as a subgroup. The *pushout* of the A_i over B is a group C together with homomorphisms $\iota_i : A_i \to C$ ($i \in I$) which agree on B, such that:

> If D is any group and $\gamma_i : A_i \to D$ are homomorphisms which agree on B, then there is a unique homomorphism $\alpha : C \to D$ such that $\gamma_i = \alpha.\iota_i$ for each $i \in I$.

By general nonsense the pushout always exists (we will construct it in a moment), and it is unique up to isomorphism over the group B.

Lemma 5.1 *Suppose the groups A_i ($i \in I$) all have B as a subgroup. Then their pushout over B is the group*

$$C = \left(\bigoplus_{i \in I} A_i \right) / E$$

where, if b is an element of B and we write b_i for the copy of b in the i-th direct factor, then E is the group generated by all the elements $b_i - b_j$ as i, j range through I and b ranges through B. The map $\iota_i : A_i \to C$ is the embedding of A_i in the direct sum, followed by the natural map to C; it is an embedding.

Proof. The group C is generated by the images of the maps ι_i, so that uniqueness of α in the definition of pushout is guaranteed. For its existence, if ι'_i is the embedding of A_i in the direct sum, then there is a unique homomorphism α' from the direct sum to D, such that $\gamma_i = \alpha'.\iota_i$ for each i. Since the γ_i agree on B, α' is zero on E, and hence it factors through the natural map as required. To confirm that ι_i is an embedding, it suffices to note that E is disjoint from the factor A_i in the direct sum. □

Since the maps ι_i are embeddings which agree on B, we can identify the A_i with subgroups of C. Hence it makes sense to describe 'internal' pushouts where the ι_i are inclusion maps, just as one has internal direct sums.

Lemma 5.2 *Let A be a group and B a subgroup.*

(a) *The group A is the (internal) pushout of subgroups A_i ($i \in I$) if and only if the A_i generate A, and for any distinct $i_0, \ldots, i_n \in I$, $A_{i_0} \cap (A_{i_1} + \ldots + A_{i_n}) = B$.*

(b) *If A is the pushout over B of its subgroups A_i ($i \in I$), then $A/B = \bigoplus_{i \in I}(A_i/B)$.*

(c) *Conversely if A/B factors as a direct sum $A/B = \bigoplus_{i \in I} C_i$, then if we put*

$$A_i \;=\; \{a \in A : a + B \in C_i\},$$

A is the pushout over B of the A_i.

Proof. (a) is clear from the construction, and then (b) follows immediately. For (c), take an arbitrary element a of A. By the direct sum decomposition

$$a + B = c_1 + \ldots + c_n$$

for some distinct $i_1, \ldots, i_n \in I$ and some $c_i \in C_i$. For each c_i, choose $a_i \in c_i$. Then there is $b \in B$ such that

$$a = a_1 + \ldots + (a_n + b).$$

The i-th term on the right is in A_i; so the A_i generate A. Suppose that this element a is also in A_{i_0} where i_0 is distinct from $i_1, \ldots, i_n$. Then $a + B$ lies in C_{i_0}. But A/B is the direct sum of the C_i, so that $a \in B$ as claimed. □

Let A be a group and B a subgroup of A; we write $t(A/B)$ for the torsion subgroup of A/B. We say that an element a of A is *torsion over B* if $a + B$ is torsion in A/B. The set of all elements of A which are torsion over B is a subgroup $t_B(A)$ of A. Let $t(A/B) = \bigoplus_p C_p$ be the primary decomposition of $t(A/B)$, and for each prime p let $t_B^p(A)$ be the group of elements a of A such that $a + B$ is in C_p. Then $t_B(A)$ is the pushout over B of the groups $t_B^p(A)$. We call $t_B(A)$, $t_B^p(A)$ respectively the *torsion-over-B* and the *p-torsion-over-B components* of A.

The next lemma describes p-heights in pushouts.

Lemma 5.3 *Let A be a group with a subgroup B such that A/B is torsion, so that A as a pushout of its subgroups $A_p = t_B^p(A)$.*

(a) *For each prime p and each element a of A_p, $ht_A^p(a) = ht_{A_p}^p(a)$.*

(b) *Let* $a = a_0 + \ldots a_n$ *be an element of* A *where for each* $i \leqslant n$, $a_i \in A_{p(i)}$ *and the* $p(i)$ *are distinct primes. Put* $p = p(0)$*, and for each* i $(1 \leqslant i \leqslant n)$ *let* $m(i)$ *be the least non-negative integer such that* $p(i)^{m(i)} a_i \in B$*. Suppose* $ht_A(a) = k < \omega$*. Then there are integers* $v_1, \ldots, v_n$ *depending only on* $p, p(1), \ldots, p(n), m(1), \ldots, m(n)$*, such that*

$$ht^p_A(a) = ht^p_{A_p}(a_0 + v_1 p(1)^{m(1)} a_1 + \ldots + v_n p(n)^{m(n)} a_n).$$

Proof. (a) Trivially $\mathrm{ht}^p_A(a) \geqslant \mathrm{ht}^p_{A_p}(a)$. Conversely suppose $p^k b = a$ with $b \in A$. Since $a \in A_p$, we know that $a + B$ is in the p-component of A/B. Then $b + B$ is also in the p-component, so $b \in A_p$.

(b) Given (a), it suffices to find the v_i so that for each i $(1 \leqslant i \leqslant n)$, p^{k+1} divides $(v_i p(i)^{m(i)} - 1)a_i$. Fixing i, put $q = p(i)$ and $m = m(i)$. Then by Euclid there are integers u, v such that $up^{k+1} + vq^m = 1$. Then

$$up^{k+1} a_i + vq^m a_i = a_i.$$

Put $v(i) = v$. □

Part II
Obstructions to categoricity

Following Shelah's recipe [12], we start with the available ways of constructing many models of a theory T over the same P-part B. In Lemma 6.1 and related results henceforth, $\mathbb{Q}$ is made a group pair by putting $\mathbb{Q}^P = 0$.

6 Copies of $\mathbb{Q}$ outside P

Lemma 6.1 *Suppose* A *is a group pair with* A/A^P *unbounded. Then for any cardinal* $\kappa > 0$, $A \preccurlyeq A \oplus^P \mathbb{Q}^{(\kappa)}$.

Proof. For each element a of A, introduce a new constant c_a; for each rational q introduce a new constant c_q. Let T be the following theory:

> The elementary diagram of A (i.e. the set of all first-order sentences true in A using the new constants c_a);
> the diagram of $\mathbb{Q}$ (i.e. the set of all atomic or negated atomic sentences true in $\mathbb{Q}$, written with the new constants c_q);
> the sentence $c_a \neq c_q$ whenever $a \neq 0 \neq q$;
> for all $q \neq 0$, the sentence $\neg P(c_q)$.

Since A/A^P is unbounded, every finite subset of T is satisfiable in A, and so by compactness T has a model B^+. Write B for the reduct of B^+ to the language of A. Then B is up to isomorphism an elementary extension of A, and B contains a copy Q of $\mathbb{Q}$ which is disjoint from B^P. Let C be a subgroup of B which contains B^P and is maximal disjoint from Q. Then $B = C \oplus \mathbb{Q}$ since $\mathbb{Q}$ is divisible (cf. [3] Theorem 21.2). Since $B^P \subseteq C$, the sum $C \oplus \mathbb{Q}$ is a group pair direct sum (by Lemma 4.1)

By the Szmielew invariants ([5] section A2) or more simply the upward Löwenheim-Skolem theorem, $\mathbb{Q} \preccurlyeq \mathbb{Q} \oplus \mathbb{Q}^{(\kappa)}$ as abelian groups, and so by Feferman-Vaught for group pairs (Lemma 4.2),

$$A \preccurlyeq B = C \oplus^P \mathbb{Q} \preccurlyeq C \oplus^P \mathbb{Q} \oplus^P \mathbb{Q}^{(\kappa)} = B \oplus^P \mathbb{Q}^{(\kappa)}.$$

But also, by Feferman-Vaught again,

$$A \subseteq A \oplus^P \mathbb{Q}^{(\kappa)} \preccurlyeq B \oplus^P \mathbb{Q}^{(\kappa)}.$$

So $A \preccurlyeq A \oplus^P \mathbb{Q}^{(\kappa)}$. □

Lemma 6.2 *Suppose A, B are group pairs with $A \subseteq B$ and $A^P = B^P$. Suppose $B/A = F$ for some torsion-free group F. Then $B \equiv A \oplus^P F$ as group pairs.*

Proof. By expressing A, B and F as parts of a single structure, we can form an ω_1-saturated elementary extension

$$A' \longrightarrow B' \longrightarrow F'$$

of the short exact sequence of group pairs

$$A \longrightarrow B \longrightarrow F.$$

Since F is torsion-free, so is F', and hence both sequences are pure exact for abelian groups. The abelian group A' is ω_1-saturated and hence pure-injective, so that the first short exact sequence splits as a sequence of abelian groups. Since $(B')^P \subseteq A'$, we have the group pair direct sum $B' = A' \oplus^P F'$. Then

$$B \equiv B' = A' \oplus^P F' \equiv A \oplus^P F,$$

using Feferman-Vaught (Lemma 4.2(b)). □

The main device of the proof of this lemma, namely putting various groups together in a structure and taking an ω_1-saturated elementary extension, will occur again. We call it *blowing up*.

Theorem 6.3 *Let A be a group pair with A/A^P unbounded. Then:*

(a) Th(A) is not (κ, κ)-categorical for any infinite κ.

(b) Th(A) is not (κ, ω)-categorical for any finite κ.

Proof. (a) We can assume A is a (κ, κ)-structure. Let A' be $A \oplus^P \mathbb{Q}^{(\omega)}$, which is also a (κ, κ)-model of Th(A) by Lemma 6.1. Let D be a subgroup of A' which contains $\mathbb{Q}^{(\omega)}$, is divisible torsion-free and disjoint from $(A')^P$, and is maximal with these properties. Then (using [3] Theorem 21.2 and Lemma 4.1 again) we can write A' as a group pair direct sum $B \oplus^P D$ with $A^P = B^P$. Note that B contains no copy of $\mathbb{Q}$ disjoint from B^P.

Let μ be any cardinal between 1 and κ inclusive, and put $C_\mu = B \oplus^P \mathbb{Q}^{(\mu)}$. Then by Lemma 4.2(a), $C_\mu \equiv A'$, so that C_μ is a (κ, κ) model of Th(A) with P-part A^P. We will show that if $1 \leqslant \mu < \nu \leqslant \kappa$ then C_μ is not isomorphic to C_ν.

For contradiction suppose there is an isomorphism $i : C_\nu \to C_\mu$. Let j be the restriction of i to $\mathbb{Q}^{(\nu)}$; we can write $j = j_1 + j_2$ where j_1 and j_2 are j followed by the projections to B and to $\mathbb{Q}^{(\mu)}$. We claim that j_2 is an embedding. For suppose not; consider some nonzero a in the kernel of j_2. The group $\mathbb{Q}j_1a$ is a subgroup of B, while B contains no copy of $\mathbb{Q}$ disjoint from B^P. Hence for some positive integer m, mj_1a lies in B^P. But then so does $mja = mj_1a + mj_2a$, contradicting the assumption that i was an isomorphism. So the claim holds and j_2 embeds $\mathbb{Q}^{(\nu)}$ into $\mathbb{Q}^{(\mu)}$. But this is impossible since $\mu < \nu$.

(b) Essentially the same argument works, giving nonisomorphic models C_μ with $1 \leqslant \mu \leqslant \omega$. □

Theorem 6.4 *Let A be a group pair and p a prime such that $\mathbb{Z}(p^\infty)^{(\omega)}$ is a subgroup of A disjoint from A^P. Then Th(A) is not (κ, λ)-categorical for any $\lambda > \kappa + \omega$.*

Proof. Let B be a (κ, λ) group pair elementarily equivalent to A. Then $B \equiv B \oplus^P \mathbb{Z}(p^\infty)^{(\mu)}$ for any infinite μ (for example using [5] Corollary 9.6.7 and Lemma A.1.6). Since A/A^P is unbounded, $B \equiv B \oplus^P \mathbb{Z}(p^\infty)^{(\mu)} \oplus^P \mathbb{Q}^{(\nu)}$ for any infinite ν. By Löwenheim-Skolem let C be an elementary substructure of B which contains B^P and has cardinality κ. By Lemma 4.2(a), $D_{\mu,\nu} = C \oplus^P \mathbb{Z}(p^\infty)^{(\mu)} \oplus^P \mathbb{Q}^{(\nu)}$ is also elementarily equivalent to B.

Then both $D_{\kappa,\lambda}$ and $D_{\lambda,\lambda}$ are (κ, λ) models of Th(A) with the same P-part. But the dimensions of the $\mathbb{F}_p$-vector spaces $D_{\kappa,\lambda}[p]$ and $D_{\lambda,\lambda}[p]$ are κ and λ respectively, so these models are not isomorphic. □

7 When groups are not divisible-plus-bounded

In this section we assume that λ is uncountable, and we violate (κ, λ)-categoricity under the assumption that A/A^P is unbounded and at least one of A and A^P is not divisible-plus-bounded. By Lemma 2.4, if a group A is not divisible-plus-bounded then $\mathrm{Th}(A)$ is not ω-stable, so we can find nonisomorphic models by manipulating the number of types realised over a countable set of elements. We show first that there are models where this number is ω throughout.

Lemma 7.1 *Let A be a group pair with A^P infinite. Then for every infinite κ there is a group pair B which is a (κ, κ) model of Th(A), and is such that the statement "Over any countable set of elements only countably many types are realised." is true for B, B^P and B/B^P.*

Proof. Let B be an Ehrenfeucht-Mostowski model of $\mathrm{Th}(A)$ with spine taken inside the P-part; let the spine be well-ordered of cardinality κ. Then in B at most countably many types are realised over any countable set X of elements ([5] Theorem 11.2.9(b)). If X is in B^P and a, b are elements of B^P realising distinct types over X in B^P, then by relativising formulas, a and b realise distinct types over X in B too; so the statement holds for B^P. Finally if X is a set of elements of B/B^P, choose representatives c_x in B so that $x = c_x + B^P$ for each $x \in X$. If $a + B^P$ and $b + B^P$ realise distinct types over X in B/B^P, then this fact can be expressed in the types of a and b over the c_x. So the statement holds for B/B^P too. □

In what follows we refer to the models constructed in Lemma 7.1 simply as *Ehrenfeucht-Mostowski models*.

Lemma 7.2 *Let J be a reduced group elementarily equivalent to $\mathbb{J}_p$ for some prime p, and suppose that in J only countably many types are realised over any countable set of elements of J. Then J is countable, and there is a pure extension J' of J of cardinality ω_1 such that J'/J is torsion-free divisible, and there are a set X of uncountably many elements of J' and an element c of J such that no two elements of X satisfy the same p.p. formulas with parameters from $\langle c \rangle$.*

Proof. Let A be an ω_1-saturated elementary extension of J. Then A is pure-injective and torsion-free, $|A/pA| = p$ and $qA = A$ for all primes $q \neq p$. By the structure theory of pure-injective groups, $A = \mathbb{J}_p \oplus \mathbb{Q}^{(\lambda)}$ for some λ. But J is reduced, so that $J \preccurlyeq \mathbb{J}_p$ and $\mathbb{J}_p$ is the pure-injective hull of J. Since

$pJ \neq J$, J contains an element c not divisible by p. If a is any element of $\mathbb{J}_p$, then a is determined by the unique sequence

$$(a_0, a_1, a_2, \ldots)$$

of integers in the interval $[0, p-1]$ such that for each integer $n \geqslant 0$, a satisfies in $\mathbb{J}_p$ the p.p. formula

$$p^{n+1} | (x - (a_0 + pa_1 + \ldots + p^n a_n)c.$$

We write this formula as $\phi_{a,n}(x)$. Then if a and b are distinct elements of J', there is a least $n \geqslant 0$ such that a satisfies $\phi_{a,n}(x)$ in $\mathbb{J}_p$ but b does not. Since at most countably many types over c are realised in J, J is countable. Also $\mathbb{J}_p/J$ is divisible (by [3] Lemma 41.8(ii)) and torsion-free (as the quotient of a torsion-free group by a pure subgroup). Since $\mathbb{J}_p$ has the cardinality of the continuum, we can choose J' to be a pure subgroup of $\mathbb{J}_p$ containing J and of cardinality ω_1. $\square$

The next lemma must surely be well known, but we don't know a reference for it.

Lemma 7.3 *Let T be a torsion group such that for infinitely many primes p the p-component T_p of T is not empty, but every T_p is bounded. Let $\hat{T}$ be the pure-injective hull of T and A a pure subgroup of $\hat{T}$ containing T. Suppose that in A only countably many types are realised over any countable set of elements. Then there is a pure extension C of A in which uncountably many types over T are realised, and C/A is torsion-free divisible.*

Proof. It suffices to show that we can find A' realising at least one more quantifier-free type over T than is realised in A. For then we can iterate to form $A'' = A^{(2)}, A^{(3)} \ldots$, taking unions at limit ordinals; let C be $A^{(\omega_1)}$. The quotients $A^{(i)}/A$ form an increasing pure chain of divisible torsion-free groups; the quotient C/A is the union of the chain, so that it is divisible torsion-free. The quantifier-free type of an element of $A^{(i)}$ over T is the same in C as it is in $A^{(i)}$.

