The nonstandard quantum plane

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1 Introduction

Let k be a field of $\mathrm{char}(k) \neq 2$, and suppose that $q \in k$ is not a root of unity. The associated *quantum plane* [5, §IV.1], denoted by $k_q[x, y]$, is defined to be the free k-algebra $k\{x, y\}$ generated by x and y, modulo the relation $yx = qxy$. The set of monomials $\{x^i y^j\}_{i,\, j \geq 0}$ is a basis for the underlying k-vector space, and for every pair $(i,\, j)$ of nonnegative integers, we have

$$y^j x^i \;=\; q^{ij} x^i y^j.$$

There is a natural action on the quantum plane by the quantum group U_q, which is defined to be the k-algebra generated by the four variables E, F, K, K^{-1}, modulo the relations:

$$\begin{array}{rcl} KK^{-1} & = & K^{-1}K \;=\; 1 \;, \\ KEK^{-1} = q^2 E\,, & & KFK^{-1} = q^{-2} F\,, \\ EF - FE & = & \dfrac{K - K^{-1}}{q - q^{-1}}\,. \end{array} \tag{1}$$

Indeed, the quantum plane $k_q[x, y]$ acquires the structure of a left U_q-module where the action of the generators is given by

$$Kx^i y^j = q^{i-j} x^i y^j, \quad Ex^i y^j = [i]\, x^{i-1} y^{j+1}, \quad Fx^i y^j = [j]\, x^{i+1} y^{j-1}\,, \tag{2}$$

and extended linearly; the coefficients are given by $[a] := \dfrac{q^a - q^{-a}}{q - q^{-1}}$. To appreciate the significance of the quantum plane and the representation theory of U_q, consult [4, 5]. This article is devoted to the model-theoretic study of the quantum plane, regarded as a U_q-module. One consequence of the main result Theorem 1.1 is Theorem 5.1, which states that, in the language of left U_q-modules, the ring of definable scalars of the quantum plane is a von Neumann regular epimorphic ring extension of the quantum group U_q.

The action of U_q given by (2) preserves the total degree $i + j$ of the monomial $cx^i y^j$, $c \in k$, so the quantum plane $k_q[x, y]$ decomposes as a U_q-module into a direct sum

$$k_q[x, y] = \bigoplus_{n \geq 0} k_q[x, y]_n,$$

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where $k_q[x,y]_n$ denotes the k-vector space of all homogenous elements in the quantum plane of degree n. Each of the components $k_q[x,y]_n$ is a simple module, i.e., irreducible representation, whose dimension over k is $n+1$. In general, a simple U_q-module V is called *finite dimensional* if the dimension of V as a k-vector space is finite. Every finite dimensional representation of U_q admits a decomposition as a direct sum of simple modules, and for every whole number n there exist (up to isomorphism) exactly two simple representations of dimension $n+1$, denoted $V_{+,n}$ and $V_{-,n}$ [4, Thm 2.6]. In this regard, the representation theory of the quantum group U_q resembles the representation theory of the Lie algebra $sl_2(k)$ where k is an algebraically closed field of characteristic 0. The principal aim of this paper is to prove for U_q the quantum analogues of results obtained by the first author [3] for the universal enveloping algebra $U(L)$ of $L = sl_2(k)$.

The simple representation $V_{+,n}$ is isomorphic to $k_q[x,y]_n$. The other simple representation $V_{-,n}$ of dimension $n+1$ is obtained by composing the action of U_q on $V_{+,n}$ with the automorphism σ (see [4, §5.2]) of U_q determined by

$$\sigma(E) = -E, \quad \sigma(F) = F, \quad \sigma(K) = -K.$$

We will also refer to the module $V_{-,n}$ as $k_q^\sigma[x,y]_n$; and to $k_q^\sigma[x,y]$ as the direct sum of one copy of each $k_q^\sigma[x,y]_n$, $n \geq 0$. This module $k_q^\sigma[x,y]$ is the U_q-module obtained by composing the action of U_q on the quantum plane with the automorphism σ. **Throughout the paper, we will denote by M the module**

$$M = k_q[x,y] \oplus k_q^\sigma[x,y],$$

obtained by taking the direct sum of one copy of each simple representation of U_q, up to isomorphism. Rather than working with the quantum plane directly, it is easier to prove the following theorem for M, and then specialize to the quantum plane in the last section of the paper. To prove the theorem, we follow the general strategy used in [3]. It is shown in [6] that a large portion of this procedure can be carried out effectively.

Theorem 1.1 *The lattice Latt(M) of pp-definable subspaces of M is complemented.*

The third section of this article is devoted to an account of this general strategy, while the second and fourth sections describe how this strategy needs to be amended to suit the case of a quantum group.

