FUSION OVER A VECTOR SPACE

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Let T_1 and T_2 be two countable strongly minimal theories with the DMP whose common theory is the theory of vector spaces over a fixed finite field. We show that $T_1 \cup T_2$ has a strongly minimal completion.

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1. Introduction

In [1] E. Hrushovski answered negatively a question posed by G. Cherlin about the existence of maximal strongly minimal sets in a countable language by constructing the *fusion* of two strongly minimal theories:

Theorem . *Let T_1 and T_2 be two countable strongly minimal theories, in disjoint languages, and with the* DMP*, the definable multiplicity property. Then $T_1 \cup T_2$ has a strong minimal completion.*

The above theorem was proved by extending Fraïssé's amalgamation procedure to a given class in which Hrushovski's "δ–function" will determine the pregeometry. In order to axiomatize the theory of the generic model, a set of representatives of rank 1 types or "codes" is chosen in a uniform way.

From now on, let F denote a fixed finite field and T_0 the theory of infinite F–vector spaces in the language $L_0 = \{0, +, -, \lambda\}_{\lambda \in F}$. In this article, we will prove the following:

Theorem 1.1. *Let T_1 and T_2 be two countable strongly minimal extensions of T_0 with the DMP, and assume that their languages L_1 and L_2 intersect in L_0. Then $T_1 \cup T_2$ has a strongly minimal completion T^μ.*

This "fusion over a vector space" was proposed by Hrushovski in [1]. In the special case where both T_1 and T_2 are 1–based this fusion was already proved by A. Hasson and M. Hils [2]. These two articles also discuss fusions over more general T_0.

Our proof uses Hrushovski's machinery. Schematically, it follows [3], which is a streamlined account of Hrushovski's aforementioned paper.

In [4] and [5] it was explained how to apply Hrushovski's method to construct "fields with black points" (see also [6]). In a similar way, the techniques exhibited here were used in [7] to construct "fields with red points" (fields with a predicate for an additive subgroup, of Morley rank 2), whose existence was conjectured in [8].

The theories T^μ, which depend on the choice of codes and of a certain function μ, have the following properties:

Theorem 1.2. *Let M be a model of T^μ.*

1. *Let tr_i denote the transcendence degree in the sense of T_i and $\dim$ the F–linear dimension. Then for every finite subset A of M we have*

$$\dim(A) \leq \mathrm{tr}_1(A) + \mathrm{tr}_2(A).$$

2. *Let N be a model of T^μ which extends M. Then $N \prec M$ if N is an elementary extension of M in the sense of T_1 and in the sense of T_2.*

It follows[a] from 1. that for every p there is a strongly minimal structure $(K, +, \odot, \otimes)$ such that $(K, +, \odot)$ and $(K, +, \otimes)$ are algebraically closed fields of characteristic p and for every transcendental x the $\odot$–powers

$$1_\odot, x, x \odot x, x \odot x \odot x, \ldots$$

are algebraically independent in the sense of $(K, +, \otimes)$, and vice versa.

2. Codes

Let us fix the following notation: T is a countable strongly minimal extension of T_0 with the DMP, $\mathbb{C}$ denotes the monster model of T, $\mathrm{tr}(a/A)$ the transcendence degree[b] of the tuple a over A, $\mathrm{MR}(p)$ the Morley rank of the type p. Thus we have

$$\mathrm{tr}(a/A) = \mathrm{MR}(\mathrm{tp}(a/A)).$$

We use

$$\phi(x) \sim^k \psi(x)$$

or $\phi(x) \sim^k_x \psi(x)$ to express that the Morley rank of the symmetric difference of ϕ and ψ is smaller than k,

[a]We will explain this at the end of the paper (p. 23).

[b]The maximal number of components of a which are algebraically independent over A.

We denote by $\langle a \rangle$ We denote by the F–vector space of dimension $\dim(a)$ spanned by the components of the n–tuple a. Subspaces of $\langle a \rangle$ can be described in terms of subspaces U of F^n as

$$Ua = \Big\{ \sum_{i=1}^{n} u_i a_i \;\Big|\; u \in U \Big\}.$$

We call a stationary type a *group type* (or *coset type*) if it is the generic type of a (coset of a) connected definable subgroup of $(\mathbb{C}^n, +)$. These properties depend only on the parallel class. So we can call a formula of Morley degree 1 a *group formula* (or *coset formula*) if it belongs to a group type (or a coset type) of the same rank.

Given a group formula $\chi(x)$ of rank k, we denote by $\mathrm{Inv}(\chi)$ the group of all $H \in \mathrm{Gl}_n(F)$ which map the generic realizations of χ to generic realizations, or, equivalently, for which $H(\chi) \sim^k \chi$. If χ is a coset formula, $\mathrm{Inv}(\chi)$ is $\mathrm{Inv}(\chi^g)$ where χ^g is the associated group formula[c].

A definable set $X \subset \mathbb{C}^n$ of rank k is *encoded* by $\varphi(x, y)$ if $n = |x|$ and there is some tuple b such that $X \sim^k \varphi(x, b)$.

A *code* c is a parameter free formula $\phi_c(x, y)$ where the variable x ranges over n_c–tuples of the home sort and y over a sort of T^{eq}, with the following properties.

- **C**(i) All non–empty[d] $\phi_c(x, b)$ have (constant) Morley rank k_c and Morley degree 1.
- **C**(ii) For every $U \leq F^{n_c}$ there is a number $k_{c,U}$ such that for every realization a of $\phi_c(x, b)$ we have:

  $$\mathrm{tr}(a/b, Ua) \leq k_{c,U}.$$

  Moreover, equality holds for generic a. (So we have $k_c = k_{c,0}$.)
- **C**(iii) $\dim(a) = n_c$ for all realizations a of $\phi_c(x, b)$. If a is generic, then $\dim(a/\operatorname{acl}(b)) = n_c$ (this is equivalent to $k_{c,U} = k_c - 1$ for all one–dimensional U).
- **C**(iv) If $\phi_c(x, b)$ and $\phi_c(x, b')$ are not empty and $\phi_c(x, b) \sim^{k_c} \phi_c(x, b')$, then $b = b'$.
- **C**(v) If some non–empty $\phi_c(x, b)$ is a coset formula, then all are. We call such a code c a *coset code*. In this case, the group $\mathrm{Inv}(\phi_c(x, b))$ does not depend on b (whenever it is defined). Hence we denote it by $\mathrm{Inv}(c)$.
- **C**(vi) For all b and m the set defined by $\phi_c(x + m, b)$ is encoded by ϕ_c.
- **C**(vii) There is a subgroup G_c of $\mathrm{Gl}_{n_c}(F)$ such that:
  a) for all $H \in G_c$ and all non–empty $\phi_c(x, b)$ there exists a (unique) b^H such that

  $$\phi_c(Hx, b) \equiv \phi_c(x, b^H).$$

[c] This is $\chi(x - m)$ for a generic realization m of $\chi(x)$.
[d] Codes where all $\phi(x, b)$ are empty will not be considered.

b) if $H \in \mathrm{Gl}_{n_c}(F) \setminus G_c$, then no non–empty $\phi_c(Hx, b)$ is encoded by ϕ_c.

Two codes c and c' are *equivalent* if for every b there is some b' such that $\phi_c(x, b) \equiv \phi_{c'}(x, b')$ and vice versa. If c is a code and $H \in \mathrm{Gl}_{n_c}(F)$, then

$$\phi_{c^H}(x, y) = \phi_c(Hx, y)$$

is also a code. $\mathbf{C}$(viia) states that c^H and c are equivalent if H lies in G_c.

Corollary 2.1. *Let $p \in \mathrm{S}(b)$ be the generic type containing $\phi_c(x, b)$. Then b is the canonical base of p.*

Proof. Immediate from $\mathbf{C}$(iv). □

A formula $\chi(x, d)$ is *simple* if it has Morley degree 1 and $\dim(a/\operatorname{acl}(d)) = |x|$ for all generic realizations a of $\chi(x, d)$. The second half of $\mathbf{C}$(iii) states that all non–empty $\phi_c(x, b)$ are simple.

Lemma 2.2. *Every simple formula $\chi(x, d)$ can be encoded by some code c.*

I.e.

$$\chi(x, d) \sim^{k_c} \phi_c(x, b_0)$$

for some parameter b_0. By $\mathbf{C}$(iv) it follows that b_0 is uniquely determined, thus $b_0 \in \mathrm{dcl}^{\mathrm{eq}}(d)$.

Proof. Set $n_c = |x|$, $k_c = \mathrm{MR}\,\chi(x, d)$ and $k_{c,U} = \mathrm{tr}(a/d, Ua)$ for a generic realization a of $\chi(x, d)$. Let p be the global type of rank k_c containing $\chi(x, d)$ and b_0 its canonical base and choose some $\phi(x, b_0) \in \mathsf{p}$ of rank k_c and degree 1. Hence, $\phi(x, b_0)$ satisfies $\chi(x, d) \sim^{k_c} \phi_c(x, b_0)$ and has property $\mathbf{C}$(iv) for all b and b' realizing $\mathrm{tp}(b_0)$. We can choose $\phi(x, b_0)$ strong enough to ensure that $\mathbf{C}$(iv) holds for all b and b'.