For each nonzero T_p, choose a nonzero cyclic direct summand C_p. List these cyclic direct summands as $(C_{p_n} : n < \omega)$, and for each n choose a generator c_n of C_{p_n}. For each n choose C'_{p_n} so that $T_{p_n} = C_{p_n} \oplus C'_{p_n}$. Then the pure-injective hull $\hat{T}$ of T is $\prod_{n<\omega}(C_{p_n} \oplus C'_{p_n}) = \prod_{n<\omega} C_{p_n} \oplus \prod_{n<\omega} C'_{p_n}$. Write $B = \bigoplus_{n<\omega} C_{p_n}$ and $D = \bigoplus_{n<\omega} C'_{p_n}$, so that $\hat{T} = \hat{B} \oplus \hat{D}$. If $b \in \hat{B}$, we write $b = (b(n) : n < \omega)$ with each $b(n)$ in C_{p_n}.

If the order of c_n is o_n, then for each k ($0 \leqslant k < o_n$) we write $\theta_{k,n}(x)$ for the p.p. formula "$o_n|(x - kc_n)$". Then for any element b of $\prod_{n<\omega} C_{p_n}$, the formulas $\theta_{k,n}$ determine the $\prod_{m\neq n} C_{p_m}$-coset of b.

We have $B \oplus D \subseteq A \subseteq \hat{B} \oplus \hat{D}$; so every element of A has the form $b + d$ with $b \in \hat{B}$ and $d \in \hat{D}$. By assumption only countably many types over $\{c_n : n < \omega\}$ are realised in A. Choose a countable set X of elements of A representing each of these types; list the elements of X as $x_i + d_i$ ($i < \omega$) with $x_i \in \hat{B}$ and $d_i \in \hat{D}$.

We will construct a matrix $(b_{mn} : m, n < \omega, m \geqslant 1)$ so that the following conditions are met:

(a) For each $m, n < \omega$ with $m \geqslant 1$, b_{mn} is a nonzero element of C_{p_n}.

(b) For each $i, j < \omega$ with $j \geqslant 1$ there is $n < \omega$ such that $jb_{1n} \neq x_i(n)$.

(c) For each $m \geqslant 2$ there is $N_m < \omega$ such that for all $n \geqslant N_m$, $b_{1n} = mb_{mn}$.

For each $m \geqslant 1$ we write $b^\star_m$ for the element of $\hat{T}$ defined by

$$b^\star_m(n) = b_{mn}, \quad {b^\star_m}'(n) = 0 \quad \text{for all } n < \omega.$$

The group A' will be the subgroup of $\hat{T}$ generated by A and all the elements $b^\star_m$ with $m \geqslant 1$. The conditions (c) ensure that for every $m \geqslant 2$, $mb^\star_m - b^\star_1 \in T \subseteq A$, so that $b^\star_1 + A$ is a divisible element of A'/A. The conditions (b) ensure that each element $jb^\star_1 + a$ with $j \geqslant 1$ and $a \in A$ realises (in $\prod_{n<\omega} C_{p_n}$) a type over $\{c_n : n < \omega\}$ that is not already realised in A, using the formulas $\theta_{k,n}$. It follows in particular that $jb^\star_1 \notin A$, and hence $b^\star_1 + A$ is a torsion-free element of A'/A. Thus $A'/A \cong \mathbb{Q}$, and since $\mathbb{Q}$ is torsion-free, this implies that A' is a pure extension of A.

It remains to find the elements $b_{m,n}$. We proceed in stages σ_k ($k < \omega$). At stage σ_k we choose a prime number $p_{f(k)}$ so that f is strictly increasing; then we choose the elements $b_{1,n}$ with $f(k-1) < n \leqslant f(k)$ so as to deal with (b) for the cases $i, j \geqslant k$ (so far as they haven't already been dealt with) by our choice of $b_{1,f(k)}$, and for (c) we find N_k and we choose b_{mn} when $m < k$ and $N_m \leqslant n \leqslant f(k)$ (where not already chosen). Thus at stage σ_k we have finitely many tasks to perform.

We begin stage σ_k by examining k (when $k > 1$) and choosing $N_k > f(k-1)$ so that no prime dividing k is among the p_n with $n \geqslant N_k$. Then we turn to (b) and assemble the pairs i, j for which (b) is not already ensured. This is a finite set of pairs. We solve the following problem at the first $n \geqslant N_k$ where it is solvable:

$b_{1,n}$ is chosen to be a nonzero element of C_{p_n} so that for each of the relevant pairs i, j, $jb_{1n} \neq x_i(n)$.

If n is chosen so that p_n is greater than any prime factor of any of the relevant j, then for any choice of nonzero $b_{1,n}$ the elements $jb_{1,n}$ will also be nonzero elements of C_{p_n}, so it's clear that by taking n large enough we can solve the problem. Having found this n, we put $f(k) = n$. It remains to deal with the requirements at (c). These are met whenever $N_m \leqslant n \leqslant f(k)$ by choosing m' such that $m'n \equiv 1$ modulo the order of C_{p_n} (which is possible by the choice of N_m) and putting $b_{mn} = m'b_{1n}$.

Finally we need to check that the types of the added elements $b_1^\star$ over $\{c_n : n < \omega\}$ remain distinct in the final structure C. Since the formulas $\theta_{k,n}$ are p.p., it suffices to check that the type of $b_1^\star$ is new in A'. If $b_{1,n} = kc_n$ then $o_n|(b_{1,n} - kc_n)$; it remains to show that $o_n|(b_1^\star - (b_{1,n} - kc_n))$. But this follows from (c) and the fact that the groups C_i ($i < N_m$), apart from C_n itself, are all of orders prime to p_n. □

Theorem 7.4 *Suppose A is a group pair, A/A^P is unbounded and A is not divisible-plus-bounded. Then $Th(A)$ is not (κ, λ)-categorical when $\omega \leqslant \kappa$ and $\omega_1 \leqslant \lambda$.*

Proof. Consider an ω_1-saturated elementary extension A' of A. By Corollary 2.3, either (a) A' has a direct summand of the form $\mathbb{J}_p$ for some prime p, or (b) A' has no such direct summand, and in this case its reduced torsion part contains nonzero p-components for infinitely many primes p, though each reduced p-component is bounded.

In case (a), choose a direct summand of the form $\mathbb{J}_p$, and add relation symbols to A' so as to express that this is a direct summand. Then take a (κ, κ) Ehrenfeucht-Mostowski model of the resulting theory. We can assume without loss that the original group pair A is the reduct of this Ehrenfeucht-Mostowski model to the language of group pairs. Now A has a direct summand which is elementarily equivalent to $\mathbb{J}_p$. Separating the divisible and reduced parts of this summand, we reach a direct summand J of A which is reduced and elementarily equivalent to $\mathbb{J}_p$, and hence is embeddable as a pure (in fact elementary) subgroup in $\mathbb{J}_p$. By the Ehrenfeucht-Mostowski construction and Lemma 4.2(c), in J there are only countably many types realised over any countable set; from the construction of $\mathbb{J}_p$ it follows that J is countable. By Lemma 7.2 there is an extension J' of J of cardinality ω_1 such that J'/J is divisible torsion-free, and J' contains an uncountable subset of elements, no two of which satisfy in J' the same p.p. formulas over J. Form the abelian group B by replacing the direct summand J by J'. Then $A \subseteq B$ as abelian groups, and $B/A \cong J'/J$. By Lemma

4.2(c) for abelian groups, B realises uncountably many types over some countable set.

Make B into a group pair by putting $B^P = A^P$. Then we are in the situation of Lemma 6.2, so $B \equiv A \oplus^P \mathbb{Q}^{(\mu)}$ for some cardinal $\mu > 0$. Now consider the two group pairs $A_1 = A \oplus^P \mathbb{Q}^{(\lambda)}$ and $B_1 = B \oplus^P \mathbb{Q}^{(\lambda)}$. These are both (κ, λ) models (bearing in mind the assumption that $\lambda \geqslant \omega_1$). Since $B \equiv A \oplus^P \mathbb{Q}^{(\mu)}$, we have $A \equiv A_1 \equiv B_1$. Also $A_1^P = A^P = B^P = B_1^P$. But in A_1 at most countably many types are realised over any countable set of elements (using Lemma 4.2(d) and the fact that this holds for both A and the ω-stable group $\mathbb{Q}^{(\lambda)}$). Since we constructed B_1 to realise uncountably many types over some countable set, $A_1 \ncong B_1$ and (κ, λ)-categoricity fails.

In case (b) we proceed similarly but using Lemma 7.3 in place of Lemma 7.2. □

Theorem 7.5 *Suppose A is a group pair and Th(A) is (κ, λ)-categorical for some κ and λ. Then:*

(a) A/A^P is divisible-plus-bounded.

(b) Either A/A^P is bounded, or A is divisible-plus-bounded.

Proof. (a) follows from (b) by Lemma 2.5(a). To prove (b) we take cases.

- When $\kappa \leqslant \lambda < \omega$, all of A, A^P and A/A^P are finite and hence bounded.
- When $\kappa < \omega = \lambda$, A/A^P is bounded by Theorem 6.3(b).
- When $\kappa < \omega < \lambda$, Th$(A)$ is uncountably categorical; hence A is ω-stable, and so divisible-plus-bounded by Theorem 2.1(a). Also in this case A^P is finite.
- When $\omega \leqslant \kappa = \lambda$, A/A^P is bounded by Theorem 6.3(a).
- When $\omega \leqslant \kappa < \lambda$, if A/A^P is not bounded then A is divisible-plus-bounded by Theorem 7.4.

□

Example 7.6 In the context of this theorem, if A/A^P is bounded then either both of A, A^P are divisible-plus-bounded or neither are, by Lemma 2.5(b,c). Clearly both can be, for example when A is finite (as in Part IV of this paper). For a not-quite-trivial example where A/A^P is bounded but neither A nor A^P is divisible-plus-bounded, take $A = \mathbb{Z}$ with $A^P = 2\mathbb{Z}$. One can show that for this example, Th(A) is (κ, λ)-categorical precisely when $\omega \leqslant \kappa = \lambda$).

Example 7.7 Let p be prime, and consider the group $A = \mathbb{Q}$ where A^P is the set of rational numbers of the form m/n with n nonzero and not divisible by p. Then A/A^P is unbounded and A is divisible. But A^P is not divisible-plus-bounded, since it is an elementary substructure of $\mathbb{J}_p$. Nevertheless Th(A) is (κ, λ)-categorical whenever $\omega \leqslant \kappa < \lambda$. One can read this off from the fact that all models of Th(A) take the form $B = C \oplus \mathbb{Q}^{(\mu)} \oplus \mathbb{Q}^{(\nu)}$ where μ, ν are any cardinals, C is the injective hull of an elementary substructure J of $\mathbb{J}_p$ and $B^P = J \oplus \mathbb{Q}^{(\mu)}$.

8 The part outside P; tight extensions

Theorem 7.5 allows us to draw out some information on the part of A that lies outside A^P.

Definition 8.1 (a) Let A be an abelian group and B a subgroup. We say that A is a *tight* extension of B if there is no nontrivial subgroup D of A disjoint from B such that $(D + B)/B$ is pure in A/B.

(b) Let A be a group pair. We say that A is *tight over* P if A as abelian group is a tight extension of A^P.

Theorem 8.2 *Let A be a group pair such that A/A^P is divisible-plus-bounded. Then:*

(a) A is a direct sum $A = C \oplus^P D$ where C is a tight extension of A^P.

(b) If moreover either A is divisible-plus-bounded or A/A^P is bounded, then in every such decomposition, $|C| \leqslant |A^P| + \omega$.

Proof. (a) By Zorn's Lemma there is a subgroup D of A which is maximal with the properties (1) D is disjoint from A^P, (2) $(D + A^P)/A^P$ is a pure subgroup of A/A^P. As a pure subgroup of a divisible-plus-bounded group, $(D + A^P)/A^P$ is a direct summand of A/A^P. Write $A/A^P = C' \oplus D$ where C' is a subgroup of A/A^P. Let C be the pre-image of C' in A. Then $A = C \oplus D$ with $A^P \subseteq C$, so that $A = C \oplus^P D$ by Lemma 4.1. If C is not a tight extension of A^P, then C' contains a direct summand disjoint from A^P, contradicting the maximality of D.

(b) Take first the case where A is divisible-plus-bounded. Write C (which is divisible-plus-bounded by Lemma 2.5(b)) as a direct sum of finite cyclic groups and groups of the forms $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$. Since C is a tight extension of A^P, each of these direct summands contains a nonzero element of A^P.

Next take the case where A/A^P is bounded. Here C/A^P is also bounded, and hence is a direct sum $\bigoplus_{i\in I} C_i$ of finite cyclic groups. For each $i \in I$ choose $c_i \in C$ so that $c_i + A^P$ generates C_i. Write r_i for the order of C_i. Then for each i, $r_i c_i \in A^P \setminus \{0\}$; for otherwise $r_i c_i = 0$, and then c_i generates a subgroup of C that contradicts tightness. We claim that if i, j are distinct members of I then the pairs $\langle r_i, r_i c_i\rangle, \langle r_j, r_j c_j\rangle$ are distinct. For otherwise we can replace C_j in the direct sum decomposition by the cyclic group of order r_i generated by $c_j - c_i + A^P$, and then we get

$$r_i(c_j - c_i) = 0$$

contradicting tightness. This proves the claim. It follows that $|I| \leqslant \omega \times |A^P|$, so that

$$|C| \leqslant |A^P| \times |C/A^P| \leqslant |A^P| \times (\omega \times |A^P|) = |A^P| + \omega$$

as required. □

Example 8.3 In general neither of the direct summands C and D is unique, even given the other. For example let A be $\mathbb{Z}(p^2) \oplus \mathbb{Z}(p)$, where the two cyclic summands are generated by c, d respectively, and let A^P be the subgroup generated by pc. Let C be the subgroup of A generated by $c + d$, with $C^P = A^P$. Let D be the subgroup of A generated by $pc + d$. Then A can also be written as a group pair direct sum with C in place of $\mathbb{Z}(p^2)$, or with D in place of $\mathbb{Z}(p)$, or both. Here Th(A) is clearly relatively categorical.

The decomposition in Theorem 8.2(a) is fundamental for the rest of this paper. The following assumption $(\star)$ holds for the rest of this section:

$(\star)$ A is a (κ, λ) group pair, $A = C \oplus^P D$, $A^P \subseteq C$ and C is tight over A.

We show that various relative categoricity conditions on A imply properties of D.

Theorem 8.4 *Assume $(\star)$. Suppose Th(A) is (κ, λ)-categorical and A/A^P is unbounded. Then D is unbounded.*

Proof. By Theorem 6.3 we know that we must have $\kappa + \omega < \lambda$. For contradiction assume that D is bounded. It follows that C/A^P is unbounded, so that by Lemma 6.1 and Lemma 4.2(a),

$$A \equiv (C \oplus^P \mathbb{Q}^{(\lambda)}) \oplus^P D.$$

Since $C \oplus^P \mathbb{Q}^{(\lambda)} \oplus^P D$ is also a (κ, λ)-model of $\mathrm{Th}(A)$, any automorphism of A^P extends to an isomorphism $i : (C \oplus^P \mathbb{Q}^{(\lambda)}) \oplus^P D \to A$. Let π_C, π_D be the projections from A to C, D respectively. Then $\pi_D i(\mathbb{Q}^{(\lambda)}) = 0$ since D is bounded, and hence $\pi_C i$ embeds $\mathbb{Q}^{(\lambda)}$ into C. But then $|C| = \lambda$, contradicting Theorems 7.5(b) and 8.2(b). $\square$

Theorem 8.5 *Assume* $(\star)$. *Suppose Th*(A) *is* (κ, λ)*-categorical and* $\kappa + \omega < \lambda$. *Then Th*$(D)$ *is* ω_1*-categorical.*

Proof. By Theorems 7.5(b) and 8.2(b), $|C| \leqslant \kappa + \omega < \lambda$ so that $|D| = \lambda$. If $D' \equiv D$ with $|D'| = \lambda$, then A is elementarily equivalent to $C \oplus^P D'$ by Lemma 4.2(a) and has the same P-part, so (κ, λ)-categoricity would make A and $C \oplus^P D'$ isomorphic. Now by Theorem 7.5 and categoricity, A/A^P is divisible-plus-bounded, hence so is D by Lemma 2.5(b). If m is the exponent of the bounded part of A/A^P, then D is a direct sum of finite cyclic groups of order $\leqslant m$ and groups of the forms $\mathbb{Z}(p^\infty)$ or $\mathbb{Q}$. There are only countably many isomorphism types of such summands, and since $\lambda > \kappa + \omega$, at least one of these types must appear at least $(\kappa + \omega)^+$ times in D.

Now the number of direct summands of the form $\mathbb{Z}(p^\infty)$ in A is the rank $\rho(A)$ of $p^m A[p]$ as a vector space over $\mathbb{F}_p$; this invariant ρ is additive in direct sums of abelian groups. Since C has cardinality $\leqslant \kappa + \omega$, $\rho(C) \leqslant \kappa + \omega$, and hence $\rho(A) = \rho(D)$ whenever $\rho(D) > \kappa + \omega$. A similar additive invariant detects from A the number of direct summands in D of the form $\mathbb{Z}(p^k)$ if this number is greater than $\kappa + \omega$. (These invariants are straightforward adaptations of Szmielew invariants—cf. [5] p. 666ff—though unlike the Szmielew invariants they are not determined by $\mathrm{Th}(A)$.)

We claim that at most one isomorphism type of summand appears infinitely often in D. For contradiction, suppose there are at least two such types, say Γ and Δ; at least one of them, say Γ, is not the type of $\mathbb{Q}$. By compactness and Lemma 4.2(a) the number of summands of type Γ can be shrunk to $\kappa + \omega$ or expanded to λ without altering $\mathrm{Th}(D)$; and likewise for Δ. So we can construct $D' \equiv D'' \equiv D$, all of cardinality λ, so that Γ appears only countably many times in D' and $(\kappa + \omega)^+$ times in D''. Then $A' = C \oplus^P D'$ and $A'' = C \oplus^P D''$ are (κ, λ)-models of $\mathrm{Th}(A)$ with the same P-part, and so by (κ, λ)-categoricity they are isomorphic. But this is impossible since A' and A'' differ in the invariant that detects Γ. The claim is proved.