The usual model-theoretic language $\mathcal{L}(U_q)$ for U_q-modules has symbols with which to express addition (in a module) and scalar multiplication (of each element of U_q on the module), as well as a constant symbol 0 for the zero element of the module. So the basic (atomic) formulas are just linear equations $r_1u_1 + \ldots + r_nu_n \doteq 0$ with scalars from U_q acting on the left. A system of linear equations can be expressed by a finite conjuction of linear equations, which will be abbreviated as $(A,B)\binom{\mathbf{u}}{\mathbf{v}} \doteq 0$, where $\mathbf{u}$ denotes the tuple $(u_1,\ldots,u_n)$, $\mathbf{v}$ the tuple $(v_1,\ldots,v_k)$, A denotes an $m\times n$ matrix and B an $m \times k$ matrix with entries from U_q. A pp ("positive primitive") formula is obtained by existentially quantifying a system of linear equations, say over the variable $\mathbf{v}$. Formally, it has the shape

$$\varphi(\mathbf{u}) = \exists \mathbf{v} \ (A,\, B) \left(\begin{array}{c} \mathbf{u} \\ \mathbf{v} \end{array} \right) \doteq 0$$

If V is a U_q-module, the set of the solutions in V to the formula $\varphi(\mathbf{v})$ is a k-subspace of V^n. If $\varphi(v)$ is a pp-formula in one free variable v, then this solution set of φ in V is denoted $\varphi(V)$ and is a typical *pp-definable subspace* of V. The collection of pp-definable subspaces of V has the structure of a modular lattice (with respect to inclusion), which is the subject of Theorem 1.1.

Let $\varphi(v)$ be a pp-formula in one free variable and consider a finite dimensional simple representation $V_{\epsilon,n}$ of U_q, where ϵ is $+$ or $-$. By virtue of the fact that $V_{\epsilon,n}$ is finite dimensional, it is easy to find a pp-formula $\psi_{\epsilon,n}(v)$ such that

$$\varphi(V_{\epsilon,n}) \oplus_k \psi_{\epsilon,n}(V_{\epsilon,n}) = V_{\epsilon,n}.$$

Theorem 1.1 states that a complementary pp-formula $\psi(v)$ may be found, which is independent of ϵ and n.

Let V be a representation of U_q. A *definable scalar* of V is a k-linear transformation $\rho_V : V \to V$, whose graph is definable in V by a pp-formula $\rho(u_1, u_2)$ in two variables,

$$V \models \forall u_1 \exists! u_2\, \rho(u_1, u_2).$$

The collection of definable scalars of V has the structure of a ring, denoted by U_V; the operations in U_V are composition and pointwise addition. There is a canonical morphism from the ring U_q to the *ring of definable scalars* U_V, which sends the element r to its action on V, defined by the pp-formula

$$u_2 = ru_1.$$

It follows from a general fact [3, Prop 7] that if the lattice of pp-definable subspaces of the U_q-module V is complemented, then the ring U_V of definable scalars of V is von Neumann regular and that the canonical ring morphism is an epimorphism. Because the quantum plane $k_q[x,y]$ is a direct summand of M, its lattice of pp-definable subspaces is also complemented. The ring of definable scalars of $k_q[x,y]$ is obtained as the quotient ring of the definable scalars of M, modulo the ideal of definable scalars that vanish in $k_q^\sigma[x,y]$. As the quantum plane is a faithful U_q-module, we may identify the quantum group U_q with a subring of its ring of definable scalars.

In general, a left module N over an associative ring R is called *pure injective* or *algebraically compact* if it is *pp-saturated*: every *pp-type* $p^+(v, A)$ (consisting only of pp-formulas) over $A \subseteq N$ which is consistent in N has a realization in N. The *Ziegler spectrum* of R is a topological space $\mathrm{Zg}(R)$ whose points are the pure-injective indecomposable left R-modules; a basis of open subsets is indexed by ordered pairs $\varphi(v)$, $\psi(v)$ of pp-formulas in one variable:

$$\mathcal{O}_{\varphi,\psi} = \{\, U \in \mathrm{Zg}(R) \,:\, U \models \exists v\, (\varphi(v) \wedge \neg\psi(v)) \,\}.$$

If V a finite dimensional representation of U_q, then it is a pure-injective U_q-module and because each $V_{\epsilon,n}$ is simple, it is an indecomposable pure-injective U_q-module and therefore a point in $\mathrm{Zg}(U_q)$.