Consider now the set X of all b of same length and sort as b_0 for which $\phi(x, y)$ satisfies $\mathbf{C}$(i), $\mathbf{C}$(ii), $\mathbf{C}$(iii) and $\mathbf{C}$(v). The latter means that $\phi(x, b)$ is a coset formula iff $\phi(x, b_0)$ is, and in this case $\mathrm{Inv}(\phi(x, b)) = \mathrm{Inv}(\phi(x, b_0))$. Let us check that X is definable by a countable disjunction of formulae. This is clear for $\mathbf{C}$(i) and $\mathbf{C}$(iii). The second part in $\mathbf{C}$(iii) is a special case of $\mathbf{C}$(ii), and the latter follows from the fact that $\mathrm{tr}(a/b, Ua) \geq k_{c,U}$ is equivalent to $\mathrm{tr}(Ua/b) \leq (k_c - k_{c,U})$ for generic a in $\phi(x, b)$. We refer to [7] for $\mathbf{C}$(v), where it is shown that the set of all b such that $\phi(x, b)$ is a group (coset) formula is definable.

All b realizing $\mathrm{tp}(b_0)$ belong to X. So a finite part $\theta(y)$ of this type implies X. Then the formula

$$\phi'_c(x, y) = \phi(x, y) \wedge \theta(y)$$

has all properties, except possibly $\mathbf{C}$(vi) and $\mathbf{C}$(vii).

Given any n_c–tuple m and parameter b, the formula $\phi'_c(x + m, b)$, if non–empty, has again rank k_c and degree 1. If a is a generic realization, then $a + m$ is a generic realization of $\phi'_c(x,b)$ and $a+m \mathop{\smile}\limits_b m$. Let u be some vector in F^{n_c} such that $\sum_i u_i a_i \in \operatorname{acl}(b,m)$. Then $\sum_i u_i(a_i+m_i) \in \operatorname{acl}(b,m)$. By independence $\sum_i u_i(a_i+m_i) \in \operatorname{acl}(b)$, which implies $u=0$. Therefore $\dim(a/\operatorname{acl}(b,m)) = n_c$ and $\phi'_c(x+m,b)$ is simple. We note also that for every U

$$\operatorname{tr}(Ua/m,b) = \operatorname{tr}(U(a+m)/m,b) = \operatorname{tr}(U(a+m)/b),$$

which implies $\operatorname{tr}(a/m,b,Ua) = k_{c,U}$.

Whence, each $\phi'_c(x+m,b)$ can be encoded by some formula $\phi'(x,y)$ which has all properties of codes except possibly $\mathbf{C}$(vi) and $\mathbf{C}$(vii). Since these properties can be expressed by a countable disjunction we conclude that there is a finite sequence of formulae $\phi_1,\ldots,\phi_r$ with all properties except possibly $\mathbf{C}$(vi) and $\mathbf{C}$(vii) which encode all formulas $\phi'_c(x+m,b)$ with m and b varying. Moreover, we may assume that for all i

$$\models\ \forall y\,\exists v,w\ \phi_i(x,y) \sim^{k_c}_x \phi'_c(x+v,w),$$

which implies that either all or none of the ϕ_i code coset formulas and if so, they have all the same invariant group $\operatorname{Inv}(\phi(x,b_0))$.

To prevent double-encoding, set

$$\theta_i(y) = \bigwedge_{j<i} \forall z\ \phi_j(x,z) \not\sim^{k_c}_x \phi_i(x,y).$$

Fix a sequence of different constants[e] $w_1,\ldots,w_r$ and define

$$\phi''_c(x,y,y') = \bigvee_{i=1}^{r} \phi_i(x,y) \wedge \theta_i(y) \wedge y' \doteq w_i.$$

$\phi''_c(x,y)$ has all properties except possibly $\mathbf{C}$(vii). To prove $\mathbf{C}$(vi) fix m and b,w such that $\phi''_c(x+m,b,w)$ is not empty. Then w equals some w_j and $\phi''_c(x+m,b,w)$ is equivalent to $\phi_j(x+m,b)$. We know that $\phi_j(x,b) \sim \phi'_c(x+m',b')$ for some m' and b'. It follows that: $\phi_j(x+m,b) \sim \phi'_c(x+(m+m'),b')$. Since $\phi'_c(x+(m+m'),b')$ can be encoded by one of the ϕ_i, property $\mathbf{C}$(vi) holds.

Only property $\mathbf{C}$(vii) remains to be obtained. Change the notation slightly and assume $\chi(x,d) \sim^{k_c} \phi''_c(x,b_0)$. Define G_c to be the set of all $A \in \mathrm{Gl}_{n_c}(F)$ such that there is some m and some realization b of $p = \operatorname{tp}(b_0)$ such that $\phi''_c(Ax,b_0) \sim^{k_c} \phi''_c(x+m,b)$. To show that G_c is a group, consider another $A' \in G_c$. Then there are m' and $b' \models p$ such that $\phi''_c(A'x,b) \sim^{k_c} \phi''_c(x+m',b')$. This yields $\phi''_c(AA'x,b_0) \sim^{k_c} \phi''_c(A'x+m,b) \equiv \phi''_c(A'(x+A'^{-1}m),b) \sim^{k_c} \phi''_c(x+(A'^{-1}m+m'),b')$, and so $AA' \in G_c$.

There is a $\rho(y) \in p$ such that for no $A \in \mathrm{Gl}_{n_c}(F) \setminus G_c$ there are some b which satisfies ρ and some tuple m with $\phi''_c(Ax,b_0) \sim^{k_c} \phi''_c(x+m,b)$, i.e.

$$\models \bigwedge_{A \in \mathrm{Gl}_{n_c}(F)\setminus G_c} \neg\rho_A(b_0),$$

[e] If T has no constants, use definable elements in a sort of T^{eq}.

where

$$\rho_A(y) = \exists z, y'\ \rho(y') \ \wedge\ \phi''_c(Ax, y) \sim^{k_c}_x \phi''_c(x+z, y').$$

Whence the formula

$$\sigma(y) = \bigwedge_{A \in G_c} \rho_A(y) \wedge \bigwedge_{A \in \mathrm{Gl}_{n_c}(F) \setminus G_c} \neg \rho_A(y)$$

is satisfied by b_0. An easy calculation shows

$$\models \forall y \ \left(\sigma(y) \to \Big(\bigwedge_{A \in G_c} \sigma^A(y) \wedge \bigwedge_{A \in \mathrm{Gl}_{n_c}(F) \setminus G_c} \neg \sigma^A(y)\Big)\right),$$

where:

$$\sigma^A(y) = \exists y'\ \sigma(y') \ \wedge\ \phi''_c(Ax, y) \sim^{k_c}_x \phi''_c(x, y').$$

Write now

$$\phi'''_c(x, y) = \phi''_c(x, y) \wedge \sigma(y).$$

It is clear that ϕ'''_c still encodes $\chi(x, d)$ and has all properties except possibly **C**(vii). For **C**(vi) assume $\phi''_c(x+m, b) \sim^{k_c} \phi''_c(x, b')$. b' satisfies ρ_A iff , $\phi''_c(Ax, b') \sim^{k_c} \phi''_c(x+m', b'')$ for some m' and some realization b'' of ρ, or, equivalently, $\phi''_c(Ax, b) \sim^{k_c} \phi''_c(x + (m' - A^{-1}m), b'')$. Therefore b satisfies ρ_A iff b' satisfies ρ_A. This implies that b satisfies σ_A iff b' satisfies σ_A. So **C**(vi) holds.

Now, **C**(vii) is satisfied by ϕ'''_c and G_c only in the weaker form that $\phi'''_c(Hx, b)$ is encoded by ϕ'''_c iff $H \in G_c$. By **C**(iv) we can define for each $A \in G_c$ a function $b \mapsto b^A$ such that

$$\phi'''_c(Ax, b) \sim^{k_c} \phi'''_c(x, b^A)$$

and set:

$$\phi_c(x, y) = \bigwedge_{A \in G_c} \phi'''_c(A^{-1}x, y^A).$$

Since $\phi_c(x, b) \sim^{k_c} \phi'''_c(x, b)$ only **C**(viia) needs to be check: Given $H \in G_c$,

$$\phi_c(Hx, b) \equiv \bigwedge_{A \in G_c} \phi'''_c(A^{-1}Hx, b^A) \equiv \bigwedge_{A \in G_c} \phi'''_c(A^{-1}x, b^{HA}) \equiv \phi_c(x, b^H).$$

□

Lemma 2.3. *There is a set C of codes with the following properties:*

***C**(viii)* *Every simple formula is encoded by a unique $c \in C$.*
***C**(ix)* *For all $c \in C$ and all $H \in \mathrm{Gl}_{n_c}(F)$ the code c^H is equivalent to some code in C.*[f]

[f] We will construct C so that every c^H is equivalent to some $c^{H'}$ which belongs to C. (We identify codes with equivalent formulas.)

