DIE BÖSE FARBE

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ABSTRACT. We construct a bad field in characteristic zero.

1. Introduction

Morley rank is a model-theoretical generalization of *Zariski dimension* which can be extended to definable sets in any mathematical structure. A structure is called ω-stable if Morley rank is ordinal-valued. Morley degree is the analogous to the number of irreducible components of maximal Zariski dimension. An early conjecture due to B. Zil'ber stated that a structure of both Morley rank and degree 1 (called a *strongly minimal set*) arose from a classical geometry: the trivial (or degenerated) one, a vector space geometry or Zariski geometry over an algebraically closed field. This conjecture was refuted in 1988 by E. Hrushovski [9], who modified in a clever way Fraïssé's original construction of the universal homogeneous model of a hereditary class of finite relational structures with the amalgamation property. This method was later divided by B. Poizat into two steps: first, the construction of a generic structure of rank ω and secondly, the *collapse* to a strongly minimal set. This procedure has been used in several applications: for example, E. Hrushovski *fused* two strongly minimal sets with the DMP in disjoint languages into a strongly minimal set [8] (*cf.* also [3]). In the aforementioned article, he also mentioned that this procedure could also be generalized to the case where both strongly minimal sets were expansions of a common vector space structure over a finite field. The fusion over a vector space was first proved by the second author and A. Hasson [7] in the 1-based case (moreover, they also studied the non-collapsed fusion of rank ω). The collapse to a strongly minimal fusion was attained finally by the first and third authors together with M. Ziegler [4]. Using similar arguments they also proved [5] the existence of a field of arbitrary prime characteristic of Morley rank 2 equipped with a definable additive subgroup of rank 1 after collapsing Poizat's *red fields* [15] of rank $\omega \cdot 2$.

A *bad field* is a field of finite Morley rank equipped with a definable non-trivial proper divisible multiplicative subgroup. They appeared first in the study of simple groups of finite Morley rank, whose Borel subgroups (i.e. maximal solvable subgroups) are of the form $K^+ \rtimes T$ with $1 < T < K^\times$. According to the *algebraicity conjecture* due to Cherlin-Zil'ber (an algebraic variation of Zil'ber's above conjecture) a simple infinite group of finite Morley rank is algebraic. The non-existence of bad fields would simplify the study of Borel subgroups. Due to a result by the fourth author [16, 17] bad fields are unlikely to exist in positive characteristic. After

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applying deep results due to J. Ax [1], B. Poizat [15] found a candidate of rank $\omega \cdot 2$ with a multiplicative subgroup of rank ω based on his construction [14] of a field of rank $\omega \cdot 2$ equipped with a definable set of rank ω. The collapse of the latter to rank 2 was explained by J. Baldwin and K. Holland in [2]. Poizat's fields refuted a conjecture of C. Berline and D. Lascar, which stated that the rank of a field (in case of an infinite ordinal) was a monomial of the form ω^α. Together with M. Ziegler, the first and third author summarized and completed [6] the above results and exhibited a simpler axiomatization. Let us remark that a field of ordinal Morley rank is algebraically closed [11].

Following Poizat's notation we denote the predicate for the multiplicative subgroup by *green*. In this article we will collapse the green fields and therefore obtain a bad field in characteristic 0. This construction is extremely related to the red collapse [5]; however we will use results due to Ax-Poizat instead of locally finiteness of $\mathbb{F}_p$-vector spaces.

2. Algebraic Lemmata

This section summarizes results coming from algebraic geometry which will be needed for our purposes. Let us fix some notation: $\mathbb{C}$ denotes an algebraically closed field of characteristic 0. A variety V will always be a subvariety of some cartensian product $(\mathbb{C}^*)^n$. A *torus* is a connected algebraic subgroup of $(\mathbb{C}^*)^n$. It is described by finitely many equations of the form: $x_1^{r_1} \cdot \ldots \cdot x_n^{r_n} = 1$. Linear dimension (as $\mathbb{Q}$-vector spaces modulo torsion) equals algebraic dimension (as varieties) for tori which we will denote as $\mathrm{l.dim}_{\mathbb{Q}}(T)$ or $\dim(T)$. Given a closed subvariety V in $(\mathbb{C}^*)^n$, its *minimal torus* is the smallest torus T such that V lies in some coset $\bar{a} \cdot T$ (with $\bar{a} \in V$). In this case, we define the *codimension* of an irreducible variety V as $\mathrm{cd}(V) := \dim(T) - \dim(V) = \mathrm{l.dim}_{\mathbb{Q}}(V) - \dim(V)$, where $\mathrm{l.dim}_{\mathbb{Q}}(V) := \dim(T)$. A subvariety $W \subseteq V$ is cd-*maximal* if $\mathrm{cd}(W') > \mathrm{cd}(W)$ for every subvariety $W \subsetneq W' \subseteq V$. Clearly, irreducible components of V and tori cosets maximally contained in V are examples of cd-maximal subvarieties.

Observe that any connected algebraic subgroup of a torus is again a torus.

We now state a result proved by B. Poizat [15, Corollaires 3.6 und 3.7]:

Theorem 2.1. *Let $V(\bar{x}, \bar{z})$ be a uniformly definable family of varieties in $(\mathbb{C}^*)^n$. There exists a finite collection of tori $\{T_0, \ldots, T_r\}$, such that for any torus $T \subseteq (\mathbb{C}^*)^n$, any member $V_{\bar{b}} = V(\bar{x}, \bar{b})$ of the family and any irreducible component W of $V_{\bar{b}} \cap \bar{a} \cdot T$ (with $\bar{a}$ in W) there is some i in $\{0, \ldots, r\}$ with $W \subseteq \bar{a} \cdot T_i$ and $\dim(T_i) - \dim(V \cap \bar{a} \cdot T_i) = \dim T - \dim W$.*

Moreover, the minimal torus of every cd*-maximal subvariety of $V_{\bar{b}}$ belongs to the collection $\{T_0, \ldots, T_r\}$.*

We will assume throughout this article that the above tori are all distinct, and $T_0 = (\mathbb{C}^*)^n$ and $T_1 = \{1\}^n$.

Note 2.2. Only the second claim of theorem 2.1 is needed for our purposes.

Corollary 2.3. *Let $V(\bar{x}, \bar{z})$ be a uniformly definable family of varities. Then:*

(1) If T is the minimal torus of $V_{\bar{b}}$, there exists some $\theta(\bar{z}) \in \mathrm{tp}(\bar{b})$, such that T is the minimal torus of every $V_{\bar{b}'}$ with $\models \theta(\bar{b}')$. In particular, there is a definable neighbourhood of $\bar{b}$ where $\mathrm{l.dim}_{\mathbb{Q}}$ *and* cd *remain constant.*

(2) *Suppose $V(\bar{x},\bar{b})$ decomposes into m irreducible components W_k with $d_k := \dim(W_k)$, $l_k := \mathrm{l.dim}_{\mathbb{Q}}(W_k)$ and $c_k := \mathrm{cd}(W_k)$. Then, there is some $\theta(\bar{z}) \in \mathrm{tp}(\bar{b})$ such that for all $\bar{b}' \models \theta$ we have that $V_{\bar{b}'}$ decomposes into exactly m irreducible components $(W'_k : 1 \leq k \leq m)$ and (possibly after permutation) $\dim(W'_k) = d_k$, $\mathrm{l.dim}_{\mathbb{Q}}(W'_k) = l_k$ and $\mathrm{cd}(W'_k) = c_k$.*

Proof. The first claim follows from 2.1, since irreducible components are cd-maximal. So their minimal tori lie in $\{T_0, \dots, T_r\}$.

The second claim follows easily from a classical result of algebraic geometry and (1), since the decomposition into irreducible components is definable. □

3. δ-Arithmetic

Let now $\mathbb{C}$ denote a large algebraically closed field of characteristic 0, i.e. a universal model of ACF_0. We consider the reduct $\mathcal{L}_{mult} := \{\cdot, 1, 0, =\} \subseteq \mathcal{L}_{Ring} = \{+, \cdot, 1, 0, =\}$ and $T_{mult} := Th_{\mathcal{L}_{mult}}(\mathbb{C})$. The prime model of T_{mult} is obtained by adjoining 0 to the set of roots of unity $\mu(\mathbb{C}) \subseteq \mathbb{C}$. The structure so obtained is $\mathcal{L}_{mult}$-isomorphic to $\mathbb{Q}/\mathbb{Z}$ (multiplicatively), after adding a new element. In particular, T_{mult} is modular non-trivial, and its geometry is projective over $\mathbb{Q}$.

Given $A \subseteq \mathbb{C}$, denote by $\langle A \rangle$ the divisible hull of $A^* := A \backslash \{0\}$ in $\mathbb{C}^*$ together with 0, equivalently, the algebraic closure of A in $\mathbb{C}$ with respect to $\mathcal{L}_{mult}$. This yields the prime model of T_{mult} over A. For a tuple $\bar{a} \in \mathbb{C}^*$ we have that $MR(\bar{a})$ in T_{mult} agrees with its linear dimension over $\mathbb{Q}$ (modulo torsion). We will write $\mathrm{l.dim}_{\mathbb{Q}}(\bar{a})$, and let $\dim(\bar{a}/B)$ denote Morley rank in ACF_0, equivalently, the dimension of its locus over B. The latter agrees with the transcendence degree of $\bar{a}$ over B.

Consider now $\langle \cdot \rangle$-closed subsets B of $\mathbb{C}$ as an $\mathcal{L}_{Ring}$-structure. Generally these structures need not be subfields of $\mathbb{C}$, however we will (after a possible Morleyzation $\mathcal{L}_{Morley}$ of ACF_0 consisting of $\mathcal{L}_{mult}$ and relational symbols) formally work with these structures. Therefore, we will consider addition on $\langle \cdot \rangle$-closed sets and refer to their fraction fields, their algebraic closure, etc. However this shall not confuse the unfortunate reader. Let $\mathcal{D}$ be the class of such structures as above described. Embeddings in $\mathcal{D}$ are elementary embeddings as $\mathcal{L}_{Ring}$-structures, i.e. $\mathcal{L}_{Morley}$-embeddings. A structure $A \in \mathcal{D}$ is *finitely generated* over $B \subseteq A$ if $A = \langle \bar{a}B \rangle$ for some finite tuple $\bar{a} \in A$. Finitely generated elements of $\mathcal{D}$ over B correspond to divisible hulls of subgroups of $\mathbb{C}^*$ of finite rank over $\langle B \rangle$ (adjoining 0). In case that $\langle AB \rangle$ is finitely generated over $\langle B \rangle$, we define

$$\delta(A/B) := 2\dim(A/B) - \mathrm{l.dim}_{\mathbb{Q}}(A/B).$$

Clearly, $\delta(A/B)$ equals $\delta(\langle AB \rangle / \langle B \rangle)$. Since T_{mult} is modular, the following holds:

Lemma 3.1.

(1) *$\delta(\bar{a}\bar{b}/C) = \delta(\bar{b}/C) + \delta(\bar{a}/\bar{b}C)$.*

(2) *Given $C \subseteq B \in \mathcal{D}$, we have that $\delta(\bar{a}/B) \leq \delta(\bar{a}/B \cap \langle C\bar{a} \rangle)$. (Submodularity)*

Lemma 3.2. *Given $\bar{a}$ and B, let W be the locus of $\bar{a}$ over $\mathrm{acl}(B)$. Then:*

$$\delta(\bar{a}/\mathrm{acl}(B)) = \dim(W) - \mathrm{cd}(W)$$

Moreover,

$$\delta(\bar{a}/B) = \dim(W) - \mathrm{cd}(W) - \mathrm{l.dim}_{\mathbb{Q}}(\langle \bar{a}B \rangle \cap \mathrm{acl}(B)/B).$$

Proof. For the first claim it suffices to observe that the smallest torus coset (over an algebraically closed set B) containing $\bar{a}$ is B-definable in $\mathcal{L}_{mult}$. Hence, its dimension equals $\mathrm{l.dim}_{\mathbb{Q}}(\bar{a}/B)$. Modularity of T_{mult} gives the second statement. □

Definition 3.3.

- Let $M \subseteq N \in \mathcal{D}$ with $\mathrm{l.dim}_{\mathbb{Q}}(N/M) = n \geq 2$. The extension N/M is *minimal prealgebraic* (of *length* n) if $\delta(N/M) = 0$ and $\delta(N'/M) > 0$ for every $N' = \langle N' \rangle \in \mathcal{D}$ with $M \subsetneq N' \subsetneq N$.
- Let $B \subseteq \mathbb{C}$. A strong type $p(\bar{x}) \in S^n(B)$ (in ACF_0) is *minimal prealgebraic*, if the extension $\langle B\bar{a} \rangle / \langle B \rangle$ is minimal prealgebraic of length n for some $\bar{a} \models p$ (in particular, $\bar{a}$ is multiplicatively independent over B).
- A formula $\varphi(\bar{x})$ of Morley degree 1 is *minimal prealgebraic* if its generic type is minimal prealgebraic.

Note 3.4.

