Complex continuations of $\mathbb{R}_{an,exp}$-definable unary functions with a diophantine application

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Abstract

Let $\mathcal{F}$ denote the field of germs at $+\infty$ of $\mathbb{R}_{an,exp}$-definable unary functions. Starting from its characterization in terms of closure conditions as given by van den Dries, Macintyre and Marker ([3], [4]), we give a similar description of its subring consisting of the germs of polynomial growth. More precisely, denoting this ring by $\mathcal{F}_{poly}$ and its unique maximal ideal by $\mathfrak{m}_{poly}$, our description picks out a subfield $\mathcal{R}_{poly}$ of representatives of the residue field of $\mathcal{F}_{poly}$ modulo $\mathfrak{m}_{poly}$. In fact, such a construction, in considerably greater generality, was already carried out in 1997 by F-V. Kuhlmann and S. Kuhlmann (unpublished, see arXiv:1206.0711v1 [math.LO]) using valuation theoretic methods, but our main aim here is to investigate the complex extensions of the functions under consideration. It turns out that $\mathcal{R}_{poly}$ consists precisely of those (germs of) $\mathbb{R}_{an,exp}$-definable unary functions that have an $\mathbb{R}_{an,exp}$-definable analytic continuation to a right half plane of $\mathbb{C}$ and we use this fact to give a different proof of the Kuhlmann result. (Roughly speaking, the (real) valuation theory is replaced by the Phragmén-Lindelöf method applied to the complex continuations.)

We then consider the analagous situation for those (germs of) functions in $\mathcal{F}$ having at most exponential growth. We briefly describe the Kuhlmann representation of the residue field and although the functions therein do not have analytic continuations to a right half plane in general, they turn out to have very good approximations that do.

There has been much research over the last ten years on diophantine properties of sets definable in o-minimal structures (see for example [15], [11], [5], [1], [6]) and in the final section of this paper we make a small contribution to this work. We apply our results to adapt a method of Pólya ([12]), as modified by Langley ([8]), and prove the following generalization of a result from [6]: let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathbb{R}_{an,exp}$-definable function such that for some $r \in \mathbb{R}$ with $0 < r < 1$, $|f(x)| \leq 2^{rx}$ for all sufficiently large x. Assume also that $f(n) \in \mathbb{Z}$ for all sufficiently large $n \in \mathbb{N}$. Then there exists a polynomial P with rational coefficients such that $f(x) = P(x)$ for all sufficiently large x.

1 Introduction

In [3] and [4] (but see also [9] for an excellent summary of the main results contained in these papers), van den Dries, Macintyre and Marker establish various quantifier elimination theorems for the theory of the real field expanded by all restricted analytic functions and the full exponential function, a structure that we henceforth denote by $\mathbb{R}_{an,exp}$ and its theory by $T_{an,exp}$. They go on to characterize the $\mathbb{R}_{an,exp}$-definable functions which, as we shall make explicit shortly, takes a particularly simple form for the germs at $+\infty$ of unary functions. Let $\mathcal{F}$ denote this collection of germs. Thus $\mathcal{F}$ consists of the collection of equivalence classes of $\mathbb{R}_{an,exp}$-definable functions $f : \mathbb{R} \to \mathbb{R}$, where two such functions are identified if they agree at all sufficiently large values of the argument. (We shall, however, usually regard elements of $\mathcal{F}$ as functions with domain (a, ∞), for sufficiently large $a \in \mathbb{R}$, rather than equivalence classes of such functions.)

Now $\mathcal{F}$ is (the domain of) an elementary extension of the structure $\mathbb{R}_{an,exp}$ (where we identify each $r \in \mathbb{R}$ with the corresponding constant function in $\mathcal{F}$) and is, in fact, the unique nonstandard model of $T_{an,exp}$ generated under the $T_{an,exp}$-definable functions by a(ny) single element of $\mathcal{F} \setminus \mathbb{R}$. The functions and relations of the language of $T_{an,exp}$ are, of course, interpreted "pointwise-eventually" in $\mathcal{F}$. Further, if $\mathbb{M}$ is any nonstandard model of $T_{an,exp}$ and α is any positive infinite element of $\mathbb{M}$, then the map sending the identity function on $\mathbb{R}$ to α extends uniquely to an elementary embedding from $\mathcal{F}$ into $\mathbb{M}$. In particular, if $h \in \mathcal{F}$ and $h(x) \to \infty$ as $x \to \infty$, then the map $\phi \to \phi \circ h$ (for $\phi \in \mathcal{F}$) is an automorphism of $\mathcal{F}$ that sends the identity function to h. This has some potential interest because it allows one to transfer certain properties of functions in $\mathcal{F}$ of polynomial growth to arbitrary functions in $\mathcal{F}$.

We now give the characterization of $\mathcal{F}$ from [3]. The operations and relations on elements of $\mathcal{F}$ mentioned below are those obtained by regarding $\mathcal{F}$ as a structure for the language of the theory $T_{an,exp}$, as just discussed. We use ι to denote the identity function on $\mathbb{R}$, as well as its germ considered as an element of $\mathcal{F}$.

Let $\mathcal{G}$ be any subset of $\mathcal{F}$ containing at least one nonzero constant function. Consider the following closure properties that $\mathcal{G}$ may or may not possess.

1.1 $\iota \in \mathcal{G}$;

1.2 for all $f, g \in \mathcal{G}$ and all $\alpha \in \mathbb{R}$, $\alpha f \in \mathcal{G}$, $f + g \in \mathcal{G}$ and $f \cdot g \in \mathcal{G}$;

1.3 for all $f \in \mathcal{G}$, if $f \neq 0$, then $1/f \in \mathcal{G}$;

1.4 for all open neighbourhoods $U \subseteq \mathbb{R}^n$ of $\bar{0} \in \mathbb{R}^n$, and all analytic functions $g : U \to \mathbb{R}$, if $f_1, \ldots, f_n \in \mathcal{G}$ and $\langle f_1(x), \ldots, f_n(x)\rangle \to \bar{0}$ as $x \to \infty$, then $g(f_1, \ldots, f_n) \in \mathcal{G}$;

1.5 for all $f \in \mathcal{G}$, if $f > 0$, then $\log(f) \in \mathcal{G}$;

1.6 for all $f \in \mathcal{G}$, and all $r \in \mathbb{R}$, if $f > 0$, then $f^r \in \mathcal{G}$;

1.7 for all $f \in \mathcal{G}$ with $f > 0$, if $\exp(f)$ is $\mathcal{G}$-bounded, i.e. if there is some $g \in \mathcal{G}$ such that $\exp(f) < g$, then $\exp(f) \in \mathcal{G}$;

1.8 for all $f \in \mathcal{G}$, $\exp(f) \in \mathcal{G}$.

It follows immediately from the explicit universal axiomatization of $T_{an,exp}$ given in [3] that if $\mathcal{G}$ satisfies 1.2-1.5 and 1.8 (which imply 1.6 and 1.7), then $\mathcal{G}$ is (the domain of) an elementary substructure of $\mathcal{F}$ and an elementary extension of $\mathbb{R}_{an,exp}$. So if $\mathcal{G}$ also satisfies 1.1 (or if it contains any nonconstant germ), then $\mathcal{G} = \mathcal{F}$.

Now consider the smallest subset of $\mathcal{F}$ (i.e. the intersection of all subsets of $\mathcal{F}$) satisfying 1.1 to 1.7. (Actually, 1.6 is implied by others. The reason for including it as a separate condition will become clear later.) Denote it by $\mathcal{R}_{poly}$ and let $\mathcal{F}_{poly}$ denote its convex closure in $\mathcal{F}$:

$$\mathcal{F}_{poly} := \{f \in \mathcal{F} : \text{for some } g \in \mathcal{R}_{poly}, |f| < g\}.$$

Then $\mathcal{F}_{poly}$ is clearly a valuation subring of the field $\mathcal{F}$ with maximal ideal

$$\mathfrak{m}_{poly} := \{f \in \mathcal{F} : \text{for all positive } g \in \mathcal{R}_{poly}, |f| < g\}.$$

The following two theorems are due to F-V. Kuhlmann and S. Kuhlmann. They appear in arXiv:1206.0711v1 [math.LO], but can also be found in S. Kuhlmann's monograph [7] (Theorem 6.44 combined with Theorem 6.46) to which I shall henceforth refer for similar results.

1.9 Theorem

The map $\phi \mapsto \phi/\mathfrak{m}_{poly}$ is an isomorphism from the field $\mathcal{R}_{poly}$ to the field $\mathcal{F}_{poly}/\mathfrak{m}_{poly}$. In other words, $\mathcal{R}_{poly}$ picks out exactly one element from each $\mathfrak{m}_{poly}$-equivalence class of $\mathcal{F}_{poly}$.

1.10 Theorem

Every $f \in \mathcal{F}_{poly}$ has polynomial growth, i.e. for some $n \in \mathbb{N}$, $|f| < \iota^n$.

Now 1.10 might seem obvious given the restriction on exponentiation imposed by 1.7. However, I know of no direct inductive argument for 1.10 and the proof in [7] requires quite careful control of the value groups and residue fields that arise during the various stages of the construction as formulated there. Our proof of 1.10 given in section 3 uses complex methods, which I now discuss.

As usual, we identify $\mathbb{C}$ with $\mathbb{R}^2$ when referring to definability of complex functions and relations in structures expanding the real field. If the structure is not mentioned, it is understood to be $\mathbb{R}_{an,exp}$.

For $a, \psi \in \mathbb{R}$ with $0 < \psi < \pi$, let $S(a, \psi)$ denote the sector

$$\{a + te^{i\theta} : t > 0, -\psi < \theta < \psi\}$$

in $\mathbb{C}$. This is clearly a definable family (as the parameters a, ψ vary as indicated) of subsets of $\mathbb{C}$. We prove the following theorem.

1.11 Theorem

Let $f \in \mathcal{F}$. The following are equivalent.

1.11.1 $f \in \mathcal{R}_{poly}$;

1.11.2 for some $\psi, a \in \mathbb{R}$ with $\frac{\pi}{2} < \psi < \pi$, there exists a definable, complex analytic function $\hat{f} : S(a, \psi) \to \mathbb{C}$ such that $\hat{f}(x) = f(x)$ for all $x \in (a, \infty)$.

Further, if 1.11.2 holds and $\frac{\pi}{2} < \psi' < \psi$, then there exists a constant c such that (possibly after increasing a) we have $|\hat{f}(z)| \leq |z|^c$ for all $z \in S(a, \psi')$. In particular, all functions in $\mathcal{R}_{poly}$ have polynomial growth, which clearly entails the same for functions in $\mathcal{F}_{poly}$ as claimed in 1.10.

In the final section of this paper we give a diophantine application. We will adapt a method of Langley who showed in [8] that if $F : S(a, \frac{\pi}{2}) \to \mathbb{C}$ is any complex analytic function such that $F(n) \in \mathbb{Z}$ for all sufficiently large $n \in \mathbb{Z}$, and if $|F(z)| = O(|z|^M \cdot 2^{|z|})$ as $|z| \to \infty$ for $z \in S(a, \frac{\pi}{2})$, then there exist polynomials $P(z), Q(z) \in \mathbb{Q}[z]$ such that $F(z) = P(z) \cdot 2^z + Q(z)$ for all $z \in S(a, \frac{\pi}{2})$. The basic idea of Langley's proof originated much earlier in a paper of Pólya ([12]), who established the corresponding result for entire functions F. Our aim is to prove a similar, but slightly weaker, result for definable functions F. This result is, however, still considerably stronger that the main result of [6] which requires a stronger growth condition on the functions involved and only applies to the expansion $\mathbb{R}_{exp}$ of the real field by exponentiation.

In order to carry out Langley's argument for definable functions we will need to analytically continue considerably more such functions than those covered by 1.11. This will be done in section 4. It turns out that there is a naturally defined subfield of $\mathcal{F}$, call it $\mathcal{R}_{subexp}$, representing the residue field of the convex subring $\mathcal{F}_{subexp}$ of $\mathcal{F}$ determined by the functions $f \in \mathcal{F}$ of subexponential growth (i.e. satisfying $f(x) = \exp(o(x))$ as $x \to \infty$). Again, this follows from results in [7], but we do need to give a different proof here in order to show that all functions in this subfield have a (not necessarily definable) analytic continuations to $S(a, \frac{\pi}{2})$ for some $a \in \mathbb{R}$.