Proof. Work inside an ω–saturated model M of T and enumerate all simple formulas χ_i, $i = 1, 2, \ldots$ with parameters in M. We need only show that all χ_i can be encoded in C. We construct C as an increasing union of finite sets $\emptyset = C_0 \subset C_1 \subset \cdots$. Assume that C_{i-1} is defined and closed under the action of $\mathrm{Gl}(F)$ in the sense of $\mathbf{C}$(ix). If χ_i can be encoded in C_{i-1}, we set $C_i = C_{i-1}$. Otherwise choose some code c' which encodes χ_i. Let $\rho(b)$ express, that $\phi_{c'}(x, b)$ cannot be encoded in C_{i-1} and define

$$\phi_c(x, y) = \phi_{c'}(x, y) \wedge \rho(y).$$

Then ϕ_c still encodes χ_i. Moreover ϕ_c determines again a code: only $\mathbf{C}$(vii) needs to be considered. So assume that $\models \rho(b)$ and let H be in $G_{c'}$. We need to show that $\models \rho(b^H)$. Otherwise $\phi_{c'}(Hx, b)$ can be encoded in C_{i-1}. Since C_{i-1} is closed under H^{-1}, also $\phi_{c'}(x, b)$ can be encoded in C_{i-1}, which is a contradiction.

Choose now a system of right representatives $A_1, \ldots, A_r$ of G_c in $\mathrm{Gl}_{n_c}(F)$ and set $C_i = C_{i-1} \cup \{c^{A_1}, \ldots, c^{A_r}\}$. $\square$

3. Difference sequences

As in the previous section, T denotes a countable strongly minimal extension of T_0 with the DMP.

Let us recall the following lemma, which will be useful to distinguish whether or not a formula determines a coset of a group, according to the independence among generic realizations.

Lemma 3.1. *Let $\phi(x)$ be a formula over B, of Morley degree 1, and e_0 and e_1 two generic B–independent realizations. If $H \in \mathrm{Gl}_n(F)$ and $e_0 \mathop{\smile\!\!\!\!\!|}_B e_0 - He_1$, then $\phi(x)$ is a coset formula and $H \in \mathrm{Inv}(\phi(x))$.*

Proof. It follows from

$$\mathrm{MR}(He_1/B, He_1 - e_0) = \mathrm{MR}(e_0/B, He_1 - e_0) = \mathrm{MR}(e_0/B) \geq \mathrm{MR}(He_1/B)$$

that e_0, He_1 and $He_1 - e_0$ are pairwise independent over B. By [9] e_0, He_1 and $He_1 - e_0$ are generic elements of B–definable cosets of a B–definable group G. Whence $\phi(x)$ is a coset formula and $HG = G$. $\square$

We fix now for every code c a number $m_c \geq 0$ such that for no $\phi_c(x, b)$ there is a Morley sequence (e_i) of length m_c and some b' from the same sort as b with $e_i \not\mathop{\smile\!\!\!\!\!|}_b b'$ for all i.

Theorem 3.2. *For every code c and any number $\mu > m_c$ there exists a parameter free formula $\Psi_c(x_0, \ldots, x_\mu)$, whose realizations are called* difference sequences *(of length μ), with the following properties.*

P(i) *If* $e'_0, \dots, e'_\mu, f$ *is a Morley sequence of* $\phi_c(x,b)$, *then* $e'_0 - f, \dots, e'_\mu - f$ *is a difference sequence.*[g]

P(ii) *For every difference sequence* $e_0, \dots, e_\mu$ *there is a unique* b *with* $\models \phi_c(e_i, b)$ *for all* i *(we call the* base *of the sequence). Furthermore,* b *is uniquely determined if* $\phi_c(e_i, b)$ *holds for at least* m_c *many* i*'s.*[h]

P(iii) *If* $e_0, \dots, e_\mu$ *is a difference sequence then so is*

$$e_0 - e_i, \dots, e_{i-1} - e_i, -e_i, e_{i+1} - e_i, \dots, e_\mu - e_i.$$

P(iv) *Let* $e_0, \dots, e_\mu$ *be a difference sequence with base* b. *We distinguish two cases: Suppose* c *is not a coset code:*

a) If e_i *is generic in* $\phi_c(x,b)$, *then* $e_i \not\downarrow_b e_i - He_j$ *for all* $H \in \mathrm{Gl}_{n_c}(F)$ *and* $i \neq j$.

Suppose c *is a coset code:*

b) $\phi_c(x,b)$ *is a group formula.*

c) $\Psi_c(e_0, \dots, e_{i-1}, e_i - e_j, e_{i+1}, \dots, e_\mu)$ *for all* $i \neq j$.[i]

d) $\Psi_c(e_0, \dots, e_{i-1}, He_i, e_{i+1}, \dots, e_\mu)$ *for all* $H \in \mathrm{Inv}(c)$.[i]

e) If e_i *is a generic realization of* $\phi_c(x,b)$, *then* $e_i \not\downarrow_b e_i - He_j$ *for all* $i \neq j$ *and* $H \in \mathrm{Gl}_{n_c}(F) \setminus \mathrm{Inv}(c)$.

P(v) *For all* $H \in G_c$

$$\Psi_c(x_0, \dots, x_\mu) \equiv \Psi_c(Hx_0, \dots, Hx_\mu).$$

The *derived* sequences of of (e_i) consist of all difference sequences obtained from (e_i) by iteration of the transformations described in **P**(iii). Note that all permutations can be derived and have the same base (by **P**(ii)). We will later use a more refined notation: if in the derivation process only indices $\leq \lambda$ are involved, then we call the resulting derivation a λ–*derivation.*

Proof. Consider the following property $\mathsf{DS}(e_0, \dots, e_\mu)$:

There is some b' *and a Morley sequence* $e'_0, \dots, e'_\mu, f'$ *of* $\phi_c(x, b')$ *such that* $e_i = e'_i - f'$.
This is clearly a partial type.

Claim: DS has all properties of Ψ_c.

Proof: Assume $e_i = e'_i - f'$ for a Morley sequence $(e'_i), f'$ of $\phi_c(x, b')$. Then (e_i) is a Morley sequence of $\phi_c(x + f', b')$ over b', f'. If $\phi_c(x + f', b') \sim \phi_c(x, b)$, then (e_i)

[g]In general b will not be the base of (e'_i) in the sense of **P**(ii).
[h]It follows that $b \in \mathrm{dcl}(e_{i_1}, \dots, e_{i_{m_c}})$ for all $0 \leq i_1 < \cdots i_{m_c} \leq \mu$.
[i]By **P**(ii) and $\mu > m_c$ this new sequence has also base b.

is a Morley sequence of $\phi_c(x,b)$.[j]

P(ii) Suppose $\models \phi_c(e_i,b'')$ for m_c–many i's. Then there exists such an i with $e_i \mathop{\smile\!\!\!\!\!\!|}_b b''$. Hence $\mathrm{MR}(\phi_c(x,b)\wedge\phi_c(x,b''))=k_c$ and therefore $b=b''$.

P(iii) Fix $i\in\{0,\ldots,\mu\}$ and note that $e'_0,\ldots,e'_{i-1},f',e'_{i+1},\ldots,e'_\mu,e'_i$ is again a Morley sequence for $\phi_c(x,b')$. Hence, the sequence

$$e'_0-e'_i,\ldots,e'_{i-1}-e'_i,f'-e'_i,e'_{i+1}-e'_i,\ldots,e'_\mu-e'_i=$$
$$e_0-e_i,\ldots,e_{i-1}-e_i,-e_i,e_{i+1}-e_i,\ldots,e_\mu-e_i$$

also satisfies DS.

P(iva) If c is not a coset code, then $\phi_c(x,b)$ is not a coset formula and the claim follows from Lemma 3.1.

P(ivb) If c is a coset code, then $\phi_c(x,b')$ is a coset formula. Since f' is a generic realization, $\phi_c(x,b)\sim\phi_c(x+f',b')$ is a group formula.

P(ivc) Extend the Morley sequence $e_0,\ldots,e_\mu$ of $\phi_c(x,b)$ by f. If $\phi_c(x,b)$ is a group formula, and $i\neq j$, then

$$e_0+f,\ldots,e_{i-1}+f,e_i-e_j+f,e_{i+1}+f,\ldots,e_\mu+f,f$$

is again a Morley sequence of $\phi_c(x,b)$. It follows that

$$e_0,\ldots,e_{i-1},e_i-e_j,e_{i+1},\ldots,e_\mu$$

realizes DS.

P(ivd) Choose f as above. If $H\in\mathrm{Inv}(c)$, then

$$e_0+f,\ldots,e_{i-1}+f,He_i+f,e_{i+1}+f,\ldots,e_\mu+f,f$$

is also a Morley sequence of $\phi_c(x,b)$. It follows that

$$e_0,\ldots,e_{i-1},He_i,e_{i+1},\ldots,e_\mu$$

realizes DS.

