Ordered o-stable groups

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May 12, 2010

Abstract

An ordered structure $M$ is called o-$\lambda$-stable if for any subset $A$ with $|A| \leq \lambda$ and for any cut in $M$ there are at most $\lambda$ 1-types over $A$ which are consistent with the cut. It is proved in the paper that an ordered o-stable group is abelian. Also there were investigated definable subsets and unary functions of o-stable groups.

1 Introduction, notations and preliminaries

Since the notion of an o-minimal structure appeared in [9] various generalizations were introduced. One of them is a notion of a weakly o-minimal structure [6, 8]. Also each o-minimal theory is quasi-o-minimal [3, 4]. It is easy to see that any cut in an o-minimal structure has only one completion up to a complete type over the model. For weakly o-minimal structures a similar result has been proved by Kulpeshov [7]: Let $M$ be a totally ordered structure. Then $M$ is weakly o-minimal iff the following holds on $M$: any cut $\langle C, D \rangle$ in $M$ has at most two complete 1-types over $M$ extending $\langle C, D \rangle$, and if a cut $\langle C, D \rangle$ in $M$ has two complete 1-types over $M$ extending $\langle C, D \rangle$, then the set of all realizations of each of these types is convex in any elementary extension of $M$.

It immediately follows from the notion of quasi-o-minimality that each cut has at most continuum many extensions up to complete types over a model.

What is common for all of these notions. That each cut has a few number of extensions. Recall that stable theories have a few types. So we can combine these things and introduce the notion of an o-stable theory: each cut in each model of this theory has a few complete types which extend it.

Let $\mathcal{M} = (M, <, \ldots)$ be a totally ordered structure, $a$ is an element of $M$ and $A, B$ subsets. As usually I write $a < A$ if $a < b$ for any $b \in A$, and $A < B$ if $a < b$ for any $a \in A$ and $b \in B$. A partition $\langle C, D \rangle$ of $M$ is called a cut if $C < D$. Given a cut $\langle C, D \rangle$ I construct a partial type $\{c < x < d : c \in C, d \in D\}$, which I also call a cut and use the same notation.
If the set $C$ is definable, then the cut is called **quasirational**, if in addition $\sup C \in M$ then the cut $\langle C, D \rangle$ is called **rational**, a non-definable cut is called **irrational**. If $C = (-\infty, c]$ I denote this cut by $c^-$, and if $C = (-\infty, c]$ I denote it by $c^+$. If $C = M$ I denote this cut $+\infty$ and sup $A$ stands for $\langle C, D \rangle$, where $C = \{ c \in M : c < \sup A \}$. If the set $C$ is definable I will sometimes distinguish cuts defined by sup $C$ and inf $D$ as: sup $C$ stands for $\langle C, D \rangle \cup \{ C(x) \}$ and inf $D$ stands for $\langle C, D \rangle \cup \{ -C(x) \}$. A cut $\langle C, D \rangle$ in an ordered group is called **non-valuational** [8, 17] if $d - c$ converges to 0 whenever $c$ and $d$ converge to sup $C$ and inf $D$ accordingly. A cut, which is not non-valuational, is called **valuational**. Observe that for a valuational cut $\langle C, D \rangle$ there is a convex non-trivial subgroup $H$ such that sup $C = \sup(a + H)$ for some $a$, and this cut is definable iff $H$ is definable. An ordered group $G$ is said to be of **non-valuational type**, if any quasirational cut is non-valuational. Note that $G$ is of non-valuational type iff there is no definable non-trivial convex subgroup in $G$.

A cut $\langle C, D \rangle$ in $(M, <)$ is called definable in $\mathcal{M}$ iff the sets $C, D$ are definable in $\mathcal{M}$. The set of all cuts $(C, D)$ definable in $\mathcal{M}$ and such that $D$ has no smallest element will be denoted by $\overline{M}$. The set $M$ can be regarded as a subset of $\overline{M}$ by identifying an element $a \in M$ with the cut $\langle (-\infty, a], (a, +\infty) \rangle$. After such an identification, $\overline{M}$ is naturally equipped with a linear ordering extending $(M, <)$: $\langle C_1, D_1 \rangle \leq \langle C_2, D_2 \rangle$ iff $C_1 \subseteq C_2$. Clearly, $(M, <)$ is a dense substructure of $\langle \overline{M}, < \rangle$.

