Steinberg’s torsion theorem in the context of groups of finite Morley rank

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Abstract. Groups of finite Morley rank generalize algebraic groups; the simple ones have even been conjectured to be algebraic. Parallel to an ambitious classification program towards this conjecture, one can try to show direct equivalents of known results on algebraic groups in the context of groups of finite Morley rank. This is what we do here with Steinberg’s theorem on centralizers of semi-simple elements.

Theorem (cf. [5, Corollary 2.16(b)]). Let $G$ be a connected group of finite Morley rank of $p^⊥$-type and $ζ ∈ G$ a $p$-element such that $ζ^{p^n} ∈ Z(G)$. Then $C(ζ)/C^o(ζ)$ has exponent dividing $p^n$.

The reader will find the argument essentially different from Steinberg’s elaborate methods, perhaps because logicians are simple-minded (the author is one). We shall need but little material; algebraically speaking the main tool is the connectedness of centralizers of tori.

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The instruments. A general reference on groups of finite Morley rank is [2], where the reader will find everything he needs to know (and actually more).

$p^⊥$-type means that $G$ has trivial $p$-unipotent subgroups, which by definition are definable, connected, nilpotent $p$-groups of bounded exponent. This mimics a characteristic-not-$p$ assumption, and [3, Theorem 4] then yields conjugacy of Sylow $p$-subgroups, which are toral-by-finite, that is, finite extensions of $p$-tori. (Quite interestingly, conjugacy is not known to hold in general.)

A $p$-torus is a finite direct sum of Prüfer $p$-groups $Z_{p^∞}$. Still about torality, [3, Corollary 3.1] shows that a $p$-element of a connected group of $p^⊥$-type is in any maximal $p$-torus of its centralizer. Decent tori are definable hulls of abelian, divisible, torsion groups. As their centralizers in a connected group are connected ([1, Theorem 1]; also [4, Corollary 3.9]), so are those of $p$-tori.

On a more methodological side, Frattini arguments are corollaries of conjugacy theorems familiar to finite group theorists: if $H ⊆ G$ is a normal subgroup and $K ≤ H$ a subgroup for which $H$ controls $G$-conjugacy, then $G = H · N_G(K)$. This will be used repeatedly.

So will actions. If $σ$ normalizes a $p$-torus $T$, then considering the map $t ↦ [t, σ]$ in view of the underlying notion of dimension (the Prüfer rank, here),
one easily finds $T = C_T(\sigma) + [T, \sigma]$ with finite intersection. If $\sigma$ has order prime to $p$, the sum is clearly direct.

The argument. Let $k = p^n$. By torality there is a maximal $p$-torus $S$ with $\zeta \in S$. A Frattini argument implies $C(\zeta) = C^o(\zeta) \cdot N_{C(\zeta)}(S)$, so lifting torsion \cite[Fact 2.5]{AltinBur} it suffices to show that if $\alpha$ is a $q$-element of $N_{C(\zeta)}(S)$ for any prime number $q$, then $\alpha^k \in C^o(\zeta)$.

If $q \neq p$, we have observed that $S = C_S(\alpha) \oplus [S, \alpha]$; in particular $C_S(\alpha) \simeq S/[S, \alpha]$ is connected. Letting $T = C^o_S(\alpha)$, one therefore has $\zeta \in T$ and $\alpha \in C(T) = C^o(T) \leq C^o(\zeta)$. So we may assume $q = p$.

By a Frattini argument, $\zeta$ normalizes a maximal $p$-torus $\Sigma$ of $C^o(\alpha)$; by torality $\alpha \in \Sigma$. One has $\Sigma = C_\Sigma(\zeta) + [\Sigma, \zeta]$. Let $T = C^o_\Sigma(\zeta)$, and write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in T, \alpha_2 \in [\Sigma, \zeta]$. Let $\Tr$ denote the norm under the action of $\langle \zeta \rangle$: $\Tr(x) = x + x^\zeta + \cdots + x^\zeta^{k-1}$. Then a quick computation yields $\alpha^k = \Tr(\alpha) = \Tr(\alpha_1) + \Tr(\alpha_2) = \alpha_1^k + 0 = \alpha_1^k$, so $\zeta \in C(T) = C^o(T) \leq C^o(\alpha_1^k) = C^o(\alpha^k)$.

But for $p$-elements $\mu$ and $\nu$ in a connected group of $p^k$-type, $\mu \in C^o(\nu)$ iff $\nu \in C^o(\mu)$: either is equivalent to having a $p$-torus containing both $\mu$ and $\nu$. It follows $\alpha^k \in C^o(\zeta)$, and we are done. \hfill \Box

References.

\begin{enumerate}
\item Robert Steinberg. Torsion in Reductive Groups. \textit{Advances in Mathematics}, \textbf{15} (1975), 63-92.
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