- The condition $\delta(N'/M) > 0$ is equivalent to $\delta(N/N') < 0$ since $0 = \delta(N/M) = \delta(N/N') + \delta(N'/M)$.
- If N/M is minimal prealgebraic and $\bar{n}$ is a multiplicative basis of N over M, then $\mathrm{stp}(\bar{n}/M)$ is minimal prealgebraic.
- Minimal prealgebraicity is invariant under parallelism class and multiplicative translation for strong types. In particular, the notion of minimal prealgebraic for stationary formulae in Definition 3.3 is well-defined.

4. Codes

We first aim to encode minimal prealgebraic extensions. Note that every strong type is the generic type of some variety. Hence, we can define the following:

Definition 4.1. A variety $V = V(\bar{x}, \bar{b})$ is a *code variety* if it is minimal prealgebraic. Equivalently, if for all $B = \langle B \rangle \ni \bar{b}$ the extension $B \subseteq \langle B\bar{a} \rangle$ is minimal prealgebraic for some B-generic $\bar{a} \in V$.

Note that a multiplicative translation of a code variety is again a code variety.

Let $M \in \mathcal{D}$ and $N = \langle M\bar{a} \rangle$ where $\mathrm{tp}(\bar{a}/M)$ is minimal prealgebraic of length n. Consider $N' = \langle N' \rangle$ with $M \subsetneq N' \subsetneq N$. Modularity of T_{mult} yields some $\mathcal{L}_{mult}$-basis $\bar{a}' \in \langle \bar{a} \rangle$ for N' over M of length m. Modulo torsion we have that $a'_j = \prod a_i^{\lambda_{ij}}$ for some $\lambda_{ij} \in \mathbb{Q}$. After substitution with suitable powers or roots, we may assume that $\lambda_{ij} \in \mathbb{Z}$ and $(\lambda_{ij})_{i<n}$ are coprime for $j < m$. Then, the equations $(\prod_{i<n} x_i^{\lambda_{ij}} = 1 : j < m)$ determine a torus T of dimension $d = n - m$. Since T is $\emptyset$-definable, it follows that $\bar{a}'$ is the canonical basis $[\bar{a}T]$ of the coset $\bar{a}T$.

Similarly, given a torus T in $(\mathbb{C}^*)^n$ of $\dim(T) = d$, the element $\bar{a}' := [\bar{a}T]$ generates a substructure $N' := \langle M\bar{a}' \rangle \subseteq N$ with $\mathrm{l.dim}_{\mathbb{Q}}(N'/M) = m = n - d$.

Lemma 4.2. *Let $V(\bar{x}, \bar{b}) \subseteq \mathbb{C}^n$ be a variety, $T \subseteq \mathbb{C}^n$ some torus and $\bar{a}$ in V generic over $B \ni \bar{b}$. Then, an element $\bar{a}_1 \in W := V \cap \bar{a}T$ is generic in W over $B[\bar{a}T]$ if and only if it is generic in $V_{\bar{b}}$.*

Proof. Note that $\bar{a}' := [\bar{a}T]$ is definable multiplicatively over any $\bar{a}_2 \in \bar{a}T$. In particular, it is definable over $\bar{a}_1 \in W$. Hence:

$$\begin{aligned}\dim(\bar{a}_1/B\bar{a}') &= \dim(\bar{a}_1\bar{a}'/B) - \dim(\bar{a}'/B) = \dim(\bar{a}_1/B) - \dim(\bar{a}'/B) \\ &\leq \dim(\bar{a}/B) - \dim(\bar{a}'/B) = \dim(\bar{a}/B\bar{a}'),\end{aligned}$$

which proves the claim. □

The following result was already stated in [15]; however we exhibit a proof for completeness since similar ideas will appear later on.

Lemma 4.3. *Let $V(\bar{x},\bar{z})$ be a uniformly definable family of varieties. If $V_{\bar{b}} = V(\bar{x},\bar{b})$ is a code variety then there is a formula $\theta(\bar{z}) \in \mathrm{tp}(\bar{b})$ such that $V_{\bar{b}_1}$ is a code variety for every $\bar{b}_1 \models \theta$.*

Proof. Let $\{T_0,\ldots,T_r\}$ be the collection of tori associated to $V(\bar{x},\bar{z})$ as in 2.1. Take some B containing $\bar{b}$. Set

$$n = \mathrm{l.\,dim}_{\mathbb{Q}}(V_{\bar{b}}) = 2k$$
$$k = \dim(V_{\bar{b}}).$$

Clearly $\langle B\bar{g}\rangle \cap \mathrm{acl}(B) = \langle B\rangle$ for B–generic $\bar{g} \in V_{\bar{b}}$.

Let $\theta(\bar{z})$ express:

(1) $\dim(V_{\bar{z}}) = k$ and $\mathrm{l.\,dim}_{\mathbb{Q}}(V_{\bar{z}}) = n$ (in particular $V_{\bar{z}} \neq \emptyset$),
(2) given generic $\bar{g}$ in $V_{\bar{z}}$, $i = 2,\ldots,r$ and W some irreducible component of $V \cap \bar{g}T_i$ of maximal dimension, then $\mathrm{cd}(W) > \dim(W)$ if $V \cap \bar{g}T_i$ is infinite.

The existence of such a formula θ follows from Corollary 2.3.

Now we show that $\models \theta(\bar{b})$. Let $\bar{g}$ in $V_{\bar{b}}$ be generic and take $T \neq T_0, T_1$ some torus with $V \cap \bar{g}T$ infinite. Choose some irreducible component $W \subseteq V \cap \bar{g}T$ of maximal dimension. By Lemma 4.2 we have that $\bar{g}$ is generic in W over $\mathrm{acl}((\bar{b},[\bar{g}T])$. Hence, $\bar{g} \notin \mathrm{acl}(\bar{b},[\bar{g}T])$ and $[\bar{g}T] \notin \mathrm{acl}(\bar{b})$. By Lemma 3.2 and miminal prealgebraicity of the extension $\langle \bar{g}\bar{b}\rangle/\langle \bar{b}\rangle$

$$\dim(W) - \mathrm{cd}(W) = \delta(\langle \bar{g}\bar{b}\rangle/\,\mathrm{acl}(\bar{b},[\bar{g}T])) = \delta(\langle \bar{g}\bar{b}\rangle/\,\mathrm{acl}(\bar{b},[\bar{g}T]) \cap \langle \bar{g}\bar{b}\rangle)) < 0.$$

Now let $\bar{b}_1 \models \theta$ and $\bar{g}$ generic in $V_{\bar{b}_1}$ over $B_1 \ni b_1$. Condition (1) in $\theta(\bar{z})$ yields $\delta(\bar{g}/B_1) = 0$. We need only show that $\delta(\bar{g}/[\bar{g}T], B_1) < 0$ for every torus T with $1 \leq d := \dim(T) \leq n-1$. Set $\bar{g}' := [\bar{g}T]$ and let W be the locus of $\bar{g}$ over $\mathrm{acl}(B_1\bar{g}')$. We consider three cases:

Case 1: $\bar{g} \in \mathrm{acl}(B_1\bar{g}')$, i.e. W is a point. Then

$$\begin{aligned}\delta(\bar{g}/B_1\bar{g}') &= -\mathrm{l.\,dim}_{\mathbb{Q}}(\bar{g}/B_1\bar{g}') = \mathrm{l.\,dim}_{\mathbb{Q}}(\bar{g}'/B_1) - \mathrm{l.\,dim}_{\mathbb{Q}}(\bar{g}\bar{g}'/B_1)\\ &= \mathrm{l.\,dim}_{\mathbb{Q}}(\bar{g}'/B_1) - \mathrm{l.\,dim}_{\mathbb{Q}}(\bar{g}/B_1) = (n-d) - n = -d < 0.\end{aligned}$$

Case 2: $\mathrm{cd}(W) = \mathrm{cd}(V_{\bar{b}_1})$. Since $W \subsetneq V_{\bar{b}_1}$ and by Lemma 3.2

$$\delta(\bar{g}/B_1\bar{g}') \leq \dim(W) - \mathrm{cd}(W) < \dim(V_{\bar{b}_1}) - \mathrm{cd}(V_{\bar{b}_1}) = 0.$$

Case 3: W is infinite and $\mathrm{cd}(W) < \mathrm{cd}(V_{\bar{b}_1})$. Choose some $W \subseteq \tilde{W} \subseteq V_{\bar{b}_1}$ irreducible maximal such that $\mathrm{cd}(\tilde{W}) \leq \mathrm{cd}(W)$. Then, $\tilde{W}$ is cd-maximal with minimal Torus T_i in the above collection. Note that $i \neq 0$ because $\mathrm{cd}(W) < \mathrm{cd}(V_{\bar{b}_1})$, and $i \neq 1$ because W is infinite. So, $\tilde{W} \subseteq V \cap \bar{g}T_i$. Take now some irreducible component $\tilde{W} \subset \tilde{W}'$ of $V \cap \bar{g}T_i$ of maximal dimension. By cd-maximality $\mathrm{cd}(\tilde{W}') > \dim(\tilde{W}')$. So

$$\delta(\bar{g}/B_1\bar{g}') \leq \dim(W) - \mathrm{cd}(W) \leq \dim(\tilde{W}) - \mathrm{cd}(\tilde{W}) \leq \dim(\tilde{W}') - \mathrm{cd}(\tilde{W}') < 0.$$

□

The above proof points out how to find — uniformly — a generic (definable) subset $\varphi_1(\bar{x}, \bar{b})$ of a code variety $V(\bar{x}, \bar{b})$ such that $\varphi_1(\bar{x}, \bar{b})$ is strongly minimal in the theory T_ω of the generic (non-collapsed) green field in case we *color* $\bar{x}$ green. Given a uniformly definable family $V(\bar{x}, \bar{z})$ of code varieties whose associated tori are $\{T_0, \ldots, T_r\}$ as in 2.1, we can define $\varphi_1(\bar{x}, \bar{z}) \subseteq V(\bar{x}, \bar{z})$ as follows:

- $\bar{a} \in V_{\bar{b}}$ realizes $\varphi_1(\bar{a}, \bar{b})$ if and only if for $i = 2, \ldots, r$ the following condition holds: In case $V_{\bar{b}} \cap \bar{a}T_i$ is infinite then $\mathrm{cd}(W) > \dim(W)$ for all irreducible components W of $V_{\bar{b}} \cap \bar{a}T_i$ of maximal dimension.

The above condition is definable by Corollary 2.3.

Lemma 4.4. *Let $(V_{\bar{z}}(\bar{x}) : \bar{z} \models \theta)$ be a family of code varieties and $\varphi_1(\bar{x}, \bar{z})$ be as above. Then $\varphi_1(\bar{x}, \bar{b})$ is generic in $V_{\bar{b}}$ for $\bar{b} \models \theta$. Moreover, for all $\bar{a} \models \varphi_1(\bar{x}, \bar{b})$ and $B \ni \bar{b}$ the following holds:*

(1) $\delta(\bar{a}/B) \leq 0$.

(2) If $\delta(\bar{a}/B) = 0$, then either $\bar{a} \in \langle B \rangle$ or $\bar{a}$ is generic in $V_{\bar{b}}$ over B.

Proof. The proof of Lemma 4.3 shows that $\varphi_1(\bar{x}, \bar{b})$ is generic in $V_{\bar{b}}$. Suppose now that $\bar{a}$ is neither generic in $V_{\bar{b}}$ nor contained in $\langle B \rangle$. We need to show that $\delta(\bar{a}/B) < 0$. Let W be its locus over $\mathrm{acl}(B)$. As in the previous proof, we consider three cases:

Case 1: $\bar{a} \in \mathrm{acl}(B)$ (equivalently, W is a single point). Then,

$$\delta(\bar{a}/B) = -\mathrm{l.dim}_{\mathbb{Q}}(\bar{a}/B) < 0.$$

Case 2: W is infinite with $\mathrm{cd}(W) = \mathrm{cd}(V_{\bar{b}})$. As above $W \subsetneq V_{\bar{b}}$, so $\delta(\bar{a}/B) \leq \dim(W) - \mathrm{cd}(W) < \dim(V_{\bar{b}}) - \mathrm{cd}(V_{\bar{b}_1}) = 0$.