P(ive) Immediate from Lemma 3.1.

P(v) If $\phi_c(Hx,b')\equiv\phi_c(x,b'')$, then $He'_0,\ldots,He'_\mu,Hf$ is a Morley sequence of $\phi_c(x,b'')$ and $(He_i)=(He'_i-Hf)$ satisfies DS.

[j]Since b is canonical.

This proves the claim.

We will take for Ψ_c a finite part of DS. Property $\mathbf{P}$(i) will hold automatically. The Properties $\mathbf{P}$(ii), $\mathbf{P}$(iva), $\mathbf{P}$(ivb), $\mathbf{P}$(ive) can be described by countable disjunctions, which follow from DS. Therefore these properties follow from a sufficiently strong part of DS, which we call Ψ'_c.

Assume c to be a non–coset code. Write

$$V_i(x_0, \ldots, x_\mu) = (x_0 - x_i, \ldots, x_{i-1} - x_i, -x_i, x_{i+1} - x_i, \ldots, x_\mu - x_i)$$

and

$$V_H(x_0, \ldots, x_\mu) = (Hx_0, \ldots, Hx_\mu).$$

Let $\mathcal{V}$ be the finite group generated by $V_0, \ldots, V_\mu$ and V_H for $H \in G_c$. The formula

$$\Psi(\bar{x}) = \bigwedge_{V \in \mathcal{V}} \Psi'_c(V(\bar{x}))$$

has now properties $\mathbf{P}$(iii) and $\mathbf{P}$(v), and it still belongs to DS, since DS satisfies $\mathbf{P}$(iii) and $\mathbf{P}$(v).

If c is a coset code, consider the group generated by $\{V_H\}_{H \in G_c}$ and the operations described in $\mathbf{P}$(ivc) and $\mathbf{P}$(ivd), and define. Ψ_c analogously. It satisfy then $\mathbf{P}$(ivc) and $\mathbf{P}$(ivd) and $\mathbf{P}$(v), and therefore[k] also $\mathbf{P}$(iii). □

We choose an appropriate Ψ_c (depending on μ) for every code c in such a way that

$$\Psi_{c^H}(x_0, \ldots) = \Psi_c(Hx_0, \ldots).$$

For two codes c and c' to be *equivalent* we also impose that

$$\Psi_c \equiv \Psi_{c'}.$$

Corollary 3.3. *Lemma 2.3 remains true if Ψ_c is also taken into account.*

Proof. This follows from $\mathbf{P}$(v) and the proof of Lemma 2.3. □

4. The δ–function

Consider now two strongly minimal theories[l] T_1 and T_2 which intersect in T_0, the theory of infinite F–vector spaces.

By considering their morleyization, we may assume that :

[k]Note that $-1 \in \mathrm{Inv}(c)$.
[l]In this section neither countability nor the DMP will be required.

QE-Assumption . Both theories T_i have quantifier elimination. Their languages L_i are relational, except for the function symbols in L_0.

We may also assume that codes ϕ_c and formulas Ψ_c for T_1 and T_2 are quantifier free, as well as T_i–types $\mathrm{tp}_i(a/B)$. This assumption will be dropped only in section 9.

Let $\mathcal{K}$ be the class of all models A of $T_1^{\forall} \cup T_2^{\forall}$. So, A is an F–vector space, which occurs at the same time as a subspace of $\mathbb{C}_1$ and as a subspace of $\mathbb{C}_2$, where $\mathbb{C}_i$ the monster model of T_i.

For finite $A \in \mathcal{K}$, define

$$\delta(A) = \mathrm{tr}_1(A) + \mathrm{tr}_2(A) - \dim A.$$

We have that:

$$\delta(0) = 0 \tag{4.1}$$

$$\delta(\langle a \rangle) \leq 1 \tag{4.2}$$

$$\delta(A + B) + \delta(A \cap B) \leq \delta(A) + \delta(B) \tag{4.3}$$

Moreover, if $\dim(A/B)$ is finite[m], then we also set

$$\delta(A/B) = \mathrm{tr}_1(A/B) + \mathrm{tr}_2(A/B) - \dim A/B.$$

In case B is finite, we have that $\delta(A/B) = \delta(A + B) - \delta(B)$.

We say that B is *strong* in A, if $B \subset A$ and $\delta(A'/B) \geq 0$ for all finite $A' \subset A$ and denote this by

$$B \leq A.$$

A proper strong extension $B \leq A$ is *minimal*, if there is no A' properly contained between B and A such that $B \leq A' \leq A$.[n]

Let $B \subset A$ and a be in A. We call a *algebraic* over B, if a is algebraic over B either in the sense of T_1 or of T_2. We call A transcendental over B, if no $a \in A \setminus B$ is algebraic over B.

Lemma 4.1. *$B \leq A$ is minimal iff $\delta(A/A') < 0$ for all A' which lie properly between B and A.*

Proof. One direction is clear, since $A' \leq A$ implies $\delta(A/A') \geq 0$. Conversely, if $\delta(A/A') \geq 0$ for some A', we may assume that $\delta(A/A')$ is maximal. Then $A' \leq A$ and A is not minimal over B. □

[m] We do not assume $B \subset A$.
[n] Note that B is strong in all $A' \subset A$.

Lemma 4.2. *Let $B \leq A$ be a minimal extension. One of the three following holds:*

(I) $\delta(A/B) = 0$ *and* $A = \langle B, a\rangle$ *for some element* $a \in A \setminus B$ *algebraic over* B *(*algebraic minimal extension*)*

(II) $\delta(A/B) = 0$*, with* A *transcendental over* B*. (*prealgebraic minimal extension*)*

(III) $\delta(A/B) = 1$ *and* $A = \langle B, a\rangle$*, for some element* a *transcendental over* B *(*transcendental minimal extension*)*

Note that in the prealgebraic case $\dim A/B \geq 2$.

Proof. Minimality implies that there is no C, properly contained between B and A with $\delta(C/B) = 0$. We distinguish two cases.

$\delta(A/B) = 0$. If there is an $a \in A \backslash B$ which is algebraic over B, then $\delta(\langle B, a\rangle/B) = 0$. Therefore $\langle B, a\rangle = A$.

$\delta(A/B) > 0$. For each $a \in A\backslash B$ it follows that $\delta(\langle B, a\rangle/B) \neq 0$. Hence $\delta(\langle B, a\rangle/B) = 1$ and therefore $\langle B, a\rangle \leq A$. By minimality $\langle B, a\rangle = A$. □

We define the class $\mathcal{K}^0 \subset \mathcal{K}$ as

$$\mathcal{K}^0 = \{M \in \mathcal{K} \mid 0 \leq M\}.$$

It is easy to see that $\mathcal{K}^0$ can be axiomatized by a set of universal $L_1 \cup L_2$–sentences. The following results are also easy.

Lemma 4.3. *Fix* M *in* K^0 *and define*

$$\mathrm{d}(A) = \min_{A \subset A' \subset M} \delta(A')$$

for all finite subspaces A *of* M*. Then* d *is (on finite subspaces) the dimension function of a pregeometry i.e.,* d *satisfies* (4.1)*,* (4.2)*,* (4.3) *and*

$$\mathrm{d}(A) \geq 0 \tag{4.4}$$

$$A \subset B \Rightarrow \mathrm{d}(A) \leq \mathrm{d}(B). \tag{4.5}$$

□

Lemma 4.4. *Let* M *be in* $\mathcal{K}^0$ *and* A *a finite subspace. Let* A' *be an extension of* A*, minimal with* $\delta(A') = \mathrm{d}(A)$*. Then* A' *is the smallest strong subspace of* M *which contains* A*. We denote it by* $\mathrm{cl}(A)$*.* □

We call $\mathrm{cl}(A)$ the *closure* of A.

For arbitrary subsets X of M we will use the notation $\delta(X) = \delta\langle X\rangle$ and $\mathrm{d}(X) = \mathrm{d}\langle X\rangle$.

Note that $\delta(A) \leq \dim(A)$.

5. Prealgebraic codes

From now on, T_1 and T_2 are two countable strongly minimal extensions of T_0 with the DMP. We assume the **QE-Assumption** of section 4, as in the next three sections 6, 7 and 8.

Choose for each T_i a set C_i of codes as in Corollary 3.3. A *prealgebraic code* $c = (c_1, c_2)$ consists of two codes $c_1 \in C_1$ and $c_2 \in C_2$ with the following properties:

- $n_c := n_{c_1} = n_{c_2} = k_{c_1} + k_{c_2}$
- For all proper, non–zero subspaces U of F^{n_c}

$$k_{c_1,U} + k_{c_2,U} + \dim U < n_c. \tag{5.1}$$

Set $m_c = \max(m_{c_1}, m_{c_2})$. Note that simplicity of the $\phi_{c_i}(x, b)$ implies that $n_c \geq 2$. Note also that for every $H \in \mathrm{Gl}_{n_c}(F)$

$$c^H = (c_1^H, c_2^H)$$

is a prealgebraic code.