A subset $A$ of $M$ is called **convex** if for any $a$ and $b \in A$ the interval $[a, b] \subseteq A$. The **length** of a convex set $A$ is defined as $\sup \{ a - b : a, b \in A \}$. A **convex component** of a set $A$ is a maximal convex subset of $A$. The **convex hull** $A^c$ of $A$ is defined as $A^c = \{ b \in M : \exists a_1, a_2 \in A (a_1 \leq b \leq a_2) \}$. An ordered structure is called **weakly o-minimal** if any definable subset consists of finitely many convex components [6, 8].

Let $P$ be some property. I say that $P$ holds **eventually in $A$** if there is an element $a \in M$ such that $a < \sup A$ and $P$ holds on $(a, \infty) \cap A$. If $A = M$, I omit it. If sets $B$ and $C$ are eventually equal in $A$ I denote this by $B \cong_A C$. Let $B \subseteq A \subseteq M$. The set $B$ is said to be **dense in $A$** if for any $a_1 < a_2$ from $A$ there is $b \in B$ with $a_1 < b < a_2$. If $A = M$ I omit it. A **dense component** of $B$ in $A$ is a maximal subset $B_0$ of $B$ which is dense in $A \cap (\inf B_0, \sup B_0)$.

Let $T$ be an $\mathcal{L}$-theory and $\phi(\vec{x}, \vec{y})$ a formula. A formula $\phi$ has the **independence property** (relative to $T$) if for all $n < \omega$ there is a model $\mathcal{M} \models T$ and two sequences $(\vec{a}_i : i < n)$ and $(\vec{b}_j : J \subseteq n)$ in $M$ such that $\mathcal{M} \models \phi(\vec{a}_i, \vec{b}_j)$ if and only if $i \in J$. A theory $T$ has the independence property if some formula has the independence property relative to $T$. 

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Notation 1.1 Let $s$ be a partial $n$-type, $A$ a set. Then

$$S^n_s(A) \triangleq \{ p \in S^n(A) : p \cup s \text{ is consistent} \}$$

Note, $s$ need not to be a partial type over $A$.

Definition 1.2

1. An ordered structure $\mathcal{M}$ is $o$-stable in $\lambda$ if for any $A \subseteq M$ with $|A| \leq \lambda$ and for any cut $\langle C, D \rangle$ in $\mathcal{M}$ there are at most $\lambda$ one-types over $A$ which are consistent with the cut $\langle C, D \rangle$, i.e. $|S^1_{\langle C,D \rangle}(A)| \leq \lambda$.

2. A theory $T$ is $o$-stable in $\lambda$ if every model of $T$ is. Sometimes I write $T$ is $o$-$\lambda$-stable.

3. $T$ is $o$-stable if there exists a $\lambda$ in which $T$ is $o$-stable.

4. $T$ is $o$-superstable if there exists a $\lambda$ such that $T$ is $o$-stable in all $\mu \geq \lambda$.

5. $T$ is strongly $o$-stable if in addition any definable cut in a model $\mathcal{M}$ of $T$ is definable in the language of pure ordering, or, equivalently, if $\sup A \in M$ for any definable subset $A$ of $\mathcal{M}$.

Definition 1.3 A totally ordered structure $\mathcal{M} = (M, <, \ldots)$ has the strict order property inside a cut, if there is a formula $\phi(x, \bar{y})$ such that for any natural number $n$ there are $\bar{a}_1, \ldots, \bar{a}_n$ and a cut $\langle C_n, D_n \rangle$ in $\mathcal{M}$ such that in some $|M|^+$-saturated elementary extension $\mathcal{N}$ of $\mathcal{M}$ the following holds:

$$\phi(\mathcal{N}, \bar{a}_1) \cap \langle C_n, D_n \rangle(\mathcal{N}) \subset \phi(\mathcal{N}, \bar{a}_2) \cap \langle C_n, D_n \rangle(\mathcal{N}) \subset \ldots \subset \phi(\mathcal{N}, \bar{a}_n) \cap \langle C_n, D_n \rangle(\mathcal{N})$$

A theory has the strict order property inside a cut if some its model has.

The following lemma is an analog of the fact that a stable theory does not have the strict order property.

Lemma 1.4 (The strict order property inside a cut) Any $o$-stable theory does not have the strict order property inside a cut.