If N is a left R-module, then the closed subset of N in $\mathrm{Zg}(R)$ is defined to be

$$\mathcal{C}\ell(N) := \bigcap_{N \models \varphi \to \psi} (\mathcal{O}_{\varphi,\psi})^c.$$

For example, the closed subset of the quantum plane $k_q[x,y]$ in $\mathrm{Zg}(U_q)$ is the closure of the points $k_q[x,y]_n = V_{+,n}$, while the closed subset of

$$M = k_q[x,y] \oplus k_q^\sigma[x,y]$$

is the closure of *all* the simple finite dimensional representations $V_{\epsilon,n}$ where $\epsilon = \pm$. It is shown in this article that all but one of the points of the closed set $\mathcal{C}\ell(k_q[x,y])$ associated to the quamtum plane is pseudo finite. These points represent nonstandard homogeneous components of the quantum plane.

2 The lattice of K-invariant pp-definable subspaces

A pp-definable subspace $\varphi(V)$ of a U_q-module V is called *K-invariant* if $K\varphi(V) \subseteq \varphi(V)$. Thus a pp-definable subspace of $M = k_q[x,y] \oplus k_q^\sigma[x,y]$ is K-invariant if for every simple representation $V_{\epsilon,n}$,

$$K\varphi(V_{\epsilon,n}) \subseteq \varphi(V_{\epsilon,n}).$$

Since the vector space $\varphi(V_{\epsilon,n})$ is finite dimensional, and K is an invertible element of U_q, we have that $K\varphi(V_{\epsilon,n}) = \varphi(V_{\epsilon,n})$, and hence that $K\varphi(M) = \varphi(M)$. The collection of K-invariant pp-definable subspacess of M forms a sublattice of the pp-definable subspaces of M. In this section, we will prove that this sublattice is complemented.

According to Equation (2), the simple representation $V_{+,n} \cong k_q[x,y]_n$ has a basis of K-eigenvectors given by the monomials $x^{n-i}y^i$. These monomials also form a basis of K-eigenvectors for the representation $k_q^\sigma[x,y]_n$. The 1-dimensional vector spaces generated by these monomials are called the *weight spaces* of $k_q[x,y]$ (resp., $k_q^\sigma[x,y]_n$). If the pp-definable subspace $\varphi(V_{\epsilon,n})$ is a direct sum of weight spaces, for every $V_{\epsilon,n}$, then the pp-definable subspace $\varphi(M)$ is clearly K-invariant. Conversely, if $\varphi(M)$ is a K-invariant pp-definable subspace, then we may multiply each of the $\varphi(V_{\epsilon,n})$ by various of the $K - \epsilon q^{n-2i}$ to see that it must be a direct sum of weight spaces.

Let U_q^{opp} denote the opposite ring of U_q, defined by reversing the multiplication in U_q. The formulas in the language $\mathcal{L}(U_q^{\text{opp}})$ of left U_q^{opp}-modules, or, equivalently, of the right U_q-modules, are expressed with the scalars from U_q acting on the right. Given a pp-formula $\varphi(\mathbf{u}) = \exists \mathbf{v}\ (A,B)\binom{\mathbf{u}}{\mathbf{v}} \doteq 0$ (with $\mathbf{u} = (u_1,\ldots,u_n)$ and $\mathbf{v} = (v_1,\ldots,v_k)$) of the language $\mathcal{L}(U_q)$, we may associate to it as in [8] a pp-formula $\varphi^*(\mathbf{u})$ of $\mathcal{L}(U_q^{\text{opp}})$ in the same variable $\mathbf{u}$, called *dual* of $\varphi(\mathbf{u})$:

$$\varphi^*(\mathbf{u}) = \exists \mathbf{w}\,(\mathbf{u},\mathbf{w}) \begin{pmatrix} I_n & 0 \\ A & B \end{pmatrix} \doteq 0,$$

where I_n denotes the $n \times n$ identity matrix. If V is a left U_q-module, then the space $V^* := \operatorname{Hom}_k(V,k)$ of functionals acquires the structure of a right U_q-module, given by $(\eta r)(v) = \eta(rv)$, for every $r \in U_q$. If $\varphi(V)$ is a pp-definable subspace of V, then $\varphi^*(V^*)$ is the subspace of V^* consisting of functionals that vanish on $\varphi(V)$. This association yields an anti-isomorphism of the lattice of pp-definable subspaces of V and that of V^*.