Notation

Unless otherwise stated, *independence* ($a \downarrow_b c$) means independent both in the sense of T_1 and T_2. If c is a prealgebraic code, a (*generic*) realization of $\phi_c(x, b)$ is a (generic) realization of both $\phi_{c_1}(x, b_1)$ and $\phi_{c_2}(x, b_2)$. A *Morley sequence* of $\phi_c(x, b)$ is a Morley sequence for both $\phi_{c_1}(x, b_1)$ and $\phi_{c_2}(x, b_2)$. Similarly, for a set X of real elements, one defines *X–generic realization* of $\phi_c(x, b)$ and *Morley sequence of $\phi_c(x, b)$ over X*. A *difference sequence* for c with basis $b = (b_1, b_2)$ is a difference sequence for c_i with basis b_i for each $i = 1, 2$.

We say c is a *coset code* if c_1 and c_2 are. We define then $\mathrm{Inv}(c) = \mathrm{Inv}(c_1) \cap \mathrm{Inv}(c_2)$.

T_1^{eq} and T_2^{eq} have only the home sort in common. So $b \in \mathrm{dcl}^{\mathrm{eq}}(A)$ (resp. $\mathrm{acl}^{\mathrm{eq}}(A)$) means that b is a pair consisting of an element in ${\mathrm{dcl}^{\mathrm{eq}}}_1(A)$ (resp. ${\mathrm{acl}^{\mathrm{eq}}}_1(A)$) and an element in ${\mathrm{dcl}^{\mathrm{eq}}}_2(A)$ (resp. ${\mathrm{acl}^{\mathrm{eq}}}_2(A)$). If M is a model of $T_1 \cup T_2$, then M^{eq} consists of imaginary elements in the sense of T_1 and in the sense of T_2.

Lemma 5.1. *Let $B \leq A$ be a prealgebraic minimal extension and $a = (a_1, \ldots, a_n)$ a basis for A over B. Then there is a prealgebraic code c and $b \in \mathrm{acl}^{\mathrm{eq}}(B)$ such that a is a generic realization of $\phi_c(x, b)$.*

Proof. Fix $i \in \{1, 2\}$. Choose $d_i \in {\mathrm{acl}^{\mathrm{eq}}}_i(B)$ such that $\mathrm{tp}_i(a/Bd_i)$ is stationary. Since A/B is transcendental, we have $\dim(a/\mathrm{acl}_i(B)) = n$. So we can find an L_i–formula $\chi_i(x) \in \mathrm{tp}_i(a/Bd_i)$ of Morley rank $k_i = \mathrm{MR}_i(a/Bd_i)$. Since A/B is transcendental, $\chi(x)$ is simple. By 2.3 there is a T_i–code $c_i \in C_i$ and $b_i \in {\mathrm{dcl}^{\mathrm{eq}}}_i(Bd_i)$ with $\chi_i(x) \sim^{k_i} \phi_{c_i}(x, b_i)$.

Set $c = (c_1, c_2)$ and $b = (b_1, b_2)$. It follows from

$$k_1 + k_2 - n = \mathrm{tr}_1(a/B) + \mathrm{tr}_2(a/B) - \dim(A/B) = \delta(A/B) = 0$$

that $n_c = k_{c_1} + k_{c_2}$. Inequality (5.1) follows from Lemma 4.1:

$$\begin{aligned} k_{c_1,U} + k_{c_2,U} - (n - \dim U) =& \mathrm{tr}_1(a/b, Ua) + \mathrm{tr}_2(a/b, Ua) - \dim(F^n/U) \\ =& \delta(A/B + Ua) < 0. \end{aligned}$$

□

Lemma 5.2. *Let $B \in \mathcal{K}$, $b \in \mathrm{acl}^{\mathrm{eq}}(B)$, c be a prealgebraic code, and a a B–generic realization of $\phi_c(x, b)$. Then $\langle B, a\rangle$ is a prealgebraic minimal extension of B.*

Note that the isomorphism type of a over B is uniquely determined.

Proof. The proof follows from the above considerations. Note that subspaces of A containing B are of the form $B + Ua$ for some subspace U of F^{n_c}. □

Lemma 5.3. *Let $B \subset A$ be in $\mathcal{K}$, c a prealgebraic code, b in $\mathrm{acl}^{\mathrm{eq}}(B)$ and $a \in A$ a realization of $\phi_c(x, b)$ in A not completely contained in B. Then*

1. $\delta(a/B) \leq 0$.
2. *If $\delta(a/B) = 0$, then a is a B–generic realization of $\phi_c(x, b)$.*

Proof. Let $Ua = \langle a\rangle \cap B$. Let $Ua = \langle a\rangle \cap B$. Since a is not contained in B, it follows that U is a proper subspace of F^{n_c}. Therefore

$$\delta(a/B) = \mathrm{tr}_1(a/B) + \mathrm{tr}_2(a/B) - (n - \dim U) \leq k_{c_1,U} + k_{c_2,U} + \dim U - n.$$

If $U \neq 0$ the right hand side is negative. If $U = 0$, we have

$$\delta(a/B) = \mathrm{tr}_1(a/B) + \mathrm{tr}_2(a/B) - n \leq k_{c_1} + k_{c_2} - n = 0.$$

So $\delta(a/B) = 0$ implies $\mathrm{tr}_i(a/B) = k_{c_i}$. □

Lemma 5.4. *Let $M \leq N$ be a strong extension of elements in $\mathcal{K}$. Given a prealgebraic code c, and natural numbers ε and r, there is some $\lambda = \lambda(\varepsilon, r, c) \geq 0$ such that for every difference sequence $e_0, \ldots, e_\mu$ in N, with basis b, and $\lambda \leq \mu$, either*

- *the basis of some λ–derived sequence of $e_0, \ldots, e_\mu$ lies in $\mathrm{dcl}^{\mathrm{eq}}(M)$,*

or

- *for every subset A of M' with $\dim A \leq \varepsilon$ the sequence $e_0, \ldots, e_\mu$ contains a Morley sequence of $\phi_c(x, b)$ over M, A of length r.*

Proof. By adding $e_0, \ldots, e_{m_c-1}$ to A, we may assume that $b \in \mathrm{dcl}^{\mathrm{eq}}(M \cup A)$. If at least (m_c+1) many of the e_i lie in the same class of N^{n_c}/M^{n_c}, we subtract one of these elements from the others and obtain a derived sequence with m_c many elements in M, which then has a base in $\mathrm{dcl}^{\mathrm{eq}}(M)$. Therefore, we may assume that each class of N^{n_c}/M^{n_c} contains at most m_c many e_i's.

Fix an A of dimension ε and set

$$d = \dim(e_0, \ldots, e_\mu / \langle M, A\rangle).$$

Then $\dim(e_0, \ldots, e_\mu/M) \le d + \varepsilon$. Thus by our assumption

$$\mu + 1 \le m_c\, |F|^{(d+\varepsilon)n_c}.$$

Consider the following sets of indices.

$$X_1 = \{i \le \mu \,|\, e_i \text{ generic over } M, A, e_0, \ldots, e_{i-1}\}$$
$$X_2 = \{i \le \mu \,|\, i \notin X_1 \,\wedge\, \dim(e_i/M, A, e_0, \ldots, e_{i-1}) > 0\}$$

It is clear that

$$d \le (|X_1| + |X_2|)\, n_c.$$

With the notation $\delta(i) = \delta(e_i/M, A, e_0, \ldots, e_{i-1})$, Lemma 5.3 implies that $\delta(i) < 0$ if $x \in X_2$, and $\delta(i) = 0$ otherwise. Since $M \le N$ we have

$$0 \le \delta(A, e_0, \ldots, e_\mu/M) = \delta(A/M) + \sum_{i=1}^{\mu} \delta(i) \le \varepsilon - |X_2|.$$

If we put the three inequalities together, we obtain

$$\mu + 1 \le m_c\, |F|^{(|X_1| n_c + \varepsilon\, n_c + \varepsilon) n_c}.$$

If μ is large enough, $|X_1| \ge r$ and $(e_i)_{i \in X_1}$ is our Morley sequence. □

6. The class $\mathcal{K}^\mu$

Choose now a function μ^* which assigns to every prealgebraic code c a natural number $\mu^*(c)$. We assume that

M(i) for every m and n there are only finitely many c with $\mu^*(c) = m$ and $n_c = n$.

The existence of such a function is ensured by the countability of C. Then we choose a function μ from prealgebraic codes to natural numbers such that

M(ii) $\mu(c) \ge \lambda(n_c, 1, c) + 1$
M(iii) $\mu(c) \ge \lambda(0, \lambda(0, m_c + 1, c) + 1, c)$
M(iv) $\mu(c) \ge \lambda(0, \mu^*(c) + 1, c)$
M(v) $\mu(c) = \mu(d)$, if c is equivalent to some d^H.[o]

From now on, all difference sequences of c will have fixed length $\mu(c) + 1$. Condition **M**(v) ensures that, if c is equivalent to d^H, and (e_i) is a difference sequence for d, then (He_i) is a difference sequence for c.