If D is unbounded, the unique type in the claim must be that of $\mathbb{Q}$, and so $\mathrm{Th}(D)$ is ω_1-categorical by Theorem 2.1(b)(ii). If D is bounded, the one type can be that of any $\mathbb{Z}(p^k)$, but the remaining summands form a finite group; again $\mathrm{Th}(D)$ is ω_1-categorical, this time by Theorem 2.1(b)(i). □

The next result on D requires a closer analysis of tightness. We will continue this analysis in section 11 below.

Definition 8.6 Let A be a group and B a subgroup. Let p be a prime and $n < \omega$. Then the *Ulm-Kaplansky invariant*, in symbols $UK_{p,n}(A, B)$, is the rank of

$$\frac{p^n A[p]}{(p^{n+1}A + B) \cap p^n A[p]} \quad \left(\cong \frac{p^n A[p] + p^{n+1}A + B}{p^{n+1}A + B} \right)$$

as vector space over $\mathbb{F}_p$. (Cf. [4] p. 61. Note that unlike the Szmielew invariants, this invariant is not necessarily first-order expressible.) If A is a group pair, we write $UK_{p,n}(A)$ for $UK_{p,n}(A, A^P)$.

Lemma 8.7 *For each prime p the Ulm-Kaplansky invariants $UK_{p,n}$ are additive in group pair direct sums, and zero in divisible groups and q-groups with $q \neq p$.*

□

Lemma 8.8 *Let A be an abelian group and B a subgroup of A such that A is a tight extension of B. Then:*

(a) For every prime p and every $k < \omega$, $p^k A[p] \subseteq p^{k+1}A + B$.

(b) (Villemaire [14]) For every prime p and every $n < \omega$, the Ulm-Kaplansky invariant $UK_{p,n}(A, B)$ is zero.

Conversely if A is a bounded abelian group and B a subgroup of A such that (a) holds, then A is a tight extension of B.

Proof. (a) For contradiction, let p be a prime, $k < \omega$ and $p^k a$ an element of $p^k A[p]$ which is not in $p^{k+1}A + B$.

We claim first that a generates a subgroup $\langle a \rangle$ of A disjoint from B. For suppose $ip^j a \in B$ where i is prime to p. By Euclid find integers x, y so that $xi + yp^{k+1} = 1$. Then

$$p^j a = xip^j a + yp^{j+k+1}a = xip^j a \in B.$$

But $p^k a$ is not in B, so $j \geqslant k + 1$ and hence $ip^j a = 0$. This proves the claim.

Second, we claim that $a+B$ generates a nontrivial pure subgroup $\langle a+B \rangle$ of A/B. Each element $p^j a + B$ with $j \leqslant k$ has p-height j in A/B since

$p^k a \notin p^{k+1} + B$. In particular $a + B$ has p-height 0 in A/B, so that $\langle a + B\rangle$ is a nontrivial group. If q is any prime $\neq p$ and h is any positive integer, then Euclid gives integers u, v such that $uq^h + vp^{k+1} = 1$, so

$$a = uq^h a + vp^{k+1} a = uq^h a,$$

and hence a has infinite q-height in A/B. This proves the claim.

The two claims together contradict the assumption that A is a tight extension of B.

Then (b) is immediate from (a) by the definition of the invariants.

For the converse, suppose A is not a tight extension of B. By Theorem 8.2(a) we can write $A = C \oplus^P D$ where C is a tight extension of B. Since A is not a tight extension of B, D is a nontrivial direct sum of finite cyclic groups. Hence D has a direct summand of the form $\mathbb{Z}(p^{k+1})$ for some prime p and $k \geqslant 0$. Let d generate D. Then $p^k d \in p^k A[p] \setminus (p^{k+1}A + B)$, contradicting (a). □

Theorem 8.9 *Assume* $(\star)$ *with* $\kappa = \lambda$. *Suppose Th*(A) *is* (κ, κ)*-categorical and* $\omega < \kappa$. *Then* D *is finite.*

Proof. Assume Th(A) is (κ, κ)-categorical with $\kappa > \omega$. By Theorem 6.3(a), A/A^P is bounded and so D is bounded. If D is infinite, then D has a direct summand that is an infinite homocyclic subgroup $\mathbb{Z}(p^k)^{(\mu)}$. By compactness,

$$\mathbb{Z}(p^k)^{(\mu)} \equiv \mathbb{Z}(p^k)^{(\omega)} \equiv \mathbb{Z}(p^k)^{(\kappa)},$$

so by Lemma 4.2(a) there are D' and D'' elementarily equivalent to D, where D' has a direct summand $\mathbb{Z}(p^k)^{(\kappa)}$ but D'' has no direct summand $\mathbb{Z}(p^k)^{(\mu)}$ with $\mu > \omega$. Write A' for $C \oplus^P D'$ and A'' for $C \oplus^P D''$. Then A' and A'' are (κ, κ)-models of Th(A) with the same P-part, and so by (κ, κ)-categoricity they are isomorphic.

We calculate the Ulm-Kaplansky invariant $UK_{p,k-1}(A')$. By Lemmas 8.7 and 8.8(b),

$$\begin{aligned} UK_{p,k-1}(A') &= UK_{p,k-1}(C) + UK_{p,k-1}(D', 0) \\ &= UK_{p,k-1}(D', 0) \\ &= p^{k-1}D'[p]/p^k D'[p] \\ &= \kappa. \end{aligned}$$

The corresponding calculation with A'' gives that $UK_{p,k-1}(A'') = \omega$. This contradicts the fact that A' is isomorphic to A''. □

9 Incompatible relative categoricities

Being (κ, λ)-categorical can sometimes prevent a theory from being (κ', λ')-categorical. The following examples are completely trivial:

Theorem 9.1 *Let T be a (κ, λ)-categorical theory of group pairs. Then under either of the following conditions, T is not also (κ', λ')-categorical:*

(a) κ is finite and $\kappa' \neq \kappa$.

(b) λ is finite and $\lambda' \neq \lambda$.

Proof. If T is (κ, λ)-categorical and (κ', λ')-categorical, then it has a (κ, λ)-model A and a (κ', λ')-model B. If κ is finite and distinct from κ', this implies $A \not\equiv B$, contradicting the completeness of T. Similarly with λ. $\square$

There is another incompatibility of this kind, involving only infinite cardinals. Like the examples above, it depends on showing that certain information is expressed in T and hence carries across from one model of T to another. But now the argument is not so trivial.

Lemma 9.2 *Suppose A is a group pair of the form $C \oplus^P D$ where $A^P \subseteq C$ and C is a tight extension of A^P. Suppose also that A/A^P is divisible-plus-bounded, and that if A/A^P is unbounded then so is D. Under these conditions Th(D) can be read off from Th(A).*

Proof. We refer to the Szmielew invariants (cf. [5] p. 666ff). The theory of the bounded direct summand of D is determined by the invariants

$$U(p, k; D) = |p^k D[p]/p^{k+1} D[p]| \in \omega \cup \{\infty\}$$

for each prime p and each $k < \omega$. To compute these invariants, we introduce an Ulm-Kaplansky invariant $UK(p, k; A)$ for group pairs by writing

$$UK(p, k; A) = p^{UK_{p,k}(A, A^P)} = |p^n A[p]/((p^{n+1}A + A^P) \cap p^n A[p])|.$$

(Cf. Definition 8.6.) Then $UK(p, k; -)$ is a Baur-Monk invariant, so it is multiplicative in direct sums (cf. [5] Lemma A.1.9). Hence

$$UK(p, k; A) = UK(p, k; C) \cdot UK(p, k; D) = UK(p, k; D)$$

since $UK(p, k; C) = 1$ by Lemma 8.8(b). But $U(p, k; D) = UK(p, k; D)$ since $D^P = \{0\}$. So we can recover $U(p, k; D)$ from Th(A).

Next we consider the divisible p-torsion part D_p of D; the relevant Szmielew invariants are

$$D(p, k; D) = |p^k D[p]| \in \omega \cup \{\infty\}$$

with $k < \omega$. For determining Th(D) we need only the values of $D(p, k; D)$ for large enough k. Now $D(p, k; D)$ is not necessarily equal to either $D(p, k; A)$ or $D(p, k; A/A^P)$, since for example there may be divisible p-torsion groups in A whose socle lies inside A^P. It suffices to use instead the Baur-Monk invariant $D^\star(p, k; -)$ where for any group pair B,

$$D^\star(p, k; B) = |p^k B[p]/(p^k B[p] \cap B^P)|.$$

Since $D^P = \{0\}$, $D^\star(p, k, D) = D(p, k; D)$ for all p and k. By Lemma 8.8(a), $D^\star(p, k; C) = 1$ when we take k so that p^k is greater than the exponent of the bounded part of A, so that $p^k C = p^\infty C$. Hence $D(p, k; D)$ for all large enough k is equal to $D^\star(p, k; A)$ and hence is determined by Th(A).

The Szmielew invariants Tf$(p, k; -)$ are not needed for determining the theory Th(D), since D is divisible-plus-bounded. The only other piece of information that we need is whether D is bounded or not. If A/A^P is bounded then clearly D is bounded; by assumption if A/A^P is unbounded then D is unbounded. □

Theorem 9.3 *Suppose T is a (κ, κ)-categorical theory of group pairs, where $\omega < \kappa$. Then if $\kappa' < \lambda'$, T is not also (κ', λ')-categorical.*

Proof. Suppose T is both (κ, κ)-categorical and (κ', λ')-categorical. Then T has a (κ, κ)-model A and a (κ', λ')-model B. By Theorem 6.3(a), A/A^P is bounded, and hence so is B/B^P since this is expressible in T. Hence by Theorem 8.2(a), A is a direct sum $A = C \oplus^P D$ where C is a tight extension of A^P, and B is a direct sum $B = C' \oplus^P D'$ where C' is a tight extension of B^P.

Since κ is infinite, so is κ'. But also κ is uncountable, and so Theorem 8.9 tells us that D is finite. By Lemma 9.2, D' is finite too. Hence by Theorems 7.5(b) and 8.2(b),

$$\kappa' = |B^P| \leqslant |C'| \leqslant |B^P| + \omega = \kappa'.$$

Since D' is finite, we have

$$\lambda' = |C'| + |D'| = \kappa'$$

contradicting the assumption that $\kappa' < \lambda'$. □

Theorem 9.4 *Suppose T is a (κ, λ)-categorical theory of group pairs, and A is a model of T of the form $C \oplus^P D$ where C is a tight extension of A^P. If A/A^P is unbounded then so is D.*

Proof. Let A' be a (κ, λ) model of T, and by Theorems 7.5 and 8.2(a) write A' as $C' \oplus^P D'$ where C' is a tight extension of A'^P. Since A/A^P is unbounded, so is A'/A'^P. By Theorem 8.4, D' is unbounded. So the assumptions of Lemma 9.2 are satisfied, and we infer that $\mathrm{Th}(D')$ is determined by T. So $\mathrm{Th}(D') = \mathrm{Th}(D)$, and hence D is unbounded too. □

10 Reduction Property

We say that T has the *Reduction Property* if for every formula $\phi(\bar{x})$ of $L(P)$ there is a formula $\phi^\star(\bar{x})$ of L such that if A is any model of T and $\bar{a}$ a tuple of elements of A^P, then

$$A \models \phi(\bar{a}) \quad \Leftrightarrow \quad A^P \models \phi^\star(\bar{a}).$$

The next result is in some sense a model-theoretic version of Lemma 1.1. The proof adapts Pillay and Shelah [10], who proved it when $\kappa = \lambda$. For any complete theory T and positive integer n, we write S^n_T for the set of complete types of T over the empty set in the variables $v_0, \ldots, v_{n-1}$.

Theorem 10.1 *Let T be a complete theory of group pairs. Suppose T is (κ, λ)-categorical for some κ and λ. Then T has the Reduction Property.*

Proof. We first claim that

> For each $n < \omega$ there is a function $\sigma : S^n_{T^P} \to S^n_T$ such that for every model B of T and every n-tuple $\bar{a}$ of elements of B^P, if $\bar{a}$ realises $p \in S^n_{T^P}$ in B^P then $\bar{a}$ realises $\sigma(p)$ in B.

Suppose first that κ is infinite. Let B, C be models of T and $\bar{b}, \bar{c}$ finite sequences in B^P, C^P respectively which realise p in B^P, C^P. We have to show that $\bar{b}, \bar{c}$ realise the same type in B, C respectively. Since $B \equiv C$, we can elementarily embed both in a single model and thus assume $B = C$. If $\kappa = \lambda$ then by Löwenheim-Skolem we can assume that B is a (κ, λ)-model. On the other hand if $\kappa < \lambda$, then we can choose a (κ, κ)-model and (by the classification above) blow it up to a (κ, λ)-model. By Theorems 7.5(b) and 8.2, $B = C \oplus^P D$ where $B^P \subseteq C$ and $|C| = |B^P| = \kappa$. Now put $C_0 = C$. By assumption $(C_0^P, \bar{b}) \equiv (C_0^P, \bar{c})$. Let a be any element of C_0^P and let $q(\bar{b}, x)$ be the type of a over $\bar{b}$ in C_0^P. Then $q(\bar{c}, x)$ relativised to P is consistent with the elementary diagram of C_0; so there exists an elementary extension C_1 of C_0 with an element d such that $(C_1^P, \bar{b}, a) \equiv (C_1^P, \bar{c}, d)$. We can choose C_1 to be of cardinality κ. Now we repeat this move back and forth, so as to build up an elementary chain $C_0 \preccurlyeq C_1 \preccurlyeq \ldots$ of length κ. It can be arranged that the elements of C_κ^P are listed as $\bar{a}$ and as $\bar{d}$ so that $(C_\kappa^P, \bar{b}, \bar{a}) \equiv (C_\kappa^P, \bar{c}, \bar{d})$. Then the map $\bar{a} \mapsto \bar{d}$ defines an automorphism of C_κ^P which takes $\bar{b}$ to $\bar{c}$. By Lemma 1.1 and (κ, λ)-categoricity, this automorphism extends to the whole of $C_\kappa \oplus^P D$, and hence $\bar{b}$ and $\bar{c}$ have the same type in $C_\kappa \oplus^P D$ and therefore also in B. This proves the claim.

If κ is finite then B^P is determined up to isomorphism by T, and it already has the property that for any two tuples which realise the same type in B^P there is an automorphism of B^P taking one to the other. So a much shorter version of the previous argument applies, and again we have the claim.

We infer the Reduction Property as follows. Consider a formula $\phi(\bar{x})$ of $L(P)$. If $\phi \in \sigma(p)$ then T implies this; so by compactness there is a formula $\theta_p \in p$ such that $\phi \in \sigma(q)$ whenever $\theta_p \in q$. Then modulo T, ϕ implies the infinite disjunction $\bigvee\{\theta_p : \phi \in \sigma(p)\}$. By compactness again, ϕ implies a finite disjunction, which will serve as $\phi^\star$. □

Theorem 10.2 *Let T be a complete theory of group pairs. Suppose there are complete theories T_1, T_2 such that every model of T has the form $A = C \oplus^P D$ where $A^P \subseteq C$, $C \models T_1$ and $D \models T_2$. If T_1 has the Reduction Property then so does T.*

Proof. Let $\phi(\bar{x})$ be a formula of $L(P)$. If $\theta(\bar{x})$ is as in Lemma 4.2(e), and the Reduction Property finds $\theta^\star(\bar{a})$, then for every tuple $\bar{a}$ in A^P,

$$C \oplus^P D \models \phi(\bar{a}) \;\Leftrightarrow\; C \models \theta(\bar{a}) \;\Leftrightarrow\; A^P \models \theta^\star(\bar{a}).$$

□

Part III
Proving categoricity

In this part we show that the conditions for categoricity described in Part II are not just necessary; they are also sufficient.

In all cases we have group pairs A, B and a pure embedding $i : A^P \to B^P$, and we lift i to a pure embedding $j : A \to B$. When i is an isomorphism, j will also be an isomorphism. The statement 'i preserves finite

p-heights from A to B' means that the p-height of each element a of A^P in A is equal to the p-height of $i(a)$ in B, where p-heights are reckoned as either finite or ∞.

11 More on tight extensions

Lemma 11.1 *Suppose A is divisible-plus-bounded and is a tight extension of its subgroup B. Then A/B is torsion.*

Proof. Choose m so that mA is divisible. Suppose for contradiction that $a \in A$ and $a + B$ has infinite order in A/B. Then a has infinite order in A, and so ma lies in a direct summand D of A of the form $\mathbb{Q}$. Since no nonzero multiple of ma lies in B, D is disjoint from B. But $(D+B)/B$ is isomorphic to $\mathbb{Q}$ and hence is a pure subgroup of A/B. This contradicts the tightness of A over B. □

Definition 11.2 When p is a prime and A is an abelian group with a subgroup C, we say that an element a of A is *p-proper* over C if $\mathrm{ht}^p_A(a) \geqslant \mathrm{ht}^p_A(a+c)$ for every $c \in C$. If A is a p-group, we shorten p-proper to *proper*.

Lemma 11.3 *Suppose p is a prime, A is a p-group, C is a subgroup of A and $a \in A$. Then a is proper over C if and only if for every element c of C,*

$$ht^p_A(a+c) = \min\{ht^p_A(a), ht^p_A(c)\}.$$

Hence if a is proper over C and c is an element of C with $ht^p_A(a) = ht^p_A(c)$, then $a + c$ is also proper over C.

Proof. Straightforward, [4] p. 61. □

Lemma 11.4 *Let A be an abelian group and B a subgroup of A such that A is a tight extension of B, and let p be prime.*

(a) If A is divisible-plus-bounded, then $p^\infty A[p] \subseteq B$.