There is another way of associating to a pp-formula $\varphi(v)$ in the language $\mathcal{L}(U_q)$ a pp-formula formula in the language of right U_q-modules. It depends on the existence of the anti-automorphism Tr of U_q determined by the values $E \mapsto F$, $F \mapsto E$, and $K \mapsto K$. This anti-automorphism is obtained by composing the anti-automorphism τ and automorphism ω given in [4, Lemma 1.2]. Quite generally, let $\alpha : R \to S$ be an isomorphism of rings. It induces an isomorphism of languages $\mathcal{L}(\alpha) : \mathcal{L}(R) \to \mathcal{L}(S)$, which sends a pp-formula φ in $\mathcal{L}(R)$ to the pp-formula $\alpha(\varphi)$ in $\mathcal{L}(S)$ obtained by replacing every occurence of the unary function symbol r with $\alpha(r)$. If N is a left R-module, then N^α is defined as the left S-module whose underlying abelian group is given by N, and the action of S by $sn := \alpha^{-1}(s)(n)$. It is readily verified that for $n \in N$, we have that

$$N \models \varphi(n) \text{ iff } N^{\alpha} \models \alpha(\varphi)(n).$$

The key operation on pp-formulae that turns out to best suit our needs is the composition of the operation $\varphi \mapsto \varphi^*$ with $\varphi \mapsto \mathrm{Tr}(\varphi)$. It will be denoted by $\varphi \mapsto \varphi^-$. The importance of this operation stems from the well-known fact that for every finite dimensional simple representation $V_{\epsilon,n}$, we have that

$$V_{\epsilon,n}^* \cong V_{\epsilon,n}^{\mathrm{Tr}}.$$

We provide a proof for lack of a reference. Recall that the most important element in U_q is the quantum Casimir element [4, §2.7]:

$$C_q = EF + \frac{q^{-1}K + K^{-1}q^{-1}}{(q-q^{-1})^2}.$$

The fundamental property of C_q is that it acts on every simple finite dimensional representation $V_{\epsilon,n}$ of U_q as multiplication by the scalar

$$C_{\epsilon,n} = \frac{q^{-1}(\epsilon q^n) + q(\epsilon q^n)^{-1}}{(q-q^{-1})^2}.$$

Furthermore, the action of the Casimir element permits us to decide when two simple finite dimensional U_q-modules are isomorphic.

Lemma 2.1 [4, Lemma 2.8] *Let V and $V'_{\epsilon,n}$ be some simple finite dimensional representations of U_q. If C_q acts on V by the same scalar as on $V'_{\epsilon,n}$, then $V \cong V'_{\epsilon,n}$.*

By the lemma, all that needs to be verified is that C_q acts by the same scalar on $V_{\epsilon,n}^*$ as it does on $V_{\epsilon,n}^{\mathrm{Tr}}$. According to the action of U_q on $V_{\epsilon,n}^*$, it holds that for every $v \in V_{\epsilon,n}$,

$$(\eta C_q)(v) = \eta(C_q v) = \eta(C_{\epsilon,n} v) = (\eta C_{\epsilon,n})(v),$$

and therefore $\eta C_q = \eta C_{\epsilon,n}$ for every $\eta \in V_{\epsilon,n}^*$. On the other hand, considering the action of U_q on $V_{\epsilon,n}^{\mathrm{Tr}}$, we have that

$$vC_q = \mathrm{Tr}(C_q)v = C_q v = C_{\epsilon,n} v = \mathrm{Tr}(C_{\epsilon,n})v = vC_{\epsilon,n},$$

for every $v \in V$; the equality $\mathrm{Tr}(C) = C$ follows from Equation (3) on [4, p. 21].

Theorem 2.2 *The rule $\varphi(M) \to \varphi^-(M)$ is an anti-isomorphism of the lattice Latt(M) of pp-definable subspaces of M.*

Proof. Let $V_{\epsilon,n}^*$ and $V_{\epsilon,n}^{\mathrm{Tr}}$ as above. By definition of M, it is enough to restrict our attention to a finite dimensional simple representation $V_{\epsilon,n}$ of U_q. Indeed, if $V \models \varphi(v) \to \psi$, then

$$V_{\epsilon,n}^{\mathrm{Tr}} = V_{\epsilon,n}^* \models \psi^*(v) \to \varphi^*(v),$$

which is equivalent to

$$V_{\epsilon,n} \models \psi^-(v) \to \varphi^-(v).$$

□

The following proposition implies that the sublattice of K-invariant pp-definable subspaces of the U_q-module M is complemented. To prove it, the argument used in [3, Lemma 13] may be adapted in the quantum case.