Case 3: W is infinite with $\mathrm{cd}(W) < \mathrm{cd}(V_{\bar{b}})$. A similar argument as in 4.3 yields the claim. Remark that the conditions on genericity and θ are now part of $\varphi_1(\bar{x}, \bar{b})$. □

Given two formulae $\varphi(\bar{x})$ and $\psi(\bar{x})$ we write $\varphi(\bar{x}) \sim \psi(\bar{x})$ if the Morley rank of their symmetric difference is less than $\mathrm{MR}(\varphi(\bar{x}))$. Therefore $\mathrm{MR}(\varphi) = \mathrm{MR}(\psi)$ and $\sim$ ist symmetric. Let p be a minimal prealgebraic (strong) type, B some set of parameters and $\bar{g} \models p|B$. Then $\bar{g}$ is a $\mathcal{L}_{mult}$-basis of the minimal prealgebraic extension $\langle B \rangle \subseteq \langle B\bar{g} \rangle$. Moreover, $\dim(\bar{g}/B) = k = \frac{n}{2}$ and $\delta(\bar{g}/[\bar{g}T], B) < 0$ for every torus T different from T_0 and T_1. Minimal prealgebraic extensions are preserved under *affine transformations* (which correspond to bases change in $\mathcal{L}_{mult}$). Same holds for strong types as well as degree 1 formulae. Let us explain this carefully: consider $\mathbb{C}^*$ as a $\mathbb{Q}$-vector space modulo torsion. Then, the group $\mathrm{GL_n}(\mathbb{Q})$ acts on the set of strong types *modulo torsion*. If X_1 and X_2 are two definable sets in $(\mathbb{C}^*)^n$ of degree 1 and $T \subseteq (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$ is an n-dimensional torus such that $(X_1 \times X_2) \cap T$ projects generically onto both X_1 and X_2, then T induces a *toric correspondence* between X_1 and X_2. The following holds:

Lemma 4.5. *Let $\varphi(\bar{x})$ be minimal prealgebraic.*

- *If there is a toric correspondence between $\varphi(\bar{x})$ and some formula $\psi(\bar{x})$ of Morley degree 1, then $\psi(\bar{x})$ is also minimal prealgebraic.*
- *Let $\bar{m} \in (\mathbb{C}^*)^n$. Then $\varphi(\bar{x} \cdot \bar{m})$ is minimal prealgebraic.* □

Definition 4.6. Let $X \subseteq (\mathbb{C}^*)^n$ be definable set of degree 1. A formula $\varphi(\bar{x}, \bar{z})$ and a torus T *encode* X if there is some $\bar{b}$ such that T induces a toric correspondence between $\varphi(\bar{x}, \bar{b})$ and X. We say that φ *encodes* X if the above correspondence is the identity (i.e. $\varphi(\bar{x}, \bar{b}) \sim X$).

Definition 4.7. A *code* α are integers n_α, k_α and a $\emptyset$-definable formula $\varphi_\alpha(\bar{x},\bar{z})$ satisfying the following:

(a) The length of $\bar{x}$ is $n_\alpha = 2k_\alpha$.
(b) $\varphi_\alpha(\bar{x},\bar{b})$ is a subset of $(\mathbb{C}^*)^{n_\alpha}$.
(c) $\varphi_\alpha(\bar{x},\bar{b})$ is either empty or has Morley rank k_α and Morley degree 1.
(d) If $\varphi_\alpha(\bar{x},\bar{b}) \neq \emptyset$, then $\varphi_\alpha(\bar{x},\bar{b})$ is minimal prealgebraic with irreducible Zariski closure $V_\alpha(\bar{x},\bar{b})$.
(e) Suppose $\varphi_\alpha(\bar{x},\bar{b}) \neq \emptyset$. Then $\delta(\bar{a}/B) \leq 0$ for every $\bar{b} \in B$ and $\bar{a} \models \varphi_\alpha(\bar{x},\bar{b})$. Moreover, $\delta(\bar{a}/B) = 0$ if and only if $\bar{a} \in \langle B\rangle$ or $\bar{a}$ is B-generic in $\varphi_\alpha(\bar{x},\bar{b})$.
(f) $\varphi_\alpha(\bar{x},\bar{z})$ encodes every multiplicative translate of $\varphi_\alpha(\bar{x},\bar{b})$.
(g) If $\emptyset \neq \varphi_\alpha(\bar{x},\bar{b}) \sim \varphi_\alpha(\bar{x},\bar{b}')$, then $\bar{b} = \bar{b}'$.

If follows from (g) that $\bar{b}$ is the canonical basis of the minimal prealgebraic type determined by $\varphi_\alpha(\bar{x},\bar{b})$.

Lemma 4.8. *Every minimal prealgebraic definable set X can be encoded by some code α.*

Proof. Let $V_\alpha(\bar{x},\bar{b})$ be the variety associated to X. Then $V_\alpha(\bar{x},\bar{b})$ is a code variety. By Lemma 4.3 there is some formula $\theta(\bar{z})$ such that $V_\alpha(\bar{x},\bar{b}')$ — and all its multiplicative translates — is a code variety for $\bar{b}' \models \theta$. Let now $\varphi_1(\bar{x},\bar{z}) \subseteq V_\alpha(\bar{x},\bar{z})$ be as in in Lemma 4.4. Note that for every multiplicative translate $V_\alpha(\bar{x}\cdot\bar{m},\bar{z})$ the corresponding translate $\varphi_1(\bar{x}\cdot\bar{m},\bar{z}) \subseteq V_\alpha(\bar{x}\cdot\bar{m},\bar{z})$ yields the claim in Lemma 4.4. Set now

$$\varphi_\alpha(\bar{x},\bar{z}\bar{z}') := V(\bar{x}\cdot\bar{z}',\bar{z}) \wedge \varphi_1(\bar{x}\cdot\bar{z}',\bar{z}) \wedge \theta(\bar{z}).$$

Therefore, φ encodes X and satisfies properties (a)-(f).

By definability of $\sim$-equivalence and elimination of imaginaries we may assume that φ also satisfies (g). □

Set now $\theta_\alpha(\bar{z}) := \exists\bar{x}\,\varphi_\alpha(\bar{x},\bar{z})$.

Lemma 4.9. *Let α and β be codes. Then there is a finite set $G(\alpha,\beta)$ of tori in $(\mathbb{C}^*)^{2n_\alpha}$ such that if $\varphi_\alpha(\bar{x},\bar{b}) \neq \emptyset$ and T induces a toric correspondence between $\varphi_\alpha(\bar{x},\bar{b})$ and $\varphi_\beta(\bar{x},\bar{b}')$, then $T \in G(\alpha,\beta)$.*

Proof. If there is no such toric correspondence between any instances of α and β, then set $G(\alpha,\beta) := \emptyset$. Otherwise, let T, $\bar{b}$ and $\bar{b}'$ as above. Let V_α (*resp.* V_β) be the family of code varieties associated to α (*resp.* β). Moreover, let $\{T_0,\ldots,T_\nu\}$ be the finite collection of tori as in 2.1 for $V_\alpha \times V_\beta$. Set $B := \mathrm{acl}(\bar{b}\bar{b}')$. Choose some B–generic point $(\bar{a},\bar{a}')$ in $(V_\alpha(\bar{x},\bar{b}) \times V_\beta(\bar{x}',\bar{b}')) \cap T$.

Let $W \subseteq (V_\alpha \times V_\beta) \cap T$ be the locus of $(\bar{a},\bar{a}')$ over B. Then T is the minimal torus of W, and $\dim(W) = \mathrm{cd}(W) = k_\alpha$.

If we show that $W \subseteq (V_\alpha \times V_\beta)$ is cd-maximal, then T lies in $\{T_0,\ldots,T_\nu\}$. Choose now some variety $W \subsetneq W' \subseteq (V_\alpha \times V_\beta)$. We may assume that W' is B–definable. If $(\bar{g},\bar{g}')$ is B–generic in W', then $\mathrm{cd}(W') = \mathrm{l.dim}_{\mathbb{Q}}(\bar{g},\bar{g}'/B) - \dim(\bar{g},\bar{g}'/B)$. Hence,

$$\begin{aligned}\mathrm{cd}(W') &= [\mathrm{l.dim}_{\mathbb{Q}}(\bar{g}/B) - \dim(\bar{g}/B)] + [\mathrm{l.dim}_{\mathbb{Q}}(\bar{g}'/B\bar{g}) - \dim(\bar{g}'/B\bar{g})]\\ &= \mathrm{cd}(W) + \mathrm{l.dim}_{\mathbb{Q}}(\bar{g}'/B\bar{g}) - \dim(\bar{g}'/B\bar{g}) > \mathrm{cd}(W) - \delta(\bar{g}'/B\bar{g}) \geq \mathrm{cd}(W).\end{aligned}$$

The proper inequality follows from $\dim(\bar{g}'/B\bar{g}) > 0$ because $W \subsetneq W'$. The last inequality follows from the fact that $\bar{g}'$ is B–generic in V_β, so it also realizes $\varphi_\beta(\bar{x},\bar{b}')$. □

Theorem 4.10. *There exists a collection $\mathcal{C}$ of codes such that every minimal prealgebraic definable set can be encoded by a unique element α in $\mathcal{C}$ and finitely many tori.*

Proof. The collection $\mathcal{C}$ will be obtained as an increasing union of finite sets constructed by recursion. Encode first all minimal prealgebraic subsets of $(\mathbb{C}^*)^n$ for every n. Fix some $n \geq 2$ and list all minimal prealgebraic subsets $(X_i : i < \omega)$ of $(\mathbb{C}^*)^n$ up to isomorphism. Let α_0 encode X_0 as in 4.8. Define $\mathcal{C}_0 = \{\alpha_0\}$.

Suppose by induction that $\mathcal{C}_i$ has been already defined encoding all X_j's with $j \leq i$. If X_{i+1} can be encoded by some element in $\mathcal{C}_i$ and some torus T, then set $\mathcal{C}_{i+1} = \mathcal{C}_i$. Otherwise find some code α_{i+1} and $\bar{b}_0$ as in 4.8 encoding X_{i+1}. Define:

$$\rho(\bar{z}) := \forall \bar{y} \left(\bigwedge_{k=0}^{i} \bigwedge_{T \in G(\alpha_k, \alpha_{i+1})} \neg \chi^T_{\alpha_k, \alpha_{i+1}}(\bar{y}, \bar{z}) \right),$$

where $\chi^T_{\alpha,\beta}(\bar{b}, \bar{b}')$ expresses that T induces a toric correspondence between $\varphi_\alpha(\bar{x}, \bar{b})$ and $\varphi_\beta(\bar{x}, \bar{b}')$ (this is a definable condition).

Now, $\varphi_{\hat{\alpha}}(\bar{x}, \bar{z}) := \varphi_{\alpha_{i+1}}(\bar{x}, \bar{z}) \wedge \rho(\bar{z})$ satisfies properties (a)–(g) in Definition 4.7. We claim that it also satisfies property (f). Let $\bar{m}$ be in $(\mathbb{C}^*)^n$ such that $\varphi_{\hat{\alpha}}(\bar{x} \cdot \bar{m}, \bar{b})$ cannot be encoded by $\varphi_{\hat{\alpha}}$. Equivalently, there are some $k \leq i$ and some torus $T \in G(\alpha_k, \alpha_{i+1})$ such that T induces a toric correspondence between $\varphi_{\alpha_k}(\bar{x}, \bar{b}_1)$ and $\varphi_{\alpha_{i+1}}(\bar{x} \cdot \bar{m}, \bar{b})$. Find $\bar{m}_1 \in (\mathbb{C}^*)^{n_\alpha}$ with $(\bar{m}_1, \bar{m}) \in T$. Then T induces a toric correspondence between $\varphi_{\alpha_k}(\bar{x} \cdot \bar{m}_1^{-1}, \bar{b}_1)$ and $\varphi_{\alpha_{i+1}}(\bar{x}, \bar{b})$. By property (f) the set $\varphi_{\alpha_k}(\bar{x} \cdot \bar{m}_1^{-1}, \bar{b}_1)$ can be encoded by φ_{α_k}, which yields a contradiction. Hence, set $\mathcal{C}_{i+1} = \mathcal{C}_i \cup \{\hat{\alpha}\}$.

Given some minimal prealgebraic definable set X, there is some X_i such that X and X_i are isomorphic. By construction, there exists a unique code $\alpha \in \mathcal{C}$ and finitely many tori which encode X_i (and hence X): finiteness of the set of tori follows from $G(\alpha, \alpha)$ being finite (see Lemma 4.9). □

Definition 4.11. A *good code* is an element of $\mathcal{C}$.

5. Difference sequences

Recall the following result due to M. Ziegler (in a more general setting) in unpublished work [18].

Lemma 5.1. *Let A be an algebraically closed subset. If the tuples $\bar{a}$, $\bar{b}$ and $\bar{a} \cdot \bar{b}$ are pairwise independent over A, then* $\mathrm{tp}(\bar{a}/A)$ *is the generic type of an A-definable coset of a torus.*

This result is extremely relevant for our purposes due to an observation by Mustafin [12], who noticed that no code variety could be a coset of a torus.

Lemma 5.2. *Let V be a code variety. Then its multiplicative stabiliser is finite. In particular, V is no coset of a torus.*

Proof. For $V = V(\bar{x}, \bar{b})$ as above, choose $T = \mathrm{stab}(V)^0$ the connected component of its multiplicative stabiliser. By construction T is a torus. Choose now generic $\bar{b}$-independent elements $\bar{g}$ in V and $\bar{a}$ in T. By definition, the element $\bar{a} \cdot \bar{g}$ is generic in V over $\bar{b}$. Therefore $\delta(\bar{a}/\bar{g}, \bar{b}) = \delta(\bar{a}/\bar{b}) = \dim(T)$. On the other hand, $\delta(\bar{a} \cdot \bar{g}/\bar{g}, \bar{b}) \leq 0$, since V is a code variety. Since $\bar{a} \cdot \bar{g}$ and $\bar{a}$ are $\bar{g}\bar{b}$-interdefinable (multiplicatively), it follows that $\dim(T) = 0$. □

Definition 5.3. Let $(\bar{e}_0, \dots, \bar{e}_\lambda)$ a sequence of length $\lambda + 1$. The i^{th} *derivation* ∂_i maps

$$(\bar{e}_0, \dots, \bar{e}_\lambda)$$

to

$$(\bar{e}_0 \cdot \bar{e}_i^{-1}, \dots, \bar{e}_{i-1} \cdot \bar{e}_i^{-1}, \bar{e}_i^{-1}, \bar{e}_{i+1} \cdot \bar{e}_i^{-1}, \dots, \bar{e}_\lambda \cdot \bar{e}_i^{-1}).$$

A sequence obtained after composing the operators $(\partial_i)_{i \leq \lambda}$ finitely many times is a *difference sequence.* If $\nu < \lambda$ and we only consider operators $(\partial_i)_{i \leq \nu}$, then we call the resulting sequence a *ν–difference sequence.*

Note 5.4. For every λ there exist only finitely many different derivations (precisely $(\lambda + 2)!$ many). Moreover, the set of derivations of a given fixed sequence is closed under permutations, since the transposition (ij) equals $\partial_j \circ \partial_i \circ \partial_j$.