However, there exist definable functions of exponential growth that have no analytic continuation to $S(a, \frac{\pi}{2})$ for any a, and to deal with these we make use of the Expansion Theorem of Miller and van den Dries (see [2]). This tells us that such functions can be well approximated (for large x) by functions of the form

$$\sum_{j=1}^{N} f_j(x) \cdot \exp(s_j x),$$

where $s_1, \ldots, s_N \in \mathbb{R}$ and each f_j is a function in $\mathcal{R}_{subexp}$, and functions of this form do have the required continuations.

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Thanks also to Salma Kuhlmann for directing me towards her monograph [7] which contains, amongst many other results, the key characterizations of the classes of functions (in the real case) that are considered here.

2 Models of T_{an}^{pow}

We now work inside an arbitrary proper elementary extension $\mathbb{M}$ of $\mathbb{R}_{an,exp}$. We always use the symbol μ to denote the infinitesimals of $\mathbb{M}$, i.e. $\mu := \{\alpha \in \mathbb{M} : \text{for all positive } r \in \mathbb{R}, |\alpha| < r\}$.

We let $\mathcal{C}$ denote the class of subsets of (the domain of) $\mathbb{M}$ satisfying the corresponding versions of 1.2, 1.3, 1.4, and 1.6. Thus $\mathcal{M} \in \mathcal{C}$ if and only if

2.1 $\mathcal{M}$ is a subfield of $\mathbb{M}$ and an $\mathbb{R}$-subalgebra of $\mathbb{M}$;

2.2 for all open neighbourhoods U of $\bar{0}$ in $\mathbb{R}^n$ and all real analytic $g : U \to \mathbb{R}$, if $\alpha_1, \ldots, \alpha_n \in \mu \cap \mathcal{M}$, then $g(\alpha, \ldots, \alpha_n) \in \mathcal{M}$, where we continue to use g to denote its extension to $\mathbb{M}$;

2.3 for all $\alpha \in \mathcal{M}$ with $\alpha > 0$, and all $r \in \mathbb{R}$, we have $\alpha^r \in \mathcal{M}$.

Consider now the structure $\mathbb{R}_{an}^{pow}$ introduced by Miller in [10]. It is the expansion of the real field by all restricted analytic functions and all power functions $x \mapsto x^r$ ($= 0$ if $x \leq 0$) for $r \in \mathbb{R}$. Denote the language of $\mathbb{R}_{an}^{pow}$ by $\mathcal{L}_{an}^{pow}$.

Then $\mathbb{R}_{an}^{pow}$ is clearly a reduct of $\mathbb{R}_{an,exp}$ and the main results of [10], namely that T_{an}^{pow} (the first order theory of $\mathbb{R}_{an}^{pow}$) has quantifier elimination and a universal axiomatization in the language $\mathcal{L}_{an}^{pow}$, imply that any $\mathcal{M} \in \mathcal{C}$

is (the domain of) a model of T_{an}^{pow} and, as such, is an $\mathcal{L}_{an}^{pow}$-elementary substructure of the $\mathcal{L}_{an}^{pow}$-reduct of $\mathbb{M}$.

With a view to proving 1.9 we now introduce further closure properties for the structures in $\mathcal{C}$.

Let $\mathcal{M} \in \mathcal{C}$. We say that an element α of $\mathbb{M}$ is **$\mathcal{M}$-bounded** if $|\alpha| < \beta$ for some $\beta \in \mathcal{M}$. We say that $\mathcal{M}$ is **log-closed** if $\log(\alpha) \in \mathcal{M}$ whenever $\alpha \in \mathcal{M}$ and $\alpha > 0$, and that $\mathcal{M}$ is **exp-closed from below** if $\exp(\alpha) \in \mathcal{M}$ whenever $\alpha \in \mathcal{M}$, $\alpha > 0$ and $\exp(\alpha)$ is $\mathcal{M}$-bounded. (Of course, these are just the corresponding versions of 1.5 and 1.7 for the present situation.)

We denote the set of all $\mathcal{M}$-bounded elements of $\mathbb{M}$ by $\mathcal{M}^*$. It is the convex closure of $\mathcal{M}$ in $\mathbb{M}$ and a valuation subring of $\mathbb{M}$. We denote its unique maximal ideal by $\mu(\mathcal{M}^*)$:

$$\mu(\mathcal{M}^*) := \{\alpha \in \mathcal{M}^* : \text{for all positive } \beta \in \mathcal{M}, |\alpha| < \beta\}.$$

However, unlike the arguments in [7], the only valuation that we shall be using is the usual one determined by $\mathbb{R}$ and its maximal ideal $\mu{=}\mu(\mathbb{R}^*)$. This valuation is denoted by ν.

For $\mathcal{M}, \mathcal{M}' \in \mathcal{C}$ with $\mathcal{M} \subseteq \mathcal{M}'$, we say that $\mathcal{M}'$ is **$\mathcal{M}$-conservative** if every $\mathcal{M}$-bounded element of $\mathcal{M}'$ is $\mathcal{M}$-infinitesimally close to an element of $\mathcal{M}$. More precisely, for all $\alpha \in \mathcal{M}' \cap \mathcal{M}^*$, there exists $\beta \in \mathcal{M}$ such that $\alpha - \beta \in \mu(\mathcal{M}^*)$.

We now come to the main lemma of this section. It can be extracted from 6.25-6.28 and 6.32 of [7], but as all the properties of the class $\mathcal{C}$ that we require follow directly from its rather simple statement, we prefer to keep the paper self-contained and include a proof.

We require a little more notation. For $\mathcal{M} \in \mathcal{C}$ and S an arbitrary subset of $\mathbb{M}$, we write $\mathcal{M}\langle S\rangle$ for the $\mathcal{L}_{an}^{pow}$-substructure of $\mathbb{M}$ generated over $\mathcal{M}$ by S. So $\mathcal{M}\langle S\rangle$ is the smallest structure in $\mathcal{C}$ (i.e. the intersection of all structures in $\mathcal{C}$) containing $\mathcal{M} \cup S$. (But if $S = \{\alpha_1, \ldots, \alpha_n\}$ we write $\mathcal{M}\langle \alpha_1, \ldots, \alpha_n\rangle$ for $\mathcal{M}\langle S\rangle$.)

2.4 Lemma (See [7])

Let $\mathcal{M} \in \mathcal{C}$ and suppose that $\mathcal{M}$ is log-closed and exp-closed from below. Then for any positive $\alpha_1, \ldots, \alpha_n \in \mathcal{M}$, we have that $\mathcal{M}\langle \exp(\alpha_1), \ldots, \exp(\alpha_n)\rangle$ is log-closed and $\mathcal{M}$-conservative.

Proof. Let V be the $\mathbb{R}$-vector subspace of $\mathcal{M}$ generated by $\alpha_1, \ldots, \alpha_n$. We may assume (by 2.3 and induction) that for no $\alpha \in V \setminus \{0\}$ do we have $\exp(\alpha) \in \mathcal{M}$. Since $\mathcal{M}$ is exp-closed from below it follows easily (using properties of ordered $\mathbb{R}$-vector spaces) that V has a basis, which we may as well take to be $\{\alpha_1, \ldots, \alpha_n\}$, such that for all positive $\gamma \in \mathcal{M}$ and all positive $r_1, \ldots, r_n \in \mathbb{R}$

$$\log(\gamma) < r_1\alpha_1 < \cdots < r_n\alpha_n. \qquad (*)$$

It now follows from the valuation inequality for polynomially bounded o-minimal theories (see e.g. [13]) that $\nu(\exp(\alpha_1)), \ldots, \nu(\exp(\alpha_n))$ generate (as an $\mathbb{R}$-vector space) the value group of $\mathcal{M}' := \mathcal{M}\langle\exp(\alpha_1), \ldots \exp(\alpha_n)\rangle$ over that of $\mathcal{M}$.

Hence, if β is any element of $\mathcal{M}'$, there exist $\alpha \in V$, $\gamma \in \mathcal{M}$ and $\epsilon \in \mu$ such that

$$\beta = \exp(\alpha) \cdot \gamma \cdot (1 + \epsilon). \qquad (**)$$

If $\beta > 0$, then necessarily $\gamma > 0$ and

$$\log(\beta) = \alpha + \log(\gamma) + \log(1 + \epsilon). \qquad (***)$$

Now $\alpha \in \mathcal{M}$ and $\log(\gamma) \in \mathcal{M}$ (as $\mathcal{M}$ is log-closed). Further, by (**), we have that $\epsilon \in \mathcal{M}'$ and hence $\log(1 + \epsilon) \in \mathcal{M}'$ by 2.2 (applied to $g(x) = \log(1 + x)$ and $U = (-\frac{1}{2}, \frac{1}{2})$, say). Thus, by (***), $\log(\beta) \in \mathcal{M}'$, and we have shown that $\mathcal{M}'$ is log-closed.

To see that $\mathcal{M}'$ is $\mathcal{M}$-conservative we set $\mathcal{M}_0 := \mathcal{M}$ and, for $j = 1, \ldots, n$, $\mathcal{M}_j := \mathcal{M}_{j-1}\langle\exp(\alpha_j)\rangle$, so that $\mathcal{M}' = \mathcal{M}_n$. We show, by induction on j, that $\mathcal{M}_j$ is $\mathcal{M}$-conservative. So let $j \geq 1$ and suppose that $\mathcal{M}_{j-1}$ is $\mathcal{M}$-conservative. (Trivially, $\mathcal{M}_0$ is $\mathcal{M}$-conservative.)

Let $\beta \in \mathcal{M}_j$ be $\mathcal{M}$-bounded. Say $B \in \mathcal{M}$ and $|\beta| < B$.

There exists an $\mathcal{L}_{an}^{pow}$-definable function $f : \mathcal{M}_j \to \mathcal{M}_j$ with parameters in $\mathcal{M}_{j-1}$ such that $f(\exp(\alpha_j)) = \beta$. Clearly we may suppose that $|f(x)| < B$ for all $x \in \mathcal{M}_j$. Now $\mathcal{M}_{j-1}$ is an $\mathcal{L}_{an}^{pow}$-elementary substructure of $\mathcal{M}_j$ so we may interpret f in $\mathcal{M}_{j-1}$, and by properties of polynomially bounded o-minimal theories it follows that for some positive $r \in \mathbb{R}$ and some $\lambda \in \mathcal{M}_{j-1}$,

$$\mathcal{M}_{j-1} \models \exists y \forall x > y(|f(x) - \lambda| < x^{-r}).$$

Now it follows from (*) (and an application of polynomial boundedness) that $\exp(\alpha_j) > \alpha$ for all $\alpha \in \mathcal{M}_{j-1}$. Therefore, upon witnessing y in $\mathcal{M}_{j-1}$ in the above, and then going up to $\mathcal{M}_j$, we obtain

$$\mathcal{M}_j \models |f(\exp(\alpha_j)) - \lambda| < \exp(-r\alpha_j).$$

Since $\exp(-r\alpha_j)$ is in $\mu(\mathcal{M}^*_{j-1})$ and $\mu(\mathcal{M}^*_{j-1}) \subseteq \mu(\mathcal{M}^*)$, it follows that $|\beta - \lambda| \in \mu(\mathcal{M}^*)$. In particular, $|\lambda| < B + 1$, so λ is $\mathcal{M}$-bounded, and hence, by our inductive hypothesis, there is some $\gamma \in \mathcal{M}$ such that $|\lambda - \gamma| \in \mu(\mathcal{M}^*)$. So $|\beta - \gamma| \leq |\beta - \lambda| + |\lambda - \gamma| \in \mu(\mathcal{M}^*)$, and we are done.

□

2.5 Corollary

Let $\mathcal{M} \in \mathcal{C}$ and suppose that $\mathcal{M}$ is log-closed and exp-closed from below. Let G be a convex additive subgroup of $\mathcal{M}$ with $1 \in G$, and set $E := \{\exp(\alpha) : \alpha \in G\}$. Then $\mathcal{M}\langle E\rangle$ is log-closed, exp-closed from below and $\mathcal{M}$-conservative.