[o]Note that every d^H can be equivalent to only one prealgebraic c.

The *class* $\mathcal{K}^\mu$ consists of all elements A of $\mathcal{K}^0$ which do not contain a difference sequence for any prealgebraic code.

Lemma 6.1. *Let $B \leq M \in \mathcal{K}^\mu$ and A/B prealgebraic minimal. Then there are only finitely many B–isomorphic copies of A strong in M.*

Proof. Let a be a basis of A/B. Choose $d \in \mathrm{acl}^{\mathrm{eq}}(B)$ such that the types $\mathrm{tp}_i(a/Bd_i)$ are stationary. It suffices to show that for all such d the partial type $\mathrm{tp}_1(a/Bd_1) \cup \mathrm{tp}_2(a/Bd_2)$ has only finitely many realizations in M. For this we choose a prealgebraic code c and $b \in \mathrm{acl}^{\mathrm{eq}}(B)$ with $\models \phi_c(a,b)$ by 5.1. We now show that $\phi_c(x,b)$ has only finitely many realizations in M. If not, there is an infinite sequence $e_0,\ldots$ of realizations such that e_i is not contained in $\langle B, e_0, \ldots, e_{i-1}\rangle$ (since the latter set is finite). Strongness of B in M yields that e_0 is a B–generic realization by 5.3. From $\delta(e_0/B) = 0$ we conclude that $\langle B, e_0\rangle \leq M$. If we proceed in this way, we see that $e_0,\ldots$ is a Morley sequence of $\phi_c(x,b)$ over B. Now $\mathbf{P}$(i) yields that $e_1 - e_0, \ldots, e_{\mu(c)+1} - e_0$ is a difference sequence of c. Contradiction. □

Corollary 6.2. *Let $B \leq M \in \mathcal{K}^\mu$ and $B \subset A$ finite with $\delta(A/B) = 0$. Then there are only finitely many $B \leq A' \subset M$, which are isomorphic to A over B.*

Note that automatically $A' \leq M$.

Proof. Decompose the extension A/B into a sequence of minimal extensions. □

Corollary 6.3. *Let X be a finite subset of $M \in \mathcal{K}^\mu$. Then the* d*–closure of X:*

$$\mathrm{cl_d}(X) = \{x \in M \mid \mathrm{d}(Xx) = \mathrm{d}(X)\}$$

is at most countable.

Proof. Note that $\mathrm{cl_d}(X)$ is the union of all $A' \subset M$ with $\mathrm{cl}(X) \subset A'$ and $\delta(A'/\mathrm{cl}(X)) = 0$. □

Lemma 6.4. *Let $M \in \mathcal{K}^\mu$, $M \leq M'$ a minimal extension and (e_i) a difference sequence for a prealgebraic code c with base $b \in \mathrm{acl}^{\mathrm{eq}}(M)$. Then c has a difference sequence (e'_i) with the same base b such that M contains $e'_0, \ldots, e'_{\mu(c)-1}$.*
In particular, $e'_{\mu(c)}$ is an M–generic realization of $\phi_c(b)$, which generates M' over M as a vector space. Also b must be in $\mathrm{dcl}^{\mathrm{eq}}(M)$.

Proof. Let e_i be any element which does not lie in M. By strongness of M in M' and Lemma 5.3, it follows that e_i is an M–generic realization of $\phi_c(x,b)$. We have $\delta(\langle M, e_i\rangle/M) = 0$ and whence $\langle M, e_i\rangle \leq M'$. By minimality $\langle M, e_i\rangle = M'$.

After permutation we may assume that $e_0, \ldots, e_{\nu-1}$ are in M and $e_\nu, \ldots, e_{\mu(c)}$ are not. Since $M \in \mathcal{K}^\mu$, it follows that $\nu \leq \mu(c)$. As above, for $i \geq \nu$, e_i is an M–generic realization of $\phi_c(x,b)$ which generates M'/M, so $e_i - H_i e_{\mu(c)} \in M$ for some $H_i \in \mathrm{Gl}_{n_c}(F)$. Therefore $e_i \mathop{\smile\!\!\!\!\!\;|}_b e_i - H_i e_{\mu(c)}$.

If c is a not coset code, it follows from $\mathbf{P}$(iva) that $i = \mu(c)$. So we have $\nu = \mu(c)$.

Suppose that c is a coset code. If $\nu \leq i < \mu(c)$, then $H_i \in \mathrm{Inv}(c)$ by $\mathbf{P}$(ive). By $\mathbf{P}$(ivc) and $\mathbf{P}$(ivd) the difference sequence

$$e_0, \ldots, e_{\nu-1}, e_\nu - H_\nu e_{\mu(c)}, \ldots, e_{\mu(c)-1} - H_{\mu(c)-1} e_{\mu(c)}, e_{\mu(c)}$$

is as stated in the claim. Note that the above sequence has same base b. □

7. Amalgamation

Theorem 7.1. *$\mathcal{K}^\mu$ (and therefore also the class of all finite elements of $\mathcal{K}^\mu$) has the amalgamation property with respect to strong embeddings.*

Proof. Consider $B \leq M$ and $B \leq A$ in $\mathcal{K}^\mu$. We want to find a strong extension $M' \in \mathcal{K}^\mu$ of M and a $B \leq A' \leq M'$ isomorphic to A over B. We may assume that A/B and M/B are minimal. We will show that either some "free amalgam" M' of M and A is in $\mathcal{K}^\mu$ or that M and A are isomorphic over B.

Case 1: A/B is algebraic. Then $A = \langle B, a\rangle$ for an element a which is (e.g.) algebraic over B in the sense of T_1 and transcendental over B in the sense of T_2. There are two (non exclusive) subcases.

Subcase 1.1: $\mathrm{tp}_1(a/B)$ is realized in M. Choose some realization a' in M. Hence, a'/B is transcendental in the sense of T_2 and $a' \mapsto a$ defines an isomorphism between $M = \langle B, a'\rangle$ and A over B.

Subcase 1.2: There is some $a' \notin M$, which realizes $\mathrm{tp}_1(a/B)$ (in the sense of T_1). Define the structure $M' = \langle M, a\rangle$ by setting a to have the same T_1–type over M as a' and being transcendental over M in the sense of T_2 i.e. M' is a *free amalgam* of A and M over B in the sense that M are A are independent over B and linearly independent[p] over B. It is easy to see that, in free amalgams, $M \leq M'$ and $A \leq M'$. By Lemma 7.2 below, M' belongs to $\mathcal{K}^\mu$.

Case 2: A/B is transcendental. We may assume that $M \cap A = B$. Since A/B is transcendental, we find $M' = M + A$ in $\mathcal{K}$, such that M and A are independent over B. So M' is a free amalgam of M and A, and M' is a minimal extension of M and of A. If $M' \in \mathcal{K}^\mu$, we are done. Otherwise, 7.3 shows that, by symmetry, we may assume that M' contains a difference sequence (e_i) of a prealgebraic code c with base $b \in \mathrm{acl}^{\mathrm{eq}}(M)$. Also by Lemma 7.2 , $\dim(M'/M) > 1$ and A/B is prealgebraic. By minimality and Lemma 6.4, we may also assume that $e_0, \ldots, e_{\mu(c)-1}$ are in M and $e_{\mu(c)}$ is an M–generic realization of $\phi_c(x, b)$, which generates M' over M. Write $e_{\mu(c)} = m + a$ for $m \in M$ and $a \in A$. Therefore $\delta(a/B) = \delta(a/M) = \delta(e_{\mu(c)}/M) = 0$.

[p] I.e. $\dim(A/B) = \dim(A/M)$.

Whence a generates A over B. We apply now Lemma 5.4 and $\mathbf{M}$(ii) to the extension (M'/A) and m and obtain two subcases:

Subcase 2.1: There is a $(\mu(c)-1)$–derived difference sequence (e'_i) with basis $b' \in \mathrm{dcl}^{\mathrm{eq}}(A)$. Since $e'_i \in M$ for $i \leq \mu(c)-1$, the base b' is in $\mathrm{dcl}^{\mathrm{eq}}(M) \cap \mathrm{dcl}^{\mathrm{eq}}(A) \subset \mathrm{acl}^{\mathrm{eq}}(B)$. Hence $e'_{\mu(c)}$ is an M–generic realization of $\phi_c(x,b')$ which generates M' over M. Again there are two cases.

Subsubcase 2.1.1: $e'_{\mu(c)} \in A$. Since $A \in \mathcal{K}^\mu$, there is an $e'_i \in M$ not in A. By minimality e'_i generates M over B and $e'_{\mu(c)} \mapsto e'_i$ defines a B-isomorphism between A and M.