Proof. Assume the contrary, that there is a formula $\phi(x, \bar{y})$ such that for any natural number $n$ there is a cut $\langle C_n, D_n \rangle$ in $\mathcal{M}$ and there is a sequence of $\text{len}(\bar{y})$-tuples $\langle \bar{a}_i^n : i < n \rangle$ such that

$$\phi(\mathcal{N}, \bar{a}_0^n) \cap \langle C_n, D_n \rangle(\mathcal{N}) \subset \phi(\mathcal{N}, \bar{a}_1^n) \cap \langle C_n, D_n \rangle(\mathcal{N}) \subset \ldots \subset \phi(\mathcal{N}, \bar{a}_n^n) \cap \langle C_n, D_n \rangle(\mathcal{N})$$
Observe that for any triple \((i, j, n)\) with \(i < j < n\) there are \(c \in C\) and \(d \in D\) such that either \(\phi(M, \bar{a}^n_i) \cap (c, \sup C) \subset \phi(M, \bar{a}^n_i) \cap (c, \sup C)\), or \(\phi(M, \bar{a}^n_j) \cap (\sup C, d) \subset \phi(M, \bar{a}^n_j) \cap (\sup C, d)\) if the set \(C\) is definable, and \(\phi(M, \bar{a}^n_i) \cap (c, d) \subset \phi(M, \bar{a}^n_j) \cap (c, d)\) if \(C\) is not definable.

I add two new \((1 + \text{len}(\bar{y}))\)-ary predicates \(P(x, \bar{y})\) and \(R(x, \bar{y})\) naming the following sets

\[
\bigcup_{n<\omega} C_n \times \{a^n_i : i < n\}, \quad \bigcup_{n<\omega} D_n \times \{a^n_i : i < n\}
\]
correspondingly. Obviously, at least one of the following two properties holds for infinitely many natural numbers:

\[
\phi(N, \bar{a}^n_i) \cap P(N, \bar{a}^n_i) \cap (C_n, D_n)(N) \subset \phi(N, \bar{a}^n_i) \cap P(N, \bar{a}^n_i) \cap (C_n, D_n)(N) \\
\subset \ldots \subset \phi(N, \bar{a}^n_{n-1}) \cap P(N, \bar{a}^n_n) \cap (C_n, D_n)(N)
\]

\[
\phi(N, \bar{a}^n_i) \cap R(N, \bar{a}^n_i) \cap (C_n, D_n)(N) \subset \phi(N, \bar{a}^n_i) \cap R(N, \bar{a}^n_i) \cap (C_n, D_n)(N) \\
\subset \ldots \subset \phi(N, \bar{a}^n_{n-1}) \cap R(N, \bar{a}^n_n) \cap (C_n, D_n)(N)
\]

Say, the first one holds. Let \(I\) be an ordered set of indices. Consider the following type \(p(\bar{x}_i : i \in I)\) which consists of formulae of the following form:

\[
\forall v_1 [P(v_1, \bar{x}_1) \to \exists v > v_1 (P(v, \bar{x}_1) \land \\
\land \phi(N, \bar{x}_i) \cap (v, \sup P(N, \bar{x}_i)) \subset \phi(N, \bar{x}_j) \cap (v, \sup P(N, \bar{x}_j))]
\]

for all \(i < j \in I\). Clearly, \(p\) is finitely consistent. Let \(N^+_i = \langle N_1, P\rangle\) be an elementary extension of \(\langle M, P\rangle\) realizing the type \(p\) by a sequence \(\langle \bar{b}_i : i \in I\rangle\).

Let \(C = P(N^+_i, \bar{b}_i)\) for some \(i \in I\) and \(D\) be the complement of \(C\). Let also \(\langle C, D\rangle\) be a cut in \(I\). Then the following type

\[
r_{\langle C, D\rangle}(x) = \{\neg \phi(x, \bar{b}_i) : i \in C\} \cup \{\phi(x, \bar{b}_i) : i \in D\}
\]
is consistent with the cut \(\langle C, D\rangle\). Thus the cut \(\langle C, D\rangle\) may have \(2^{|I|}\) extensions and the elementary theory \(T\) of \(\mathcal{M}\) is not \(|I|\)-stable. Since \(I\) is arbitrary, so \(T\) is not o-stable.