Proof. Obviously if $\beta \in \mathcal{M}\langle E\rangle$ then $\beta \in \mathcal{M}\langle \exp(\alpha_1), \dots, \exp(\alpha_n)\rangle$ for some $n \in \mathbb{N}$ and some positive $\alpha_1, \dots, \alpha_n \in G$. So it follows immediately from 2.4 that $\mathcal{M}\langle E\rangle$ is log-closed and $\mathcal{M}$-conservative. Further, if $\beta > 0$ and $\exp(\beta)$ is $\mathcal{M}\langle E\rangle$-bounded, then we may assume that $\alpha_1, \dots, \alpha_n$ have been chosen so that $\exp(\beta)$ is $\mathcal{M}\langle \exp(\alpha_1), \dots, \exp(\alpha_n)\rangle$-bounded. We now want to show that $\exp(\beta) \in \mathcal{M}\langle E\rangle$.

Since all elements of $\mathcal{M}\langle \exp(\alpha_1), \dots, \exp(\alpha_n)\rangle$ are bounded by $B \cdot \exp(\alpha)$ for some $\mathbb{R}$-linear combination α of $\alpha_1, \dots, \alpha_n$, and some positive $B \in \mathcal{M}$, it follows that β, being positive, is itself bounded by $\log(B) + \alpha$, and is therefore $\mathcal{M}$-bounded. Thus $|\beta - \gamma| \in \mu(\mathcal{M}^*)$, for some $\gamma \in \mathcal{M}$ (which we may assume is nonnegative), since $\mathcal{M}\langle E\rangle$ is $\mathcal{M}$-conservative. Now if $\log(B) \notin G$ then $\alpha_1, \dots, \alpha_n < \log(B)$ (as G is convex), whence $E \subseteq \mathcal{M}$ (because $\mathcal{M}$ is exp-closed from below) and the result we are proving is trivial. So we may assume that $\log(B)$, and hence $\log(B) + \alpha$, are in G. But clearly $0 \leq \gamma < \log(B) + \alpha + 1$, so $\gamma \in G$. But $\beta = \gamma + \epsilon$ for some $\epsilon \in \mu(\mathcal{M}^*) \cap \mathcal{M}\langle E\rangle$. Further, $\exp(\epsilon) \in \mathcal{M}\langle E\rangle$ by 2.2 (with $U = (-1, 1)$ and $g(x) = \exp(x)$) and $\exp(\gamma) \in E \subseteq \mathcal{M}\langle E\rangle$. So $\exp(\beta) = \exp(\gamma) \cdot \exp(\epsilon) \in \mathcal{M}\langle E\rangle$, and we have shown that $\mathcal{M}\langle E\rangle$ is exp-closed from below.

□

2.6 Corollary

Let $\mathcal{M} \in \mathcal{C}$ and suppose that $\mathcal{M}$ is log-closed and exp-closed from below. Let $\mathcal{M}^\dagger$ be the smallest element of $\mathcal{C}$ containing $\mathcal{M}$ and closed under (full) exponentiation. Then $\mathcal{M}^\dagger$ is log-closed and $\mathcal{M}$-conservative. Further, $\mathcal{M}^\dagger \models T_{an,exp}$ and $\mathcal{M}^\dagger$ is an elementary substructure of $\mathbb{M}$ for the language of $T_{an,exp}$.

Proof. The log-closedness and $\mathcal{M}$-conservativity of $\mathcal{M}^\dagger$ follow by iterating 2.5 (let $\mathcal{M}_0 := \mathcal{M}$ and $\mathcal{M}_{j+1} := \mathcal{M}_j\langle E_j\rangle$ where $E_j := \{\exp(\alpha) : \alpha \in \mathcal{M}_j\}$). The remaining statements are consequences of the quantifier elimination and universal axiomatization of $T_{an,exp}$ as found in [3].

$\square$

Theorem 1.9 now follows easily: just take $\mathbb{M} = \mathcal{F}$ and $\mathcal{M} = \mathcal{R}_{poly}$ in 2.6. Then, as discussed in section 1, $\mathcal{M}^\dagger = \mathcal{F}$.

3 Complex continuation of functions in $\mathcal{F}$: the definable case

Our aim in this section is to show that a function $f \in \mathcal{F}$ lies in $\mathcal{R}_{poly}$ if and only if it has a definable, complex analytic continuation to a function $\hat{f} : S(a, \psi) \to \mathbb{C}$ for some $a, \psi \in \mathbb{R}$ with $\frac{\pi}{2} < \psi < \pi$. (Here, as throughout this section, definability is with respect to the structure $\mathbb{R}_{an,exp}$ (in the full language of $T_{an,exp}$), and I also remind the reader that $S(a, \psi)$ denotes the sector $\{a + te^{i\theta} : t > 0, -\psi < \theta < \psi\}$.)

Firstly, however, we establish some a priori properties that definable, complex analytic functions $F : S(0, \psi) \to \mathbb{C}$ possess. (It will be a triviality to translate such properties to functions with domain $S(a, \psi)$ for arbitrary $a \in \mathbb{R}$.)

3.1 Definition

For $0 < \psi < \pi$, we denote by $\mathcal{H}_\psi$ the class of all definable, complex analytic functions with domain the sector $S(0, \psi)$ which take real values on the positive real axis.

Consider some $F \in \mathcal{H}_\psi$, where $\frac{\pi}{2} < \psi < \pi$. Assume that F has no zeros in $S(0, \psi)$ and that $F(x) > 0$ for $x > 0$. Then there exist continuous functions $R : S(0, \psi) \to \mathbb{R}$, $\theta : S(0, \psi) \to \mathbb{R}$, with $\theta(x) = 0$ for positive real x, such that $F(z) = R(z) \cos\theta(z) + iR(z) \sin\theta(z)$ for all $z \in S(0, \psi)$. Since $R(z) = |F(z)|$

(for $z \in S(0,\psi)$) it easily follows that R is definable. Hence so is $\cos\theta$ (and $\sin\theta$). We now show that θ itself is definable.

Obviously θ has total variation at most 2π on each connected component of the definable set $\mathbb{C} \setminus \{z \in S(0,\psi) : \cos\theta(z) = \pm 1\}$ (and zero variation on each connected component of its complement). It follows that θ is bounded since there are only finitely many such components. Further, if Λ is a connected component of the set $\mathbb{C} \setminus \{z \in S(0,\psi) : \cos\theta(z) = \pm 1\}$, then for all $z \in \Lambda$ there exists a unique $x_z \in (0, 2\pi)$ such that $\cos x_z = \cos\theta(z)$. Since θ is continuous it follows that there is some $N \in \mathbb{N}$ (depending only on Λ) such that $\theta(z) = 2N\pi + x_z$ for all $z \in \Lambda$. Thus $\theta \restriction \Lambda$ is definable. (The function $z \mapsto x_z$ is definable since the restriction of the function cos to any bounded interval is definable.). Also, since $\theta \restriction \Lambda'$ is obviously definable for each connected component Λ' of the set $\{z \in S(0,\psi) : \cos\theta(z) = \pm 1\}$, it follows that $\theta : S(0,\psi) \to \mathbb{R}$ is definable. We can now prove

3.2 Lemma

Let $\frac{\pi}{2} < \psi < \pi$, $F \in \mathcal{H}_\psi$ and assume that F is nowhere vanishing and positive on the positive real axis. Then $\log F \in \mathcal{H}_\psi$ (where we take the branch of the logarithm that is real on the positive real axis). Further, $\log F$ has bounded imaginary part.

Proof. We have $\log F(z) = \log R(z) + i\theta(z)$ with R, θ chosen as in the discussion above. □

3.3 Corollary

Let $\frac{\pi}{2} < \psi' < \psi < \pi$, $F \in \mathcal{H}_\psi$, and suppose that F is bounded on the set $\{z \in S(0,\psi) : |z| \leq 1\}$. Then there exists a constant B such that $|F(z)| \leq B(1+|z|)^B$ for all $z \in S(0,\psi')$.

Proof. We may assume that F is not identically zero. By o-minimality and the fact that its set of zeros is discrete, F has only finitely many zeros which, since F is real on the positive real axis, are symmetrically distributed about the real axis. Let P be the monic polynomial vanishing precisely on the zeros of F (and with the correct multiplicities). Then $F/P \in \mathcal{H}_\psi$ (note that P has real coefficients) and F/P is nowhere vanishing. Further, F/P is also bounded on the set $\{z \in S(0,\psi) : |z| \leq 1\}$ and clearly, if we establish the conclusion of the corollary for the function F/P, then it would follow for F. So, we may as well assume that F is nowhere vanishing and, by multiplying it by -1 if necessary, that $F(x) > 0$ for all $x > 0$. Thus, by 3.2, we have that $\log F \in \mathcal{H}_\psi$ and that $\log F$ has bounded imaginary part, with bound K say.

Now there exists a positive constant c such that for any $z_0 \in S(0, \psi')$, the closure of the disk $\Delta_{z_0} := \Delta(z_0; 2c|z_0|)$ centred at z_0 and of radius $2c|z_0|$, is contained in the sector $S(0, \psi)$. (For example, $c = \frac{1}{2}\sin(\psi - \psi')$ will work.) It follows, by applying the Borel-Cartheodory theorem (see [14] section 5.5) to the function $\log F(z) - \log F(z_0)$ (the imaginary part of which is bounded by $2K$ on the disk Δ_{z_0}), that

$$|\log F(z) - \log F(z_0)| \leq 6K$$

for all $z \in \Delta(z_0; c|z_0|)$, from which it follows that

$$|F(z)| \leq |F(z_0)| \cdot e^{6K}$$

for all $z \in \Delta(z_0; c|z_0|)$. In particular, for any $w \in S(0, \psi')$ with $|w| \leq 1$, and any $n \in \mathbb{N}$, we have

$$|F((1+c)^{n+1} \cdot w)| \leq |F((1+c)^n \cdot w)| \cdot e^{6K} \qquad (*)$$

Now let $z \in S(0, \psi')$ satisfy $|z| > 1$. Choose $n_0 \in \mathbb{N}$ such that $(1+c)^{n_0} < |z| \leq (1+c)^{n_0+1}$ and set $w_0 := z \cdot (1+c)^{-(n_0+1)}$. Then $|w_0| \leq 1$ and $w_0 \in S(0, \psi')$. Iterating (*) now yields

$$|F(z)| = |F((1+c)^{n_0+1} \cdot w_0)| \leq |F(w_0)| \cdot e^{(n_0+1)\cdot 6K}.$$

However, $n_0 < \frac{\log|z|}{\log(1+c)}$ so, choosing B greater than both $\frac{6K}{\log(1+c)}$ and $e^{6K} \cdot \sup\{|F(z)| : z \in S(0, \psi), |z| \leq 1\}$ we obtain $|F(z)| \leq B(1+|z|)^B$ (for all $z \in S(0, \psi')$) as required. □

3.4 Corollary

Let $\frac{\pi}{2} < \psi' < \psi < \pi$, $F \in \mathcal{H}_\psi$, and assume that F does not vanish on $S(0, \psi)$. Assume further that both F and $1/F$ are bounded on the set $\{z \in S(0, \psi) : |z| \leq 1\}$. Suppose that $F(x) \to \alpha$ as $x \to \infty$ (for $x \in \mathbb{R}$), where $\alpha \in \mathbb{R} \cup \{\pm\infty\}$. Then $F(z) \to \alpha$ ($= \infty$ if $\alpha = \pm\infty$) as $|z| \to \infty$ for $z \in S(0, \psi')$.

Proof. Choose ψ'' such that $\frac{\pi}{2} < \psi' < \psi'' < \psi < \pi$. Let $l := \{te^{i\psi''} : t > 0\}$. Assume first that F is bounded on l. By o-minimality there exists $\beta \in \mathbb{C}$ such that $F(z) \to \beta$ as $|z| \to \infty$ for $z \in l$. Since F is real on the positive real axis we have that $F(z) \to \bar{\beta}$ as $|z| \to \infty$ for $z \in \bar{l} := \{te^{-i\psi''} : t > 0\}$, where the bar denotes complex conjugation. It now follows from 3.3 and the Phragmén-Lindelof method (see [14] sections 5.11 and 5.14) that $\beta = \bar{\beta} = \alpha$ and, indeed, that $F(z) \to \alpha$ as $|z| \to \infty$ for $z \in S(0, \psi')$.