Proposition 2.3 *If φ is a K-invariant pp-formula, then φ^- is also K-invariant and for every simple finite dimensional representation $V_{\epsilon,n}$,*

$$\varphi(V_{\epsilon,n}) \oplus \varphi^-(V_{\epsilon,n}) = V_{\epsilon,n}.$$

3 The general strategy

Like the universal enveloping algebra $U(L)$ of $L = sl_2(k)$, the quantum group U_q is a left Ore domain and so admits a field of fractions, denoted by $Q = Q(U_q)$. The U_q-module Q is indecomposable and *pp-simple*: for every pp-formula $\varphi(v)$ in one variable, the pp-definable subspace $\varphi(Q)$ equals either Q or 0. Thus the lattice of pp-formulas in the language of U_q-modules may be partitioned into the filter $\mathcal{F}$ of *high* formulas - those formulas for which $\varphi(Q) = Q$ - and the ideal $\mathcal{I}$ of *low* formulas - those formulas for which $\varphi(Q) = 0$. It is a general fact about left Ore domains that a pp-formula is high if and only if it is implied by some divisibility pp-formula $r|u$, with $r \in U_q$ nonzero. On the other hand, every annihilator pp-formula $sv \doteq 0$, with $s \in U_q$ nonzero, is necessarily low, because it defines in Q the zero subspace. Because U_q is also a right Ore domain, every low formula in the language of left U_q-modules implies some annihilator pp-formula $sv \doteq 0$.

A pp-formula $\varphi(v)$ is called *uniformly bounded,* with bound $n_\varphi \geq 0$, if for every finite dimensional simple representation $V_{\epsilon,n}$ of U_q,

$$\dim_k \varphi(V_{\epsilon,n}) \leq n_\varphi.$$

Dually, we define a pp-formula $\psi(v)$ to be *uniformly cobounded,* if there is a bound on the k-dimension of the quotient $V_{\epsilon,n}/\psi(V_{\epsilon,n})$, for every $V_{\epsilon,n}$. The next section of the article is devoted to a proof of the following fundamental result.

Theorem 3.1 *If s in U_q is nonzero, and $\varphi(v)$ is the annihilator formula $sv \doteq 0$, then there is a uniformly cobounded formula $\psi(v)$ such that the pp-definable subspace $\psi(M)$ is K-invariant, and*

$$\varphi(M) \cap \psi(M) = 0.$$

Let us note, assuming the theorem, how every low pp-formula $\varphi(v)$ is uniformly bounded. Since any pp-definable subspace $\varphi(M)$ defined by a low formula is contained in a pp-definable subspace defined by an annihilator pp-formula, we may assume that $\varphi(v)$ is an annihilator pp-formula. The bound n_φ is then obtained by taking the bound on the k-dimension of $V_{\epsilon,n}/\psi(V_{\epsilon,n})$, where $\psi(v)$ is the formula given by Theorem 3.1. This implies that every high formula $\varphi(v)$ is uniformly cobounded, because $\varphi(v)$ is implied by a divisibility pp-formula $s|v$, and the k-dimension of $V_{\epsilon,n}/sV_{\epsilon,n}$ is bounded by n_ψ, where $\psi(v)$ is the annihilator pp-formula $sv \doteq 0$.

Because the anti-isomorphism $\varphi \mapsto \varphi^-$ associates to a divisibility pp-formula $s|v$ the annihilator pp-formula $\mathrm{Tr}(s)v \doteq 0$ - and vice versa - we see that $\varphi(v)$ is high if and only if $\varphi^-(v)$ is low. The following proposition is proved *mutatis mutandis* as in [3, Props 16-18], using a highest pseudoweight argument.

Proposition 3.2 *If $\varphi(v)$ is a low pp-formula for which the pp-definable subspace $\varphi(M)$ is K-invariant, then the interval $[0, \varphi(M)]$ of the lattice Latt(M) is complemented.*

Given Theorem 3.1 and Proposition 3.2, we may now proceed with the proof of Theorem 1.1 as in [3]. If $\varphi(M)$ is defined by a high formula, then $\varphi^-(v)$ is low, so that we may use Theorem 3.1 to obtain a high formula $\psi(v)$ such that $\psi(M)$ is K-invariant and

$$\varphi^-(M) \cap \psi(M) = 0.$$

Applying the anti-automorphism $\varphi \mapsto \varphi^-$ once more gives that

$$\varphi(M) + \psi^-(M) = M.$$

Now $\psi^-(M)$ is a K-invariant subspace defined by a low pp-formula, so Proposition 3.2 implies that the interval $[0, \psi^-(M)]$ is complemented. A complement of $\varphi(M) \cap \psi^-(M)$ in $\psi^-(M)$ then serves as a complement of $\varphi(M)$ in M.

If, on the other hand, the pp-definable subspace $\varphi(M)$ is defined by a low pp-formula, then we may apply the preceding argument to obtain a complement $\psi(M)$ of $\varphi^-(M)$ in M, and apply the anti-automorphism $\varphi \mapsto \varphi^-$ to see that $\psi^-(M)$ is then a complement of $\varphi(M)$ in M.