Fix for each $\alpha \in \mathcal{C}$ a positive integer m_α such that every $\bar{b} \models \theta_\alpha$ lies in the definable closure of some (*any*) Morley sequence of $\varphi_\alpha(\bar{x}, \bar{b})$ of length m_α.

Theorem 5.5. *For every α in $\mathcal{C}$ and $\lambda \geq m_\alpha$ there is some formula $\psi_\alpha(\bar{x}_0, \dots, \bar{x}_\lambda)$ (whose realizations will be called* difference sequences*) satisfying the following:*

(h) *If $\models \psi_\alpha(\bar{e}_0, \dots, \bar{e}_\lambda)$, then $\bar{e}_i \neq \bar{e}_j$ for $i \neq j$.*

(i) *Given $\bar{b} \models \theta_\alpha$ and a Morley sequence $\{\bar{e}_0, \dots, \bar{e}_\lambda, \bar{f}\}$ for $\varphi_\alpha(\bar{x}, \bar{b})$, then*

$$\models \psi_\alpha(\bar{e}_0 \cdot \bar{f}^{-1}, \dots, \bar{e}_\lambda \cdot \bar{f}^{-1}).$$

(j) *For any realization $(\bar{e}_0, \dots, \bar{e}_\lambda)$ von ψ_α there exists a unique $\bar{b}$ with $\models \varphi_\alpha(\bar{e}_i, \bar{b})$ for $i = 0, \dots, \lambda$. Moreover, $\bar{b}$ lies in the definable closure of any segment of length m_α of the $\bar{e}_i$'s. Hence, $\bar{b}$ is called the* canonical parameter *of the sequence $\bar{e}_0, \dots, \bar{e}_\lambda$.*

(k) *If $\models \psi_\alpha(\bar{e}_0, \dots, \bar{e}_\lambda)$, then $\models \psi_\alpha(\bar{e}_0, \dots, \bar{e}_{\lambda'})$ for each $m_\alpha \leq \lambda' < \lambda$.*

(l) *Let $i \neq j$ and $(\bar{e}_0, \dots, \bar{e}_\lambda)$ be a realization of ψ_α with canonical parameter $\bar{b}$ as in (j). If there is some T in $G(\alpha, \alpha)$ and $\bar{e}'_j$ with $(\bar{e}_j, \bar{e}'_j) \in T$, then $\bar{e}_i \not\perp_{\bar{b}} \bar{e}'_j \cdot \bar{e}_i^{-1}$ in case $\bar{e}_i$ is a generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$.*

(m) *If $\models \psi_\alpha(\bar{e}_0, \dots, \bar{e}_\lambda)$, then $\models \psi_\alpha(\partial_i(\bar{e}_0, \dots, \bar{e}_\lambda))$ for $i \in \{0, \dots, \lambda\}$.*

Proof. We find $\psi_\alpha(\bar{x}_0, \dots, \bar{x}_\lambda)$ inductively in λ. Consider the following type-definable property $\Sigma(\bar{e}_0, \dots, \bar{e}_\lambda)$:

> there exist some $\bar{b}'$ and a Morley sequence $\bar{e}'_0, \dots, \bar{e}'_\lambda, \bar{f}$ of $\varphi_\alpha(\bar{x}, \bar{b}')$
> with $\bar{e}_i = \bar{e}'_i \cdot \bar{f}^{-1}$.

It is easy to see that Σ has properties (h)–(k) and (m). Note that $(\bar{e}_i : i \leq \lambda)$ is a Morley sequence over $\bar{b}'\bar{f}$. In particular, its canonical parameter $\bar{b}$ lies in $\mathrm{dcl}(\bar{b}'\bar{f})$. Let $T \in G(\alpha, \alpha)$ and $(\bar{e}_j, \bar{e}^*_j) \in T$. Then $\bar{e}^*_j \in \mathrm{acl}(\bar{e}_j)$, so $\bar{e}^*_j \perp_{\bar{b}} \bar{e}_i$ for $i \neq j$. If $\bar{e}_i \perp_{\bar{b}} \bar{e}^*_j \cdot \bar{e}_i^{-1}$, then $\bar{e}_i^{-1}$, $\bar{e}^*_j$ and $\bar{e}^*_j \cdot \bar{e}_i^{-1}$ will determine a pairwise $\bar{b}$–independent triple, since

$$\mathrm{MR}(\bar{e}^*_j/\bar{b}, \bar{e}^*_j \cdot \bar{e}_i^{-1}) = \mathrm{MR}(\bar{e}_i^{-1}/\bar{b}, \bar{e}^*_j \cdot \bar{e}_i^{-1}) = \mathrm{MR}(\bar{e}_i^{-1}/\bar{b}) = \mathrm{MR}(\bar{e}_j/\bar{b}) = \mathrm{MR}(\bar{e}^*_j/\bar{b})$$

so $\bar{e}^*_j \perp_{\bar{b}} \bar{e}^*_j \cdot \bar{e}_i^{-1}$. By Lemma 5.1 the type $\mathrm{tp}(\bar{e}_i^{-1}/\bar{b})$ will be the generic type of some torus coset. Likewise for $\mathrm{tp}(\bar{e}_i/\bar{b})$. This contradicts Lemma 5.2 since $\bar{e}_i$ is generic in $\varphi_\alpha(\bar{x}, \bar{b})$. Therefore, property (l) also holds.

Find now by compactness a finite set ψ_0 in Σ which implies (h)-(l). Define

$$\psi_\alpha(\bar{x}_0, \dots, \bar{x}_\lambda) := \bigwedge_{\partial \text{ derivation}} \psi_0(\partial(\bar{x}_0, \dots, \bar{x}_\lambda)).$$

□

6. Green up!

From now on we will consider an extension $\mathcal{L}^* := \mathcal{L}_{Morley} \cup \{\ddot{\mathrm{U}}\}$, where $\ddot{\mathrm{U}}$ is a new unary predicate which determines the *green coloring* (from the german word *grün*). An $\mathcal{L}^*$-structure is a pair $(A, \ddot{\mathrm{U}}(A))$ consisting of a structure A in $\mathcal{D}$ (i.e. $A = \langle A \rangle \subseteq \mathbb{C}$) and a divisible torsion-free subgroup $\ddot{\mathrm{U}}(A)$ of $A \setminus \{0\}$, that is, a $\mathbb{Q}$-vector space. Given two $\mathcal{L}^*$-structures B and A, we write $B \subseteq A$ in case $B \subset A$ as elements of $\mathcal{D}$ and $\ddot{\mathrm{U}}(A) \cap B = \ddot{\mathrm{U}}(B)$.

The δ-function introduced in Section 3 will be modified accordingly: Given an $\mathcal{L}^*$-structure A finitely generated with respect to $\langle\cdot\rangle$, set

$$\delta(A) := 2\dim(A) - \mathrm{l.}\dim_{\mathbb{Q}}(\ddot{\mathrm{U}}(A)).$$

If $B \subseteq A$ and A is f.g. over B, or more generally both $\mathrm{l.}\dim_{\mathbb{Q}}(\ddot{\mathrm{U}}(A)/\ddot{\mathrm{U}}(B))$ and $\dim(A/B)$ are finite, define

$$\delta(A/B) := 2\dim(A/B) - \mathrm{l.}\dim_{\mathbb{Q}}(\ddot{\mathrm{U}}(A)/\ddot{\mathrm{U}}(B)). \tag{6.1}$$

This agrees with Poizat's context (*cf.* [15, Section 3]). Hence,*bases*, *generated*, *linearly (in)dependent* refer to the underlying $\mathcal{L}_{mult}$-structure. Similarly, *generic*, *Morley sequence*, *(in)dependent*, *transcendental* or *algebraic* refer to the theory ACF_0, unless otherwise specified. From now no, $\mathrm{acl}(M)$ denotes the field theoretical algebraic closure of M.

Note 6.1. Let $B \subseteq A$ as above. If A has a green basis over B, that is $A = \langle \ddot{\mathrm{U}}(A)B \rangle$, then the previous definition agrees with the definition of $\delta(A/B)$ in Section 3. In particular, property (e) in Definition 4.7 holds for green realizations of a code instance $\varphi_\alpha(\bar{x}, \bar{b})$.

Given $B \subseteq A$, we say that B is *strong* in A, if $\delta(A'/B) \geq 0$ for every $A' = \langle A' \rangle \subset A$ f.g over B. We denote it by $B \leq A$. If there is some $\bar{b} \in B$ generating B, then we write $\bar{b} \leq A$ in case $B \leq A$. Similarly, set $\delta(\bar{a}/\bar{b}) := \delta(\langle \bar{a}\bar{b} \rangle / \langle \bar{b} \rangle)$.

Lemma 6.2. *Suppose all structures can be embedded into a common $\mathcal{L}^*$-structure M. Then:*

(1) For $B \subseteq C \subseteq A$, we have $\delta(A/B) = \delta(C/B) + \delta(A/C)$.
(2) $\delta(\langle AB \rangle / B) \leq \delta(A/A \cap B)$. (Submodularity)
(3) If $C \leq M$ and $C' \leq M$, then $C \cap C' \leq M$.
(4) For every $A \subseteq M$ there exists a unique $A \subseteq C = \langle C \rangle \leq M$ minimal such. We call such a set the strong closure *of A (in M) and denote it by $\mathrm{cl}_M(A)$.*
(5) If $(A_i)_{i<\alpha}$ is an increasing sequence with $A_i \leq K$ for all i, then $\bigcup_i A_i \leq M$.

Now consider the class of $\mathcal{L}^*$-Structures $\mathcal{K} := \{M \mid \emptyset \leq M\}$. Unlike in [15] we are not interested in $\mathcal{L}^*$-structures whose underlying $\mathcal{L}_{Ring}$-structure is an algebraically closed field but mere expansions of structures in $\mathcal{D}$ with hereditarily non-negative predimension function δ.

Assumption 6.3. From now on, δ will be as in in (6.1). A realization of a code or a difference sequence will consist exclusively of green elements, unless otherwise specified. Likewise, a *minimal prealgebraic* extension $M \leq N$ in $\mathcal{K}$ is a minimal prealgebraic extension of structures N/M in $\mathcal{D}$ such that N has a green basis over M.

Given a strong extension $B \leq A$ in $\mathcal{K}$ with $\operatorname{l.dim}_{\mathbb{Q}}(A/B) < \infty$, we can find a decomposition $B = A_0 \leq A_1 \leq \ldots \leq A_{n-1} \leq A_n = A$ such that A_{i+1}/A_i is minimal strong for all $i < n$. Note that a strong extension $M \leq N$ in $\mathcal{K}$ ($M \subsetneq N$) is *minimal (strong)* if there is no $M' = \langle M' \rangle$ with $M < M' \leq N$. The following result is easy to prove.

Lemma 6.4. *Let $B \leq A$ be a minimal extension. One of the following cases holds:*

(1) algebraic: *$\ddot{\mathrm{U}}(A) = \ddot{\mathrm{U}}(B)$ and $A = \langle Ba \rangle$ for some $a \in \operatorname{acl}(B) \setminus B$. Then, $\delta(A/B) = 0$.*

(2) white generic: *$\ddot{\mathrm{U}}(A) = \ddot{\mathrm{U}}(B)$ and $A = \langle Ba \rangle$ for some element a transcendental over B. Then, $\delta(A/B) = 2$.*

(3) green generic: *A contains a basis consisting of a green singleton a over B. Moreover, a is transcendental over B and $\delta(A/B) = 1$.*

(4) minimal prealgebraic: *$B \leq A$ is minimal prealgebraic as in 6.3, that is, A contains a green basis $\bar{a}$ over B such that the (strong) type of $\bar{a}$ over B is minimal prealgebraic. In this case $\delta(A/B) = 0$.*

The class $\mathcal{K}$ can be easily axiomatized as shown in [15]:

Theorem 6.5. *Let $(M, \ddot{\mathrm{U}}(M))$ be an $\mathcal{L}^*$-expansion of an algebraically closed field of characteristic 0. Then $M \in \mathcal{K}$ if and only if the following (definable) conditions hold:*

(1) $\ddot{\mathrm{U}}(M)$ is a trosion-free divisible multiplicative subgroup of M.

(2) $\ddot{\mathrm{U}}(M)$ has no non-trivial algebraic points.

(3) For every $\emptyset$-definable variety $V(\bar{x}) \subseteq (\mathbb{C}^*)^{2n+1}$ of dimension n with associated tori $\{T_0, \ldots, T_r\}$ as in 2.1, then $\bar{a} \in T_i$ for some $i > 0$ for all $\bar{a} \in V \cap \ddot{\mathrm{U}}(M)$.