Now if F is not bounded on l, then $|F(z)| \to \infty$ as $|z| \to \infty$ for $z \in l$ (by o-minimality). Since F has no zeros, we have that $1/F \in \mathcal{H}_\psi$ and $(1/F)(z) \to 0$ as $|z| \to \infty$ for $z \in l$. Applying the above argument to $1/F$ and $\alpha = 0$, we see that $(1/F)(z) \to 0$ as $|z| \to \infty$ for $z \in S(0, \psi')$. Thus it must have been the case that $\alpha = \pm\infty$, and we have $F(z) \to \infty$ as $|z| \to \infty$ for $z \in S(0, \psi')$, as required. □

We now show that each function $f \in \mathcal{R}_{poly}$ has a definable, complex analytic continuation to some right half plane $\{z \in \mathbb{C} : Re(z) > a\}$. It is clearly sufficient to show that there exist $a, \psi \in \mathbb{R}$, with $\frac{\pi}{2} < \psi < \pi$, and some $F \in \mathcal{H}_\psi$ such that $F(x) = f(a + x)$ for all positive $x \in \mathbb{R}$. For then the function $\hat{f} : S(a, \frac{\pi}{2}) \to \mathbb{C}$ given by $\hat{f}(z) = F(z - a)$ is the required continuation. So let $\tilde{\mathcal{F}}$ denote the collection of all such f. That is

$$\tilde{\mathcal{F}} := \{f \in \mathcal{F} : \exists a \in \mathbb{R}, \psi \in (\frac{\pi}{2}, \pi), F \in \mathcal{H}_\psi, \forall x > 0\, F(x) = f(a + x)\}.$$

We must show that $\tilde{\mathcal{F}}$ satisfies 1.1 to 1.7 (for $\mathcal{G} = \tilde{\mathcal{F}}$).

Now 1.1 and 1.2 are clear, although for 1.2 we should remark that if $F_1 \in \mathcal{H}_{\psi_1}$, $a_1 \in \mathbb{R}$ and $F_2 \in \mathcal{H}_{\psi_2}$, $a_2 \in \mathbb{R}$ work for $f_1, f_2 \in \tilde{\mathcal{F}}$, then we may assume that $a_1 = a_2$ and $\psi_1 = \psi_2$: just take $\psi = \min(\psi_1, \psi_2)$, and if $a_1 < a_2$ change $F_1(z)$ to $F_1(a_2 - a_1 + z)$. Then, for example, $F_1 + F_2, a_2, \psi$ work for $f_1 + f_2$.

A similar observation works for 1.3 and 1.5: we may always translate any (nonzero) $F \in \mathcal{H}_\psi$ so that so that it has no zeros in $S(0, \psi)$, and then $1/F \in \mathcal{H}_\psi$ and, by 3.2, $\log F \in \mathcal{H}_\psi$.

We now deal with 1.4. Let $n \geq 1$ and $r > 0$ and suppose that $g : U \to \mathbb{R}$ is a real analytic function, where U is an open set in $\mathbb{R}^n$ with $\{\bar{x} \in \mathbb{R}^n : ||\bar{x}|| \leq r\} \subseteq U$. (We use the supnorm, $||\bar{x}|| = ||\langle x_1, \ldots, x_n\rangle|| := \max\{|x_1|, \ldots, |x_n|\}$ for real or complex $x_1, \ldots, x_n$.)

Now choose $r_0 > 0$ with $r_0 \leq r$ such that g extends to a complex analytic function (which we also denote by g) from the polydisk $\{\bar{z} \in \mathbb{C} : ||\bar{z}|| \leq r_0\}$ to $\mathbb{C}$. Let $f_1 \ldots, f_n \in \tilde{\mathcal{F}}$ be such that $f_j(x) \to 0$ as $x \to \infty$ for each $j = 1, \ldots n$. By applying a similar remark to that used in establishing 1.2 above, choose

$a, \psi \in \mathbb{R}$, with $\frac{\pi}{2} < \psi < \pi$, and $F_1, \ldots, F_n \in \mathcal{H}_\psi$ such that $F_j(x) = f_j(a+x)$ for all $x > 0$ and $j = 1, \ldots n$. We have to extend the function $g(f_1, \ldots, f_n)$.

We may clearly assume (by induction) that no f_j is (eventually) the zero function. So by increasing a if necessary we may assume also that the hypotheses of 3.4 are satisfied for each F_j (and with $\alpha = 0$). Thus by decreasing ψ slightly we have that each $F_j(z)$ converges to 0 on $S(0, \psi)$ as $|z| \to \infty$. In particular, there exists $R > 0$ such that $|F_j(z)| < r_0$ for all $z \in S(0, \psi)$ with $|z| > R$. Now choose $R' > R$ so that the disk $\Delta(0, R)$ and the sector $S(R', \psi)$ have no points in common. Then $|F_j(R'+z)| < r_0$ for all $z \in S(0, \psi)$ and all $j = 1, \ldots, n$. It follows that $g(F_1(R'+z), \ldots, F_n(R'+z))$ is a well-defined, complex analytic function of $z \in S(0, \psi')$ and it is clearly definable (because the real and imaginary parts of g are restricted analytic functions, and $F_1, \ldots, F_n \in \mathcal{H}_{\psi'}$). Hence it is in $\mathcal{H}_{\psi'}$. Further, $g(F_1(R'+x), \ldots, F_n(R'+x)) = g(f_1(a+R'+x), \ldots, f_n(a+R'+x))$ for all $x > 0$. This shows that $g(f_1, \ldots, f_n) \in \tilde{\mathcal{F}}$, and so $\tilde{\mathcal{F}}$ satisfies 1.4.

Finally, 1.7 will follow easily from the following

3.5 Lemma

Let $\frac{\pi}{2} < \psi < \pi$ and suppose that $F \in \mathcal{H}_\psi$. Assume that $|F(x)| \le C \cdot \log x$ for all sufficiently large $x \in \mathbb{R}$, where C is some positive constant. Then for any ψ' with $\frac{\pi}{2} < \psi' < \psi$, there exists $b \ge 0$ such that the imaginary part of F is bounded on $S(b, \psi')$.

Proof. We may assume that F is not constant. Since the derivative of a definable, complex analytic function is definable (via the usual ϵ-δ definition), we have that $F' \in \mathcal{H}_\psi$. Further, it is an easy consequence of the given bound on F (and o-minimality) that $|F'(x)| \le \frac{C+1}{x}$ for all sufficiently large $x \in \mathbb{R}$. Now choose $a \ge 0$ large enough so that $F'(z)$ does not vanish for $z \in S(a, \psi)$ and so that both $|zF'(z)|$ and $1/|zF'(z)|$ are bounded for $z \in S(a, \psi)$ with $|z - a| \le 1$. Now the boundedness of $xF'(x)$ for sufficiently large $x \in \mathbb{R}$ (and o-minimality) implies that $xF'(x)$ converges to a finite limit as $x \to \infty$, and so by reducing ψ slightly, we see from 3.4 that $zF'(z)$ converges, as $|z| \to \infty$ for $z \in S(a, \psi)$, to this same limit. In particular, there exists a constant $K \in \mathbb{R}$ such that (possibly after increasing a)

$$|F'(z)| \le \frac{K}{|z|} \quad \text{for all} \quad z \in S(a, \psi). \qquad (*)$$

Now consider a point $z_0 = a + Re^{i\theta_0} \in S(a, \psi)$ where $R > 0$ and $-\psi < \theta_0 < \psi$.

Let Γ denote the directed arc of the circle centred at a and radius R joining the point $a + R$ to z_0. Then

$$\begin{aligned}
|F(z_0) - F(a+R)| &= |\int_\Gamma F'(z)dz| \\
&= |\int_0^{\theta_0} F'(a+Re^{i\theta}) \cdot i \cdot R \cdot e^{i\theta} d\theta| \\
&\leq R \cdot \int_0^{\theta_0} |F'(a+Re^{i\theta})| d\theta \\
&\leq RK \cdot \int_0^{\theta_0} \frac{d\theta}{|a+Re^{i\theta}|} \quad \text{by (*)} \\
&\leq \frac{RK|\theta_0|}{R-a} \quad \text{for} \quad R > a \\
&\leq 2K\pi \quad \text{for} \quad R > 2a.
\end{aligned}$$

However, since $F(a+R)$ is real, the imaginary part of $F(z_0)$ is bounded by $|F(z_0) - F(a+R)|$, and hence by $2K\pi$, for all $z \in S(b, \psi)$, provided that b is chosen large enough so that $b > a$ and the sector $S(b, \psi)$ does not intersect the disk $\Delta(a, 2a)$. □

Now to show that 1.7 holds for $\tilde{\mathcal{F}}$, let $f, g \in \tilde{\mathcal{F}}$ with $0 < f$ and $\exp(f) < g$. We choose ψ with $\frac{\pi}{2} < \psi < \pi$, $a \in \mathbb{R}$ and $F, G \in \mathcal{H}_\psi$ so that $f(a+x) = F(x)$ and $g(a+x) = G(x)$ for all $x > 0$. We may also assume that a, F, G have been chosen so that $G(z)$ is bounded on $\{z \in S(a,\psi) : |z| \leq 1\}$. Then by 3.3 (after reducing ψ slightly if necessary) we may assume that $|G|$ is bounded by a polynomial in $|z|$ on $S(a, \psi)$ and hence that $0 < f(x) < C \cdot \log x$ for all sufficiently large $x \in \mathbb{R}$ and some constant $C > 0$. By reducing ψ again (though, of course, still maintaining that $\frac{\pi}{2} < \psi$) it follows from 3.5 that for some $b > 0$, the imaginary part of $F(z)$ is bounded for $z \in S(b, \psi)$.

Thus $F(x+iy) = \sigma(x,y) + i\tau(x,y)$, where σ, τ are definable real functions of x, y for $x + iy \in S(b, \psi)$ and where τ is bounded. Now $\sin \restriction [-X, X]$ and $\cos \restriction [-X, X]$ are restricted analytic functions (for any fixed $X > 0$) from which it follows that $\cos \tau(x,y)$ and $\sin \tau(x,y)$ are definable functions of x, y for $x + iy \in S(b, \psi)$, and hence that $\exp(F(z))$ is a definable function of z for $z \in S(b, \psi)$. But then $\exp(F(z+b))$ is a definable function of z for $z \in S(0, \psi)$, and since it is equal to $\exp(f(a+b+z))$ for z real and positive, it follows that $\exp(f) \in \tilde{\mathcal{F}}$, as required.

3.6 Theorem

$\mathcal{R}_{poly} = \tilde{\mathcal{F}}$ and all functions in $\mathcal{F}_{poly}$ do indeed have polynomial growth (cf 1.10).

Proof. We have just shown that $\tilde{\mathcal{F}}$ satisfies all the closure conditions 1.1-1.7, so $\mathcal{R}_{poly} \subseteq \tilde{\mathcal{F}}$ since $\mathcal{R}_{poly}$ is the smallest subcollection of $\mathcal{F}$ satisfying these conditions. Further, it follows from 3.3 that every function in $\tilde{\mathcal{F}}$ is polynomially bounded, i.e. it lies in $\mathcal{F}_{poly}$. So $\mathcal{R}_{poly} \subseteq \tilde{\mathcal{F}} \subseteq \mathcal{F}_{poly}$.

It remains to show that $\tilde{\mathcal{F}} \subseteq \mathcal{R}_{poly}$, so let $f \in \tilde{\mathcal{F}}$. Then f is certainly $\mathcal{R}_{poly}$-bounded, since it is polynomially bounded. So, by 1.9, there is some $h \in \mathcal{R}_{poly}$ such that $h - f \in \mathfrak{m}_{poly}$. Then f and h are both in $\tilde{\mathcal{F}}$, so if $f \neq h$ then $(f-h)^{-1}$ would be too (by 1.2 and 1.3 for $\tilde{\mathcal{F}}$). But this is impossible since $(f-h)^{-1} \notin \mathcal{F}_{poly}$. So $f = h$ and $f \in \mathcal{R}_{poly}$ as required. □

Clearly 1.11 now follows from 3.6.