Subsubcase 2.1.2: $e'_{\mu(c)} \notin A$. Then $e'_{\mu(c)}$ is an A–generic realization of $\phi_c(x,b')$. Write $e'_{\mu(c)} = m' + a'$ for $m' \in M$ and $a' \in A$. Since $e'_{\mu(c)}$, m' and a' are pairwise independent over b', then, for $i = 1, 2$, $\phi_{c_i}(x, b'_i)$ is a coset formula by [9] and whence a group formula by $\mathbf{C}$(v) and $\mathbf{P}$(ivb). It follows that $-m'$ and a' are generics of the same Bb'_i–definable coset of a Bb'_i–definable connected group. Thus they have the same type over B. As above m' generates M over B and a' generates A over B. So the map $a' \mapsto -m'$ defines an isomorphism between A and M over B.

Subcase 2.2: $e_0, \ldots, e_{\mu(c)-1}$ contains a B, m–generic realization of $\phi_c(x,b)$, say e_0. For $i = 1, 2$, e_0 and $e_{\mu(c)}$ have the same T_i–type over B, m, b_i. Whence $e_0 - m$ and a have the same T_i–type over B, m, b_i, a forteriori over B. Whence $a \mapsto e_0 - m$ defines a B–isomorphism between A and M.

□

Lemma 7.2. *Let $M \in \mathcal{K}^\mu$, $M \leq M'$ and $\dim(M'/M) = 1$. Then, $M' \in \mathcal{K}^\mu$.*

Proof. Assume $M' \notin \mathcal{K}^\mu$ and (e_i) is a difference sequence in M' for a prealgebraic code c with base b witnessing this fact. Since $\dim(M'/M) = 1$ and $n_c \geq 2$, no e_i is an M–generic realization. By the choice of $\mu(c)$ and Lemma 5.4 we may assume that $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$. By Lemma 5.3 we conclude that all e_i lie in M. Contradiction. □

Lemma 7.3. *Let M' be a free amalgam of M and A over B and (e_i) a difference sequence in M'. Then there is a derived sequence with base in* $\mathrm{acl}^{\mathrm{eq}}(M)$ *or a derived sequence with base in* $\mathrm{acl}^{\mathrm{eq}}(A)$.

Actually we find the base in $\mathrm{dcl}^{\mathrm{eq}}(M)$, $\mathrm{dcl}^{\mathrm{eq}}(A)$ or $\mathrm{acl}^{\mathrm{eq}}(B)$.

Proof. Let b be the base of $s = (e_i)$. If no derivation has a base in $\mathrm{dcl}^{\mathrm{eq}}(M)$, Lemma 5.4 and $\mathbf{M}$(iii) yield a subsequence s' of length $\lambda(0, m_c + 1, c) + 1$ which is a Morley sequence of $\phi_c(x,b)$ over M. Again by 5.4, applied to M'/A, if there is no derivation with base in $\mathrm{dcl}^{\mathrm{eq}}(A)$, there is a subsequence s'' of s' of length

$m_c + 1$, say $e_0, \ldots, e_{m_c}$, which is also a Morley sequence of $\phi_c(x, b)$ over A. Set $E = \{e_0, \ldots, e_{m_c - 1}\}$. Hence, $b \in \mathrm{dcl}^{\mathrm{eq}}(E)$ and

$$e_{m_c} \mathop{\downarrow}_{b} M, E \ , \qquad\qquad e_{m_c} \mathop{\downarrow}_{b} A, E \ .$$

Write every $e \in E$ as the sum of an element of M and an element of A. Define E_M to be the set of all elements in M which occur as summands, and likewise E_A, and set $E' = E_M \cup E_A$. Then also $b \in \mathrm{dcl}^{\mathrm{eq}}(E')$ and, since E' and E are interdefinable over M and as well as over A, we have

$$e_{m_c} \mathop{\downarrow}_{b} M, E' \ , \qquad\qquad e_{m_c} \mathop{\downarrow}_{b} A, E' \ ,$$

which implies

$$e_{m_c} \mathop{\downarrow}_{B,E'} M \ , \qquad\qquad e_{m_c} \mathop{\downarrow}_{B,E'} A \ .$$

Furthermore

$$M \mathop{\downarrow}_{B,E'} A \ .$$

Write $e_{m_c} = m + a$ for $m \in M$ and $a \in A$. Then e_{m_c}, m, and a are pairwise independent over B, E'. Fix $i = 1, 2$. Then $\phi_{c_i}(x, b_i)$ is a group formula for a definable group G_i and b_i is the canonical parameter of G_i. Moreover, a is a generic element of an $\mathrm{acl^{eq}}_i(B, E')$–definable coset of G_i and b_i is definable from the canonical base of $p = \mathrm{tp}_i(a/\,\mathrm{acl^{eq}}_i(B, E'))$. Note that $a \downarrow_{B,E_A} E'$. So the canonical base of p is in $\mathrm{acl^{eq}}_i(A)$, hence $b \in \mathrm{acl}^{\mathrm{eq}}(A)$. By symmetry $b \in \mathrm{acl}^{\mathrm{eq}}(M)$, and since M and A are independent over B, this yields $b \in \mathrm{acl}^{\mathrm{eq}}(B)$. □

We call $M \in \mathcal{K}^\mu$ *rich*, if for all finite $B \le M$ and all finite $B \le A \in \mathcal{K}^\mu$ there is an $B \le A' \le M$, which is B–isomorphic to A. We will show in the next section (8.3) that rich structures are models of $T_1 \cup T_2$.

Corollary 7.4. *There is a unique countable rich structure K^μ. All rich structures are $(L_1 \cup L_2)_{\infty,\omega}$–equivalent.* □

8. The theory T^μ

Lemma 8.1. *Let $M \in \mathcal{K}^\mu$, $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$, c a prealgebraic code and M' a prealgebraic minimal extension of M, generated by an M–generic realization a of $\phi_c(x, b)$ as in 5.2. If M' does not belong to $\mathcal{K}^\mu$, one of the following is true.*

(a) M' contains a difference sequence (e_i) for c whose elements but one lie in M.
(b) M' contains a difference sequence for a prealgebraic code c' with base b' which contains a Morley sequence of $\phi_{c'}(x, b')$ over M of length $\mu^*(c') + 1$.

Proof. If $M' \notin \mathcal{K}^\mu$ there is a difference sequence (e'_i) in M' for a prealgebraic code c' with base b'. If case (b) does not occur, by $\mathbf{M}$(iv) and Lemma 5.4 we may assume that $b' \in \mathrm{dcl}^{\mathrm{eq}}(M)$ and furthermore that (e'_i) is as in Lemma 6.4. So $n_{c'} = n_c = \dim(M'/M)$ and we have $He'_{\mu(c')} + m = a$ for some $H \in \mathrm{Gl}_{n_c}(F)$ and $m \in M$. By $\mathbf{C}$(vi) there is a $d \in \mathrm{dcl}^{\mathrm{eq}}(M)$ with $\phi_{c_i}(x + m, b_i) \sim^{k_{c_i}} \phi_{c_i}(x, d_i)$ $(i = 1, 2)$. Then $He'_{\mu(c')}$ is an M–generic realization of $\phi_c(x, d)$, i.e. $e'_{\mu(c')}$ is an M–generic realization of $\phi_{c^H}(x, d)$. By $\mathbf{C}$(ix) there is a prealgebraic code c'' which is equivalent to c^H. We have $\phi_{c^H}(x, d) \equiv \phi_{c''}(x, b'')$ for some $b'' \in \mathrm{dcl}^{\mathrm{eq}}(M)$. By $\mathbf{C}$(viii) and $\mathbf{C}$(iv) we conclude $c'' = c'$ and $b'' = b'$.

Finally note that (e'_i) is a difference sequence for c^H. So $(e_i) = (He'_i)$ is the desired difference sequence for c as in (a). □

Corollary 8.2.

1. *Let c be a prealgebraic code. That a structure $M \in \mathcal{K}$ contains no difference sequence for c can by expressed by a single sentence α_c.*
2. *Let c be a prealgebraic code, $M \in \mathcal{K}^\mu$ a model of $T_1 \cup T_2$. That no extension of M in $\mathcal{K}^\mu$ is generated by a generic realization of some $\phi_c(x, b)$ with $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$ can be expressed by an sentence β_c.*
3. *Let $M \in \mathcal{K}^\mu$ be a model of $T_1 \cup T_2$. That M has no prealgebraic minimal extension in $\mathcal{K}^\mu$ can be expressed by a set of sentences.*

Proof. 1. Let $\alpha_c = \neg\exists x_0, \ldots, x_{\mu(c)} \big(\Psi_{c_1}(x_0, \ldots, x_{\mu(c)}) \wedge \Psi_{c_2}(x_0, \ldots, x_{\mu(c)})\big)$.

2. Fix $i = 1, 2$ and let M be a submodel of $\mathbb{C}_i$. Let $m \in M$, $\phi(x, m)$ an L_i–formula of Morley rank k and degree 1, and $a \in \mathbb{C}_i$ be an M–generic realization of $\phi(x, m)$. There is a uniform way to translate a quantifier free property $\psi(a, m)$ of a, m into a quantifier free property $\psi^*(m)$ of m: Set

$$\psi^*(y) = \mathrm{MR}_x\big(\phi(x, y) \wedge \psi(x, y)\big) \doteq k$$

This shows that, if $M \in \mathcal{K}$ and a is an M–generic realization of $\phi_c(x, b)$, then any $L_1 \cup L_2$–sentence α about $\langle M, a\rangle$ can be translated into an $L_1 \cup L_2$–sentence $\alpha^c(b)$ about M.