(b) If $a \in A \setminus B$, $pa \in B$ and a is proper over B, then $ht^p_A(pa) = ht^p_A(a) + 1$.

Proof. (a) Suppose for contradiction that a is an element of $A[p] \setminus B$ which has infinite p-height in A. Let kp^m be the exponent of the bounded part of A, with k prime to p. Inductively choose elements a_i ($i < \omega$) of infinite p-height in A so that $pa_{i+1} = a_i$ and $p^{i+1}a_i = 0$, as follows. Put $a_0 = a$. When a_i has been chosen, let c be an element of A such that $p^{m+1}c = a_i$.

Then $p^m c$ has p-height $\geqslant m$ in A. By Euclid there are integers u, v such that $uk + vp^{i+2} = 1$. Then

$$p^m c = ukp^m c + vp^{m+i+2} c = kp^m(uc) + vp^{i+1} a_i = kp^m(uc).$$

So $p^m c$ is in the divisible part of A. Put $a_{i+1} = p^m c$. When the a_i are defined, let D be the subgroup of A generated by $\{a_0, a_1, \ldots\}$. Then D is nontrivial, divisible and disjoint from B, so that A can be split as $C \oplus^P D$ where $B \subseteq C$. So D is a pure subgroup of $A/B = C/B \oplus^P D$, contradicting tightness.

(b) Let k be the p-height of a in A. When k is infinite the statement is trivial (recalling that we put $\infty + 1 = \infty$). When k is finite, for contradiction let d be an element of A such that $a = p^k d \in A \setminus B$ and $p^{k+1} d \in B$, and suppose $p^{k+1} d$ has p-height $> k + 1$ in A. Then there is $b \in A$ such that $p^{k+2} b = p^{k+1} d$. So by Lemma 8.8(a), $p^{k+1} b - p^k d \in p^k A[p] \subseteq p^{k+1} A + B$, hence the B-coset of $p^k d$ contains an element of p-height $\geqslant k + 1$, contradicting the properness of $p^k d$ over B. □

Corollary 11.5 *Suppose A is a group pair such that A/A^P is divisible-plus-bounded, A is a tight extension of A^P, B is a group pair and $B \equiv A$. Assume also that if A/A^P is unbounded then A is divisible-plus-bounded. Then:*

(a) If B can be written as $C \oplus^P D$ where $B^P \subseteq C$ and C is a tight extension of B^P, then $D = \mathbb{Q}^{(\mu)}$ where $\mu \geqslant 0$; if A/A^P is bounded then $\mu = 0$.

(b) B can be written as in (a).

In particular if A is a tight extension of A^P, A/A^P is bounded and $A = B \oplus^P D$ with B a tight extension of A^P, then $D = 0$.

Proof. Since $B \equiv A$, $B/B^P \equiv A/A^P$ and hence B/B^P is also divisible-plus-bounded. So by Theorem 8.2(a) we can write $B = C \oplus^P D$ where C is a tight extension of B^P and D is disjoint from B^P. This takes care of (b), and it remains to show that D must be as described in (a).

Now D is a pure subgroup of B/B^P, so by Lemma 2.5(b), D is divisible-plus-bounded. Hence it is a direct sum of summands of the following forms:

(a) A cyclic p-group for some prime p.

(b) $\mathbb{Z}(p^\infty)$ for some prime p.

(c) $\mathbb{Q}$.

To prove the Corollary we show that D has no summands of either of the forms (a) and (b), and that when A/A^P is bounded, D has no summands of the form (c) either.

For (a), suppose that D has a direct summand of the form $\mathbb{Z}(p^{k+1})$, generated by an element d. Then $p^k d \in p^k B[p] \setminus (p^{k+1}B + B^P)$. It follows that $p^k B[p] \not\subseteq p^{k+1}B + B^P$. But this is a first-order property of B and hence of A, contradicting Lemma 8.8(a). Hence case (a) is impossible.

For (b), suppose that D has a direct summand D_1 of the form $\mathbb{Z}(p^\infty)$. Then B/B^P is unbounded, and so by assumption A is divisible-plus-bounded. Let kp^m be the exponent of the bounded part of A, where k is prime to p. This exponent can be read off from the Szmielew invariants of A, so it is also the exponent of the bounded part of B since $A \equiv B$. Choose d of order p^{m+1} in D_1. Then d can be written as kc with $c \in D_1$. Thus B contains an element $kp^m c \in B[p] \setminus B^P$. Since $A \equiv B$, A contains an element $kp^m a \in A[p] \setminus A^P$. But every element of the form $kp^m a$ lies in the divisible part of A and hence has infinite p-height. This contradicts Lemma 11.4(a). Hence case (b) is impossible.

If A/A^P is bounded, then so is B/B^P, and it follows that D has no summands of the form $\mathbb{Q}$. □

Lemma 11.6 *Suppose A is an abelian group with a subgroup B.*

(a) If p is a prime and A/B is a bounded p-group, then for every $a \in A \setminus B$ the coset $a + B$ contains a p-proper element.

(b) If A is divisible-plus-bounded, then for every prime p, every coset of B in A contains a p-proper element.

Proof. (a) The number of p-heights in A/B is finite, and if $a \in A \setminus B$ then $\mathrm{ht}^p_A(a) \leqslant \mathrm{ht}^p_{A/B}(a + B) \neq \infty$.

(b) For each prime p the set of values $\mathrm{ht}^p_A(a)$ ($a \in A$) is finite. □

12 The lifting

The main result of this section has the following form. Given group pairs A, B and an embedding $i : A^P \to B^P$, we extend i to a group pair embedding from A into B. This result is needed for showing that under certain conditions a theory of group pairs is (κ, λ)-categorical.

Our main tool is the Kaplansky-Mackey extension lemma, [4] Lemma 77.1, suitably adapted. In that lemma the groups are reduced p-groups.

There is no way for us to assume that our groups are p-groups, or that they are reduced. In the case where A is divisible-plus-bounded, we can write A as a direct sum of its torsion and its torsion-free parts; but we don't know that this decomposition carries over to A^P, and we still have the divisible parts to take care of. In the case where A/B is bounded, we don't even know that the torsion part of A is a direct summand.

Since we can't separate out the primary components in A, we will take second best and separate them in A/A^P. The effect is that we decompose A as a pushout of extensions A_p of A^P where each A_p/A^P is a p-group. We do the same for B, and our version of Kaplansky-Mackey gives us embeddings from A_p to B_p over the given $i : A^P \to B^P$. The pushout property allows us to combine these embeddings into a single embedding of A into B.

We still have to generalise Kaplansky-Mackey to a situation where the ground groups are not p-groups and the A_p/A^P may have divisible components. The following example makes it unlikely that we can handle the bounded part and the divisible part separately; so we need to adjust Kaplansky-Mackey to handle both simultaneously.

Example 12.1 Let p be a prime. Put $A = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^2)$. In A let a be an element of $\mathbb{Z}(p^\infty)$ of order p and b an element of $\mathbb{Z}(p^2)$ of order p. Make A into a group pair by taking for A^P the subgroup generated by $a + b$. Then A is divisible-plus-bounded, and A is tight over A^P. But we can't split A into two group pair summands, one divisible and one bounded. Also note that $\mathbb{Z}(p^\infty) + A^P$ is not tight over A^P; the whole divisible part becomes a separate summand. So fibring over A^P won't help either.

But we can assume that A and B are tight over A^P and B^P respectively, and this gives us the required legroom to adapt Kaplansky-Mackey to our situation.

Theorem 12.2 *Let p be a prime. Suppose A and B are groups and C, D are subgroups of A, B such that A/C and B/D are p-groups. Assume A is a tight extension of C and B is a tight extension of D. Let $i : C \to D$ be an embedding which preserves p-heights from A to B. Assume either*

(a) A/C and B/D are bounded groups, or

(b) A and B are divisible-plus-bounded groups.

Then there is an embedding $j : A \to B$ which extends i and preserves p-heights from A to B. When i is an isomorphism, j can be taken to be an isomorphism.

Proof. We define, by induction on α, an increasing chain of embeddings $i_\alpha : A_\alpha \to B_\alpha$ ($\alpha \leqslant \xi$) such that

- $A_0 = C$, $B_0 = D$ and $A_\xi = A$.
- Each i_α preserves finite p-heights from A to B.

Since $i : C \to D$ is an embedding which preserves finite p-heights from A to B, we can put $i_0 = i$. At limit ordinals we take unions.

We define $i_{\alpha+1} : A_{\alpha+1} \to B_{\alpha+1}$, assuming that $i_\alpha : A_\alpha \to B_\alpha$ has been defined and $A_\alpha \neq A$. Since A/A_α is a p-group, there is some element $a \in A \setminus A_\alpha$ with $pa \in A_\alpha$. By Lemma 11.6, (a) or (b) according as (a) or (b) holds above, we can assume that a is proper over A_α. Put $k = \mathrm{ht}^p_A(a)$.

Now there are two cases, according as k is finite or infinite.

If k is finite, then since A is tight over A_α, Lemma 11.4(b) tells us that $\mathrm{ht}^p_A(pa) = k + 1$. Since i_α preserves finite p-heights from A to B, $i_\alpha(pa)$ has p-height $k + 1$ in B. Choose b in B so that $pb = i_\alpha(pa)$ and $\mathrm{ht}^p_B(b) = k$.

If k is infinite, then we are in assumption (b). Certainly pa has infinite p-height in A. Using the facts that i_α preserves finite p-heights from A to B, and that elements of large enough p-height have infinite p-height, we can choose b in B so that $pb = i_\alpha(pa)$ and $\mathrm{ht}^p_B(b) = \infty = k$.

Either way, we have chosen b in B so that $pb = i_\alpha(pa)$ and $\mathrm{ht}^p_B(b) = \mathrm{ht}^p_A(a)$.

Claim A. The element b is proper over B_α. This is trivial if k is infinite. If k is finite and the claim fails, then there is $b' \in B_\alpha$ such that $b + b'$ has p-height $> k$, where necessarily b' has p-height k and so does $a' = i_\alpha^{-1}(b')$. Since a was proper over A_α, the same holds for $a + a'$ by Lemma 11.3, and hence by tightness and Lemma 8.8(a), $p(a + a')$ has p-height $k + 1$. But $p(b+b')$ has p-height $> k+1$; since $i_\alpha(p(a+a')) = p(b+b')$, this contradicts the induction assumption that i_α preserves finite p-heights from A to B. The claim is proved.

Claim B. For every $a' \in A_\alpha$ and every positive integer m,

$$\mathrm{ht}^p_A(ma + a') = \mathrm{ht}^p_B(mb + i_\alpha a').$$

Since A/A_α and B/B_α are p-groups and pa, pb are in A_α, B_α, it suffices to prove the claim when $m = 1$. But in this case the claim follows at once from Lemma 11.3 and the fact that a and b are proper. The claim is proved.

Claim C. For every $a' \in A_\alpha$ and every positive integer m,

$$ma + a' = 0 \quad \text{if and only if} \quad mb + i_\alpha(a') = 0.$$

If $ma + a' = 0$ then $ma \in A_\alpha$, so that $m = pn$ for some n. Then $a' = -ma = -npa$ and $i_\alpha(a') = -ni_\alpha(pa) = -mb$. So the left equation implies the right, and vice versa by symmetry, proving the claim.

We define $A_{\alpha+1}$ to be the subgroup of A generated by A_α and a, and $B_{\alpha+1}$ likewise in B with b. We define $i_{\alpha+1}$ by

$$i_{\alpha+1}(ma + a') = mb + i_\alpha a'$$

for all integers m and all $a' \in A_\alpha$. By Claim C this defines an embedding from $A_{\alpha+1}$ to $B_{\alpha+1}$ which extends i_α. By Claim B, $i_{\alpha+1}$ preserves finite p-heights from A to B.

We put $j = i_\xi$ when $A_\xi = A$. By construction j is an embedding which preserves finite p-heights from A to B.

Suppose $i : C \to D$ is an isomorphism. If there is an element b of $B \setminus j(A)$ with $pb \in j(A)$, then we can continue the construction using j^{-1} to extend the domain of j. This is absurd, and hence there is no such element. Thus j is an isomorphism. □

To combine the extensions given by Theorem 12.2 at the separate primes, we use pushouts.

Theorem 12.3 *Suppose*

(a) A and B are group pairs,

(b) A^P and B^P are both divisible-plus-bounded,

(c) A and B are tight extensions of A^P and B^P respectively,

(d) $i : A^P \to B^P$ is an embedding which preserves finite q-heights from A to B for every prime q, and

(e) either both A/A^P and B/B^P are bounded, or both A and B are divisible-plus-bounded.

Then there is a pure embedding $j : A \to B$ which extends i. When i is an isomorphism, j can be taken to be an isomorphism.

Proof. The groups A/A^P and B/B^P are both torsion; when A and B are divisible-plus-bounded, this follows from Lemma 11.1. We can write A/A^P as a direct sum of its p-components for each prime p. By Lemma 5.2 this direct sum makes A the pushout of subgroups A_p (p prime) over A^P, where A_p/A^P is a p-group. Likewise the decomposition of B/B^P makes B the pushout of subgroups B_p (p prime) over B^P. Each A_p is uniquely determined as the set of elements a of A such that $p^k a \in A^P$ for some $k < \omega$; and likewise with the B_p's.

Consider a prime p for which $A_p \neq A^P$. Now by Lemma 5.3(a), p-heights in A_p are the same as in A, and likewise for B_p. So the assumption (d) implies that i preserves finite p-heights from A_p to B_p. Hence by Theorem 12.2 there is an embedding $j_p : A_p \to B_p$ which extends i and preserves p-heights from A^P to B^P. The pushout property (section 5) amalgamates these maps j_p into a single embedding $j : A \to B$ extending i. When i was an isomorphism, each of the j_p is an isomorphism (again by Theorem 12.2), and so j is an isomorphism.

We show that even when j is not an isomorphism, it is a pure embedding. It suffices to show that for each p, j preserves p-heights from A to B. Fix the prime p and an element a of A. Since j is an embedding, $\mathrm{ht}^p_A(a) \leqslant \mathrm{ht}^p_B(ja)$. We must show equality when $\mathrm{ht}^p_A(a)$ is finite. Now $a = a_0 + \ldots + a_n$ where each a_i is in some $A_{p(i)}$ for a distinct prime $p(i)$ and $p(0) = p$. Also for each i there is a least $m(i)$ such that $p(i)^{m(i)} a(i) \in A^P$. Since j is an embedding, there is a corresponding decomposition $ja = b_0 + \ldots + b_n$ where for each i, $m(i)$ is the least non-negative integer such that $p(i)^{m(i)} b_i \in B^P$. Then Lemma 5.3(b), together with the fact that j_p preserves p-heights from A_p to B_p, shows that $\mathrm{ht}^p_A(a) = \mathrm{ht}^p_B(ja)$ as required. □

13 The decomposition of the theory

The next theorem applies the results of the previous section to theories of group pairs.

Theorem 13.1 *Let T be a complete theory of group pairs which has the Reduction Property. Let A and B be models of T, and suppose $A = C \oplus^P D$ and $B = C' \oplus^P D'$ where C, C' are tight extensions of A^P, B^P respectively. Suppose also that either T/T^P is bounded or T is divisible-plus-bounded. Then every isomorphism $i : A^P \to B^P$ extends to an isomorphism from C to C'.*

Proof. Let i be an isomorphism from A^P to B^P. By the Reduction Property, i preserves finite q-heights from A to B, for all primes q. Hence i also preserves finite q-heights from C to C', since C and C' are direct summands of A and B respectively. By Theorem 12.3 it follows that i extends to an isomorphism from C to C'. □

The following definition makes sense for any first-order language with $\times$ in place of $\oplus^P$, though we will use it only for the case of group pairs.

Definition 13.2 Let T_1 and T_2 be complete theories in $L(P)$, and suppose T_2 is disjoint from P. Let T be a complete theory in $L(P)$. We write

$$T = T_1 \oplus T_2$$

to mean that if A_1 and A_2 are models of T_1 and T_2 respectively, then $A_1 \oplus^P A_2$ is a model of T; and moreover every model of T is of this form.

Note that by Lemma 4.2(a), if A_1 and A_2 have complete theories T_1 and T_2 respectively, then the theory $T = \mathrm{Th}(A_1 \oplus^P A_2)$ depends only on T_1 and T_2. But without further argument we can't infer that every model of T has this form.

Theorem 13.3 *Let T be a complete theory of group pairs, and suppose T is (κ, λ)-categorical for some κ and λ. Then there are theories T_1 and T_2 with the following properties:*

(a) $T = T_1 \oplus T_2$*.*

(b) If A is any model of T of the form $A = C \oplus^P D$ where C is a tight extension of A^P, then $T_1 = Th(C)$ and $T_2 = Th(D)$.

Hence T_1 and T_2 are determined uniquely by T and the property (b).

Proof. By Theorem 7.5, T/T^P is divisible-plus-bounded. Let A be any model of T. Then by Theorem 8.2(a), A has a decomposition as $A = C \oplus^P D$ where C is a tight extension of A^P. Put $T_1 = \mathrm{Th}(C)$ and $T_2 = \mathrm{Th}(D)$.

We show that every model of T is of the required form. Let B be any model of T; then B has a decomposition $B = C' \oplus^P D'$. By saturation arguments we can blow up A and B to isomorphic elementary extensions $\tilde{A}$ and $\tilde{B}$. We can arrange that $\tilde{A} = \tilde{C} \oplus^P \tilde{D}$ where $\tilde{C}$, $\tilde{D}$ are elementary extensions of the group pairs C, D respectively; but in general there is no guarantee that $\tilde{C}$, $\tilde{D}$ are tight extensions of $\tilde{A}^P$, $\tilde{B}^P$.