4 Complex continuation of functions in $\mathcal{F}$: the non-definable case.

We now want to find complex continuations (to a right half plane) of functions $f \in \mathcal{F}$ which have greater than polynomial growth. We know from the results of the last section that such continuations cannot be definable. We first investigate the subexponential case. To this end, let L denote the subcollection of $\mathcal{R}_{poly}$ consisting of the functions of sublinear growth:

$$L := \{g \in \mathcal{R}_{poly} : \frac{g(x)}{x} \to 0 \quad \text{as} \quad x \to \infty\},$$

and define

$$\mathcal{F}_{subexp} := \{f \in \mathcal{F} : \text{ for some } \ g \in L, \ |f| < \exp(g)\}.$$

Now, working within the $\mathcal{L}_{an}^{pow}$-structure $\mathcal{F}$, let

$$\mathcal{R}_{subexp} := \mathcal{R}_{poly}\langle\{\exp(g) : g \in L\}\rangle.$$

We have the following analogue of 1.9.

4.1 Theorem (F-V. Kuhlmann and S. Kuhlmann, see [7])

The structure $\mathcal{R}_{subexp}$ is log-closed and exp-closed from below (in $\mathcal{F}$). Further, $\mathcal{F}$ is $\mathcal{R}_{subexp}$-conservative. Thus $\mathcal{R}_{subexp}$ is a copy of the residue field of the valuation ring $\mathcal{F}_{subexp}$.

Proof. This follows immediately from 2.5 with $\mathbb{M} = \mathcal{F}$, $\mathcal{M} = \mathcal{R}_{poly}$ and $G = L$. □

We shall prove the following

4.2 Theorem

Let $f \in \mathcal{R}_{subexp}$. Then there exists $a \in \mathbb{R}$ such that f has a complex analytic continuation to $S(a, \frac{\pi}{2})$. Further, denoting this continuation by $\hat{f}$, we have that if $f(x) \to 0$ as $x \to \infty$, then $\hat{f}(z) \to 0$ as $|z| \to \infty$ for $z \in S(a, \frac{\pi}{2})$.

Our proof relies heavily on the characterization of the $\mathcal{L}_{an}^{pow}$-definable, unary functions as given by Miller in [10]. In fact we need a version that is uniform in parameters and this can be found in [2]. We apply the result-the Expansion Theorem from section 4 of [2]- in the following form.

4.3 Proposition (van den Dries, Miller [2])

Let $\mathcal{M}_1$, $\mathcal{M}_2$ be models of the theory T_{an}^{pow} with $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Let $t \in \mathcal{M}_2$ be greater than every element of $\mathcal{M}_1$ and let $\omega \in \mathcal{M}_1\langle t\rangle$. Then there exist

(i) a real analytic function $F : (-1-\epsilon, 1+\epsilon)^m \times (-\epsilon, \epsilon)^d \to \mathbb{R}$ for some positive $\epsilon \in \mathbb{R}$ and some $m, d \in \mathbb{N}$ (and we use the same symbol, F, to denote its interpretation in $\mathcal{M}_1$ and in $\mathcal{M}_2$),

(ii) real numbers $r_0, \ldots, r_d$, with $r_1, \ldots, r_d$ positive, and

(iii) elements $\alpha, \beta_1, \ldots, \beta_m, \gamma$ of $\mathcal{M}_1$ with $\alpha > 0, -1 \leq \beta_1, \ldots, \beta_m \leq 1$, and $\gamma \geq 1$ such that $F(\beta_1, \ldots, \beta_m, 0, \ldots, 0) \neq 0$ and

$$\omega = \alpha \cdot t^{r_0} \cdot F(\beta_1, \ldots, \beta_m, (\frac{\gamma}{t})^{r_1}, \ldots, (\frac{\gamma}{t})^{r_d}).$$

(I have applied the van den Dries-Miller Expansion Theorem (at $+\infty$ rather than at 0^+) in the following way. First of all, choose an $\mathcal{L}_{an}^{pow}$-definable (without parameters) $(p+1)$-ary function G and $\alpha_1, \ldots, \alpha_p \in \mathcal{M}_1$ such that $G(\alpha_1, \ldots, \alpha_p, t) = \omega$. Working in $\mathbb{R}_{an}^{pow}$, apply the Expansion Theorem with $f = G$ and $A = \mathbb{R}^p$ to obtain a partition, $A_1, \ldots, A_l$ say, of $\mathbb{R}^p$ such that G has a uniform expansion on each A_i. (We may ignore the “eventually 0” case.) Now interpret the A_i's in $\mathcal{M}_1$ and choose the $i_0 \in \{1, \ldots, l\}$ such that $\langle \alpha_1, \ldots, \alpha_p\rangle \in A_{i_0}$. Now, back in $\mathbb{R}_{an}^{pow}$, choose the data as given in (1), (2) and (3) of the definition of a function having a uniform expansion on A_{i_0} (the first definition of section 4 of [2]). Then interpreting the resulting definable maps, $a, b_1, \dots, b_m, c$ in $\mathcal{M}_2$, I have set $\alpha := a(\alpha_1, \dots \alpha_p)$, $\gamma := c(\alpha_1, \dots \alpha_p)$ and $\beta_j := b_j(\alpha_1, \dots \alpha_p)$ for $j = 1, \dots m$.)

4.4 Remark

We may simplify the representation of ω in 4.3(iii) as follows. For each $j = 1, \dots, m$, let β_j^0 be the standard part of β_j, i.e. β_j^0 is the unique real numbers satisfying $(\beta_j - \beta_j^0) \in \mu$. Then by substituting $X_j + \beta_j^0$ for the j'th variable X_j of F and replacing β_j by $\beta_j - \beta_j^0$, we may assume (provided that we restrict the domain of F to the box $(-\epsilon, \epsilon)^{m+d}$) that $\beta_1, \dots, \beta_m \in \mu$. Note also that $(\frac{\gamma}{t})^{r_1}, \dots, (\frac{\gamma}{t})^{r_d} \in \mu$.

Before proceeding with the proof of 4.2 we require the following version of the Phragmén-Lindelöf method.

4.5 Lemma

Let $g \in L$, $g > 0$, and let $\hat{g} : S(0, \psi) \to \mathbb{C}$ be the definable, complex analytic continuation of a suitable translate of g, as given by 3.6 (for some $\frac{\pi}{2} < \psi < \pi$). Suppose that $g(x) \to \infty$ as $x \to \infty$ (but recall that $g(x)/x \to 0$ as $x \to \infty$). Then $Re(\hat{g}(z)) \to \infty$ as $|z| \to \infty$ for $z \in S(0, \frac{\pi}{2})$.

Proof. If the conclusion were false, then for some $C \in \mathbb{R}$ we would have $Re(\hat{g}(z)) \leq C$ for an unbounded set of $z \in S(0, \frac{\pi}{2})$. Since $Re(\hat{g})$ is definable and $Re(\hat{g}(z)) = Re(\hat{g}(\bar{z}))$ for all $z \in S(0, \frac{\pi}{2})$ (the bar denotes complex conjugation), it follows that there exists a definable function $\eta : [0, \infty) \to S(0, \frac{\pi}{2}) : t \mapsto \eta_1(t) + i\eta_2(t)$, with η_2 positive, such that $Re(\hat{g}(\eta(t))) \leq C$ and $Re(\hat{g}(\overline{\eta(t)})) \leq C$ for all $t > 0$, and, further, $\max\{\eta_1(t), \eta_2(t)\} \to \infty$ as $t \to \infty$. By o-minimality we may assume η_1 and η_2 are both continuous and monotonic (or constant).

Now consider the region $\Sigma \subseteq S(0, \frac{\pi}{2})$ bounded by the two curves $\{\eta(t) : t > 0\}$, $\{\overline{\eta(t)} : t > 0\}$ and by the vertical line $\{\eta_1(0) + iy : -\eta_2(0) \leq y \leq \eta_2(0)\}$ and which contains the interval $(\eta_1(0), \infty)$ of the real axis. Let $G(z) := \exp(\hat{g}(z))$. Then $G(z)$ is bounded for z on the boundary of Σ. Further, we may clearly suppose (because g, and therefore (exercise) any translate of g, lies in L) that, for sufficiently large $b > 0$, the function $F : S(0, \psi) \to \mathbb{C}$ given by $F(z) := \hat{g}(z + b)/(z + b)$ satisfies the hypotheses of of 3.4 with $\alpha = 0$. It follows that $F(z) \to 0$ as $|z| \to \infty$ for $z \in S(0, \psi)$ (possibly after reducing ψ slightly). In particular, $Re(\hat{g}(z)) = o(|z|)$, and hence $G(z) = \exp(o(|z|))$, as $|z| \to \infty$ for $z \in S(0, \frac{\pi}{2})$. We now apply the Phragmén-Lindelöf method in a more subtle form than was used in the proof of 3.4. For this see section 5.6.2

of [14] and the comments immediately preceeding this section. The method tells us that G must in fact be bounded (by C) throughout Σ. In particular, $G(x)$ is bounded for $x \in (\eta_1(0), \infty)$. However, this contradicts the facts that for some $a \in \mathbb{R}$ we have that $G(x) = \exp(g(x+a))$ for all $x > 0$ and, by hypothesis, $g(x) \to \infty$ as $x \to \infty$. $\square$

Proof of 4.2 Let $\omega \in \mathcal{R}_{subexp}$. Then there exists $k \in \mathbb{N}$ such that $\omega \in \mathcal{R}_{poly}\langle \exp(g_1), \ldots, \exp(g_k)\rangle$ for some $g_1, \ldots, g_k \in L$. Just as in the proof of 2.4, we may assume that for all positive $s_0, \ldots, s_k \in \mathbb{R}$ we have

$$s_0 \log(\iota) < s_1 g_1 < \cdots < s_k g_k \qquad (*)$$

(Recall that every element of $\mathcal{R}_{poly}$ is bounded by some (finite) power of the identity function $\iota : \mathbb{R} \to \mathbb{R} : x \mapsto x$. I will also use, without further mention, the immediate consequence of (*) and the polynomial boundedness of the theory T_{an}^{pow}, that for any subset $Q \subseteq \{1, \ldots, k\}$, the value group of $\mathcal{R}_{poly}\langle\{\exp(g_q) : q \in Q\}\rangle$ is generated over $\nu(\mathcal{R}_{poly})$, as an $\mathbb{R}$-vector space, by $\{\nu(\exp(g_q)) : q \in Q\}$.)

We may assume, by induction, that the required result holds for all $h \in \mathcal{R}_{poly}\langle \exp(g_1), \ldots, \exp(g_{k-1}\rangle$. That is, for some $a > 0$, the function h has a complex analytic continuation, $\hat{h}$ say, to $S(a, \frac{\pi}{2})$, and if $h(x) \to 0$ as $x \to \infty$, then $\hat{h}(z) \to 0$ as $|z| \to \infty$ for $z \in S(a, \frac{\pi}{2})$. (The case $k = 0$ is covered by the results of section 3.)

We now apply 4.3 and 4.4 with $\mathcal{M}_1 = \mathcal{R}_{poly}\langle \exp(g_1), \ldots, \exp(g_{k-1})\rangle$, $\mathcal{M}_2 = \mathcal{R}_{poly}\langle \exp(g_1), \ldots, \exp(g_k)\rangle$ and $t = \exp(g_k)$.

Thus, for some $\alpha, \gamma \in \mathcal{M}_1$ with $\gamma \geq 1$, some $\beta_1, \ldots, \beta_m \in \mathcal{M}_1 \cap \mu$, some positive $r_1, \ldots, r_d \in \mathbb{R}$ and some $r_0 \in \mathbb{R}$ we have that

$$\omega = \alpha \cdot \exp(r_0 g_k) \cdot F(\beta_1, \ldots, \beta_m, (\gamma \exp(-g_k))^{r_1}, \ldots, (\gamma \exp(-g_k))^{r_d}) \quad (**)$$

where F is (the extension to $\mathcal{M}_2$ of) a real analytic function mapping $(-\epsilon, \epsilon)^{m+d}$ to $\mathbb{R}$ for some positive $\epsilon \in \mathbb{R}$. We also have that $F(\beta_1, \ldots, \beta_m, 0, \ldots, 0)$ is a nonzero element of $\mathcal{M}_1$. Further, we may suppose that F extends to a complex analytic function on $\{\bar{z} \in \mathbb{C}^{m+d} : ||\bar{z}|| \leq \epsilon\}$.