4 Homogeneous elements of degree 0

The quantum group U_q may be equipped with the structure of a $\mathbb{Z}$-graded algebra by assigning degrees as follows: $\deg(E) = 1$, $\deg(F) = -1$, and $\deg(K) = \deg(K^{-1}) = 0$. Then the relations given in Equation (1) are all homogenous, so that U_q becomes a $\mathbb{Z}$-graded algebra such that for every $m \in \mathbb{Z}$, the m-th homogenous component of U_q, denoted U_q^m, is the vector space spanned by $\left\{E^i K^l F^j : i, j \in , l \in \mathbb{Z},\ i - j = m\right\}$. This follows from the quantum version of the Boincaré -Birkhoff-Witt Theorem [4, Thm 1.5]. In fact, we have

$$K\,(E^i K^l F^j)\,K^{-1} = q^{2(i-j)}(E^i K^l F^j), \tag{3}$$

so that $U_q = \bigoplus_{m \in \mathbb{Z}} U_q^m$ and the grading is the same as the eigenspace decomposition for the action of K on U_q by conjugation.

Remark 4.1 (see [5, Lemma VI.4.2]) *Let u be an element of U_q. Then $u \in U_q^0$ if and only if u commutes with K.*

Proof. If $u \in U_q^0$, then Equation (3) implies that u commutes with K. For the converse, write $u = \sum c_{i,j} E^i K^l F^j$ (for some $i, j \in \mathbb{N}$ and $l \in \mathbb{Z}$). If u commutes with K, then by the quantum PBK Theorem the coefficient $q^{2(i-j)}$ in Equation (3) must equal 1: as q is not a root of unity, $i - j = 0$. □

The remark implies that U_q^0 contains the center of U_q. Repeated application of the Relations (1) shows that every element in U_q^0 is a polynomial in EF, K, and K^{-1} with coefficients in k. Since these elements of U_q commute we see that the subring U_q^0 is a finitely generated commutative k-algebra $U_q^0 = k[C_q, K, K^{-1}]$.

Lemma 4.2 *If $r \in U_q^0$, then the annihilator formula $\varphi(v) = (rv \doteq 0)$ is uniformly bounded.*

Proof. Express $r = r(C_q, K, K^{-1})$ as a polynomial in C_q, K, and K^{-1} over k. By factoring out a suitably large power of K^{-1}, we have that $r = K^{-t}\bar{r}(C_q, K)$, where $\bar{r}$ is a polynomial in C_q and K over k. Since the axioms for a left U_q-module imply that the pp-formulae $rv \doteq 0$ and $\bar{r}v \doteq 0$ are equivalent, we may assume, without loss of generality, that $r = \bar{r}$ is a polynomial in C_q and K over k.

Since r commutes with K, the pp-definable subspace $\varphi(V_{\epsilon,n})$ is K-invariant for every finite dimensional simple representation $V_{\epsilon,n}$. The pp-definable subspace $\varphi(V_{\epsilon,n})$ is therefore a sum of weight spaces,

$$V^p_{\epsilon,n} = \{v_p \in V_{\epsilon,n}\, Kv_p = \epsilon q^{n-2p} v_p\},$$

$p = 0, \ldots, n$, of $V_{\epsilon,n}$. By considering the action of C_q and K on $V^p_{\epsilon,n}$, we see that the pp-definable subspace $\varphi(V_{\epsilon,n})$ will contain $V^p_{\epsilon,n}$ if and only if

$$r(C_{\epsilon,n}, \epsilon q^{n-2p}) \; = \; 0.$$

Because q is not a root of unity, the exponential map $p \mapsto q^p$ is a monomorphism as is the linear function $p \mapsto n - 2p$. We may deduce that the number of solutions p is bounded by the degree of K in the polynomial r, that is, $\dim_k \varphi(V_{\epsilon,n}) \leq \deg_K(r)$. □

To generalize Lemma 4.2 for every element in U_q, we may apply the following property of the m-th homogeneous component of U_q :

$$\forall m > 0,\, U^m_q = E^m\, U^0_q,$$

$$\forall m < 0,\, U^m_q = U^0_q\, F^m.$$

It may established by repeated application of Equations (1). The following result then follows using a quantum version of the argument used in [3, Lemma 21].

Proposition 4.3 *Let $s \in U_q$ and consider the corresponding annihilator formula $\varphi(v) = (sv \doteq 0)$. There are an element r in U^0_q and a natural number m such that for every simple finite dimensional representation $V_{\epsilon,n}$ of U_q,*

$$\varphi(V_{\epsilon,n}) \cap F^m V_{\epsilon,n} \cap rV_{\epsilon,n} \cap E^m V_{\epsilon,n} = 0.$$

An immediate consequence of Proposition 4.3 is the proof of Theorem 3.1. The uniformly cobounded formula $\psi(v)$ is given by the conjunction of $F^m|v$, $r|v$, and $E^m|v$. It clearly defines a K-invariant subspace of M.