By Corollary 11.5 the blowup $\tilde{C}$ of C has the form $C_0 \oplus^P \mathbb{Q}^{(\mu)}$ where C_0 is a tight extension of $\tilde{C}^P$, and $\mu = 0$ unless A/A^P is unbounded. If A/A^P is unbounded, then by Theorem 9.4, D and hence also $\tilde{D}$ will be unbounded, so $\mathrm{Th}(\tilde{C})$ and $\mathrm{Th}(\tilde{D})$ are not affected by transferring the $\mathbb{Q}^{(\mu)}$ to $\tilde{D}$. Assume this done, so that again $\tilde{C}$ is a tight extension of $\tilde{A}^P$. Do likewise with

$\tilde{B}$. Now by Theorem 13.1 any isomorphism from $\tilde{A}^P$ to $\tilde{B}^P$ extends to an isomorphism from $\tilde{C}$ to $\tilde{C}'$. Hence

$$C \equiv \tilde{C} \equiv \tilde{C}' \equiv C'.$$

Also

$$D \equiv \tilde{D} \equiv \tilde{D}' \equiv D'$$

where the middle equivalence is by Theorem 9.2.

We have shown that if B is any structure elementarily equivalent to A, then B is the direct sum of a model of Th(C) and a model of Th(D); this establishes (a). But we also showed that any decomposition of B as in Theorem 8.2(a) yields the same two theories Th(C) and Th(D), proving (b). □

14 The possible relative categoricity spectra

We can now characterise all possible relative categoricity spectra for abelian group pairs, beginning with the spectrum illustrated by Example 3.8.

Theorem 14.1 *Let T be a complete theory of group pairs. Then the following are equivalent:*

(a) The relative categoricity spectrum RCspec(T) is the class of all pairs (κ, κ) where κ is infinite.

(b) T is (κ, κ)-categorical for some uncountable κ.

(c) T is relatively categorical and has infinite models.

(d) T has the Reduction Property and has infinite models, T/T^P is bounded; and if A is a model of T of the form $A = C \oplus^P D$ where C is a tight extension of A^P, then D is finite.

Proof. First we prove (a) ⇒ (b) ⇒ (d) ⇒ (a). The first implication is immediate.

We prove (b) ⇒ (d). Assume (b). Then by Theorem 10.1, T has the Reduction Property; and by Theorem 6.3(a), T/T^P is bounded. The statement about D follows from Theorem 8.9 and Lemma 9.2. Also T clearly has infinite models.

We prove (d) ⇒ (a). Assume (d). Let κ be an infinite cardinal. Since T has infinite models, T has a (κ, κ)-model A. Suppose B is a (κ, κ)-model of T and $i : A^P \to B^P$ an isomorphism. By Theorem 8.2(a) we can write A, B as $A = C \oplus^P D$ and $B = C' \oplus^P D'$ where C, C' are tight extensions of A^P, B^P respectively. By Theorem 13.1(a), i extends to an isomorphism $j : C \to C'$. By Lemma 9.2, D and D' have the same complete theory T_D, and the assumption in (d) implies that models of T_D are finite; so D is isomorphic to D'. Combining isomorphisms, we have an isomorphism from A to B extending i. This shows that RCspec(T) includes all the pairs of cardinals described in (a).

We must show that it includes no other pairs. It suffices to show that if A is any (κ, λ)-model of T then $\kappa = \lambda$. As in the previous paragraph, we can write A as $C \oplus^P D$ where C is a tight extension of A^P. By the finiteness of D, the boundedness of A/A^P and Theorem 8.2(b),

$$\lambda = |A| = |C| \leqslant |A^P| + \omega = \kappa$$

as required.

We prove (d) ⇒ (c). Assume (d), and let A, B be models of T with $A^P = B^P$. As in the previous paragraph, we can write A, B as $A = C \oplus^P D$ and $B = C' \oplus^P D'$ where C, C' are tight extensions of A^P, B^P respectively. It follows, as in the proof of (d) ⇒ (a), that the identity on A^P extends to an isomorphism from A to B.

Finally we prove (c) ⇒ (b). Assume (b). Since T has infinite models, it has an (ω_1, ω_1)-model A. Suppose B is another (ω_1, ω_1)-model of T with A^P isomorphic to B^P. Then by relative categoricity, every isomorphism from A^P to B^P extends to an isomorphism from A to B. □

Next we take the spectrum illustrated by Example 3.6.

Theorem 14.2 *Let T be a complete theory of group pairs. Then the following are equivalent:*

- *(a) The relative categoricity spectrum RCspec(T) is the class of all pairs (κ, λ) where either $\omega \leqslant \kappa < \lambda$ or $\omega = \kappa = \lambda$.*
- *(b) T is (ω, ω)-categorical and (κ, λ)-categorical for some uncountable $\kappa < \lambda$.*
- *(c) T has the Reduction Property and has infinite models, T/T^P is bounded; and if A is a model of T of the form $A = C \oplus^P D$ where C is a tight extension of A^P, then D is ω_1-categorical.*

Proof. We prove (a) ⇒ (b) ⇒ (c) ⇒ (a). The first implication is immediate.

We prove (b) ⇒ (c). Assume (b). Then by Theorem 10.1, T has the Reduction Property; and by Theorem 6.3(a), T/T^P is bounded. Also T clearly has infinite models. The statement about D follows from Theorem 8.5 and Lemma 9.2.

We prove (c) ⇒ (a). Assume (c). Let (κ, λ) be as in the spectrum in (a). Since T has infinite models, T has a (κ, κ)-model A_0. By Theorem 8.2(a) we can write A_0 as $A_0 = C \oplus^P D_0$ where C is a tight extension of A^P. By (c), D_0 is infinite, so it has an elementary extension D of cardinality λ. Then by Lemma 4.2(a), $A = C \oplus^P D$ is a (κ, λ)-model of T.

Now suppose B is also a (κ, λ)-model of T and $i : A^P \to B^P$ an isomorphism. By Theorem 8.2(a) we can write B as $B = C' \oplus^P D'$ where C' is a tight extension of B^P. By Theorem 13.1(a), i extends to an isomorphism $j : C \to C'$. By Lemma 9.2, D and D' are elementarily equivalent. In view of Theorem 8.2(b), D' must have cardinality λ. If λ is uncountable, then it follows from (c) that D is isomorphic to D' since their theory is ω_1-categorical. If $\lambda = \omega$ then it follows from (c) and Theorem 2.1(c) that D is isomorphic to D', since T/T^P is bounded and hence both D and D' are bounded. Either way, combining isomorphisms, we have an isomorphism from A to B extending i.

We must show that if κ is uncountable then T is not (κ, κ)-categorical. This is immediate from Theorem 9.3. □

Example 3.7 illustrates the next theorem.

Theorem 14.3 *Let T be a complete theory of group pairs. Then the following are equivalent:*

(a) The relative categoricity spectrum RCspec(T) consists of the pair (ω, ω).

(b) T has the Reduction Property and has infinite models, T/T^P is bounded; and if A is a model of T of the form $A = C \oplus^P D$ where C is a tight extension of A^P, then D is bounded but not ω_1-categorical.

Proof. We prove (a) ⇒ (b). Assume (a). Then by Theorem 10.1, T has the Reduction Property; and by Theorem 6.3(a), T/T^P is bounded. Also T clearly has infinite models. By Lemma 9.2 the theory of D depends only on T. From the implication (c) ⇒ (a) of the previous theorem we know that if D was ω_1-categorical then the spectrum would contain (ω, ω_1), contradicting our present assumption (a). Since T/T^P is bounded, it follows that D is bounded but not ω_1-categorical.

We prove (b) ⇒ (a). Assume (b). Since T has infinite models, T has an (ω, ω)-model A. Suppose B is also an (ω, ω)-model of T and $i : A^P \to B^P$ is an isomorphism. By Theorem 8.2(a) we can write B as $B = C' \oplus^P D'$ where C' is a tight extension of B^P. By Theorem 13.1(a), i extends to an isomorphism $j : C \to C'$. By Lemma 9.2, D and D' are elementarily equivalent. If D and D' are finite, then it follows at once that D is isomorphic to D'. If D and D' are infinite, then they are isomorphic by Theorem 2.1(c), since T/T^P is bounded and hence both D and D' are bounded. Either way, combining isomorphisms, we have an isomorphism from A to B extending i.

We must show that if λ is uncountable and $\kappa \leqslant \lambda$ then T is not (κ, λ)-categorical. Let A be a (κ, λ)-model of T, and decompose A as $C \oplus^P D$ where C is a tight extension of C. Since D is bounded but not ω_1-categorical, it has two infinite homocyclic components $\mathbb{Z}(p^m)^{(\mu)}$ and $\mathbb{Z}(q^n)^{(\nu)}$ where p, q are prime and $p^m \neq q^n$. By the Szmielew invariants, the complete theory of A is unaltered if we change μ and ν to any two infinite cardinals. So let B_1 be the result of putting $\mu = \omega$ and $\nu = \lambda$, and B_2 the result of putting $\mu = \lambda$ and $\nu = \omega$. Since λ is uncountable, the dimensions of $p^{m-1}B_1[p]$ and $p^{m-1}B_2[p]$ as $\mathbb{F}_p$-vector spaces are different, and so B_1 and B_2 are not isomorphic. But they are (κ, λ)-models of T with the same P-part. □

Finally Example 3.5 illustrates the following theorem.

Theorem 14.4 *Let T be a complete theory of group pairs. Then the following are equivalent:*

- *(a) The relative categoricity spectrum RCspec(T) is the class of all pairs (κ, λ) where $\omega \leqslant \kappa < \lambda$.*
- *(b) T is (κ, λ)-categorical for some uncountable $\kappa < \lambda$ but not (ω, ω)-categorical.*
- *(c) T has the Reduction Property and has infinite models, T/T^P is unbounded, T is divisible-plus-bounded; and if T has a model $A = C \oplus^P D$ where C is a tight extension of A^P then D is unbounded ω_1-categorical.*

Proof. **Proof**. We prove (a) ⇒ (b) ⇒ (c) ⇒ (a). The first implication is immediate.

We prove (b) ⇒ (c). Assume (b). Then by Theorem 10.1, T has the Reduction Property. Also T clearly has infinite models. By Theorem 7.5, T/T^P is divisible-plus-bounded. It follows from Theorem 8.5 and Lemma 9.2 that D in (c) is ω_1-categorical. But then the implication (c) ⇒ (a) of Theorem 14.2 tells us that T/T^P is unbounded, since otherwise T would be (ω, ω)-categorical. Then by Theorem 9.4, D is unbounded. From Theorem 7.4, T is divisible-plus-bounded.

We prove (c) $\Rightarrow$ (a). Assume (c). Suppose $\omega \leqslant \kappa < \lambda$. Then as in the proof of (c) $\Rightarrow$ (a) of Theorem 14.2, T has a (κ, λ)-model A of the form $A = C \oplus^P D$ where C is a tight extension of A^P and D is unbounded ω_1-categorical.

Now suppose B is also a (κ, λ)-model of T and $i : A^P \to B^P$ an isomorphism. By Theorem 8.2(a) we can write B as $B = C' \oplus^P D'$ where C' is a tight extension of B^P. By Theorem 13.1(a), i extends to an isomorphism $j : C \to C'$. By Lemma 9.2, D and D' are elementarily equivalent. In view of Theorem 8.2(b), D and D' must both have cardinality λ, and hence they are isomorphic. Combining isomorphisms, we have an isomorphism from A to B extending i.

We must show that T is not (κ, κ)-categorical for any infinite κ. For uncountable κ this follows from Theorem 9.3. If T was (ω, ω)-categorical then by Theorem 14.2(b)$\Rightarrow$(c), T/T^P would be bounded. □

Theorem 14.5 *The possible relative categoricity spectra of group pairs are the four spectra above, together with the following spectra:*

(i) $\{(m, n)\}$ *where* m, n *are positive finite numbers and* m *divides* n*.*

(ii) $\{(m, \omega)\}$ *where* m *is finite.*

(iii) $\{(m, \kappa) : \kappa$ *infinite*$\}$.

(iv) $\{(m, \kappa) : \kappa$ *uncountable*$\}$.

Proof. First we show that all cases occur. Examples for the four preceding spectra are given in the preceding discussion. When m, n are positive finite numbers and $mk = n$, take A to be $\mathbb{Z}(n)$ and A^P to be kA. For the remaining cases we can take $m = 1$, thus. Putting $A = \mathbb{Z}(2)^{(\omega)} \oplus \mathbb{Z}(3)^{(\omega)}$ and $A^P = 0$ gives the spectrum $\{(1, \omega)\}$. Putting $A = \mathbb{Z}(2)^{(\omega)}$ and $A^P = 0$ gives the spectrum $\{(m, \kappa) : \kappa$ infinite$\}$. Putting $A = \mathbb{Q}$ and $A^P = 0$ gives the spectrum $\{(m, \kappa) : \kappa$ uncountable $\}$.

It remains to show that no other spectra occur. Suppose the theory T is (κ, λ)-categorical. Assume first that κ is infinite. If $\kappa < \lambda$ then we are in the spectrum of Theorem 14.2 if T is (ω, ω)-categorical, and in the spectrum of Theorem 14.4 otherwise. If $\kappa = \lambda$, then we are in the spectrum of Theorem 14.1 if κ is uncountable, and in the spectrum of Theorem 14.3 if T is not (μ, μ)-categorical for any uncountable μ.

Next assume κ is finite. If λ is finite, then κ divides λ by Lagrange's Theorem, and all models of T are (κ, λ)-models, so we are in case (i). In the remaining cases, T determines A^P up to isomorphism, so (κ, λ)-categoricity implies λ-categoricity. So T is either ω-categorical or ω_1-categorical or both, giving (ii)–(iv). □

Corollary 14.6 *Let T be a complete theory of group pairs.*

(a) If T is (κ, λ)-categorical for some pair (κ, λ) of infinite cardinals with $\kappa < \lambda$, then it is (κ, λ)-categorical for all such pairs.

(b) If T is (κ, κ)-categorical for some uncountable cardinal κ, then T is relatively categorical.

□

Corollary 14.6(b) is not true for arbitrary theories of pairs of structures; see [13]. It's reasonable to conjecture that Corollary 14.6(a) has counterexamples too in the general case.

Corollary 14.7 *Let T be a theory of group pairs which is (κ, λ)-categorical for some κ and λ with $\lambda > \omega$. Let A_1 and A_2 be models of T with $|A_1| \leqslant |A_2|$ and $|A_2^P| < |A_2|$, and let $f : A_1^P \to A_2^P$ be an elementary embedding. Then f extends to an elementary embedding of A_1 into A_2.*

Proof. As above, we can write $A_i = C_i \oplus^P D_i$ where C_i is a tight extension of A_i^P, for $i = 1, 2$. By Theorem 12.3, f extends to a pure embedding $h : C_1 \to C_2$. But $C_1 \equiv C_2$ by Theorem 13.3, and hence h is an elementary embedding ([5] Corollary A.1.3). Since λ is uncountable and $|A_2^P| < |A_2|$, we are in the situation of one of Theorems 14.2 and 14.4. In each of these cases, D_1 and D_2 are elementarily equivalent ω_1-categorical structures and $|A_2| = |D_2|$. Since $|D_1| \leqslant |A_1| \leqslant |A_2| = |D_2|$, it follows that there is an elementary embedding of D_1 into D_2. Combining these elementary embeddings by Lemma 4.2(b) gives the required elementary embedding from A_1 to A_2. □

15 Comparison with Shelah [12]

Like us, Shelah considers $L(P)$-structures A where the 1-ary relation symbol P picks out an L-substructure A^P, the *P-part* of A. He assumes we consider models of a complete first-order theory T in $L(P)$. Our L is countable, but Shelah doesn't assume this.

Shelah's Hypothesis 1.0 is that $|A| = |A^P|$. He doesn't assume this in general; he marks with * those results that do assume it.

Shelah's Hypothesis 1.1, alias Hypothesis A, is the Reduction Property. This property holds for all (κ, κ)-categorical theories for reasons of general model theory [10]. Our proof used mostly general theory, but also a direct sum decomposition. Does the Reduction Property hold for (κ, λ)-categorical pairs in general?

Shelah's Hypothesis 1.2, alias Hypothesis B, is the definability of types over the P-part. More precisely he assumes that for every model A of T, every tuple $\bar{a}$ in A and every formula $\phi(\bar{x}, \bar{y})$ of $L(P)$ there are a formula $\psi(\bar{y})$ of L and a tuple $\bar{c}$ in A^P such that for every tuple $\bar{b}$ in A^P,

$$A \models \phi(\bar{a}, \bar{b}) \;\Leftrightarrow\; A^P \models \psi(\bar{b}, \bar{c}).$$

In our case Hypothesis B is guaranteed by the stability of T (e.g. [5] Theorem 6.7.8).

Shelah defines: a subset X of a model A of T is *complete* if whenever $A \models (\exists \bar{x} \subseteq P)\phi(\bar{x}, \bar{a})$ with $\bar{a}$ in X, there is $\bar{c}$ in $X \cap A^P$ such that $A \models \psi(\bar{c}, \bar{a})$. Hypothesis C is that for every model A of T and every complete set $X \subseteq A$, the cardinality of

$$\{\text{tp}(\bar{b}/X) : \bar{b} \cap \mathbb{M}^P = \emptyset, X \cup \{\bar{b}\} \text{ is complete }\}$$

is at most $|X|^{|L|}$. Hypothesis C holds in our case because A/A^P is finite or ω-stable under any relative categoricity assumption.

Shelah defines a notion of a set being 'stable'. He shows that if all sets in models of T are stable then Hypothesis C holds, and he proves (using diamonds) that in general relative categoricity assumptions imply that every model of T is stable.