Now by our inductive hypothesis, we know that $\alpha, \gamma, \beta_1, \ldots, \beta_m$, and g_k all have the required complex analytic extensions, $\hat{\alpha}, \hat{\gamma}, \hat{\beta_1}, \ldots, \hat{\beta_m}, \hat{g_k} : S(a, \frac{\pi}{2}) \to \mathbb{C}$, for sufficiently large $a > 0$. (Of course, the inductive hypothesis is not required here for g_k since it is in $\mathcal{R}_{poly}$).) Further, since each β_j is in μ, i.e. $\beta_j(x) \to 0$ as $x \to \infty$, our inductive hypothesis also tells us that

$\beta_j(z) \to 0$ as $|z| \to \infty$ for $z \in S(a, \frac{\pi}{2})$. So to obtain the required continuation of ω it only remains to show that $\hat{\gamma}(z)\exp(-\hat{g_k}(z)) \to 0$ as $|z| \to \infty$, for then we may increase a so that $\hat{\gamma}(z) \neq 0$ and $||\langle \hat{\beta_1}(z), \ldots, \hat{\beta_m}(z), (\hat{\gamma}(z)\exp(-\hat{g_k}(z)))^{r_1}, \ldots, (\hat{\gamma}(z)\exp(-\hat{g_k}(z)))^{r_d}\rangle|| < \epsilon$ for all $z \in S(a, \frac{\pi}{2})$, and then (**) naturally provides the required continuation.

Let us first observe that $\gamma(x)\exp(-g_k(x)) \to 0$ as $x \to \infty$ (by (*) and the parenthetical comments immediately following (*)). Thus $g_k(x) - \log(\gamma(x)) \to \infty$ as $x \to \infty$. We shall be done if we can show that $Re(\hat{g_k}(z) - \log(\hat{\gamma}(z)) \to \infty$ as $|z| \to \infty$ (for $z \in S(a, \frac{\pi}{2})$). Unfortunately, we cannot apply 4.5 directly because the function $g_k - \log(\gamma)$ may not lie in $\mathcal{R}_{poly}$, and hence not in L. However, $\mathcal{M}_1$ is log-closed and $\mathcal{R}_{poly}$-conservative (by 2.4 with $\mathbb{M} = \mathcal{F}$ and $\mathcal{M} = \mathcal{R}_{poly}$). Hence $\log(\gamma) \in \mathcal{M}_1$ and since

$$0 \leq \log(\gamma) < g_k \in L \subseteq \mathcal{R}_{poly} \qquad (***)$$

we also have that $\log(\gamma)$ is an $\mathcal{R}_{poly}$-bounded element of $\mathcal{M}_1$. So there exists a (unique) element, δ say, of $\mathcal{R}_{poly}$ such that $\delta - \log(\gamma) \in \mathfrak{m}_{poly}$. In fact, we only require from this that $\delta(x) - \log(\gamma(x)) \to 0$ as $x \to \infty$, that $\delta \in L$ (using (***)) and, upon setting $g := g_k - \delta$, that $g \in L$ and $g(x) \to \infty$ as $x \to \infty$. So now we may apply 4.5 (after suitably translating g so that it is defined on $S(0, \frac{\pi}{2})$) to conclude that $Re(\hat{g}(z)) \to \infty$ as $|z| \to \infty$ for $z \in S(a, \frac{\pi}{2})$.

We now invoke the principle of analytic continuation. This implies that the operation $f \mapsto \hat{f}$, of extending the functions in $\mathcal{M}_1$ analytically to $S(a, \frac{\pi}{2})$, commutes with the $\mathbb{R}$-algebra operations, with log and with exp (where defined). Thus, for example, we have that $\hat{g} = \hat{g_k} - \hat{\delta}$ and $\hat{h} = \hat{\delta} - \log(\hat{\gamma})$, where $h := \delta - \log(\gamma)$. So, $Re(\hat{g_k}(z) - \hat{\delta}(z)) \to \infty$ and, by a further use of the inductive hypothesis, $\hat{\delta}(z) - \log(\hat{\gamma}(z)) \to 0$ as $|z| \to \infty$, for $z \in S(a, \frac{\pi}{2})$. Clearly these facts imply that $Re(\hat{g_k}(z) - \log(\hat{\gamma}(z))) \to \infty$ as $|z| \to \infty$, for $z \in S(a, \frac{\pi}{2})$, which completes the first half of the proof. Notice that the argument shows that for any $\eta \in \mathcal{M}_1$ and any positive $r \in \mathbb{R}$

$$\hat{\eta}(z)\exp(-r\hat{g_k}(z)) \to 0 \text{ as } |z| \to \infty \text{ for } z \in S(a, \frac{\pi}{2}) \qquad (\dagger).$$

For the second half of the proof, we keep the same notation and now assume that $\omega(x) \to 0$ as $x \to \infty$ (i.e. $\omega \in \mu$), and we must show that $\hat{\omega}(z) \to 0$ as $|z| \to \infty$ for $z \in S(a, \frac{\pi}{2})$.

By Schwarz's Lemma (see section 5.2 of [14]) and induction on d, there exist a positive $K \in \mathbb{R}$ such that for all $\langle w_1, \ldots, w_m, z_1, \ldots, z_d\rangle \in (-\epsilon, \epsilon)^{m+d}$

$$|F(w_1, \ldots, w_m, z_1, \ldots, z_d) - F(w_1, \ldots, w_m, 0, \ldots, 0)| \leq K \cdot (|z_1| + \cdots + |z_d|) \quad (\dagger\dagger)$$

Applying this (for real arguments) in the structure $\mathcal{M}_2$, we see immediately from $(*)$ and the fact that $F(\beta_1, \ldots, \beta_m, 0, \ldots, 0)$ is a nonzero element of $\mathcal{M}_1$, that $r_0 \leq 0$.

I do the harder case that $r_0 = 0$, after which the reader will easily be able to complete the details of the case that $r_0 < 0$.

Thus, if $r_0 = 0$ then $(\dagger\dagger)$ and $(*)$ imply that $\nu(\omega) = \nu(\alpha \cdot \beta)$, where $\beta := F(\beta_1, \ldots, \beta_m, 0, \ldots, 0)$. (Recall that ν denotes the usual valuation of $\mathcal{F}$.) So $\alpha \cdot \beta \in \mu$. But then, by our inductive hypothesis, $\hat{\alpha}(z) \cdot \hat{\beta}(z) \to 0$ as $|z| \to 0$ for $z \in S(a, \frac{\pi}{2})$. Now by $(**)$ and $(\dagger\dagger)$ we obtain that for all $z \in S(a, \frac{\pi}{2})$, $|\hat{\omega}(z)|$ is bounded by

$$|\hat{\alpha}(z)\hat{\beta}(z)| + K|\hat{\alpha}(z)|(|(\hat{\gamma}(z)\exp(-\hat{g}_k(z)))^{r_1}| + \ldots + |(\hat{\gamma}(z)\exp(-\hat{g}_k(z)))^{r_d}|).$$

We have already seen that the first main summand here converges to 0 as $|z| \to \infty$ (for $z \in S(a, \frac{\pi}{2})$). That the second one does too follows immediately from $(\dagger)$. Thus $\hat{\omega}(z) \to 0$ as $|z| \to \infty$ (for $z \in S(a, \frac{\pi}{2})$) as required.

This completes the proof of 4.2.

We now take our final step up the $\mathcal{F}$-hierarchy, namely to the functions of exponential growth:

4.6 Definition

(i) $\mathcal{F}_{exp} := \{f \in \mathcal{F} : \text{ for some } c \in \mathbb{R}, |f| \leq \exp(c \cdot \iota)\}$.

(ii) $\mathcal{R}_{exp} := \mathcal{R}_{subexp}\langle \exp(\iota) \rangle$.

Clearly $\mathcal{R}_{exp} \subseteq \mathcal{F}_{\mathrm{exp}}$ and $\mathcal{F}_{exp}$ is the convex closure of $\mathcal{R}_{exp}$ in $\mathcal{F}$. Further, we have the following

4.7 Theorem (F-V. Kuhlmann and S. Kuhlmann, see [7])

$\mathcal{R}_{exp}$ is log-closed, exp-closed from below and $\mathcal{F}$ is $\mathcal{R}_{exp}$-conservative. Thus $\mathcal{R}_{exp}$ is (isomorphic to) the residue field of the valuation ring $\mathcal{F}_{exp}$. In particular, for all $f \in \mathcal{F}_{exp}$ and all positive $R \in \mathbb{R}$, there exist $g \in \mathcal{R}_{exp}$ and $B \in \mathbb{R}$ such that $|f(x) - g(x)| < \exp(-Rx)$ for all $x > B$.

Proof. Let $G := \{f \in \mathcal{R}_{poly} : \text{for some } c \in \mathbb{R}, |f| < c \cdot \iota\}$. Then by 2.5 (with $\mathbb{M} = \mathcal{F}$ and $\mathcal{M} = \mathcal{R}_{poly}$) we have that $\mathcal{R}_{poly}\langle\{\exp(f) : f \in G\}\rangle$ is log-closed and exp-closed from below. Further, every element f of G may clearly be written (uniquely) in the form $f = g + r\iota$ where $r \in \mathbb{R}$ and $g \in L$ (cf. the first paragraph of section 4), and it follows immediately from this that $\mathcal{R}_{poly}\langle\{\exp(f) : f \in G\}\rangle = \mathcal{R}_{subexp}\langle\exp(\iota)\rangle$, and hence that $\mathcal{R}_{exp}$ is log-closed and exp-closed from below.

So we may now apply 2.6 with $\mathcal{M} = \mathcal{R}_{exp}$ (and $\mathbb{M} = \mathcal{F}$). Since (as previously observed) $\mathcal{M}^\dagger = \mathcal{F}$, it follows that $\mathcal{F}$ is $\mathcal{R}_{exp}$-conservative. □

The following result will be needed in our final section.

4.8 Corollary Let $f \in \mathcal{F}_{exp}$. Then for any positive $R \in \mathbb{R}$, there exist $N \in \mathbb{N}$, $s_1, \ldots, s_N \in \mathbb{R}$, $f_1, \ldots, f_N \in \mathcal{R}_{subexp}$ and $B \in \mathbb{R}$ such that for all $x > B$,

$$|f(x) - \sum_{j=1}^{N} f_j(x) \cdot \exp(s_j x)| < \exp(-Rx).$$

Proof. By 4.7 we may assume that $f \in \mathcal{R}_{exp}$. We apply 4.3 and 4.4 with $\mathcal{M}_1 = \mathcal{R}_{subexp}$, $\mathcal{M}_2 = \mathcal{F}$ and $t = \exp(\iota)$, so that $f \in \mathcal{M}_1\langle t\rangle$. Thus we obtain $r_0, \ldots, r_d \in \mathbb{R}$, with $r_1, \ldots, r_d$ positive, elements $\alpha, \beta_1, \ldots, \beta_m, \gamma \in \mathcal{M}_1$ with $\alpha > 0$, $\gamma \geq 1$, and $\beta_1, \ldots, \beta_m \in \mu$ such that (in $\mathcal{M}_2$)

$$f = \alpha \cdot t^{r_0} \cdot F(\beta_1, \ldots, \beta_m, (\frac{\gamma}{t})^{r_1}, \ldots, (\frac{\gamma}{t})^{r_d}) \qquad (*)$$

where F is the interpretation in $\mathcal{M}_2$ of some real analytic function (also denoted) $F : (-\epsilon, \epsilon)^{m+d} \to \mathbb{R}$, for some $\epsilon \in \mathbb{R}$ with $0 < \epsilon < 1$.