5 Ideals of definable scalars

Before discussing the ring of definable scalars of the quantum plane, let us review some of the consequences of Theorem 1.1 for the representation theory of U_q. These results are recounted without proof, which in all cases is entirely analogous to that given in [3] for the universal enveloping algebra $U(L)$ of $L = sl_2(k)$, when k is an algebraically closed field of characteristic 0.

Let U'_q be the ring of definable scalars of the U_q-module M. If r belongs to U'_q, then the pp-definable subspace rM is complemented by some pp-definable subspace $\psi(M)$ of M

$$rM \oplus_k \psi(M) = M.$$

If $e \in U'_q$ is the idempotent projection onto rM with respect to this decomposition, then

$$M \models \forall v(\psi(v) \leftrightarrow (ev \doteq 0)),$$

and $rU'_q = eU'_q$. Similarly, define $e_0 \in U'_q$ to be the idempotent projection onto the pp-definable subspace $\varphi(M)$ defined by $\varphi(v) = (Ev \doteq 0)$, with respect to the decomposition

$$\varphi(M) \oplus_k FM = M.$$

For every simple finite dimensional representation $V_{\epsilon,n}$ of U_q, $e_0 V_{\epsilon,n}$ is the highest weight space. If $I_0 \subseteq U'_q$ denotes the two-sided ideal generated by e_0, then, as in [3], I_0 consists of all the elements $r \in U'_q$ for which the formula $r|v$ is is uniformly bounded, and U'_q/I_0 is isomorphic to the field of fractions Q of U_q.

As in the general case of a von Neumann regular ring, the Ziegler spectrum $\mathrm{Zg}(U'_q)$ of U'_q consists of the injective indecomposable U'_q-modules where the open subsets in $\mathrm{Zg}(U'_q)$ are in bijective correspondence with the two-sided ideals of U'_q according to the rule

$$I \mapsto \mathcal{O}(I) := \{E \in \mathrm{Zg}(U'_q) : IE \neq 0\}.$$

If $\varphi(v)$ is a pp-formula in $\mathcal{L}(U'_q)$, then there is a complementary pp-formula $\psi(v)$ such that that

$$\varphi(M) \oplus_k \psi(M) = M.$$

If $e \in U'_q$ is the projection onto $\varphi(M)$ with respect to this decomposition, then it may be easily checked that in $\mathrm{Zg}(U'_q)$,

$$\mathcal{O}_{\varphi, v \doteq 0} = \mathcal{O}(I),$$

where I is the two-sided ideal of U'_q generated by e.

As in the case of the universal enveloping algebra of $sl_2(k)$,

$$\mathrm{Zg}(U'_q) = \mathcal{O}(I_0) \,\dot{\cup}\, \{Q\}.$$

The open subset $\mathcal{O}(I_0)$ forms a compact totally disconnected subspace of $\mathrm{Zg}(U'_q)$, and the subset of finite dimensional simple representations $V_{\epsilon,n}$ is a dense and discrete open subset of $\mathrm{Zg}(U'_q)$. It follows that if I_1 and I_2 are two-sided ideals of U'_q contained in I_0, then $I_1 = I_2$ if and only if the corresponding open subsets $\mathcal{O}(I_1)$ and $\mathcal{O}(I_2)$ contain the same finite dimensional representations.

If $V \in \mathrm{Zg}(U'_q)$ is not Q, then $I_0 V \prec V$ is a simple U'_q-module which is an elementary substructure of V regardless of whether V is considered as a structure for the language of U_q-modules or U'_q-modules. An indecomposable representation V in $\mathrm{Zg}(U'_q)$ is finite dimensional if and only if $I_0 V = V$.

A U_q-module V is said to be *pseudo-finite dimensional* if it satisfies all the first order sentences of the language of U_q-modules satisfied by every finite dimensional module. A U_q-module V is pseudo-finite if and only if it is a U'_q-module and $I_0 V \prec V$. Every pseudo-finite dimensional representation V is elementary equivalent to a direct sum

$$V \equiv \bigoplus_{W \in \mathcal{C}\ell(V)} I_0 W,$$

where every $I_0 W$ is a pseudo-finite dimensional simple representation of U'_q. This is an elementary version of [4, Theorem 2.9].

Let φ_+ be the sum of the following pp-formulae

$$Kv = qv, \quad Kv = v;$$

the pp-definable subspace $\varphi_+(M)$ of M is K-invariant and is complemented by the pp-definable subspace $\varphi_+^-(M)$ of M

$$\varphi_+(M) \oplus_k \varphi_+^-(M) = M.$$

So, define e_+ to be the idempotent projection onto $\varphi_+(M)$, with respect to this decomposition. Then $e_+V_{\epsilon,n} \neq 0$ if and only if $\epsilon = +$. If we denote by I_+ the ideal generated by the idempotent e_+, then $\mathcal{O}(I_+)$ is the unique open subset of $\mathrm{Zg}(U_q')$ contained in $\mathcal{O}(I_0)$ that contains all the indecomposable summands $V_{+,n}$ of the quantum plane, and none of the $V_{-,n}$.