Shelah's Question D is whether for every pair of models $A \preccurlyeq B$, the set $A \cup B^P$ is stable. When the answer is Yes, he shows that every suitably saturated model E of T^P is the P-part of a model A of T which is prime over E. Presumably in our context the words 'suitably saturated' can be dropped; but to prove this will need some closer correlations between abelian notions and stability notions than we have room for in this paper.

Shelah is also concerned with the number of models not isomorphic over a given P-part. Write $I(T, \kappa, \lambda)$ for the supremum, over families of (κ, λ)-models of T with the same P-part, of the number of isomorphism types of models over the P-part. Then our calculations show that when T has (κ, λ)-models,

- If $\kappa = \lambda = \omega_\alpha$ with $\alpha \geqslant 0$, and T is not bounded over P, then $I(T, \kappa, \lambda) \geqslant |\omega + \alpha|$. (Theorem 6.3(a) with the models C_μ, $1 \leqslant \mu \leqslant \lambda$).

- If $\kappa < \omega = \lambda$ and T is not bounded over P, then $I(T, \kappa, \lambda) \geqslant \omega$. (Theorem 6.3(b) similarly.)
- If $\kappa = \omega_\alpha < \lambda = \omega_\beta$ and T has models with $\mathbb{Z}(p^\infty)^{(\omega)}$ a direct summand disjoint from P, then $I(T, \kappa, \lambda) \geqslant |\beta - \alpha|$. (Theorem 6.4, considering the models $D_{\mu,\lambda}$ with $\kappa \leqslant \mu \leqslant \lambda$.)
- If $T = T_1 \oplus T_2$ where T_1 has models that are tight over P and and T_2 is disjoint from P, then $I(T, \kappa, \lambda)$ is at least the number of isomorphism types of models of T_2 of cardinality λ.

Part IV
Finite groups

A finite group pair is relatively categorical if and only if its p-components are relatively categorical for each prime p. When $p \neq 2$ we will describe the relatively categorical finite p-group pairs. The case where $p = 2$ is more complicated, and our description in this case will be complete only when the P-part A^P is a characteristic subgroup of A.

16 Preliminaries

If the group pair A is finite, then $\mathrm{Th}(A)$ determines A up to isomorphism. So in particular the relative categoricity of $\mathrm{Th}(A)$ is the same thing as the relative categoricity of A.

Lemma 16.1 *Let p be a prime and A a finite p-group pair. Then the following are equivalent:*

(a) A is relatively categorical.

(b) Every automorphism of A^P extends to an automorphism of A.

(c) Every automorphism of A^P extends to an endomorphism of A.

(d) A has the Reduction Property.

(e) $A = C \oplus^P D$ where $A^P \subseteq C$, and C is relatively categorical and a tight extension of A^P.

Proof. (a) implies (b) by Lemma 1.1 and (d) by Theorem 10.1. By Theorems 7.5 and 8.2, (a) implies that A has a decomposition $A = C \oplus^P D$ where $A^P \subseteq C$, C is a tight extension of A^P and C has the Reduction Property.

Since A is finite, (b) and (d) are equivalent; clearly (b) implies (a). This shows that (a), (b) and (d) are equivalent. The implication (d) $\Rightarrow$ (a), applied to C in the previous paragraph, proves that C is relatively categorical and hence that (a) implies (e). The implication from (e) to (a) is clear.

Clearly (b) implies (c). For the converse, suppose β is an automorphism of A^P which extends to an endomorphism α of A. By Fitting's Lemma (e.g. Jacobson [7] p. 113), there is an abelian group direct sum decomposition $A = A_1 \oplus A_2$ such that α is an automorphism on A_1 and nilpotent on A_2. Then $A^P \subseteq A_1$. The automorphism which agrees with α on A_1 and with the identity on A_2 is an automorphism of A extending β. □

In view of Lemma 16.1 we can use 'relatively categorical' henceforth as meaning the purely group-theoretic statement that every automorphism of A^P extends to an automorphism of A.

Lemma 16.2 *Let A be a finite p-group pair where $B = A^P$ is cyclic, generated by an element $b \neq 0$. Then A is a tight extension of B if and only if A can be written as $A_1 \oplus^P \ldots \oplus^P A_n$ where*

(a) each A_i is cyclic, generated by a nonzero element a_i of order p^{r_i}, and $b = p^{s_1}a_1 + \ldots + p^{s_n}a_n$ where $0 \leqslant s_i < r_i$ for each i;

(b) $s_i < s_j$ whenever $1 \leqslant i < j \leqslant n$;

(c) $1 \leqslant r_i - s_i < r_j - s_j$ whenever $1 \leqslant i < j \leqslant n$.

When A is a tight extension of B, the sequence $(s_1, \ldots, s_n; r_1, \ldots, r_n)$ is uniquely determined by A.

Proof. Suppose first that A is a tight extension of B. Write A as a direct sum of nonzero cyclic groups. We can arrange that b is of the form $p^{s_1}a_1 + \ldots + p^{s_n}a_n$ by multiplying each generator a_i by a suitable integer prime to p. Then tightness guarantees that $s_i < r_i$ for each i. Order the summands so that $r_1 \leqslant \ldots \leqslant r_n$.

Suppose $i < j$ and $s_i \geqslant s_j$. Then we can replace a_j by $a_j + p^{s_i - s_j}a_i$; this reduces $A_i \cap B$ to 0 and contradicts tightness (by Corollary 11.5). So (b) is proved.

We turn to (c). Let $i < j$ and for contradiction suppose $r_i - s_i \geqslant r_j - s_j$. Then we can replace a_i by $a_i + p^{s_j - s_i}a_j$, since by supposition $r_j - (s_j - s_i) \leqslant$

r_i. This reduces $A_j \cap B$ to 0 and again contradicts tightness. So (c) is proved. Note that (b) and (c) together imply that $r_j \geqslant r_i + 2$ whenever $i < j$.

Conversely suppose A is a direct sum of cyclic groups with (a)–(c) holding. Write $c_1, \ldots, c_n$ for the generators of the cyclic summands of A. We show that for every $k \leqslant r_n$, $p^k A[p] \subseteq p^{k+1}A + B$; by Lemma 8.8 this implies that A is a tight extension of B.

Suppose there is some element a of $p^k A[p] \setminus p^{k+1}A[p]$. Then $k = r_i - 1$ for some unique i, and by subtracting suitable elements of $p^{k+1}A$ we can suppose that a generates the socle $A_i[p]$. Consider the element $d = p^{r_i - s_i - 1}b$ of B, (which exists since $r_i - s_i - 1 \geqslant 0$ by (a)). For each $j < i$, $r_j - s_j - 1 \geqslant r_i - s_i$ by (c) again, and hence the element d lies in $A_i \oplus \ldots \oplus A_n$. Then for some suitable m prime to p, $ma = d + a'$ where a' lies in $A_{i+1} \oplus \ldots \oplus A_n$. Now if $i < j$ then by (b), $s_i < s_j$. It follows that a' has p-height at least $r_i - s_i - 1 + s_{i+1} \geqslant r_i \geqslant k + 1$, so that $a' \in p^{k+1}A$ as required.

The numbers r_i are recoverable from the Szmielew invariants U of the group A in the usual way. The number s_1 is the minimum p-height in A of elements of A^P. Then $s_2 - s_1$ is the minimum p-height in $p^{s_1}A$ of elements of $p^{s_1}A \cap A^P$; and so on. Hence the sequences of r_i's and s_i's are recoverable from the group pair A. □

Definition 16.3 We refer to the sequence $(s_1, \ldots, s_n; r_1, \ldots, r_n)$ in Lemma 16.2 as the *ticket* of the group pair A.

Lemma 16.4 *Let A be a finite p-group pair.*

(a) Suppose A is a group pair direct sum, $A_1 \oplus^P \ldots \oplus^P A_n$. If A is relatively categorical then so is each A_i.

(b) Suppose $A = C \oplus^P D$ where $A^P \subseteq C$ and C is a tight extension of A^P. Then A is relatively categorical if and only if C is relatively categorical.

Proof. (a) Assume A is relatively categorical and let α be an automorphism of A_i^P. Extend α to the whole of A^P by taking it to be the identity on each A_j^P with $j \neq i$. By assumption α extends to an automorphism β of the whole of A. Let γ be $\beta \restriction A_i$ followed by projection onto A_i along the remaining direct summands. Then γ is an endomorphism of A_i which extends α, so A_i is relatively categorical by Lemma 16.1(c).

(b) is then clear. Note that by Theorem 8.2(a), A can always be decomposed in this form. □

17 Automorphisms

Definition 17.1 Fixing a prime p, we consider an abelian p-group $A = A_1 \oplus \ldots \oplus A_n$ where each A_i has generator a_i of order p^{r_i} with $r_1 \leqslant \ldots \leqslant r_n$. By an *elementary automorphism* of A (with respect to the given decomposition) we mean an automorphism α of A with one of the following two forms:

(a) For some i and some m prime to p, $\alpha a_i = m a_i$, and $\alpha a_k = a_k$ for all $k \neq i$.

(b) For some i and j with $i \neq j$, $\alpha a_i = a_i + p^h m a_j$ where m is prime to p and $r_j - h \leqslant r_i$; and $\alpha a_k = a_k$ for all $k \neq i$.

(We use 'elementary' here in the sense of linear algebra; of course all automorphisms are elementary embeddings in the model-theoretic sense.)

Lemma 17.2 *If A is as in Definition 17.1, then every automorphism of A is a product of elementary automorphisms. Also every elementary automorphism of the form (b) is a power of the elementary automorphism where $m = 1$ and $h = \max(0, r_j - r_i)$.*

Proof. The second sentence is immediate. For the first, use Gaussian elimination with the obvious adjustments. If α is an automorphism of A, we write $\alpha a_i = \sum_j m_{ij} a_j$ for each i; here each m_{ij} is a unique integer modulo p^{r_j}. If M is the matrix (m_{ij}), we write M^{-1} for the matrix of the inverse automorphism α^{-1}. Since αa_n has order p^{r_n}, there is at least one j with a_j of order p^{r_n} and m_{nj} prime to p. So elementary column operations on the matrix (m_{ij}) bring the last row to the form $(0, \ldots, 0, 1)$, and then elementary row operations bring the final column to the form $(0, \ldots, 0, 1)^T$. Applying the same argument to $n - 1$, $n - 2 \ldots$ in place of n, there are matrices P, Q of elementary automorphisms such that $P(m_{ij})Q$ is the unit matrix, and hence $(m_{ij}) = P^{-1}Q^{-1}$. The righthand side of this equation is the matrix of a product of elementary automorphisms; hence so is the lefthand. □

Definition 17.3 Let A be a group pair with $A^P = B$. We say that A is *separated* if A has a group pair direct sum decomposition $A = A_1 \oplus^P \ldots \oplus^P A_k$ such that each A_i^P is cyclic.

Lemma 17.4 *Suppose the finite p-group pair A is separated as in Definition 17.3, and each A_i^P is generated by an element b_i of order p^{s_i}. Then A is relatively categorical if and only if:*

for all $i \neq j$ ($1 \leqslant i, j \leqslant k$), there is a group homomorphism $g : A_i \to A_j$ taking b_i to $p^h b_j$ where $h = \max(0, s_j - s_i)$.

Proof. Write B for A^P. First we show that if A is relatively categorical then the condition holds. Let α be the automorphism of B which takes b_i to $b_i + p^h b_j$ and fixes each $b_{i'}$ ($i' \neq i$). Then α extends to an automorphism α^+ of A. The automorphism $\alpha^+ \restriction A_i$ takes b_i to $b_i + p^h b_j$, and so $\alpha^+ \restriction A_i$ followed by projection onto A_j takes b_i to $p^h b_j$ as required.

Second, suppose the condition holds. To show that A is relatively categorical, it suffices to show that each elementary automorphism of B lifts to A. The one-dimensional automorphisms ((a) in Definition 17.1) lift immediately; multiply A by the same scalar as B. Suppose next that $i \neq j$ and α is an automorphism of B which takes b_i to $b_i + p^h m b_j$ where $s_j - h \leqslant s_i$ and m is prime to p. By assumption there is a group homomorphism β from A_i to A_j which takes b_i to $p^{\max(0, s_j - s_i)} b_j$. Suppose $p^h m = m' p^{\max(0, s_j - s_i)}$. (It must have this form, since $h \geqslant s_j - s_i$ and $h \geqslant 0$.) Then $m'\beta$ is a homomorphism from A_i to A_j taking b_i to $p^h m b_j$. Counting $m'\beta$ as zero on all elements outside A_i, the endomorphism $1_A + m'\beta$ of A is an automorphism extending α. □

Now when A is a separated group pair, we can combine the lemmas above and read off necessary and sufficient conditions for A to be relatively categorical, in terms of the tickets of the direct summands of A. By Lemma 16.4(b) there is no loss in assuming that A is a tight extension of A^P, so that all the direct summands of A contain nontrivial cyclic subgroups of A^P and hence are tight extensions of their P-parts.

Theorem 17.5 *Let A be a separated p-group pair and $\bigoplus_{i \in I} A_i$ a decomposition of A as a direct sum of indecomposable group pairs. Assume A is a tight extension of A^P, and for each $i \in I$ let $\tau_i = (s_{i,1}, \ldots, s_{i,n_i}; r_{i,1}, \ldots, r_{i,n_i})$ be the ticket of A_i. Then A is relatively categorical if and only if for all $i \neq j$ in I, and for each k' ($1 \leqslant k' \leqslant n_j$), either $r_{j,k'} - s_{j,k'} \leqslant h$ or there is k ($1 \leqslant k \leqslant n_i$) such that*

(a) $s_{i,k} \leqslant s_{j,k'} + h$ *and*

(b) $r_{i,k} - s_{i,k} \geqslant r_{j,k'} - s_{j,k'} - h$

where $h = \max(0, (r_{j,n_j} - s_{j,n_j}) - (r_{i,n_i} - s_{i,n_i}))$.

Proof. For each i let b_i be a generator of A_i^P as in Lemma 16.2; note that the order of b_i is $p^{r_{i,n_i} - s_{i,n_i}}$. By Lemma 17.4 it suffices to show that the condition above is equivalent to: There is an abelian group homomorphism $g : A_i \to A_j$ taking b_i to $p^h b_j$.

The condition is sufficient. Suppose the condition holds. If k' is such that $r_{j,k'} - s_{j,k'} \leqslant h$, then $p^h b_j$ has zero k'-th component and we can ignore it. For each k' where this inequality fails, choose a k as in the condition, and define a homomorphism $\alpha_{k'}$ from the subgroup $\mathbb{Z}a_{i,k}$ to A_j by putting

$$\alpha_{k'}(a_{i,k}) = p^{s_{j,k'}+h-s_{i,k}} a_{j,k'}.$$

The element of A_j is well-defined by (a). To ensure that $\alpha_{k'}$ is a homomorphism we need to know that the order of $\alpha_{k'}(a_{i,k})$ is at most $p^{r_{i,k}}$, in other words that

$$r_{j,k'} - s_{j,k'} - h + s_{i,k} \leqslant r_{i,k}.$$

But (b) guarantees precisely this. So the homomorphisms $\alpha_{k'}$ are well-defined. Now define $\beta : A_i \to A_j$ to be the sum $\sum_{1 \leq k' \leq n_j} \alpha_{k'}$. One can check that $\beta(b_i) = p^h b_j$.

The condition is necessary. Suppose there is a homomorphism $\alpha : A_i \to A_j$ such that $\alpha(b_i) = p^h b_j$, and consider some k' ($1 \leqslant k' \leqslant n_j$). Let β be α followed by projection onto $\mathbb{Z}a_{j,k'}$. If $r_{j,k'} - sj, k' \leqslant h$ then $\beta(b_i) = 0;$. If not then $\beta(b_i)$ has p-height $s_{j,k'} + h$. Write $\beta(a_{i,k}) = p^{\ell_k} m_k a_{j,k'}$ for each k ($1 \leqslant k \leqslant n_i$), where each m_k is prime to p. Partition $\{1, \ldots, n_i\}$ into $K_1 \cup K_2$ where $k \in K_2$ if and only if $s_{i,k} + \ell_k > s_{j,k'} + h$. Let $\gamma : A_i \to A_j$ be the homomorphism that acts on each $a_{i,k}$ like β if $k \in K_1$ and as zero if $k \in K_2$. Then $\beta(b_i) - \gamma(b_i)$ has p-height $> s_{j,k'} + h$; it follows that K_1 is not empty and $\beta(b_i)$ has the same p-height as $\gamma(b_i)$, so $\beta(b_i) = m\gamma(b_i)$ for some integer m prime to p. Finally let $\delta : A_i \to A_j$ be $m\gamma$. Then $s_{i,k} + \ell_k \leqslant s_{j,k'} + h$ for each $k \in K_1$, and in particular (a) holds for each such k.

Since δ is a homomorphism, the order of some $a_{i,k}$ ($k \in K_1$) is at least that of $p^{\ell_k} m_k a_{j,k'}$, in other words $r_{i,k} \geqslant r_{j,k'} - \ell_k$. Then

$$r_{i,k} \leqslant r_{j,k'} - \ell_k \leqslant r_{j,k'} + s_{i,k} - (s_{j,k'} + h)$$

which immediately gives (b). □

Corollary 17.6 *Under the hypotheses of the theorem, suppose that each A_i is cyclic. Then for each i the numbers $r_{i,k}$ and $s_{i,k}$ reduce to single numbers r_i and s_i, and the conditions (a), (b) reduce to*

(a) $s_i \leqslant s_j + h$ *and*

(b) $r_i - s_i \geqslant r_j - s_j + h$

where $h = \max(0, (r_j - s_j) - (r_i - s_i))$. □

A small part of the information in Theorem 17.5 extends to the non-separated case.