Note that (2) follows from (1) and (3).

The class $\mathcal{K}$ has the amalgamation property with respect to strong embeddings. Moreover, it has the JEP and contains only countably many f.g. structures up to isomorphism. Hence, the Fraïssé-Hrushovski limit M_ω of the subcollection of f.g. structures in $(\mathcal{K}, \leq)$ exists. We call M_ω the *generic model* of $\mathcal{K}$. Let T_ω be the $\mathcal{L}^*$-theory of M_ω. Recall the following result from [15]:

Theorem 6.6. *The generic model M_ω is ω-saturated. Its theory T_ω is ω-stable of Morley rank $\omega \cdot 2$. Moreover, $\ddot{\mathrm{U}}(M_\omega)$ has Morley rank ω.*

7. Green counts

This section contains the main result of a combinatorial flavour, which will be extremely useful in order to show that the generic model M_ω can be collapsed into a finite rank one.

Definition 7.1. Let A and M be elements in $\mathcal{K}$ with a common strong substructure B. An $\mathcal{L}^*$-structure M' in $\mathcal{K}$ is an *amalgam* of A and M over B if A and M are strongly embedded in M' over B and $M' = \langle M, A \rangle$ (after identification of A and M with their images under their respective embeddings). In case M and A (or rather, their images in M') are algebraically independent over B and $M \cap A = B$, then M' is a *free amalgam*.

Following [15] we obtain the following:

Lemma 7.2. *Given M, A and B in $\mathcal{K}$ with $B \leq A$ and $B \leq M$. Then there is an amalgam M' in $\mathcal{K}$ of A and M over B such that A and M are algebraically independent over B. If B is algebraically closed in A or in M, then the amalgam can chosen to be a free amalgam.*

The following Lemma yields a lower bound for the length of a difference sequence for a good code in order to recover a Morley segment inside the sequence over a strong subset of parameters.

Lemma 7.3. *For every code α and every natural number n there exists a positive integer $\lambda_\alpha(n) = \lambda \geq 0$ such that given any strong extension $M \leq N$ in $\mathcal{K}$ and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_\mu)$ for α in N with canonical parameter $\bar{b}$, we have that if $\mu \geq \lambda$ then either the canonical parameter of a λ-derived sequence of $(\bar{e}_0, \ldots, \bar{e}_\mu)$ lies in $\operatorname{acl}(M)$, or the sequence $(\bar{e}_0, \ldots, \bar{e}_\mu)$ contains a Morley subsequence for $\varphi_\alpha(\bar{x}, \bar{b})$ over M of length n.*

Proof. Given $(\bar{e}_0, \ldots, \bar{e}_\mu)$ as above such that first part of the statement does not hold, define:

$$\begin{aligned} X_1 &= \{i \in [m_\alpha, \mu] : \bar{e}_i \text{ generic over } M \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\}\}, \\ X_2 &= \{i \in [m_\alpha, \mu] : \bar{e}_i \subseteq \langle M \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\}\rangle\}, \\ X_3 &= [m_\alpha, \mu] \setminus (X_1 \cup X_2). \end{aligned}$$

After possible permutation of the set of indices, we may assume that $X_1 < X_3 < X_2$ (note that we may have indices of X_2 go to X_3 and indices of X_3 land in X_1). Since $\bar{b} \in \operatorname{dcl}(\bar{e}_0, \ldots, \bar{e}_{m_\alpha - 1})$, then by Property (e)

$$\begin{aligned} \delta(\bar{e}_i / M, \bar{e}_0, \ldots, \bar{e}_{i-1}) &\leq -1 \quad \text{for } i \in X_3 \text{ and} \\ \delta(\bar{e}_i / M, \bar{e}_0, \ldots, \bar{e}_{i-1}) &= 0 \quad \text{for } i \in X_1 \cup X_2. \end{aligned}$$

It follows from $M \leq N$ that

$$\begin{aligned} 0 \leq \delta(\bar{e}_0, \ldots, \bar{e}_\mu / M) &\leq \delta(\bar{e}_0, \ldots, \bar{e}_{m_\alpha - 1} / M) + \sum_{i=m_\alpha}^{\mu} \delta(\bar{e}_i / M, \bar{e}_0, \ldots, \bar{e}_{i-1}) \\ &\leq m_\alpha n_\alpha + (-1)|X_3| \ , \end{aligned}$$

therefore $|X_3| \leq m_\alpha n_\alpha$.

Let now $r = m_\alpha + |X_1| + |X_3|$, and $s = r(n_\alpha + 1)$. Take $I \subseteq \{\bar{e}_r, \ldots, \bar{e}_\mu\}$ of cardinality $|I| = rn_\alpha + 1$. To simplify the notation, assume that $I = \{r, \ldots, s\}$. Choose varieties $W_0 \subset V_0, \ldots, W_t \subset V_t$ (with the V_i's irreducible) such that $\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_s)$ equals $\bigcup_{i \leq t}(V_i \setminus W_i)$. Let $T_0, \ldots, T_\ell$ be the associated tori to the V_i's as in 2.1. The point $(\bar{e}_0, \ldots, \bar{e}_s)$ lies in some $V_{i_0} \setminus W_{i_0}$ for some $i_0 \leq t$. Let W be its locus over $\operatorname{acl}(M)$. Choose $W \subseteq W' \subseteq V_{i_0}$ maximal such that $\operatorname{cd}(W') \leq \operatorname{cd}(W)$. By construction W' is cd-maximal, so there is some $j \in \{0, \ldots \ell\}$ such that T_j is its minimal torus. Fix some $\bar{m}$ in $W' \subseteq \bar{m}T_j \cap V_{i_0}$. We may assume that $\bar{m} \in \operatorname{acl}(M)$, since $W \subseteq \bar{m}T_j$ by $\operatorname{acl}(M)$-definability of W'. Choose $(\bar{a}_0, \ldots, \bar{a}_s)$ a generic point of W' over $\operatorname{acl}(M)$ and paint it green. Then, the point lies in $V_{i_0} \setminus W_{i_0}$, since $(\bar{e}_0, \ldots, \bar{e}_s)$ was an specialization of $(\bar{a}_0, \ldots, \bar{a}_s)$ and $V_{i_0} \setminus W_{i_0}$ is Zariski open in its closure.

Therefore $\psi_\alpha(\bar{a}_0, \ldots, \bar{a}_s)$ holds. Note that

$$\begin{aligned} r \cdot n_\alpha &\geq \mathrm{l.\,dim}_{\mathbb{Q}}(\bar{e}_0, \ldots, \bar{e}_{r-1}/M) = \mathrm{l.\,dim}_{\mathbb{Q}}(\bar{e}_0, \ldots, \bar{e}_s/M) \geq \mathrm{cd}(W) \geq \mathrm{cd}(W') \\ &= \sum_{i \leq s} \mathrm{l.\,dim}_{\mathbb{Q}}(\bar{a}_i/\operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1}) - \dim(\bar{a}_i/\operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1}) \\ &\geq \sum_{r \leq i \leq s} \mathrm{l.\,dim}_{\mathbb{Q}}(\bar{a}_i/\operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1}) - \dim(\bar{a}_i/\operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1}). \end{aligned}$$

Property (e) yields that $\delta(\bar{a}_i/\operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1}) \leq 0$ for $i \geq r \geq m_\alpha$, that is,

$$2\dim(\bar{a}_i/\operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1}) \leq \mathrm{l.\,dim}_{\mathbb{Q}}(\bar{a}_i/\operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1}).$$

Hence, if $\bar{a}_i \notin \langle \operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1} \rangle$ then

$$\mathrm{l.\,dim}_{\mathbb{Q}}(\bar{a}_i/\operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1}) - \dim(\bar{a}_i/\operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{i-1}) \geq 1.$$

Therefore, there is some $t \in \{r, \ldots, s\}$ with $\bar{a}_t \in \langle \operatorname{acl}(M), \bar{a}_0, \ldots, \bar{a}_{t-1} \rangle$. The linear dependence will be determined by the coset $\bar{m}T_j$. So $\bar{m}T_j$ also determines that $\bar{e}_t \in \langle \operatorname{acl}(M), \bar{e}_0, \ldots, \bar{e}_{t-1} \rangle$.

Consider now all possible pairs (t, j) as above. This determines a $(rn_\alpha+1)(\ell+1)$-coloring of all $(rn_\alpha + 1)$-subsets of $\{r, \ldots, \mu\}$. By (finite) Ramsey's theorem, there is some number μ_0, such that for $\mu \geq \mu_0$ there is a monochromatic subset $I \subseteq \{r, \ldots, \mu\}$ of cardinality $|I| \geq m_\alpha + rn_\alpha + 1$. Equivalently, there is some $t \in \{r, \ldots, s\}$ and some $j \leq \ell$ with

$$\bar{e}_{i_t} \in \langle \operatorname{acl}(M), \bar{e}_0, \ldots, \bar{e}_{r-1}, \bar{e}_{i_r}, \bar{e}_{i_{r+1}}, \ldots, \bar{e}_{i_{t-1}} \rangle,$$

for all $i_r < \cdots < i_s$ in I. Moreover, the linear dependence comes from some $\bar{m}T_j$ with $\bar{m}$ in $\operatorname{acl}(M)$ (note that the tuple $\bar{m} \in \operatorname{acl}(M)$ may change). Let γ_i be the $(t+i)^{\text{th}}$ element in I. For $i > 0$ we have that $\bar{e}_{\gamma_i}\bar{e}_{\gamma_0}^{-1}$ lies in $\operatorname{acl}(M)$. Hence, the canonical parameter of the difference sequence lies in

$$\operatorname{acl}(\partial_{\gamma_0}(\bar{e}_0, \ldots, \bar{e}_\mu)) \subseteq \operatorname{acl}(M),$$

which contradicts our assumption.

Hence, there is an upper bound for μ depending on r, hence a lower bound for X_1 depending on μ. □

8. Prealgebraicity go home!

Choose now finite-to-one functions $\mu^*, \mu : \mathcal{C} \to \mathbb{N}$ such that:

$$\begin{aligned} \mu^*(\alpha) &\geq n_\alpha k_\alpha + 1, \\ \mu^*(\alpha) &\geq \lambda_\alpha(m_\alpha + 1), \\ \mu(\alpha) &\geq \lambda_\alpha(\mu^*(\alpha)). \end{aligned}$$

where λ_α is the function obtained in Lemma 7.3.

Definition 8.1. Let $\mathcal{K}^\mu$ be the subcollection of elements M in $\mathcal{K}$ such that no good code α has a (green) difference sequence in M of length at least $\mu(\alpha) + 1$.

The class $\mathcal{K}^\mu$ is universally axiomatizable relative to ACF_0. We want now to obtain a theory T^μ whose models lie in $\mathcal{K}^\mu$. Moreover, green prealgebraic extensions of strong subsets will become algebraic in T^μ. In order to ensure that T^μ is complete, we will impose that every good code attains the maximal number of realizations and moreover describe all this in an elementary way.

Lemma 8.2. *Let $M \in \mathcal{K}^\mu$ and $M' \in \mathcal{K}$ be a minimal prealgebraic extension of M not in $\mathcal{K}^\mu$. Given a good code α and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$ in M' with canonical parameter $\bar{b}$ in $\operatorname{acl}(M)$, then there exists some i such that all $\bar{e}_j$'s with $i \neq j$ lie in M and $\bar{e}_i$ is an M-generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$ which generates M' over M.*

Proof. We may assume that M is algebraically close, for otherwise we may define an $\mathcal{L}^*$-structure on $\operatorname{acl}(M)$,by setting $\ddot{\mathrm{U}}(\operatorname{acl}(M)) = \ddot{\mathrm{U}}(M)$; minimality of the extension M'/M yields that M is algebraically closed in M' so we could replace M' by $\langle M' \operatorname{acl}(M)\rangle$. Since $M \leq M'$, then it follows from property (e) that for each j either $\bar{e}_j$ lies in M or is a generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$ over M. Since $M \in K^\mu$, there must be some generic $\bar{e}_i$. By minimality of the extension, it follows that $M' = \langle M\bar{e}_i\rangle$. Suppose there is another M-generic realization $\bar{e}_j$ different from $\bar{e}_i$. Then $M' = \langle M\bar{e}_i\rangle = \langle M\bar{e}_j\rangle$, so there is some tuple $\bar{m}$ in M and a toric correspondence induced by some $T \in G(\alpha, \alpha)$ with

$$(\bar{e}_i \cdot \bar{m}, \bar{e}_j) \in T.$$

Let $\bar{e}'_j := \bar{e}_i \cdot \bar{m}$. In particular, $\bar{e}'_j \cdot \bar{e}_i^{-1} \in M$. Since $\bar{e}_i$ is M-generic, then

$$\bar{e}_i \mathop{\bigcup\!\!\!\!\!\!\smile}_{\bar{b}} \bar{e}'_j \cdot \bar{e}_i^{-1},$$

which contradicts property (l) of a difference sequence. □

Corollary 8.3. *Let $M \in \mathcal{K}^\mu$ and $M' \in \mathcal{K}$ be a minimal extension of M. If $\operatorname{l.dim}_{\mathbb{Q}}(M'/M) = 1$, then M' lies also in $\mathcal{K}^\mu$. Otherwise, M'/M is minimal prealgebraic, and M' does not lie in $\mathcal{K}^\mu$ if and only if there is some good $\alpha \in \mathcal{C}$ and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$ for α in M' such that one of the following holds:*

a) *$\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1}$ lie in M and $\langle M, \bar{e}_{\mu(\alpha)}\rangle = M'$. Moreover, α is the unique good code describing the extension $M' \geq M$.*
b) *There is some subsequence of length $\mu^*(\alpha)$ which is a Morley sequence for $\varphi_\alpha(\bar{x}, \bar{b})$ over $M\bar{b}$, where $\bar{b}$ is the canonical parameter of $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$.*