Similarly, let $\varphi_- = \sigma(\varphi_+)$ be the pp-formula gotten by applying the automorphism σ to the scalar in φ_+. It is the sum of the pp-formulae

$$Kv = -qv, \quad Kv = -v.$$

If we define $e_- = \sigma(e_+)$ to be the idempotent projection onto $\varphi_-(M)$, with respect to the corresponding K-invariant decomposition of M, then $e_-V_{\epsilon,n} \neq 0$ if and only if $\epsilon = -$. Then the ideal I_- generated by e_- is nothing more than $\sigma(I)$. Here we are making tacit use of the fact that $M^\sigma \cong M$, and if $r \in U_q'$ is a definable scalar of M, represented say by the pp-formula $\rho(u, v)$, then the pp-formula $\sigma(\rho)$ also defines an element $\sigma(r) \in U_q'$. As both of the ideals I_- and I_+ are contained in I_0, and the open subsets associated to I_0 and $I_- + I_+$ both contain all the finite dimensional points of $\mathrm{Zg}(U_q')$, we conclude that

$$I_- + I_+ = I_0.$$

Since $\mathcal{O}(I_-) \cap \mathcal{O}(I_+) = \mathcal{O}(I_- \cap I_+)$ contains no finite dimensional points, the sum must be direct.

Theorem 5.1 *The lattice Latt($k_q[x, y]$) of pp-definable subspaces of the quantum plane is complemented. The ring of definable scalars of $k_q[x, y]$ may be identified with the von Neumann regular ring U_q'/I_-. The canonical morphism $\rho : U_q \to U_q'/I_-$ is an epimorphism of rings with* 0 *kernel.*

Proof. The first statement is clear, because if $\varphi(v)$ is a pp-formula, then any pp-formula $\psi(v)$ that defines a complementary subspace of $\varphi(M)$ in M, will also define a complementary subspace of $\varphi(k_q[x, y])$ in $k_q[x, y]$.

To prove the second statement, let us first note that any definable scalar r of M that vanishes on $k_q[x, y]$ must belong to I_0. This is because $k_q[x, y]$ contains finite dimensional indecomposable summands of arbitrarily large k-dimension. The formula $r|v$ cannot therefore be uniformly cobounded. It is therefore uniformly bounded, and $r \in I_0$. Since I_- consists of the elements of I_0 that vanish on $k_q[x, y]$, our claim is established.

Since every element of U_q', acts definably on $k_q[x, y]$, there is a canonical morphism of rings from U_q' to the ring U_q'' of definable scalars of $k_q[x, y]$. The lattice of pp-definable subspaces of $k_q[x, y]$ is the same whether we consider $k_q[x, y]$ as a U_q-module or a U_q'-module. Since it is complemented, the canonical morphism is an epimorphism of rings. If we can show that this ring epimorphism is onto, it will follow that U_q'' is isomorphic in the obvious way to the quotient ring U_q'/I_-. That the kernel of ρ is 0 follows from the fact that the quantum plane is a faithful representation of U_q. This follows from Proposition 4.3, because if $s \in U_q$ is nonzero, then the annihilator formula $sv \doteq 0$ is uniformly bounded.

Let us note, quite generally, that if $\rho : R \to S$ is a ring epimorphism, with R von Neumann regular, then it must be onto. Replacing R with the von Neumann regular ring $R/\text{Ker } \rho$, we may assume without loss of generality that $\text{Ker } \rho = 0$. As ρ is an epimorphism, the morphism $\mu : S \otimes_R S \to S$ of abelian groups, given by $s \otimes r \mapsto sr$, is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R \otimes_R S & \xrightarrow{\rho \otimes S} & S \otimes_R S & \longrightarrow & S/R \otimes_R S & \longrightarrow & 0 \\ & & & \searrow & \downarrow \mu & & & & \\ & & & & S. & & & & \end{array}$$

The top row is exact, because every short exact sequence of R-modules is pure exact. Since the vertical and diagonal arrows are both isomorphisms, it follows that the monomorphism $\rho \otimes S$ in the top row is also an isomorphism, and hence that $S/R \otimes_R S = 0$. But this implies that $S/R = 0$, because the morphism

$$S/R \otimes \rho : S/R \otimes_R R \ \to \ S/R \otimes_R S$$

is a monomorphism. □

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