Theorem 17.7 *Let A be a relatively categorical finite p-group pair, and suppose that $A^P = B_1 \oplus^P B_2$ where B_1 is a cyclic group of order p^r, generated by an element b of p-height s in A. Then every element of B_2 of order $\leqslant p^r$ has p-height $\geqslant s$ in A.*

Proof. Suppose c is an element of B_2 of order $\leqslant p^r$. Then there is an automorphism β of B which takes b to $b + c$. By assumption β extends to an automorphism α of A. Since b and $\alpha(b) = b + c$ both have p-height s, c has p-height $\geqslant s$. □

When is a relatively categorical finite p-group pair separated?

Theorem 17.8 *Let A be a relatively categorical finite p-group pair.*

(a) If $p \neq 2$ then A is separated.

(b) If $p = 2$ then A splits as a group pair direct sum of relatively categorical group pairs $A_1 \oplus^P A_2$ where A_1 is separated and A_2^P is a direct sum of cyclic groups of pairwise distinct orders.

Proof. Write $B = A^P$ as a direct sum of cyclic groups, $B_1 \oplus \ldots \oplus B_n$ with generators $b_1, \ldots, b_n$.

(a) We assume $p \neq 2$. We will show that A is a group pair direct sum $A_1 \oplus^P A_2$ with $B_1 \subseteq A_1$ and $\bigoplus_{j \neq 1} B_j \subseteq A_2$, where each of A_1 and A_2 are relatively categorical. Induction completes the argument.

We first show that projection onto B_1 along the other direct summands of B is a linear combination of automorphisms of B. Let β be the automorphism taking b_1 to $2b_1$ and each other generator b_j to b_j. Then the required projection is $\beta - 1_B$. By relative categoricity, β extends to an automorphism α of A. Then $\gamma = \alpha - 1_A$ is an endomorphism of A extending the projection. By Fitting's Lemma the group A has a direct sum decomposition $A = A_1 \oplus A_2$ such that γ is the sum of an automorphism of A_1 and a nilpotent endomorphism of A_2. Clearly then $B_1 \subseteq A_1$ and $\bigoplus_{j \neq 1} B_j \subseteq A_2$, so $A = A_1 \oplus A_2$ is a group pair direct sum. Both A_1 and A_2 are relatively categorical by Lemma 16.4(a).

(b) Suppose now that $p = 2$. We show that the construction of (a) allows us to separate off one cyclic summand B_i provided that there is $j \neq i$ such that B_i and B_j have the same order. Let α transpose b_i and b_j, fixing the other generators of B pointwise. Let β take b_i to $b_i + b_j$ and fix all generators

b_k ($k \neq i$) pointwise. Then $\alpha(\beta - 1_B)$ is the projection of B onto B_i along the other direct summands, and the argument proceeds as before. $\square$

Example 17.9 The following example shows that a relatively categorical finite 2-group pair need not be separated. Let A be $\mathbb{Z}(4) \oplus \mathbb{Z}(16)$, and a_1, a_2 generators of the two summands. Let $B = A^P$ be the subgroup generated by the two elements $a_1 + 2a_2$ and $2a_1$; then B is $B_1 \oplus^P B_2$ where $a_1 + 2a_2$ generates B_1 and $2a_1$ generates B_2. If C is a direct summand of A containing $a_1 + 2a_2$, then C contains $8a_2$ and so must contain an element c of A with $8c = 8a_2$. All such elements c have the form $m_1a_1 + m_2a_2$ where m_1, m_2 are odd. Hence C contains $2m'a_1 + 2a_2$ for some odd m', and thus also $(2m' - 1)a_1$. It follows quickly that $C = A$.

We confirm that A is relatively categorical. It suffices to check that each elementary automorphism of B extends to one of A. The only automorphisms of B of the form (a) are scalar multiplication by m, where m is an odd integer. Each such automorphism extends to the automorphism multiplying each element of A by m. Next there is the automorphism

$$a_1 + 2a_2 \mapsto 3a_1 + 2a_2, \quad 2a_1 \mapsto 2a_1.$$

This extends to the automorphism

$$a_1 \mapsto 3a_1, \quad a_2 \mapsto a_2.$$

Finally there is the automorphism

$$a_1 + 2a_2 \mapsto a_1 + 2a_2, \quad 2a_1 \mapsto 2a_1 + 8a_2.$$

This extends to the automorphism

$$a_1 \mapsto a_1 + 4a_2, \quad a_2 \mapsto 15a_2$$

of A.

18 Characteristic subgroups

Definition 18.1 Recall that a subgroup B of a group A is *characteristic* if for every automorphism α of A, $\alpha \restriction B$ is an automorphism of B. We call a group pair A *characteristic* if A^P is a characteristic subgroup of the group A.

Definition 18.2 We fix some conventions for this section. The p-group pair A is a group direct sum $A_1 \oplus \ldots \oplus A_n$ where each A_i is a nonzero direct sum of cyclic groups of order p^{r_i}, and $r_1 < \ldots < r_n$. Decomposing each

A_k into a direct sum of cyclic groups, we can choose a generator for each of these cyclic groups; the set of these generators for all A_k will be called the *chosen basis*. The chosen basis is not fixed; we can re-choose it. We write $B = A^P$. For each i ($1 \leqslant i \leqslant n$) we define two integers s_i, t_i in the following way. If $B \cap A_i = 0$, set $s_i = r_i$; otherwise s_i is the least nonnegative integer such that $A_i \cap B$ contains an element of p-height s_i. If $B \subseteq A_1 \oplus \ldots \oplus A_{i-1} \oplus A_{i+1} \oplus \ldots \oplus A_n$, set $t_i = r_i$; otherwise, t_i is the least nonnegative integer such that there exists an element $c_1 + \ldots + c_i + \ldots + c_n$ of the group B, with c_j in A_j for each j, such that c_i is an element of p-height t_i. It is clear that $0 \leqslant t_i \leqslant s_i \leqslant r_i$.

Lemma 18.3 *Suppose B is a characteristic subgroup of A. Then:*

(a) For each i, $B \cap A_i = p^{s_i} A_i$.

(b) If $j < i$, then $t_j \leqslant s_j \leqslant t_i \leqslant s_i$.

(c) If $i < j$, then $r_i - t_i \leqslant r_j - s_j$.

(d) For each i, $s_i \leqslant t_i + 1$; if $p \neq 2$ or A_i has rank > 1 then $t_i = s_i$.

Proof. Fix i and j. By the definition of t_i, there exists an element $c = c_1 + \ldots + c_n$ of the group B, with $c_j \in A_j$ for each j, such that the p-height of c_i is equal to t_i. We can arrange the chosen basis so that it contains an element $a_i \in A_i$ for which $p^{t_i} a_i = c_i$. Let $a' \in A_j$, and let $h = \max(0, r_j - r_i)$. In each of the cases

1. $i \neq j$, a' is an element of the chosen basis;
2. $i = j$, $a' = a_i$ and $p \neq 2$;
3. $i = j$, A_i has rank > 1 and a' is an element of the chosen basis distinct from a_i;
4. $i = j$, $a' = pa_i$

there is an automorphism α of A which takes a_i to $a_i + p^h a'$ and fixes each other element of the chosen basis. Since B is a characteristic subgroup of A, the element $b = p^{t_i+h} a' = \alpha(c) - c$ is contained in $B \cap A_j$.

In the first three cases put $b = p^{t_i+h} a' \in B \cap A_j$. Then either b is an element of p-height $t_i + h$ and then $s_j \leqslant t_i + h$, or else $b = p^{t_i+h} a' = 0$ so that $t_i + h \geqslant r_j \geqslant s_j$. Similarly, in the fourth case $b = p^{t_i+1} a_i \in B \cap A_i$ is an element of p-height $t_i + 1$, and then $s_i \leqslant t_i + 1$, or $b = p^{t_i+1} a_i = 0$, which means that $t_i + 1 \geqslant r_i \geqslant s_i$. To complete the proof of (b), (c), (d) of this Lemma it suffices to recall that $t_i \leqslant s_i$ for each i.

To prove (a), arrange the chosen basis so that it contains an element a_i in A_i such that $p^{s_i} a_i \in B$. Let a_i' be another element of the chosen basis that lies in A_i. Then there is an automorphism β of A that transposes a_i and a_i', and fixes each other element of the chosen basis. Then $\beta(p^{s_i} a_i) = p^{s_i} a_i'$, and this element is in B since B is a characteristic subgroup. Since $A_i \cap B$ is a group, we infer that $p^{s_i} A_i \subseteq B$. The converse inclusion $B \cap A_i \subseteq p^{s_i} A_i$ is an immediate corollary of the definition of s_i. □

Corollary 18.4 *The order of each element of B is at most $p^{r_n - t_n}$. If $b = b_1 + \ldots + b_n$, with $b_j \in A_j$ for each j, is an element of B, and its component b_n in A_n has the least possible p-height t_n, then the order of b is equal to $p^{r_n - t_n}$. Hence the cyclic group generated by b is a direct summand of B.*

Proof. Let $c = c_1 + \ldots + c_n$ be an element of the group B, with c_i belonging to A_i for each i. By the definition of t_i, $c_i \in p^{t_i} A_i$, hence $p^{r_i - t_i} c_i = 0$. By (c) of Lemma 18.3, $r_i - t_i \leqslant r_n - s_n \leqslant r_n - t_n$ for each $i < n$ and consequently $p^{r_n - t_n} c_i = 0$. Thus $p^{r_n - t_n} c = 0$ for every element $c \in B$. On the other hand, $p^{r_n - t_n - 1} b_n \in A_n$ is an element of p-height $r_n - t_n - 1 + t_n = r_n - 1$, which implies that $p^{r_n - t_n - 1} b_n \neq 0$ and consequently $p^{r_n - t_n - 1} b \neq 0$. □

From the preceding lemmas we can read off a characterisation of the finite relatively categorical characteristic p-group pairs that are separated (bearing in mind that when $p \neq 2$, all finite relatively categorical p-group pairs are separated).

Theorem 18.5 *The following are equivalent, for any finite p-group pair:*

- *(a) A is a relatively categorical characteristic p-group pair which is a group pair direct sum of cyclic group pairs.*
- *(b) For some s, $A^P = p^s A$.*
- *(c) In the notation of Definition 18.2, $s_n = t_n$.*
- *(d) Either $B = 0$, or for some s, $0 \neq p^s A \subseteq B$ and $pB \subseteq p^{s+1} A$.*

If $p \neq 2$, we can leave out the condition that A is a group pair direct sum of cyclics.

Remark 18.6 The condition (d) is technical; we shall need it below.

Proof. (a) ⇒ (c): Assume (a) and recall Definition 18.2. Since A is a group pair direct sum of cyclic group pairs, we can choose the decomposition $A = A_1 \oplus^P \ldots \oplus^P A_n$ so that each A_i is a group pair direct sum of cyclic group pairs. Then $B = (B \cap A_1) \oplus^P \ldots \oplus^P (B_n \cap A_n)$, and it follows that the component $b_n \in A_n$ of any element $b = b_1 + \ldots + b_n \in B$ is contained in the group $B \cap A_n$, which, by Lemma 18.3(a), coincides with $p^{s_n}A$. Therefore $t_n = s_n$.

(c) ⇒ (b). Let $s_n = t_n$ and let $a_n \in A_n$ be an element of the chosen basis. Then $b = p^{s_n}a_n \in B$, and the p-height of its component $p^{s_n}a_n$ in A_n is equal to $s_n = t_n$. By Corollary 18.4, the cyclic group D generated by b is a direct summand of B; let B' be any complementary direct summand, so that $B = D \oplus B'$. Again by Corollary 18.4, the order of any element $b' \in B' \subseteq B$ is not greater than the order of $b = p^{s_n}a_n$. Therefore there is an automorphism β of the group B which takes b to $b + b' = p^{s_n}a_n + b'$. By relative categoricity, there exists an automorphism α of A such that $p^{s_n}\alpha(a_n) = \alpha(b) = \beta(b) = p^{s_n}a_n + b'$; therefore, $b' \in p^{s_n}A$. Thus B' is contained in $p^{s_n}A$, and, since $p^{s_n}a_n$ also belongs to $p^{s_n}A$, we obtain that $B \subseteq p^{s_n}A$.

On the other hand, by (b) of Lemma 18.3, $s_i \leqslant s_n$ for each $i \leqslant n$, and consequently $p^{s_n}A_i \subseteq p^{s_i}A_i \subseteq B$. Hence, $p^{s_n}A = p^{s_n}A_1 \oplus \ldots \oplus p^{s_n}A_n \subseteq B$.

(b) ⇒ (a) and (b) ⇒ (d) are immediate.

(d) ⇒ (c). If $B = 0$ then $t_n = s_n = r_n$. Suppose $0 \neq p^sA \subseteq B$, $pB \subseteq p^{s+1}A$. Then $s < r_n$ and $0 \neq p^sA_n \subseteq B \cap A_n = p^{s_n}A_n$; it follows that $s_n \leqslant s < r_n$. If $t_n \neq s_n$, then $t_n = s_n - 1$. There is an element $c_1 + \ldots + c_n$ of the group B, with c_j in A_j for each j, such that c_n has p-height t_n. Then pc_n has p-height $t_n + 1 = s_n < r_n$; on the other hand, $pc_n \in p^{s+1}A_n$, and we obtain that $s + 1 \leqslant s_n \leqslant s$. This contradiction proves that the assumption $t_n \neq s_n$ was erroneous.

Finally if $p \neq 2$, then by Lemma 18.3 (d), each A_i is a group pair direct summand of A with $A_i^P = p^{s_i}A_i$, so a group decomposition of A_i into a direct sum of cyclics is in fact a group pair decomposition. □

We remark that the finite group pairs satisfying Theorem 18.5(b) are exactly those investigated in Evans, Hodges and Hodkinson [2], which characterised those group pairs A of this form for which A is coordinatisable over A^P.

Let us turn now to the case $t_n \neq s_n$; by Lemma 18.3 (d), this is possible only if $p = 2$ and A_n has rank 1. To the end of the section A is a finite relatively categorical characteristic 2-group pair and $B = A^P$. We denote by A' the direct sum $A_1 \oplus^P \ldots \oplus^P A_{n-1}$ and by B' the intersection $B \cap A'$.

We always assume that $t_n \neq s_n$. Since the integers s_{n-1} and t_n play an essential role in the following argument, we write them s and t for brevity; by Lemma 18.3(b), $s \leqslant t$.

Lemma 18.7 *There exists a subset Γ of the set $\{1, \dots, n-1\}$, such that:*

(a) if $i \in \Gamma$, then $t_i \neq s_i$ and consequently A_i has rank 1;

(b) generators a_i of the cyclic groups A_i $(i \in \Gamma \cup \{n\})$ can be chosen so that the element $b = 2^t a_n + \sum_{i \in \Gamma} 2^{t_i} a_i$ belongs to B.

Proof. By the definition of $t = t_n$, there exists an element $b' = b_1 + \dots + b_n \in B$, with $b_i \in A_i$ for each i, such that b_n has 2-height t. Denote by Γ the set of all indices $i < n$, such that $b_i \notin 2^{s_i} A_i \subset B$; observe that $t_i \neq s_i$ for every $i \in \Gamma \cup \{n\}$. Then the element $b = b_n + \sum_{i \in \Gamma} b_i$ also belongs to B, and it is clear that for each $i \in \Gamma \cup \{n\}$ the element b_i has 2-height $< s_i$; since the 2-height of the (nonzero) component in A_i of an element of B cannot be smaller than t_i, and $t_i \geqslant s_i - 1$ by (d) of Lemma 18.3, we find that $t_i = s_i - 1$ and that b_i is an element of 2-height t_i. If $i \in \Gamma \cup \{n\}$, then by (d) of Lemma 18.3 the group A_i has rank 1, and we can choose a generator a_i of A_i so that $b_i = 2^{t_i} a_i$; in particular, $b_n = 2^t a_n$. □

Lemma 18.8 *The group B decomposes into the direct sum of the group $B' = B \cap A'$ and the cyclic group D generated by the element b which was defined in Lemma* 18.7.

Proof. Let $c = c_1 + \dots + c_n \in B$, where $c_i \in A_i$ for each i. Then the 2-height of the element $c_n \in A_n$ is not greater than the 2-height t of the element b_n, and, since the group A_n is cyclic, there is an integer m such that $c_n = m b_n$. Then obviously $c - mb \in B \cap ((A_1 \oplus \dots \oplus A_{n-1})) = B'$. Thus, $B = B' + D$, and this sum is direct because the component b_n of the element b in A_n has the same order as the element b itself. □

Lemma 18.9 *The group pair (A', B') is a relatively categorical group pair.*

Proof. It is obvious that the group B' is a characteristic subgroup of the group A'. Further, any automorphism β' of the group B' extends to an automorphism β of the direct sum $B = B' \oplus D$, which in its turn extends to an automorphism α of the group A, because B is relatively categorical in A. The composition of α with the projection $A = A' \oplus A_n \to A'$ is an endomorphism of A', and its restriction to B' coincides with β'; by Lemma 16.1, A' is a relatively categorical group pair. □

Lemma 18.10 *The group $2B'$ is contained in the group $2^{t+1}A'$.*

Proof. Let c be an arbitrary element of $B' = A' \cap B$; there exists an automorphism β of the group $B = B' \oplus D$ which takes b to $b + c$ and which is the identity on B'. Since A is a relatively categorical group pair, there is an automorphism α of the group A which extends β. Unfortunately, we do not know the images $\alpha(2^{t_i}a_i)$, $i \in \Gamma$, because $2^{t_i}a_i \notin B'$. But $2^{t_i+1}a_i = 2^{s_i}a_i \in B'$, and it follows that $\alpha(2^{t_i+1}a_i) = \beta(2^{t_i+1}a_i) = 2^{t_i+1}a_i$ for every $i \in \Gamma$. Therefore

$$2c = \beta(2b) - 2b = \alpha(2b) - 2b = (\alpha(2^{t+1}a_n) + \textstyle\sum_{i\in\Gamma} \alpha(2^{t_i+1}a_i)) - \\ -(2^{t+1}a_n + \textstyle\sum_{i\in\Gamma} 2^{t_i+1}a_i) = 2^{t+1}(\alpha(a_n) - a_n) \in A' \cap 2^{t+1}A = 2^{t+1}A'.$$

Lemma 18.11 *The group B' is equal to the group $2^sA'$.*

Proof. First suppose $s = s_{n-1} < r_{n-1}$. Then $0 \neq 2^sA_{n-1} \subseteq 2^sA'$. Further, $2^sA' \subseteq B'$ because, by Lemma 18.3, $s_i \leqslant s = s_{n-1}$ for each $i < n$, and $2^sA_i \subseteq 2^{s_i}A_i = B \cap A_i$. By Lemma 18.10, $2B' \subseteq 2^{t+1}A'$; but $s \leqslant t$, therefore $2B' \subseteq 2^{s+1}A'$. Now it follows from (d) $\Rightarrow$ (b) of Theorem 18.5 that $B' = 2^sA'$.