Proof. Consider first the case $\operatorname{l.dim}_{\mathbb{Q}}(M'/M) = 1$. If $\ddot{\mathrm{U}}(M') = \ddot{\mathrm{U}}(M)$, then there are no new green difference sequences in M', so $M' \in \mathcal{K}^\mu$. Otherwise, $\operatorname{l.dim}_{\mathbb{Q}}(\ddot{\mathrm{U}}(M')/M) = 1$ and $M' = \langle \ddot{\mathrm{U}}(M'), M\rangle$ (the green generic case). Suppose M' is not in K^μ, then there is a good code α and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$ for α in M' witnessing this fact. Let $\bar{b}$ be the canonical parameter of some derived sequence and $\bar{e}$ be a generic element over $M\bar{b}$. Then

$$\operatorname{l.dim}_{\mathbb{Q}}(M'/M) \geq \operatorname{l.dim}_{\mathbb{Q}}(\bar{e}/M\bar{b}) \geq 2,$$

which contradicts our assumption. By Lemma 8.2 there is no such derived sequence. However, Lemma 7.3 yields a contradiction since $\mu(\alpha) \geq \lambda_\alpha(\mu^*(\alpha))$ and $\mu^*(\alpha) \geq 1$.

Let now M' be a minimal prealgebraic extension of M. If a) or b) hold, then clearly M' does not belong to K^μ. Conversely, if M' is not in K^μ, then there is some good code α and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$ for α in M' witnessing this fact. Let $\bar{b}$ be its canonical parameter. If we may choose the difference sequence such that $\bar{b}$ lies in $\operatorname{acl}(M)$, then case a) holds by Lemma 8.2. Otherwise no difference sequence has canonical parameter in $\operatorname{acl}(M)$, which yields case b) because $\mu(\alpha) \geq \lambda_\alpha(\mu^*(\alpha))$ by Lemma 7.3.

In order to show that α is uniquely determined, consider another good code α' different from α with $M' = \langle M, \bar{e}_{\mu(\alpha')}\rangle$. Then $n_\alpha = n_{\alpha'} = \operatorname{l.dim}_{\mathbb{Q}}(M'/M)$ and the locus of $(\bar{e}_{\mu(\alpha)}, \bar{e}_{\mu(\alpha')})$ over M determines a coset of a torus in $G(\alpha, \alpha')$. Since both α and α' are good, $G(\alpha, \alpha') = \emptyset$ by construction, obtaining hence the desired contradiction. □

Corollary 8.4. *Given a good code α there is a $\forall\exists$-sentence χ_α such that every structure M in $\mathcal{K}^\mu$ satisfies χ_α if and only if it contains no minimal prealgebraic extension in $\mathcal{K}^\mu$ given by α.*

Proof. Let $\alpha \in \mathcal{C}$, M in $\mathcal{K}^\mu$ and $\bar{b} \in M$ such that a generic realization $\bar{a}$ of $\varphi_\alpha(\bar{x}, \bar{b})$ generates a minimal prealgebraic extension $M[\bar{a}] := \langle M\bar{a}\rangle$ over M. If $M[\bar{a}]$ does not belong to $\mathcal{K}^\mu$, either case a) or b) in 8.3 hold.

a) is equivalent to the existence of some good code α' and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha')})$ in M', whose first $\mu(\alpha')$ many elements (and hence its canonical parameter) lie in M and $M[\bar{a}] = \langle M\bar{e}_{\mu(\alpha')}\rangle$. By uniqueness, $\alpha = \alpha'$. Since $M \leq M[\bar{a}]$, it follows that $\bar{e}_{\mu(\alpha)}$ is M-generic. Therefore, $\bar{a}$ can be mapped to $\bar{e}_{\mu(\alpha)}$ over M by some $\mathbb{Q}$–basis change. Equivalently, there is some green tuple $\bar{m} \in M$ and some torus $T \in G(\alpha, \alpha)$ such that T induces a toric correspondence between $\psi_\alpha(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1}, \bar{x})$ and $\varphi_\alpha(\bar{x} \cdot \bar{m}, \bar{b})$. Finiteness of $G(\alpha, \alpha)$ allows us to express this fact by an existential formula over $\bar{b}$.

b) implies that there is some good code β and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\beta)})$ in $M[\bar{a}]$ with $\mu^*(\beta)$ many M-linearly independent elements. We need only consider finitely many such $\beta's$ since

$$n_\beta \mu^*(\beta) \leq \operatorname{l.dim}_{\mathbb{Q}}(M[\bar{a}]/M) = n_\alpha.$$

Assume first the following

Assumption: ψ_β equals $V_1 \setminus W_1$, where V_1 is a irreducible variety and $W_1 \subsetneq V_1$ is a proper subvariety, both $\operatorname{acl}(\emptyset)$-definable.

Let $V_0 = V_\alpha(\bar{x}, \bar{b})$ be the corresponding code variety for $\varphi_\alpha(\bar{x}, \bar{b})$, that is, the locus of $\bar{a}$ over $\operatorname{acl}(M)$. Set $V = V_0 \times V_1$ and let $\{T_0, \ldots, T_r\}$ be the tori associated to V as in 2.1. Take $W \subseteq V$ to be the locus of $(\bar{a}, \bar{e}_0, \ldots, \bar{e}_{\mu(\beta)})$ over $\operatorname{acl}(M)$. Note that $\operatorname{cd}(W) = \operatorname{cd}(V_0) = k_\alpha$ by 3.2. Since W projects generically onto V_0, then $\operatorname{cd}(W') \geq \operatorname{cd}(W)$ for all $W' \supseteq W$. Let now T be the minimal torus of W and $\bar{m} \in M$ such that $W \subseteq \bar{m}T$. The coset $\bar{m}T$ contains hence the green tuple $(\bar{a}, \bar{e}_0, \ldots, \bar{e}_{\mu(\beta)})$.

We call a torus coset $\bar{c}T$ *gay* (as in colorful, let the distinction be made) if it contains a green tuple . In case T is given by the equations $\prod x_i^{\lambda_{ij}} = 1$ $(j = 1, \ldots, d)$, it is easy to see that $\bar{c}T$ is gay if and only if $c'_j := \prod c_i^{\lambda_{ij}}$ is green for $j = 1, \ldots, d$. If $\bar{c}T$ is gay and $T \subseteq T'$ (that is, T' lies on top of T), then $\bar{c}T'$ is also gay.

Choose now some M-definable W' maximal such that $\operatorname{cd}(W') = \operatorname{cd}(W)$ containing W. Hence, W' is cd-maximal and its minimal torus equals some T_i. Moreover, $W' \subseteq \bar{m}T_i$. Since $\bar{m}T \subseteq \bar{m}T_i$ and $\bar{m}T$ is gay, so is $\bar{m}T_i$. Let $(\bar{a}^*, \bar{e}_0^*, \ldots, \bar{e}_{\mu(\beta)}^*)$ be generic in W' over M. We may assume that $\bar{a}^* = \bar{a}$ since they have the same M-type. It follows from $\operatorname{cd}(W') = \operatorname{cd}(W) = \operatorname{cd}(V_0)$ that

$$\begin{aligned}\operatorname{l.dim}_{\mathbb{Q}}(\bar{a}, \bar{e}_0^*, \ldots, \bar{e}_{\mu(\beta)}^*/\operatorname{acl}(M)) - \dim(\bar{a}, \bar{e}_0^*, \ldots, \bar{e}_{\mu(\beta)}^*/\operatorname{acl}(M)) \\ = \operatorname{l.dim}_{\mathbb{Q}}(\bar{a}/\operatorname{acl}(M)) - \dim(\bar{a}/\operatorname{acl}(M))\end{aligned}$$

so

$$\text{l.}\dim_{\mathbb{Q}}(\bar{e}^*_0,\dots,\bar{e}^*_{\mu(\beta)}/M\bar{a}) = \dim(\bar{e}^*_0,\dots,\bar{e}^*_{\mu(\beta)}/M\bar{a}) =: \ell.$$

Choose now an $\mathcal{L}_{mult}$-basis $f_0,\dots,f_{\ell-1}$ in $(\bar{e}^*_0,\dots,\bar{e}^*_{\mu(\beta)})$ over $M\bar{a}$. The elements $(f_0,\dots,f_{\ell-1})$ are hence algebraically independent over $M\bar{a}$. Gayness of $\bar{m}T_i$ yields a structure N in $\mathcal{K}$ if we paint $(\bar{a}^*,\bar{e}^*_{\leq\mu(\beta)})$ green (after closing it under $\langle\cdot\rangle$). Note that

$$N = \langle M\bar{a}\bar{e}^*_0,\dots,\bar{e}^*_{\mu(\beta)}\rangle = \langle M[\bar{a}]f_0,\dots,f_{\ell-1}\rangle,$$

where $f_0,\dots,f_{\ell-1}$ is a tuple of green independent generic elements. Set $F_i := \langle M\bar{a}f_0,\dots,f_{i-1}\rangle$ and observe that $F_i \leq F_{i+1}$ gives a tower of green generic extensions for $0 \leq i \leq \ell-1$. By Collorary 8.3 (repeatedly) we have that:

(*) $M[\bar{a}] \in \mathcal{K}^\mu$ if and only if $N \in \mathcal{K}^\mu$.

Now, $(\bar{e}_0,\dots,\bar{e}_{\mu(\beta)})$ is a specialization of $(\bar{e}^*_0,\dots,\bar{e}^*_{\mu(\beta)})$, both lying in F. By assumption ψ_β is Zariski open in F, so $\models \psi_\beta(\bar{e}^*_0,\dots,\bar{e}^*_{\mu(\beta)})$ since $(\bar{e}_0,\dots,\bar{e}_{\mu(\beta)})$ realizes ψ_β. Therefore, the existence of a green difference sequence for β (of length $\mu(\beta)+1$) implies the existence of another one in $N = M[\bar{a}][\bar{f}]$, which may be obtained by only finitely many possibilities. Conversely, it suffices to ensure the existence of $(\bar{e}^*,\dots,\bar{e}^*_{\mu(\beta)}) \in N$ to conclude that $M[\bar{a}] \notin \mathcal{K}^\mu$ by (*). Consider the following definable conditions:

There is a tuple $\bar{m} \in M$ and an irreducible component W' of $V \cap \bar{m}T_i$ (where $V = V_0 \times V_1$ and $V_0 = V_\alpha(\bar{x},\bar{b})$ is as above) such that:

(1) The coset $\bar{m}T_i$ is gay.
(2) W' projects generically onto V_0.
(3) $\mathrm{cd}(W') = \mathrm{cd}(V_0)$.
(4) $\models \psi_\beta(\bar{e}^*_0,\dots,\bar{e}^*_{\mu(\beta)})$ for generic $(\bar{a}^*,\bar{e}^*_0,\dots,\bar{e}^*_{\mu(\beta)})$ in W'.

Therefore, we obtain an existential sentence over $\bar{b}$ for each T_i. The disjunction of all these formulae yields the desired sentence.

For the general case, decompose ψ_β into a finite union of locally closed sets $V_i \setminus W_i$ (for $1 \leq i \leq t$). We proceed as above for each i and form the disjunction of all the sentences so obtained. □

9. Fraïssé à la verte

This section shows that $\mathcal{K}^\mu$ has the amalgamation property with respect to strong embeddings. Hence, we obtain a *rich* field as in [14]. Work done in previous sections yields now the following key result:

Lemma 9.1. *Let A, B and M be structures in $\mathcal{K}^\mu$, where B is a common strong substructure of both A and M. Let M' be their free amalgam over B and consider a difference sequence $(\bar{e}_0,\dots,\bar{e}_{\mu(\alpha)})$ in M' for some good code α. Then there is some derived sequence whose canonical parameter lies either in* $\mathrm{acl}(M)$ *or in* $\mathrm{acl}(A)$.