Now let a positive $R \in \mathbb{R}$ be given and choose $p \in \mathbb{N}$ so that $p \cdot \min\{r_1, \ldots, r_d\} - r_0 > R$. Let $\overline{a} \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})^m$. Then by applying Taylor's theorem around $\overline{0} \in \mathbb{R}^d$ (with a suitable form of the remainder), we have that for all $\overline{y} = \langle y_1, \ldots, y_d\rangle \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})^d$

$$|F(\overline{a}, \overline{y}) - \sum_{|\lambda| \leq p} \frac{F_\lambda(\overline{a}, \overline{0})}{\lambda!} \overline{y}^\lambda| \leq C \cdot \max\{\prod_{l=1}^{d} |y_l|^{\lambda_l} : |\lambda| = p + 1\} \qquad (**)$$

where C is some positive constant (depending only on F and p). (I have used the usual multi-index notation for $\lambda = \langle \lambda_1, \ldots, \lambda_d \rangle \in \mathbb{N}$ and the derivatives F_λ of F are taken with respect to the last d variables. Also, 0^0 is taken to be 1.)

We now apply (**) in $\mathcal{M}_2$ with $\overline{a} = \langle \beta_1, \ldots, \beta_m \rangle$ and $\overline{y} = \langle (\frac{\gamma}{t})^{r_1}, \ldots, (\frac{\gamma}{t})^{r_d} \rangle$ (which are both tuples of infinitesimals). After multiplying (**) by $\alpha \cdot t^{r_0}$ (which is positive) and using (*) we obtain

$$|f - \sum_{|\lambda| \leq p} f^\lambda \cdot t^{s_\lambda}| \leq A \cdot t^{s_{\lambda'}} \qquad (***)$$

where, for $\lambda \in \mathbb{N}^d$,

$$s_\lambda := r_0 - (r_1\lambda_1 + \cdots + r_d\lambda_d) \in \mathbb{R}$$

and if $|\lambda| \leq p$,

$$f^\lambda := \alpha \cdot \frac{F_\lambda(\beta_1, \ldots, \beta_m, \overline{0})}{\lambda!} \in \mathcal{M}_1.$$

Further,

$$A := \alpha \cdot C \cdot \gamma^{r_0 - s_{\lambda'}} \in \mathcal{M}_1$$

where λ' is chosen so that $|\lambda'| = p + 1$ and $(\frac{\gamma}{t})^{r_0 - s_{\lambda'}} \geq (\frac{\gamma}{t})^{r_0 - s_\lambda}$ for each λ with $|\lambda| = p + 1$.

Now $s_{\lambda'} \leq r_0 - (p+1)\min\{r_1, \ldots, r_d\} < -R - r_q$ for some $q = 1, \ldots, d$, by the choice of p. But $A \cdot t^{-r_q} < 1$ (since $A \in \mathcal{M}_1$) so we see from (***) that

$$|f - \sum_{|\lambda| \leq p} f^\lambda \cdot t^{s_\lambda}| \leq t^{-R}$$

which, upon noting the definition of equality and inequality in $\mathcal{M}_2$ $(= \mathcal{F})$ (!), clearly implies the required result. $\square$

As we have already suggested, we cannot replace $\mathcal{R}_{subexp}$ by $\mathcal{R}_{exp}$ in 4.2. For example let $g(x) = \exp(\sqrt{x} - x)$ for $x > 3$. Then $g \in \mathcal{R}_{exp}$ and $g(x) \to 0$ as $x \to \infty$, but the analytic continuation $\hat{g}$ of g to $S(a, \frac{\pi}{2})$ (for any $a \geq 3$) is given by $\hat{g}(z) = \exp(\sqrt{z} - z)$ (where the square root is taken to be positive on the positive real axis), and $|\hat{g}(a + 1 + it)| \to \infty$ as $t \to \infty$. So it is certainly not the case that $\hat{g}(z) \to 0$ as $|z| \to \infty$ for $z \in S(a, \psi)$, and hence the second part of 4.2 fails.

It is now easy to produce an example for which the first part of 4.2 fails: just let $F : \Delta(0; 1) \to \mathbb{C}$ be any analytic function on the unit disk in $\mathbb{C}$ having $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ as a natural boundary. Then $F \circ g \in \mathcal{R}_{exp}$ (note that $|g(x)| < \frac{1}{2}$ for $x > 3$) but any analytic continuation of $F \circ g$ to a sector $S(a, \frac{\pi}{2})$ would result in a continuation of F across S^1.

However, we do have the following result. I only sketch the proof since it will not be needed (and, indeed, is not strong enough) for our diophantine application.

4.2 Theorem
Let $f \in \mathcal{R}_{exp}$ and let $\psi \in \mathbb{R}$ satisfy $0 < \psi < \frac{\pi}{2}$. Then there exists $a \in \mathbb{R}$ such that f has a complex analytic continuation to $S(a, \psi)$. Further, denoting this continuation by $\hat{f}$, we have that if $f(x) \to 0$ as $x \to \infty$, then $\hat{f}(z) \to 0$ as $|z| \to \infty$ for $z \in S(a, \psi)$.

Proof. With f, ψ as in the hypotheses, let $r = \frac{2\psi}{\pi}$, $\rho_0(x) = x^r$ and $\rho_1(x) = x^{\frac{1}{r}}$ for $x > 0$. Then $0 < r < 1$ and one easily checks that $f \circ \rho_0 \in \mathcal{R}_{subexp}$. So by 4.2 there exists an analytic continuation, $H : S(a, \frac{\pi}{2}) \to \mathbb{C}$ say, of $f \circ \rho_0$ (for some $a > 0$). Now the analytic continuation $\hat{\rho}_1$ of ρ_1 to $S(a, \psi)$ takes values in $S(a, \frac{\pi}{2})$, and since $H \circ \rho_1(x) = f(x)$ for all $x > 0$, the function $H \circ \hat{\rho}_1 : S(a, \psi) \to \mathbb{C}$ is the required continuation.

I leave the proof of the second part of the theorem as an exercise. □

5 Diophantine questions

In this section we consider the problem of characterizing those functions in $\mathcal{F}$ that take integral values for all sufficiently large integral arguments.

Let $f \in \mathcal{F}$ be such a function. Say $f(a) \in \mathbb{Z}$ for all $a \geq B$. (The letter a, as well as n and j, will always range over $\mathbb{N}$ from now on.) Following Pólya ([12]), we consider the n^{th}-difference function $\Delta^n f$ defined for all n and all $a \geq B$ by

$$\Delta^n f(a) := \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!} f(a+j) \qquad (*)$$

(One easily checks that Δ^n is the n^{th} iterate of the difference operator Δ defined by $\Delta f(a) := f(a+1) - f(a)$.)

One expects that Δf grows more slowly than f (e.g. if f is a polynomial then Δf is a polynomial of one degree lower) and the idea is to use additional analytic information about f in order to show that for fixed $a \geq B$, $\Delta^n f(a) \to 0$ as $n \to \infty$. Since, obviously, $\Delta^n f(a)$ is integer valued as a function of n, it follows that it must be 0 for sufficiently large n. But this easily implies, as Pólya observed, that f is a polynomial (on $\mathbb{N} \cap (B, \infty)$ and hence, in our o-minimal situation, on (B, ∞).).

Notice, however, that the function $f(x) = 2^x$ is a fixed point for the difference operator and so provides a natural limitation of the method. I shall prove the following

5.1 Theorem

Let $f \in \mathcal{F}$ and assume that $f(n) \in \mathbb{Z}$ for all sufficiently large n. Suppose that r is a real number satisfying $0 < r < 1$ and that $|f(x)| \leq 2^{rx}$ for all sufficiently large x. Then there exists a polynomial P such that $f(x) = P(x)$ for all sufficiently large x.

We will use 4.8, but a difficulty arises: the fact that two functions, f and g say, have the property that $|f(x) - g(x)|$ is small for large x does not imply that $|\Delta^n f(a) - \Delta^n g(a)|$ is small for large n if a is fixed. However, it will be small if a and n are of the same order of magnitude. So we require a modification of Pólya's observation above. The following will suffice.

5.2 Lemma

Let $f : \mathbb{N} \to \mathbb{R}$ be any function and let K be a positive integer. Assume that for all sufficiently large a, we have that $\Delta^n f(a) = 0$ for all n satisfying $Ka < n \leq K(a+1) + 1$. Then there exists a polynomial P such that $f(x) = P(x)$ for all sufficiently large $x \in \mathbb{N}$.

Proof. Choose $B \in \mathbb{N}$ so that $\Delta^n f(a) = 0$ for all $a \geq B$ and all n with $Ka < n \leq K(a+1) + 1$.

Now fix an arbitrary a with $a \geq B$. Let P_a be the polynomial (over $\mathbb{R}$) of degree at most Ka such that $f(a+x) = P_a(x)$ for $x = 0, \dots, Ka$. Then it follows from (*) that $\Delta^n f(a) = \Delta^n P_a(0)$ for $n = 0, \dots, Ka$. However, since, as remarked above, the difference operator Δ reduces the degree of polynomials, it follows that $\Delta P_a(0) = 0$ for all $n > Ka$, so in fact we have that $\Delta^n f(a) = \Delta^n P_a(0)$ for $n = 0, \dots, K(a+1) + 1$. So using (*) again together with induction on n we see that

$$f(a+n) = P_a(n) \quad \text{for} \quad n = 0, \ldots, K(a+1)+1 \qquad (\dagger)_a$$

In particular, $(\dagger)_a$ implies that $f(a+n+1) = P_a(n+1)$ for $n = 0, \ldots, K(a+1)$, whereas $(\dagger)_{a+1}$ implies that $f(a+1+n) = P_{a+1}(n)$ for $n = 0, \ldots, K(a+2)+1$. Hence $P_a(n+1) - P_{a+1}(n) = 0$ for $n = 0, \ldots, K(a+1)$. However, $P_a(x+1) - P_{a+1}(x)$ is a polynomial in x of degree at most $K(a+1)$, so it must be identically zero. Since this is true for all $a \geq B$ it follows by induction that $P_a(x) = P_B(x+a-B)$ for all $x \in \mathbb{R}$ and all $a \geq B$. But if we now apply $(\dagger)_a$ for an abitrary integer $a \geq B$ and $n = 0$, we see that $f(a) = P_a(0) = P_B(a-B)$, which completes the proof upon setting $P(x) := P_B(x-B)$. □

We require two further lemmas.

5.3 Lemma

Let $f \in \mathcal{R}_{subexp}$. Then for any $\epsilon > 0$, there exists a positive integer a such that f has a complex analytic continuation $\hat{f} : S(a, \frac{\pi}{2}) \to \mathbb{C}$ that satisfies $|\hat{f}(z)| < 2^{\epsilon|z|}$ for all $z \in S(a, \frac{\pi}{2})$.

Proof. We may assume that $f(x) > 0$ for all sufficiently large x.

Let $\phi(x) := \frac{\log f(x)}{x}$.

Since $f \in \mathcal{R}_{subexp}$, we have that $\phi \in \mathcal{R}_{subexp}$ by 4.1 and hence, by 4.2, there exist complex analytic continuations $\hat{f}, \hat{\phi} : S(a, \frac{\pi}{2}) \to \mathbb{C}$ for some sufficiently large a. Further, since $\phi(x) \to 0$ as $x \to \infty$ (by definition of $\mathcal{F}_{subexp}$, which contains $\mathcal{R}_{subexp}$) it follows from 4.2 that $\hat{\phi}(z) \to 0$ as $|z| \to \infty$ for $z \in S(a, \frac{\pi}{2})$. Let $\epsilon > 0$ be given and increase a so that $|\hat{\phi}(z)| < \frac{\epsilon}{2}$ for all $z \in S(a, \frac{\pi}{2})$.

Now, by analytic continuation, $\hat{\phi}(z) = \frac{\log \hat{f}(z)}{z}$ for all $z \in S(a, \frac{\pi}{2})$ and hence, for each such z we have

$$\log |\hat{f}(z)| = Re(\log \hat{f}(z)) \leq |\log \hat{f}(z)| = |z\hat{\phi}(z)| < \frac{\epsilon}{2}|z|,$$

and hence that $|\hat{f}(z)| < \exp(\frac{\epsilon}{2}|z|) < 2^{\epsilon|z|}$, as required. □

5.4 Lemma

Let $f \in \mathcal{R}_{subexp}$, $r \in \mathbb{R}$ with $r < 1$, and set $g(x) := f(x) \cdot 2^{rx}$. Let K be a positive integer satisfying $K > (1 - \frac{1}{2^{1-r}})^{-1} + 2$. Then for all sufficiently large a and all n satisfying $Ka < n \leq K(a+1)+1$ we have $|\Delta^n g(a)| < \frac{1}{a}$.