Next suppose $s = r_{n-1}$. Then for each $i < n-1$, $r_i - t_i \leqslant r_{n-1} - s = 0$. This means that if $c = c_1 + \ldots + c_{n-1}$ is an element of B', with $c_j \in A_j$ for each j, then $c_1 = \ldots = c_{n-2} = 0$ and $c = c_{n-1} \in A_{n-1} \cap B = p^{s_{n-1}}A_{n-1} = 0$. Therefore $B' = 0$, and we have again $B' = 0 = 2^sA'$. □

We can now calculate s_i, t_i and obtain information about the set Γ.

Lemma 18.12 *(a) If $i < n$ then $s_i = \min(r_i, s)$. If $i \notin \Gamma$ then $t_i = s_i$. If $i \in \Gamma$ then $t_i = s_i - 1$. Thus Γ consists of all the indices i such that $t_i \neq s_i$.*

(b) If $t < r_{n-1}$ then $s = t$. If $t \geqslant r_{n-1}$ then $s = r_{n-1}$ or $s = r_{n-1} - 1$.

(c) If $i \in \Gamma$ and $r_i \geqslant s$, then $r_{i-1} < s$ or $i = 1$.

(d) If $i \in \Gamma$ and $r_i < s < r_{n-1}$, then $r_{i+1} > s$.

(e) If $n - 1 \in \Gamma$ and $s < r_{n-1}$, then $t < r_n - 2$.

Proof. (a) By Lemma 18.11, $B \cap A' = 2^sA'$ and consequently $B \cap A_i = 2^sA_i$ for each $i < n$; therefore, $s_i = \min(r_i, s)$. If $i \in \Gamma$, then $s_i \neq t_i$ and so $t_i = s_i - 1$. We have seen that B is generated by the group $2^sA'$ and the element $b = 2^ta_n + \sum_{i\in\Gamma} 2^{t_i}a_i$. Hence every element of B is the sum $2^ta' + q(2^ta_n + \sum_{i\in\Gamma} 2^{t_i}a_i)$, with $a' \in A'$, $q \in \mathbb{Z}$, and for $j \notin \Gamma \cup \{n\}$ its component in A_j is contained in $2^sA_j = 2^{s_j}A_j$, which means that $t_j = s_j$.

(b) If $t < r_{n-1}$, then $0 \neq 2^t A' \subseteq 2^s A'$ and $2B' \subseteq 2^{t+1} A'$; using once more Theorem 18.5, we obtain that $B' = 2^t A'$. Thus $0 \neq 2^t A' = B' = 2^s A'$, which implies that $s = t$. If $t \geqslant r_{n-1}$, then $2^{s+1} A_{n-1} = 2 \cdot 2^s A_{n-1} = 2(A_{n-1} \cap B') \subseteq 2B' \subseteq 2^{t+1} A' = 0$, which means that $s + 1 \geqslant r_{n-1} \geqslant s$.

(c), (d), (e) Let $i \in \Gamma$, $i < n$. If both r_{i-1} and r_i are $\geqslant s$, we have $s = s_{i-1} \leqslant t_i = s_i - 1 = s - 1$; contradiction. If both r_i and r_{i+1} are $\leqslant s$, we have $1 = s_i - t_i = r_i - t_i \leqslant r_{i+1} - s_{i+1} = 0$; contradiction. If $n - 1 \in \Gamma$ and $s < r_{n-1}$, then $2 \leqslant r_{n-1} - (s - 1) = r_{n-1} - t_{n-1} \leqslant r_n - s_n < r_n - t_n = r_n - t$. □

The following result is now obvious.

Lemma 18.13 *There are only the following variants for the set $\Gamma \subseteq \{1, \ldots, n-1\}$ and the integers s, t, t_i $(i \in \Gamma)$:*

(a) $\Gamma = \{m - 1, m\},\ r_{m-1} < s < r_m,\ t_{m-1} = r_{m-1} - 1,\ t_m = s - 1,\ t = s.$

(b) $\Gamma = \{m - 1\},\ r_{m-1} < s < r_m,\ t_{m-1} = r_{m-1} - 1,\ t = s.$

(c) $\Gamma = \{m\},\ s \leqslant r_m,\ r_{m-1} < s$ *or* $m = 1,\ t_m = s - 1,\ t = s.$

(d) $\Gamma = \{n - 2, n - 1\},\ r_{n-2} < s = r_{n-1} - 1,\ t_{n-1} = r_{n-1} - 2,\ t_{n-2} = r_{n-2} - 1,\ r_{n-1} \leqslant t < r_n - 2.$

(e) $\Gamma = \{n - 1\},\ r_{n-2} < s = r_{n-1} - 1,\ t_{n-1} = r_{n-1} - 2,\ r_{n-1} \leqslant t < r_n - 2.$

Bringing together all the preceding results, we obtain the complete description of those finite relatively categorical characteristic p-group pairs which are not group pair direct sums of cyclics.

Theorem 18.14 *Let A be a 2-group pair which is a group direct sum $A_1 \oplus \ldots \oplus A_n$ where each A_i is a nonzero direct sum of cyclic groups of order 2^{r_i}, and $r_1 < \ldots < r_n$. Further, let $s \leqslant r_{n-1}$, $\Gamma \subseteq \{1, \ldots, n - 1\}$, t, t_i satisfy the requirements of one of the items of Lemma 18.13. Assume that for each $i \in \Gamma \cup \{n\}$ the group A_i is cyclic; let a_i be a generator of this group. If $B = A^P$ is the direct sum of the group $2^s(A_1 \oplus \ldots \oplus A_{n-1})$ and the cyclic group generated by the element $b = 2^t a_n + \sum_{i \in \Gamma} 2^{t_i} a_i$, then A is a relatively categorical characteristic group pair. Conversely, any finite relatively categorical characteristic p-group pair which is not a group pair direct sum of cyclics can be obtained in this way.*

Proof. The converse statement is in fact already proved: by Theorem 18.5 if a finite relatively categorical characteristic p-group pair is not a group pair direct sum of cyclics, then $p = 2$, $t_n \neq s_n$, and Lemmas 18.7 – 18.13

show that this group pair has the structure described in the first part of Theorem. Therefore, it remains to check that in all cases B is a characteristic subgroup of A and that every automorphism of B can be extended to an automorphism of A. We shall consider only the most complicated cases (a), (d), because the same argument (and even a part of it) works in the three other cases.

Case (a). B is the direct sum of the group $2^t A'$ and the cyclic group generated by the element $b = 2^{r_{m-1}-1}a_{m-1} + 2^{t-1}a_m + 2^t a_n$, where $1 < m < n$, $r_{m-1} < t < r_m$. Note that $r_n - r_m \geqslant 2$ because $r_m - (t-1) = r_m - t_m \leqslant r_n - s_n = r_n - (t+1)$.

First we show that B is a characteristic subgroup of A. Since the group $C = 2^t A' + 2^{t+1} A_n$ is contained in B and is characteristic in A, it is sufficient to check that every elementary automorphism α of the group A takes the element b into the coset $b + C$.

We can assume that the chosen basis of A contains the elements a_{m-1}, a_m, a_n. If an automorphism of A does not move a_{m-1}, a_m, a_n, then it does not move b. Any elementary automorphism of A which moves one of the elements a_{m-1}, a_m, a_n is a specialisation of one of the automorphisms α, β, γ, such that

$$\begin{aligned}
&\alpha(a_{m-1}) = a_{m-1} + c_i, \quad \alpha(a_m) = a_m + c_j, \quad \alpha(a_n) = a_n + c_k;\\
&\beta(a_{m-1}) = a_{m-1} + 2^{h_1}c_u, \quad \beta(a_m) = a_m + 2^{h_2}c_v, \quad \beta(a_n) = a_n;\\
&\gamma(a_{m-1}) = (2x+1)a_{m-1}, \quad \gamma(a_m) = (2y+1)a_m, \quad \gamma(a_n) = (2z+1)a_n,
\end{aligned}$$

where $i < m-1,\ j < m,\ k < n,\ u > m-1,\ v > m,\ c_i \in A_i,\ c_j \in A_j,\ c_k \in A_k,\ c_u \in A_u,\ c_v \in A_v,\ h_1 = r_u - r_{m-1},\ h_2 = r_v - r_m,\ x, y, z \in \mathbb{Z}$. We have:

$$\begin{aligned}
&\alpha(b) - b = 2^{r_{m-1}-1}c_i + 2^{t-1}c_j + 2^t c_k = 2^t c_k \in 2^t A' \subseteq C,\\
&\beta(b) - b = 2^{h_1 + r_{m-1}-1}c_u + 2^{h_2+t-1}c_v = 2^{r_u-1}c_u + 2^{r_v - r_m + t-1}c_v \in C,\\
&\gamma(b) - b = x \cdot 2^{r_{m-1}}a_{m-1} + y \cdot 2^t a_m + z \cdot 2^{t+1}a_n \in C,
\end{aligned}$$

because $r_{m-1} - 1 \geqslant r_i,\ t-1 \geqslant r_{m-1} \geqslant r_j,\ r_u - 1 \geqslant r_m - 1 \geqslant t,\ r_v - r_m + t - 1 \geqslant 1 + t - 1 = t$ for $v < n$ and $r_n - r_m + t - 1 \geqslant 2 + t - 1 = t + 1$. Thus B is a characteristic subgroup of A.

Now we prove that A is a relatively categorical group pair. The elements b, $2^t a_m$, $2^t a$, where a runs through all elements of the chosen basis in $A_{m+1} \oplus \ldots \oplus A_{n-1}$, constitute a basis of B, which we shall call the chosen basis of B. We must prove that each elementary automorphism of B can be extended to A. Fix an element $a \in A_q$ of the chosen basis of A, where

$m < q < n$, and consider the automorphisms φ, χ, ψ of B, which do not move any elements of the chosen basis of B except $b,\ 2^t a_m,\ 2^t a$, and which act on $b,\ 2^t a_m,\ 2^t a$ in the following way:

$$\begin{aligned}
&\varphi(b) = b + 2^t c_i, \quad \varphi(2^t a_m) = 2^t a_m, \quad \varphi(2^t a) = 2^t a + 2^t c_j;\\
&\psi(b) = b, \quad \psi(2^t a_m) = 2^t a_m + 2^{h_1} 2^t c_u + x \cdot 2^{h_2} b,\\
&\qquad\qquad\qquad\qquad \psi(2^t a) = 2^t a + 2^{h_3} 2^t c_v + y \cdot 2^{h_4} b;\\
&\chi(b) = (2x+1)b, \quad \chi(2^t a_m) = (2y+1)2^t a_m, \quad \chi(2^t a) = (2z+1)2^t a,
\end{aligned}$$

where $m \leqslant i < n,\ m \leqslant j \leqslant q,\ m < u < n,\ q < v < n,\ c_i \in A_i,$ $c_j \in A_j,\ c_u \in A_u,\ c_v \in A_v,\ h_1 = r_u - r_m,\ h_2 = r_n - r_m,\ h_3 = r_v - r_q,$ $h_4 = r_n - r_q,\ x, y, z \in \mathbb{Z}$. Besides, we require that if $j = q$, then c_j is not contained in the cyclic group generated by a (otherwise φ is not necessarily an automorphism). Each elementary automorphism of B can be obtained as a specialisation of one of these automorphisms for an appropriate choice of parameters $q,\ a,\ c_i$ etc. Therefore it is sufficient to observe that the automorphisms of A which fix all elements of the chosen basis of A except $a_m,\ a,\ a_n$ and which act on $a_m,\ a,\ a_n$ by the following rules:

$$\begin{aligned}
&a_m \to a_m, \quad a \to a + c_j, \quad a_n \to a_n + c_i;\\
&a_m \to (1 + 2^{h_2 - 1}x)a_m + 2^{h_1} c_u + 2^{h_2} x a_n,\\
&\qquad a \to a + 2^{h_3} c_v + y(2^{h_4 - 1} a_m + 2^{h_4} a_n);\\
&\qquad\qquad a_n \to (1 - 2^{h_2 - 1}x)a_n - 2^{h_1 - 1} c_u - 2^{h_2 - 2} x a_m,\\
&a_m \to (2y+1)a_m, \quad a \to (2z+1)a, \quad a_n \to (2x+1)a_n + (x - y)a_m,
\end{aligned}$$

extend respectively $\varphi,\ \psi,\ \chi$ (note that obviously $h_1, h_4 \geqslant 1,\ h_2 \geqslant 2$).

Case (d). B is the group generated by the elements $b_1 = 2^s a_{n-1}$ and $b = 2^{t_{n-2}} a_{n-2} + 2^{t_{n-1}} a_{n-1} + 2^t a_n$, where $t_{n-2} = r_{n-2} - 1 < r_{n-1} - 2 = t_{n-1} = s - 1$, $r_{n-1} \leqslant t < r_n - 2$. We check that B is a characteristic subgroup of A, i.e., that each elementary automorphism of A takes b and b_1 into B. Note first of all that $2^{t+1} a_n = 2b - b_1 \in B$.

Assume that the chosen basis of A contains the elements a_{n-2}, a_{n-1}, a_n. If an automorphism of A does not move a_{n-2}, a_{n-1}, a_n, then it does not move any element of B. Any elementary automorphism of A which moves one of the elements a_{n-2}, a_{n-1}, a_n is a specialisation of one of the automorphisms α, β, γ, such that

$$\begin{aligned}
&\alpha(a_{n-2}) = a_{n-2} + c_i, \quad \alpha(a_{n-1}) = a_{n-1} + c_j, \quad \alpha(a_n) = a_n + c_k;\\
&\beta(a_{n-2}) = a_{n-2} + 2^{h_1} x a_{n-1} + 2^{h_2} y a_n,\\
&\qquad\qquad\qquad \beta(a_{n-1}) = a_{n-1} + 2^{h_3} z a_n, \quad \beta(a_n) = a_n;\\
&\gamma(a_{n-2}) = (2x+1)a_{n-2}, \quad \gamma(a_{n-1}) = (2y+1)a_{n-1}, \quad \gamma(a_n) = (2z+1)a_n,
\end{aligned}$$

where $x, y, z \in \mathbb{Z}$, $h_1 = r_{n-1} - r_{n-2}$, $h_2 = r_n - r_{n-2}$, $h_3 = r_n - r_{n-1}$, $c_i \in A_i$, $c_j \in A_j$, $c_k \in A_k$, $i < n-2$, $j < n-1$, $k < n$. We have:

$$\alpha(b) = b + 2^{t_{n-2}} c_i + 2^{t_{n-1}} c_j + 2^t c_k = b, \quad \alpha(b_1) = b_1 + 2^s c_j = b_1,$$

because $t_{n-2} \geqslant r_{n-3} \geqslant r_i$, $t_{n-1} \geqslant r_{n-2} \geqslant r_j$, $t \geqslant r_{n-1} \geqslant r_k$;

$$\begin{aligned}\beta(b) &= b + 2^{t_{n-2}+h_1} x a_{n-1} + (2^{t_{n-2}+h_2} y + 2^{t_{n-1}+h_3} z) a_n = \\ &= b + x b_1 + (2^{r_n-2-t} y + 2^{r_n-3-t} z) \cdot 2^{t+1} a_n \in B, \\ \beta(b_1) &= b_1 + 2^{s+h_3} z a_n = b_1 + 2^{r_n-t-2} z \cdot 2^{t+1} a_n \in B,\end{aligned}$$

because $r_n - t - 2 > 0$; finally,

$$\begin{aligned}\gamma(b) &= b + 2^{t_{n-2}+1} x a_{n-2} + 2^s y a_{n-1} + 2^{t+1} z a_n = b + (y - z) b_1 + 2zb, \\ \gamma(b_1) &= b_1 + 2y \cdot 2^s a_{n-1} = b_1,\end{aligned}$$

because $t_{n-2} + 1 = r_{n-2}$, $s + 1 = r_{n-1}$. Thus B is a characteristic subgroup of A.

To show that A is a relatively categorical group pair, we must show that the elementary automorphisms

$$b \to b + b_1,\ b_1 \to b_1; \quad b \to (2z+1)b,\ b_1 \to b_1; \quad b \to b,\ b_1 \to b_1 + 2^{r_n-t-1} b$$

of the group B can be extended to A. But we have just seen that the first two of them are restrictions of γ respectively for $y = 1$, $z = 0$ and for $y = z$. The endomorphism of A which takes a_{n-1} to $a_{n-1} + 2^{r_n - r_{n-1}} a_n$, a_n to $(1 - 2^{r_n-t-2}) a_n$ and fixes all other elements of the chosen basis of A extends the third elementary automorphism of B; since $r_n - t - 2 > 0$, the integer $(1 - 2^{r_n-t-2})$ is odd, which means that the above endomorphism of A is an automorphism.

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