Proof. Suppose the statement does not hold. Then, find $\bar{e}_0,\bar{e}_1,\dots,\bar{e}_{m_\alpha}$ of length $m_\alpha+1$ which is a Morley sequence over both M and A after applying Lemma 7.3 twice by choice of $\mu(\alpha)$ and $\mu^*(\alpha)$. Let $E = \{\bar{e}_0,\dots,\bar{e}_{m_{\alpha-1}}\}$. The canonical parameter $\bar{b}$ lies in $\mathrm{dcl}(E)$ and

$$\bar{e}_{m_\alpha} \mathop{\smile\!\!\!\!\!\!|}_{\bar{b}} ME \qquad \text{und} \qquad \bar{e}_{m_\alpha} \mathop{\smile\!\!\!\!\!\!|}_{\bar{b}} AE\,.$$

Decompose each tuple in E as a product of a green tuple in M and one in A and define E_M (*resp.* E_A) to be the collection of these factors in M (*resp.* A). Set $E' = E_M \cup E_A$. Then $\bar{b} \in \mathrm{dcl}(E')$ and by interdefinability of E und E' over M (*resp.* over A), we conclude that

$$\bar{e}_{m_\alpha} \underset{\bar{b}}{\perp} ME' \qquad \text{und} \qquad \bar{e}_{m_\alpha} \underset{\bar{b}}{\perp} AE',$$

and

$$\bar{e}_{m_\alpha} \underset{BE'}{\perp} M \qquad \text{und} \qquad \bar{e}_{m_\alpha} \underset{BE'}{\perp} A.$$

Let $\bar{e}_{m_\alpha} = \bar{m} \cdot \bar{a}$ with $\bar{m}$ in M and $\bar{a}$ in A. Since $M \perp_B A$ then $M \perp_{BE'} A$, so $\{\bar{e}_{m_\alpha}, \bar{m}, \bar{a}\}$ is a pairwise BE'-independent triple. This contradicts Lemma 5.2 by Lemma 5.1, since $\mathrm{stp}(\bar{e}_{m_\alpha}/BE')$ will be the generic type of a torus coset. □

An embedding B in A is *strong* if the image of B in A is a strong substructure.

Theorem 9.2. *$\mathcal{K}^\mu$ has the amalgamation property with respect to strong embeddings.*

Proof. let $B \leq M$ and $B \leq A$ be structures in $\mathcal{K}^\mu$. We need to find a strong extension M' of M in $\mathcal{K}^\mu$ with $B \leq A' \leq M'$, where A and A' are B-isomorphic. Decomposing both $B \leq A$ and $B \leq M$ in minimal extensions, we may reduce it to the case where both A and M are minimal extensions of B. If any of them is algebraic, add the corresponding elements to the $\langle\cdot\rangle$-closure (we obtain a new structure in $\mathcal{K}^\mu$ since there are no new green points).

Otherwise, we may consider the free amalgam M' of M and A over B by Lemma 7.2. If $M' \in K^\mu$, we are done. Otherwise, we need only show that M and A are B-isomorphic. It follows from Corollary 8.3 that both M and A are minimal prealgebraic over B. Note that only the first case in Lemma 9.1 may occur, so there is a good code α and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$ in M' with canonical parameter $\bar{b}$. By symmetry we may assume that $\bar{b}$ lies in $\mathrm{acl}(M)$. After possible permutations we may assume by Lemma 8.2 that $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1}$ lie in M and $\bar{e}_{\mu(\alpha)}$ is an M-generic realization of α.

Case 1. There is some $(\mu(\alpha) - 1)$-derived difference sequence with canonical parameter in $\mathrm{acl}(B)$.

Work hence with the above sequence, which we will still denote by $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$. It usffices to show that $\bar{e}_{\mu(\alpha)}$ lies in A. Otherwise, $\bar{e}_{\mu(\alpha)}$ is generic in $\varphi_\alpha(\bar{x}, \bar{b})$ over A and M, so independent from A and from M over B. Find two green tuples $\bar{a} \in A$ and $\bar{m} \in M$ with $\bar{e}_{\mu(\alpha)} = \bar{m} \cdot \bar{a}$. Observe that $\bar{e}_{\mu(\alpha)}$, $\bar{a}$ and $\bar{m}$ are pairwise B-independent, whihc contradicts Lemma 5.2 by Lemma 5.1. Minimal prealgebraicity of A over B implies that $A = \langle B, \bar{e}_{\mu(\alpha)} \rangle$. Since $A \in K^\mu$, there is some $\bar{e}_i$ in $M \setminus B$. Since $B \leq M$, then $\bar{e}_i$ is B-generic by property (e). Hence, the map $\bar{e}_i \mapsto \bar{e}_{\mu(\alpha)}$ induces a B-isomorphism between A and M.

Case 2. No $(\mu(\alpha) - 1)$-derivation has canonical parameter in $\mathrm{acl}(B)$.

As above decompose $\bar{e}_{\mu(\alpha)} = \bar{m} \cdot \bar{a}$ with $\bar{m} \in M$ and $\bar{a} \in A$ both green tuples. Since $\bar{e}_{\mu(\alpha)}$ is M-generic, then $0 = \delta(\bar{e}_{\mu(\alpha)}/M) = \delta(\bar{a}/M)$, so $\bar{a}$ generates A over B. Apply now Lemma 7.3 to $B \leq M'$ and $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$. There is some Morley segment of length $\mu^*(\alpha)$ over $B\bar{b}$.

Since

$$\mu^*(\alpha) \geq n_\alpha k_\alpha + 1 > n_\alpha \geq MR(\bar{m}/B\bar{b}),$$

there is some $\bar{e}_i$ in M with $\bar{m} \mathop{\smile\!\!\!\!\!\!|}_{B\bar{b}} \bar{e}_i$. In particular, $\bar{e}_{\mu(\alpha)}$ and $\bar{e}_i$ have the same type over $B\bar{b}\bar{m}$, and so do $\bar{a} = \bar{m} \cdot \bar{e}_{\mu(\alpha)}^{-1}$ and $\bar{m} \cdot \bar{e}_i^{-1}$. By minimality, $\bar{a} \mapsto \bar{m} \cdot \bar{e}_i^{-1}$ induces a B-isomorphism between A and M. □

Using notation developed by Poizat [14] we say that an $\mathcal{L}^*$-structure M in $\mathcal{K}^\mu$ is *rich* if for every f.g. $B \leq M$ and every f.g. strong extension $B \leq A$ in $\mathcal{K}^\mu$ there is some strong substructure $A' \leq M$ with $A \simeq_B A'$. Since every algebraic strong extension of an element in $\mathcal{K}^\mu$ lies again in $\mathcal{K}^\mu$, it follows that rich structures are algebraically closed fields.

Corollary 9.3. *There is a (unique upto isomorphism) countable rich structure in $\mathcal{K}^\mu$. Moreover, all rich structures are $\mathcal{L}_{\infty,\omega}$-equivalent.*

10. Axioms for T^μ

Recall that given $\bar{a}, B \subseteq M \in \mathcal{K}$ we say that B is strong in M if $\langle B \rangle \leq M$ and we denote by $\delta(\bar{a}/B)$ the quantity $\delta(\langle B\bar{a}\rangle/\langle B\rangle)$. Let T^μ be the elementary theory of rich fields in $\mathcal{K}^\mu$. We will show in this section that T^μ is axiomatizable and model-complete.

Definition 10.1. Let $M \models T^\mu$ and B be some subset of M. Denote by $\mathrm{cl}_d^M(B)$ the collection of all f.g. $A \subseteq M$ with $\delta(A/\mathrm{cl}(B)) = 0$. Set $d_M(A/B) := d(A/B) := \delta(\mathrm{cl}(\langle A, B\rangle)/\mathrm{cl}(B))$.

Note that $\mathrm{cl}_d^M(B) = \{a \in M : d(a/B) = 0\}$.

It is easy to see [15] that:

Lemma 10.2. *For any structure in $\mathcal{K}$ the following holds:*

(1) $d(\bar{a}\bar{c}/B) = d(\bar{a}/B\bar{c}) + d(\bar{c}/B)$.

(2) The closure operator cl_d defines a pregeometry over the set $\ddot{\mathrm{U}}$ of green points whose associated dimension function is d.

Lemma 10.3. *Let $e \geq 0$, a subset $B = \langle B \rangle \leq M \in \mathcal{K}$ and a tuple $\bar{a} \in M$. Then:*

(1) If $\delta(\bar{a}/B) = e$, then there is some existential $\mathcal{L}^*$-formula $\tau_\delta(\bar{x}, \bar{z})$ and a tuple $\bar{b} \in B$ such that:
- *$\models \tau_\delta(\bar{a}, \bar{b})$,*
- *for every $\bar{a}'$ and $\bar{b}' \in B' \subseteq M' \in \mathcal{K}$ with $\models \tau_\delta(\bar{a}', \bar{b}')$, then $\delta(\bar{a}'/B') \leq e$.*

(2) If $d(\bar{a}/B) = e$, then there is an existential $\mathcal{L}^*$-formula $\tau_d(\bar{x}, \bar{z})$ and a $\bar{b} \in B$ such that:
- *$\models \tau_d(\bar{a}, \bar{b})$,*
- *for every $\bar{a}'$ and alle $\bar{b}' \in M' \in \mathcal{K}$ with $\models \tau_d(\bar{a}', \bar{b}')$, then $d(\bar{a}'/\bar{b}') \leq e$.*

Proof. We need only prove part (1). Hence, choose $\bar{a} \in M$ and $B \leq M$ as above. Let $B = A_0 \leq A_1 \leq \ldots \leq A_n = \langle B\bar{a}\rangle =: A$ be the decomposition of $B \leq A$ into minimal strong embeddings. Then $e = \delta(\bar{a}/B) = \sum_{i=1}^n \delta(A_i/A_{i-1})$. Therefore we may assume that $n = 1$, that is, $B \leq A$ is minimal strong. Four cases may occur by Lemma 6.4. Cases (1)–(3) are easy so we may hence consider case (4), that is, a minimal prealgebraic extension. Let $\bar{c}$ be a green basis for A/B and $\bar{b} \in B$ with $\mathrm{l.dim}_{\mathbb{Q}}(\bar{a}\bar{c}/B) = \mathrm{l.dim}_{\mathbb{Q}}(\bar{a}\bar{c}/\bar{b})$ and $\bar{a}\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{\bar{b}} B$. Let $\alpha \in \mathcal{C}$ be unique encoding a/B. Choose some quantifier-free $\mathcal{L}^*$-formula $\tilde{\tau}(\bar{x}, \bar{y}, \bar{z})$ with $\models \tilde{\tau}(\bar{a}, \bar{c}, \bar{b})$ such that:
- The tuples $\bar{a}$ and $\bar{c}$ are interdefinable over $\bar{b}$ (explicitly given).
- $\models \tilde{\tau}(\bar{x}, \bar{y}, \bar{z}) \to \bigwedge_i \ddot{\mathrm{U}}(y_i)$.

- $\bar{c}$ realizes some $\varphi_\alpha(\bar{y}, \bar{b}_1)$, where $\bar{b}_1$ lies in $\operatorname{acl}(\bar{b})$ (explicitly given).

By property (e) the formula $\tau_\delta(\bar{x}, \bar{z}) := \exists \bar{y}\, \tilde{\tau}(\bar{x}, \bar{y}, \bar{z})$ satisfies the required conditions, because $\delta(\bar{a}'/\bar{b}') = \delta(\bar{c}'/\bar{b}') \leq \delta(\bar{c}'/\operatorname{acl}(\bar{b}') \leq 0$ if $\models \tilde{\tau}(\bar{a}', \bar{c}', \bar{b}')$.

The general case may be reduced to the minimal one by considering some tuple to express the decomposition into minimal extensions. □

Given $M \subseteq N$ both elements in $\mathcal{K}$ such that M is $\mathcal{L}^*$-existentially closed in N, it follows from 10.3 that $M \leq N$: otherwise, there is a tuple $\bar{a} \in M$ with $d_M(\bar{a}) > d_N(\bar{a}) = e$. Choose some τ_d as in 10.3 such that $N \models \tau_d(\bar{a}, \bar{b})$ for some $\bar{b} \in \langle\emptyset\rangle \subseteq M$. Hence, $M \models \tau_d(\bar{a}, \bar{b})$ (since M is existentially closed in N), which contradicts $d_M(\bar{a}/\bar{b}) > e$. Hence, the following holds:

Lemma 10.4. *If M is an elementary submodel of $N \models T^\mu$, then M is strong in N.* □

Define now an $\mathcal{L}^*$-theory $\tilde{T}^\mu$ as follows:

(1) Every model lies in $\mathcal{K}^\mu$.
(2) Every model is an algebraically closed field of characteristic 0.
(3) Given a model M, a good code α and a parameter $\bar{b}$ in M, then there is no extension of M in $\mathcal{K}^\mu$ given by a green M-generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$.
(4) In some ω-saturated model there are infinitely many d-independent green generic elements.

Poizat [15] axiomatized universally condition "$\emptyset \leq M$": since $\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_\nu)$ are quantifier-free, then axiom (1) is universal. Both ACF_0 and hence (2) are inductively axiomatizable. Corollary 8.4 yields the inductively axiomatization of (3), and the $\exists\forall$-axiomatization of (4) follows from Lemma 10.3.

A key result is the following:

Theorem 10.5. *An $\mathcal{L}^*$-structure M is rich if and only if it is an ω-saturated model of $\tilde{T}^\mu$. In particular, $\tilde{T}^\mu = T^\mu$ and $\tilde{T}^\mu$ is complete.*

Proof. The proof is divided into two parts: first, we show that every ω-saturated model of $\tilde{T}^\mu$ is rich in $\mathcal{K}^\mu$. Then show that all rich structures are models of $\tilde{T}^\mu$, which yields ω-saturation of rich structures because they are all ∞-equivalent by Corollary 9.3.