Proof. Let f, r be as in the hypotheses and set $\epsilon = \min\{\frac{1-r}{2}, \frac{1}{2}\log_2(1+\frac{1}{4K})\}$ so that $0 < \epsilon < \frac{1}{2}$. Choose $N_0 \in \mathbb{N}$ large enough so that the conclusion of 5.3 holds for all $a \geq N_0-3$. Now fix $a > N_0+2$ and n with $Ka < n \leq K(a+1)+1$.

Following Pólya's method ([12]) we use the following formula, which is easily derived from (*) using the residue theorem:

$$\Delta^n g(a) = \frac{1}{2\pi i}\int_\Gamma \frac{n!\cdot 2^{r(z+N_0)}\cdot \hat{f}(z+N_0)\,dz}{(z-(a-N_0))(z-(a-N_0+1))\cdots(z-(a-N_0+n))} \quad (**)$$

where Γ is any simple closed contour in $\mathbb{C}$ (orientated in the anticlockwise direction) surrounding the points $a-N_0, a-N_0+1, \ldots, a-N_0+n$ and such that $Re(z) \geq -2$ for all z on Γ.

We take Γ to be the contour C_n from Lemma 3.2 of Langley's paper [8] (with $s = 2$, $\mu = 0$). That is, Γ consists of the arc Ω_n of the circle $\{z \in \mathbb{C} : |z| = 2n\}$ from the point $-2 - i\sqrt{4n^2-1}$ to the point $-2 + i\sqrt{4n^2-1}$ (traversed in the anticlockwise direction), followed by the straight line segment T_n from $-2 + i\sqrt{4n^2-1}$ down to $-2 - i\sqrt{4n^2-1}$. (Note that $0 \leq a - N_0 < a - N_0 + 1 < \cdots < a - N_0 + n < 2n$, and hence these points do lie within C_n.)

We complete the proof by showing that the integral is bounded in modulus by $\frac{1}{a}$ if a is sufficiently large. We note here that $K \geq 2$ and so $n \geq 2a$.

Now, for z on Ω_n we have, for each $j = 0, \ldots n$, $|z-(a-N_0+j)| \geq 2n - a + N_0 - j \geq (2n-j)(1-\frac{a}{n}) \geq (2n-j)(1-\frac{1}{K})$ and hence

$$|\prod_{j=0}^{n}(z-(a-N_0+j)| \geq (1-\frac{1}{K})^{n+1}\cdot\prod_{j=0}^{n}(2n-j).$$

But $\prod_{j=0}^{n}(2n-j) \geq n!\cdot 2^{2n-1}$ for $n \geq 2$ and so the integrand I_z in (**) satisfies (for z on Ω_n)

$$|I_z| \leq \frac{2^{rN_0}\cdot|2^{rz}|\cdot|\hat{f}(z+N_0)|}{2^{2n-1}\cdot(1-\frac{1}{K})^{n+1}}.$$

However, assuming for the moment that $r \geq 0$, we have that $|2^{rz}| \leq 2^{r|z|} = 2^{2rn}$. Further, taking into account the conclusion of 5.3, we see that $|\hat{f}(z+N_0)| \leq 2^{\epsilon|z+N_0|} \leq 2^{\epsilon N_0}\cdot 2^{2\epsilon n}$.

So, letting A and s denote the constants $\frac{2^{(r+\epsilon)N_0+1}}{(1-\frac{1}{K})}$ and $\frac{2^{2(r+\epsilon)}}{4\cdot(1-\frac{1}{K})}$ respectively, we obtain the bound $|I_z| \leq A\cdot s^n$.

Now $\epsilon \leq \frac{1-r}{2}$ and $K > (1 - \frac{1}{2^{1-r}})^{-1}$ which easily imply that $0 < s < 1$. Further, since the length of the contour Ω_n is $2\pi n$, we obtain

$$|\frac{1}{2\pi i}\int_{\Omega_n} I_z dz| \leq A \cdot n \cdot s^n \leq \frac{1}{n} \leq \frac{1}{2a}$$

for sufficiently large a. Recall that this is under the assumption that $r \geq 0$. However, if $r < 0$ then the term $|2^{rz}|$ is bounded everywhere on C_n by the constant $2^{2|r|}$ and the same argument succeeds even more readily.

It remains to show that $|\frac{1}{2\pi i}\int_{T_n} I_z dz| < \frac{1}{2a}$ for sufficiently large a.

For this we observe that for z on T_n and $j = 1, \ldots, n$, we have that $|z - (a - N_0 + j)| \geq |-2 - (a - N_0 + j)| = a + j + 2 - N_0 \geq j(1 + \frac{a-N_0}{j}) \geq j(1 + \frac{a}{n} - \frac{N_0}{n})$.

However, from the upper bound for n in the hypotheses of the lemma it clearly follows that $\frac{a}{n} \geq \frac{1}{2K}$ for sufficiently large a, and hence that $|z - (a - N_0 + j)| \geq j(1 + \frac{1}{3K})$ for sufficiently large a. Since also $|z - (a - N_0 + j)| \geq 1$ for $j = 0$, we arrive at the lower bound of $n! \cdot (1 + \frac{1}{3K})^n$ for the modulus of the denominator of the integrand I_z (for z on T_n).

As for the numerator of I_z, we note that $|2^{r(z+N_0)}| \leq c$ and $|\hat{f}(z + N_0)| \leq c2^{2\epsilon n}$ (for z on T_n) for some constant c. Since $\epsilon \leq \frac{1}{2}\log_2(1 + \frac{1}{4K})$ we obtain the upper bound $c^2 \cdot n! \cdot (1 + \frac{1}{4K})^n$ for the modulus of the numerator of I_z (for z on T_n). Letting $\sigma := (1 + \frac{1}{4K})(1 + \frac{1}{3K})^{-1}$, so that $0 < \sigma < 1$, we see that, for some constant c',

$$|\frac{1}{2\pi i}\int_{T_n} I_z dz| \leq c' \cdot n \cdot \sigma^n < \frac{1}{n} \leq \frac{1}{2a}$$

for sufficiently large a, as required.

□

We can now complete the proof of 5.1 as follows.

Let $f \in \mathcal{F}$ and r be as in the hypotheses. Let $K = [(1 - \frac{1}{2^{1-r}})^{-1} + 3]$ and apply 4.8 with $R = 2K$. With the notation of 4.8, let

$$H(x) := f(x) - \sum_{j=0}^{N} f_j(x) \cdot \exp(s_j x)$$

so that for sufficiently large x, say for $x \geq B$, we have that $|H(x)| < \exp(-2Kx)$.

Since Δ^n is a linear operator we have, for any positive integers a, n,

$$\Delta^n f(a) = \Delta^n H(a) + \sum_{j=0}^{N} \Delta^n g_j(a) \qquad (\dagger)$$

where $g_j(x) := f_j(x) \cdot \exp(s_j x)$ for $j = 1, \ldots, n$.

We want to show that $\Delta^n f(a) = 0$ for sufficiently large a and for all n satisfying $a < n \leq K(a+1)+1$, for then 5.1 follows from 5.2.

Since $\Delta^n f(a) \in \mathbb{Z}$ for all a and n, it is sufficient to show that $|\Delta^n f(a)| < 1$ for a, n in the stated range.

Assume that $B+2 < a < n \leq K(a+1)+1$. Then the first term in ($\dagger$) may be estimated directly from the formula (*):

$$\begin{aligned}
|\Delta^n H(a))| &\leq \sum_{j=0}^{n} |H(a+j)| \cdot \frac{n!}{j!(n-j)!} \\
&\leq \sum_{j=0}^{n} \exp(-2K(a+j)) \cdot \frac{n!}{j!(n-j)!} \\
&\leq \exp(-2Ka) \cdot \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \\
&= \exp(-2Ka) \cdot 2^n \\
&\leq \exp(-2Ka) \cdot 2^{K(a+1)+1} \\
&< \frac{1}{2} \quad \text{since } a, K \geq 2.
\end{aligned}$$

In view of ($\dagger$), it only remains to show that $|\sum_{j=0}^{N} \Delta^n g_j(a)| < \frac{1}{2}$.

Let $t_j = \frac{s_j}{\log 2}$ for $j = 0, \ldots N$, so that $g_j(x) = f_j(x) \cdot 2^{t_j x}$.

Since we may clearly assume that the t_j's are pairwise distinct, the growth condition on f implies that $t_j \leq r$ for $j = 0, \ldots N$. Further, since each f_j is in $\mathcal{R}_{subexp}$, we may apply 5.4 to g_j. So for all sufficiently large a, say for $a > B_j$, and all n with $a < n \leq K(a+1)+1$, we have $|\Delta^n g_j(a)| < \frac{1}{a}$ (note that $K \geq (1 - \frac{1}{2^{1-t_j}})^{-1} + 2$ because $t_j \leq r$).

It follows that for $a > \max\{B+2, B_0, \ldots, B_N, 2N+3\}$ and $Ka < n \leq K(a+1)+1$ we have that $|\sum_{j=0}^{N} \Delta^n g_j(a)| < \frac{1}{2}$, as required, and the proof of 5.1 is now complete.

As a final remark, I would guess that the growth of the function f in the statement of Theorem 5.1 can be weakened to the condition that $f(x)\cdot 2^{-x} \to 0$ as $x \to \infty$. In fact, it seems reasonable to go further and make the following

Conjecture

Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathbb{R}_{an,exp}$-definable and suppose that $f(n) \in \mathbb{Z}$ for all sufficiently large positive integers n. Assume further that for some $r > 0$ we have that $|f(x)| < \exp(rx)$ for all sufficiently large $x \in \mathbb{R}$. Then there exist a polynomial $P(x, y_1, \ldots, y_m)$ with rational coefficients, and positive real algebraic integers $a_1, \ldots, a_m$ such that $f(x) = P(x, a_1^x, \ldots, a_m^x)$ for all sufficiently large x.

References

[1] L. A. Butler, *Some cases of Wilkie's conjecture*, Bulletin of the London Math. Soc., 44 (2012), 642-660

[2] L. van den Dries and C. Miller, *Extending Tamm's theorem*, Ann. Inst. Fourier, Grenoble 44, 5 (1994),1367-1395.

[3] L. van den Dries, A. Macintyre and D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, Ann. of Math., 140 (1994), 183-205.

[4] L. van den Dries, A. Macintyre and D. Marker, *Logarithmic-exponential power series*, J. London Math. Soc. (2), 56 (1997), 417-434.

[5] G. O. Jones and M. E. M. Thomas, *The density of algebraic points on certain Pfaffian surfaces*, Q. J. Math, 63, no. 3, (2012), 637-651.

[6] G. O. Jones, M. E. M. Thomas and A. J. Wilkie, *Integer-valued definable functions*, Bull. London Math. Soc. 44 (6), (2012), 1285-1291.

[7] S. Kuhlmann, *Ordered Exponential Fields*, The Fields Institute Monograph Series, Vol. 12, AMS Publications, 164 pages (January 2000).

[8] J. K. Langley, *Integer-valued analytic functions in a half-plane*, Comput. Methods Funct. Theory, 7 (2007), 433-442.

[9] C. Miller, *Review of [3] and [4]*, The Bulletin of Symbolic Logic, Vol. 6, No. 2 (Jun., 2000), 213-216.

[10] Chris Miller, *Expansions of the real field with power functions*, Ann. Pure and App. Log., 68 (1994), 79-94.

[11] J. Pila and A. J. Wilkie, *The rational points of a definable set*, Duke Math. J., 133, No.3 (2006), 591-616.

[12] G. Pólya, *Über ganze ganzwertige Funktionen*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, (1921), 1-10.

[13] P. Speissegger, *Lectures on o-minimality*, in 'Lectures on algebraic model theory', Bradd Hart, Matthew Valeriote, editors, Fields Institute Monographs, AMS (2002), 47-65.

[14] E. C. Titchmarsh, *The theory of functions*, second edition (1939) (reprinted 1997) OUP.

[15] A. J. Wilkie, *Diophantine properties of sets definable in o-minimal structures*, J. Symbolic Logic, 69(3) (2004), 851861.