Now there is only a finite set C_c of codes c' which can occur in (b) of 8.1 since $(\mu^*(c') + 1)n_{c'} \leq \dim(M'/M) = n_c$. So set

$$\beta_c = \forall y_c\ \alpha^c_c(y_c) \wedge \bigwedge_{c' \in C_c} \forall y_{c'}\ \alpha^c_{c'}(y_{c'}).$$

The variables $y_c, y_{c'}$ are understood to range over appropriate sorts of M^{eq}.

3. This follows from 2. and Lemma 5.1. □

We now introduce the theory T^μ described by the following axioms, which by the above are elementarily expressible.

Axioms of T^μ M is model of T^μ iff

(i) $M \in \mathcal{K}^\mu$
(ii) M is a model of $T_1 \cup T_2$
(iii) No prealgebraic minimal extension of M belongs to $\mathcal{K}^\mu$.

Theorem 8.3. *Rich structures are exactly the* ω*–saturated models of* T^μ*.*

Proof. Let M be an ω–saturated model of T^μ. In order to show that M is rich, we consider a finite strong subspace B of M and a minimal extension $A \in \mathcal{K}^\mu$ of B. We want to find a copy $B \leq A' \leq M$ of A/B.

case (I): A/B is algebraic. Since M is a model of $T_1 \cup T_2$, it has no proper algebraic extension in $\mathcal{K}$. So A' exists by 7.1.

case (II): A/B is prealgebraic. Since M has no prealgebraic minimal extension, 7.1 forces to obtain a copy of A in M.

case (III): A/B is transcendental. Since A/B is generated by a transcendental element we have to find an $a' \in M$ which is transcendental over B such that $\langle B, a' \rangle \leq M$. Since this equivalent to realize a partial type, and since M is ω–saturated, it suffices to find a' in an elementary extension M' of M. Choose M' uncountable. By 6.3 $\mathrm{cl_d}(B) \leq M'$ is countable. For every $a' \in M' \setminus \mathrm{cl_d}(B)$, we have $\delta(a'/B) = 1$ and $\langle B, a' \rangle \leq M'$.

Assume now that M is rich. We show first that M is a model of T^μ.

Axiom (ii): By Lemma 7.2 there are elements in $\mathcal{K}^\mu$ of arbitrary finite dimension. So M is infinite and we need only show that M is algebraically closed in the sense of T_1 and of T_2.

Let a be an element in $\mathrm{acl}_1(M)$ and transcendental over M in the sense of T_2. Therefore, a is 1–algebraic over a finite subset B of M. We may assume that $B \leq M$. Since (by Lemma 7.2) $B \leq \langle B, a \rangle \in \mathcal{K}^\mu$, there is a copy of a over B in M. This implies that M acl_1–closed. Likewise M is algebraically closed in the sense of T_2.

Axiom (iii): Let M' be a prealgebraic minimal extension generated by an M–generic realization a of $\phi_c(x, b)$. Assume $M' \in \mathcal{K}^\mu$. Choose a finite subspace $C_0 \leq M$ with $b \in \mathrm{dcl}^{\mathrm{eq}}(C_0)$. Then $C_0 \leq \langle C_0, a \rangle$. Since M is rich, M contains a copy e_0 of a over C_0 with $C_1 = \langle C_0, e_0 \rangle \leq M$. Continuing this way we obtain an infinite Morley sequence

$e_0, e_1, \ldots$ of $\phi_c(x, b)$. By $\mathbf{P}$(i), $e_1 - e_0, \ldots, e_{\mu(c)+1} - e_0$ is a difference sequence for c.

Choose an ω–saturated $M' \equiv M$. By the above we know that M' is rich. Since $M' \equiv_{\infty,\omega} M$, this implies that M is ω–saturated. □

9. Proof of the Theorem

In this section quantifier elimination for T_1 and T_2 will no longer be required. Hence, replace in the class $\mathcal{K}$ embeddings by elementary maps in the sense of T_1 and in the sense of T_2, which we call *bi-elementary* maps.

Corollary 9.1. *T^μ is complete. Two tuples a and a' in two models M and M' have the same type iff there is bi-elementary bijection*

$$f : \mathrm{cl}(a) \to \mathrm{cl}(a')$$

which maps a to a'.

Proof. K^μ is a model of T^μ. So is T^μ consistent. Let M be any model of T^μ. By theorem 8.3 there is a rich $M' \equiv M$. So $M' \equiv_{\infty,\omega} K^\mu$, which proves completeness.

To prove the second statement choose ω–saturated elementary extensions $M \prec N$ and $M' \prec N'$. It is easy to see[q] that $M \leq N$ and $M' \leq N'$, so "cl" does not increase.

Since M' and N' are rich, f is even ∞, ω–elementary.

For the converse suppose that a and a' have the same type. There is a bi-elementary map $f : \mathrm{cl}(a) \to M'$ which maps a onto a'. We write A' for $f(\mathrm{cl}(a))$. Then $\mathrm{d}(a) = \delta(\mathrm{cl}(a)) = \delta(A')$. It follows $\mathrm{d}(a') \leq \mathrm{d}(a)$ and $\mathrm{d}(a') = \mathrm{d}(a)$ by symmetry. A' has, like $\mathrm{cl}(a)$, no proper subset A'' which contains a' and with $\delta(A'') = \mathrm{d}(a')$. This implies $A' = \mathrm{cl}(a')$. □

Theorem 9.2. *T^μ is strongly–minimal and* d *is the dimension function of the natural pregeometry on models of T^μ, i.e.*

$$\mathrm{MR}(a/B) = \mathrm{d}(a/B).$$

Proof. Let a be a single element. Types $\mathrm{tp}(a/B)$ with $\mathrm{d}(a/B) = 0$ are algebraic by Corollary 6.2. It follows from 9.1, that there is only one type with $\mathrm{d}(a/B) = 1$.[r]

[q] If $M \not\leq N$, there is a tuple $a \in N$ with $\delta(a/M) < 0$. We find a finite $B \leq M$ with $\delta(a/B) < 0$. This is witnessed by the truth of an $L_1 \cup L_2$–formula $\phi(a, \bar{b})$. However, $\phi(x, \bar{b})$ is not satisfiable in M, whence $M \not\prec N$.

[r] This is the type of elements a which are transcendental over $\mathrm{cl}(B)$ and for which $\langle \mathrm{cl}(B), a \rangle$ is strong in the considered model.

This implies strong minimality. The rest of the claim follows from the fact that d describes the algebraic closure. □

This completes the proof of 1.1.

Proof. [Proof of Theorem 1.2, 2.] Let M be an elementary submodel of N in the sense of T_1 and T_2. By Corollary 9.1 we need only show that M is strong in N. Suppose not and pick a smallest extension $M \subset H \subset N$ with negative $\delta(H/M)$. We may decompose H/M into a sequence $M \leq K \subset H$, where $\delta(K/M) = 0$ and $H = \langle K, a\rangle$ for some element a with $\delta(a/K) = -1$. Since M is a model of Axiom (iii), we have $M = K$. a is algebraic over M in the sense of T_1 (and T_2), whence by Axiom (ii) we have $a \in M$. Contradiction.

Corollary 9.3. *If T_1 and T_2 are model-complete, then T^μ is also model-complete.*

We now prove the last remark of the introduction. Let T_1 and T_2 be both the theory of algebraically closed fields of characteristic p formulated in $L_1 = \{+, \odot\}$ and $L_2 = \{+, \otimes\}$. Let T^μ be a fusion over

T_0, the theory of $\mathbb{F}_p$–vector spaces. Let x be transcendental (in the sense of T^μ), x_i the i–th power in the sense of T_1 and $X = \{x_i \mid i \in \mathbb{N}\}$. Let S be any subset of X. Then $\dim(S) = |S|$ and $\mathrm{tr}_1(S) \leq 1$. It follows from Theorem 1.2, 1. that $\mathrm{tr}_2(S) \geq |S| - 1$. We claim that $\mathrm{tr}_2(S) = |S|$, which is clear for $S = \{x_0\}$. Assume the contrary. Then, for some $n > 0$, we have $\mathrm{tr}_2(x_1 \ldots, x_n/x_0) < n$. But x_{n+1} is also transcendental, therefore it has the same type as x. So $\mathrm{tr}_2(x_{n+1}, \ldots, x_{(n+1)n}/x_0) < n$. It follows

$$\mathrm{tr}_2(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{(n+1)n}/x_0) < 2n - 1,$$

which is impossible.

Remark 9.4. E. Hrushovski stated in [1] that the DMP survives the fusion. M. Hils explained a proof of this fact to us, which shows also that T^μ *has the DMP.* □

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