Hence, let M be an ω-saturated model of $\tilde{T}^\mu$, a finite subset $B \leq M$ and $A \geq B$ a f.g. structure in $\mathcal{K}^\mu$. We need to embedd A in M strong over B. We may assume that A/B is minimal strong. By Lemma 6.4 there are four possibilities:

A/B is algebraic. we are done by axiom (2).

A/B is minimal prealgebraic. Consider the free amalgam M' of M and A over B. Moreover, let α be the good code encoding A/B. Axiom (3) implies that M' is not in $\mathcal{K}^\mu$. Since $\mathcal{K}^\mu$ has the amalgamation property with respect to strong embeddings by Theorem 9.2, then A must be already embeddable in M over B.

A/B is green generic, i.e. generated by a green transcendental element a. Axiom (4) and ω-saturation imply that M contains infinitely many green d-independent generic elements $(g_i)_{i\in\mathbb{N}}$. Since $d(B) = e < \infty$, there is some $i \in \mathbb{N}$ with $d(g_i/B) = 1$. Hence, $\langle Bg_i\rangle/B$ is a green generic extension (which lies in M), and the map $a \mapsto e_i$ yields a strong embedding of A in M over B.

A/B is white generic, i.e. generated by a white generic element w over B and $\ddot{\mathrm{U}}(A) = \ddot{\mathrm{U}}(B)$. It is easy to see that the sum w' of two B-generic d-independent green elements g_1 and g_2 is white B-generic, that is, $d(w'/B) = 2$. As above we can find such elements g_1, g_2 in M, so we are done.

Now, let M be a rich structure in $\mathcal{K}^\mu$. In order to show that $M \models T^\mu$, we first show that M is an algebraically closed field. Let $a \in \mathrm{acl}(M)$ and $B \leq M$ f.g. with $a \in \mathrm{acl}(B)$. If a is green, then a in B, because $B \leq M$. Otherwise, a is white and $\ddot{\mathrm{U}}(B) = \ddot{\mathrm{U}}(\langle Ba \rangle)$. Hence, the extension $\langle Ba \rangle \geq B$ is in $\mathcal{K}^\mu$, so we can realize it in M over B.

For axiom (3), consider a good code α and some parameter $\bar{b} \in M$. Let $\bar{a}$ be an M-generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$. If $\langle M\bar{a} \rangle \in \mathcal{K}^\mu$, then choose some $B_0 \leq M$ containing $\bar{b}$. Therefore, $\langle B_0 \bar{a} \rangle \in \mathcal{K}^\mu$ and by richness of M, we can find some $\bar{a}_0 \in M$ with $\langle B_0 \bar{a} \rangle \cong \langle B_0 \bar{a}_0 \rangle =: B_1 \leq M$. Iterating this process with B_1, we can find a sequence $B_0 \leq B_1 \leq B_2 \leq \dots$ in M, such that $B_{i+1} := \langle B_i \bar{a}_i \rangle \cong \langle B_i \bar{a} \rangle$. Then $\bar{a}_1, \bar{a}_2, \dots$ is a sufficiently long green Morley sequence for $\varphi_\alpha(\bar{x}, \bar{b})$, whose difference realizes ψ_α, contradicting $M \in \mathcal{K}^\mu$.

Axiom (4) is satisfied by M by richness. □

Remark that given $A \subseteq M \in \mathcal{K}$, we define $\mathrm{cl}(A)$ as the smallest $\langle \cdot \rangle$-closed subset containing A with $\mathrm{cl}(A) \leq M$. If A is f.g., so is $\mathrm{cl}(A)$.

Corollary 10.6. *T^μ is complete. Two tuples $\bar{a}$ and $\bar{a}'$ in two models M and M' have the same type if and only if there is some $\mathcal{L}^*$-isomorphism f from $\mathrm{cl}_M(\bar{a})$ to $\mathrm{cl}_{M'}(\bar{a}')$, mapping $\bar{a}$ to $\bar{a}'$.*

Proof. The rich field obtained in Section 9 is a model of T^μ by Theorem 10.5. Given two models M and M' of T^μ, we may replace them by elementary extension and assume that they are ω-saturated. Hence they are rich by Theorem 10.5, hence elementarily equivalent by Corollary 9.3. So are M and M'.

In order to prove the second statement, suppose that both M and M' are ω-saturated, since by Lemma 10.4 the closure $\mathrm{cl}_M(\bar{a})$ does not change. Hence, M and M' are rich structures and the map $f : \mathrm{cl}_M(\bar{a}) \to \mathrm{cl}_{M'}(\bar{a}')$ induces a back-and-forth system, so f is elementary.

Suppose now that $\bar{a}$ in M has the same type as $\bar{a}'$ in M'. Since $\mathrm{cl}(\bar{a})$ is in the algebraic closure (in T^μ) of $\bar{a}$, there is an elementary map f from $\mathrm{cl}(\bar{a})$ to M' by ω-saturation, mapping $\bar{a}$ onto $\bar{a}'$. Let $A' = f(\mathrm{cl}(\bar{a}))$. Hence $A' \leq M'$ because A' has the same type as $\mathrm{cl}(\bar{a}')$. Therefore, $A' = \mathrm{cl}(\bar{a}')$. □

Corollary 10.7. *The theory T^μ is model-complete.*

Proof. We will give a straight-foward proof due to M. Ziegler. We need only show that given any two models M and N of T^μ with $M \subseteq N$ then M is strong in N. This implies that $\mathrm{cl}_M(\bar{a}) = \mathrm{cl}_N(\bar{a})$ for any $\bar{a}$ in M. So the inclusion is elementary by Corollary 10.6. In particular, we will show the following:

Claim. *If $M \models T^\mu$ and $M \subseteq N \in \mathcal{K}^\mu$, then M is strong in N.*

Otherwise, choose some $M \subseteq N_1$ with minimal $\mathrm{l.dim}_{\mathbb{Q}}(N_1/M) = e$. Since $M = \mathrm{acl}(M)$, then $e \geq 2$. Minimality of e implies that $\delta(N_0/M) \geq 0$ for every $N_0 = \langle N_0 \rangle$ with $M \subseteq N_0 \subsetneq N_1$. In particular, $M \leq N_0$. Choose now some N_0 with $\mathrm{l.dim}_{\mathbb{Q}}(N_0/M) = e - 1$.

$$-1 \geq \delta(N_1/M) = \delta(N_1/N_0) + \delta(N_0/M)$$

and $\delta(N_1/N_0) \geq -1$ imply that $\delta(N_0/M) \leq 0$. Since $M \leq N_0$, the extension N_0/M is prealgebraic in $\mathcal{K}^\mu$ (i.e. a tower of algebraic and minimal prealgebraic extensions). Hence, find N'/M minimal prealgebraic with $M \leq N' \leq N_0$, which contradicts axiom (3). □

Note 10.8. One can show that axiom (4) follows already from (1)–(3), by aproximating a green generic extension by suitable prealgebraic extensions. On the other hand, the ∀∃-axiomatization follows from model-completeness (Corollary 10.7) by general model-theoretical arguments.

11. Ranks

We show in this section that T^μ has Morley rank 2. Let acl^μ denote the algebraic closure in models of T^μ. All model-theoretical notions refer exclusively to T^μ, which we will emphazise with μ if necessary. We will show that $\mathrm{acl}^\mu(\bar{a})$ is the union of all minimal prealgebraic extensions of $\mathrm{cl}(\bar{a})$.

Lemma 11.1. *Both closures* acl^μ *and* cl_d^M *agree in models of* T^μ.

Proof. If B is f.g., so is $\mathrm{cl}(B)$ and contained in $\mathrm{acl}^\mu(B)$. Hence, we may assume that B is f.g. and strong in $M \models T^\mu$.

First, show that $\mathrm{cl}_d^M(B) \subseteq \mathrm{acl}^\mu(B)$. Let $A \subset M$ be f.g. with $\delta(A/B) = 0$. Since $\mathrm{l.dim}_{\mathbb{Q}}(A/B)$ is finite, then we may decompose A/B into a finite sequence of minimal extensions. If $A' \supseteq B$ is such that $\delta(A'/B) = 0$, then $A' \leq M$ because $B \leq M$. Hence, we may assume that A/B is minimal. By Lemma 6.4 one of the following holds:

i) A is algebraic over B (in the field reduct). Hence, $A \subseteq \mathrm{acl}^\mu(B)$.
ii) A is minimal prealgebraic over B. Choose a good code α and some parameter $\bar{b}$ in $\mathrm{acl}(B)$ encoding A/B by Theorem 4.10. Then, $A = \langle B\bar{a}\rangle$ for some generic green realization $\bar{a} \models \varphi_\alpha(\bar{x}, \bar{b})$. We need only show that all green realizations of $\varphi_\alpha(\bar{x}, \bar{b})$ lie already in M, for otherwise there is some $M \preccurlyeq N$ and $\bar{a}' \in N$ not completely contained in M. Hence, $\bar{a}'$ is generic over M cotradicting axiom (3).

For the other inclusion, choose some $a \in M \setminus \mathrm{cl}_d^M(B)$. Set $A = \mathrm{cl}(B, a)$ and observe that $\delta(A/B) > 0$. Decompose now A/B in minimal extensions $B \leq A_0 \leq A_1 \leq \ldots \leq A_n = A$. Then there is some $i < n$ with $\delta(A_{i+1}/A_i) > 0$. By Lemma 6.4 we obtain $\mathrm{l.dim}_{\mathbb{Q}}(A_{i+1}/A_i) = 1$ so the extension A_{i+1}/A_i is either white or green generic. Corollary 8.3 implies that the free amalgam of A_{i+1} and every strong extension of A_i lies in $\mathcal{K}^\mu$. Richness of M implies that there are infinitely many $A' \leq M$, isomorphic to A_{i+1} over A_i. Moreover, they all have the same type over A_i by Corollary 10.6. So, $A_{i+1} \not\subseteq \mathrm{acl}^\mu(A_i)$, hence $a \notin \mathrm{acl}^\mu(B)$, because A_{i+1} is algebraic over $\langle B, a\rangle$ and $B \subseteq A_i$. □

Theorem 11.2. T^μ *has Morley rank* 2 *and is uncountably categorical. The white generic type has Morley rank* 2 *and the green generic one has Morley rank* 1. *Algebraic closure is determined by* cl_d*-closure in any model of* T^μ. *Moreover, for all* $\bar{a}$ *and* B *we have that :*

$$\mathrm{MR}(\bar{a}/B) = \mathrm{U}(\bar{a}/B) = d(\bar{a}/B).$$

Proof. Let M be an ω-saturated model of T^μ, seen as subset of the monster model of T^μ. We compute $\mathrm{MR}(a/M)$ for elements a coming from the monster model. Since

$$0 \leq d(a/M) \leq \delta(a/M) \leq 2.$$

there are four cases to consider:

$d(a/M) = 0$: By Lemma 10.4, we have that $a \in \mathrm{acl}^\mu(M) = M$. So $\mathrm{MR}(a/M) = 0$.

$d(a/M) = 1$ and a is green: Then $\langle Ma \rangle$ is strong in the monster model and $\mathrm{tp}(a/M)$ is the type of the *green generic* element by Corollary 10.6. Since all other green generic types are algebraic, then $\mathrm{MR}(a/M) = 1$, so $\ddot{\mathrm{U}}$ defines a strongly minimal set.

$d(a/M) = 1$ with a white: There must be some green point $c \in \mathrm{cl}(\langle M, a \rangle) \setminus M$. Hence, $\langle Mc \rangle \leq \mathrm{cl}(Ma)$ and $d(a/Mc) = 0$. Therefore, a and c are T^μ-interalgebraic over M and by the above case, $\mathrm{MR}(a/M) = \mathrm{MR}(c/M) = 1$.

$d(a/M) = 2$: Then $\ddot{\mathrm{U}}(\langle Ma \rangle) = \ddot{\mathrm{U}}(M)$ and $\langle Ma \rangle$ is strong in the monster model. Corollary 10.6 implies that $\mathrm{tp}(a/M)$ is the type of the *white generic element*, that is, the *generic type* of the field. Since all other types have Morley rank at most 1, then $\mathrm{MR}(a/M) \leq 2$. Now, $\ddot{\mathrm{U}}(M)$ is an infinite group of infinite index in M. Therefore, $\mathrm{MR}(a/M) = 2$ and T^μ has Morley rank 2.

For the last statement, $\mathrm{MR}(\bar{a}/B) = \mathrm{U}(\bar{a}B) = d(\bar{a}/B)$, recall that $\mathrm{MR} = \mathrm{U}$ holds on all $\aleph_1$-categorical theories. So Morley rank is additive and the above shows that $\mathrm{MR}(a/B) = d(a/B)$ for elements. Since d is also additive, then we are done. □

Note 11.3. For every natural number $r \geq 2$ there is some field of rank r with a multiplicative green subgroup of rank $r-1$. Work with the following predimension function (see [6]):

$$\delta(A) = r \dim(A) - 1. \dim_{\mathbb{Q}}(\ddot{\mathrm{U}}(A)).$$

Note 11.4. Rigidity results due to J. Kirby [10] may be useful to prove that there is a field of finite Morley rank of characteristic 0 equipped with a non-algebraic subgroup of a semiabelian variety.

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