Another property of o-stable theories is the following:

**Fact 1.5** [15] An o-stable theory does not have the independence property.

**Lemma 1.4** and Fact 1.5 implies the following criterion of o-stability.

**Theorem 1.6** Let a language \(L\) contain binary symbol ‘\(<\)’ and a theory \(T\) of \(L\) include axioms saying that ‘\(<\)’ is a total order. The theory \(T\) is o-stable if and only if it has no the independence property and the strict order property inside a cut.
Proof. The direction ‘only if’ follows from Lemma 1.4 and Fact 1.5. So I assume that a theory \( T \) has no both the independence property and the strict order property inside a cut. If \( T \) is not \( o \)-stable, similar to stability there is a model \( \mathcal{M} \models T \), a cut \( \langle C, D \rangle \) in \( \mathcal{M} \) and a formula \( \varphi(x; \bar{y}) \) such that there are \( 2^{\lvert M \rvert} \) \( \varphi \)-types over \( M \) which are consistent with the cut \( \langle C, D \rangle \). I add a new unary relational symbol \( P \) naming the set \( C \). Let \( \langle N, P \rangle \succ \langle M, P \rangle \) be sufficiently saturated, and both \( \alpha \) and \( \beta \) be such that they realize the cut \( \langle C, D \rangle \) and \( \alpha < \sup P < \beta \). Note that each \( \varphi \)-type \( p \) which is consistent with the cut \( \langle C, D \rangle \) is also consistent with the cut \( \sup(P(N)) \) in the model \( N \).

Let \( \theta(x; \bar{y}, \alpha, \beta) \equiv \varphi(x; \bar{y}) \land \alpha < x < \beta \), then there are \( 2^{\lvert M \rvert} \) \( \theta \)-types over \( M \). The formula \( \theta \) has no the independence property, then by Shelah’s results \( \theta \) has the strict order property: there is a sequence \( (\bar{b}_i : i < \omega) \) of elements of \( M \) such that \( \theta(N; \bar{b}_i, \alpha, \beta) \subset \theta(N; \bar{b}_j, \alpha, \beta) \) iff \( i < j \). Thus to obtain a contradiction it is sufficient to show that \( \neg \theta(x; \bar{b}_i, \alpha, \beta) \land \theta(x; \bar{b}_j, \alpha, \beta) \) is consistent with the cut \( \sup(P(N)) \) for any \( i < j \). Indeed, then \( \neg \varphi(x; \bar{b}_i) \land \varphi(x; \bar{b}_j) \) is consistent with the cut \( \sup(P(N)) \) if \( i < j \), which contradicts to \( o \)-stability.

Let \( i < j \). If \( \neg \theta(x; \bar{b}_i, \alpha, \beta) \land \theta(x; \bar{b}_j, \alpha, \beta) \) is not consistent with the cut \( \sup(P(N)) \) then there is \( \gamma \in P(N) \) such that

\[
\theta(N; \bar{b}_i, \alpha, \beta) \cap (\gamma, \sup(P(N))) = \theta(N; \bar{b}_j, \alpha, \beta) \cap (\gamma, \sup(P(N)))
\]

or that is the same, then there is \( \gamma \in P(N) \) such that

\[
\varphi(N; \bar{b}_i) \cap (\gamma, \sup(P(N))) = \varphi(N; \bar{b}_j) \cap (\gamma, \sup(P(N)))
\]

Then there is \( c \in C \) such that

\[
\varphi(M; \bar{b}_i) \cap (c, \sup(C)) = \varphi(M; \bar{b}_j) \cap (c, \sup(C))
\]

Similarly it can be shown that there is \( d \in D \) such that

\[
\varphi(M; \bar{b}_i) \cap (\sup(C), d) = \varphi(M; \bar{b}_j) \cap (\sup(C), d)
\]

Thus \( \varphi(M; \bar{b}_i) \cap (c, d) = \varphi(M; \bar{b}_j) \cap (c, d) \). This contradicts to existence of \( \alpha \) and \( \beta \) in \( N \).

Let \( T \) be a theory of a language \( \mathcal{L} \), and \( \mathcal{M} \prec N \) two models of \( T \) such that \( N \) is \( |M|^+ \)-saturated. For any formula \( \phi(x, \bar{a}) \) with parameters \( \bar{a} \) in \( N \) I add a new relational symbol \( P_{\phi(x, \bar{a})}(\bar{x}) \) interpreted by \( P_{\phi(x, \bar{a})}(M) = \phi(N, \bar{a}) \cap M^k \) in order to form language \( \mathcal{L}^* \).

**Fact 1.7** [13] Let \( T \) be a theory of a language \( \mathcal{L} \) without the independence property. Then the expansion \( \mathcal{M}^* \) of \( \mathcal{M} \) as an \( \mathcal{L}^* \)-structure admits quantifier elimination. In particular, \( \mathcal{M}^* \) does not have the independence property. 

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Lemma 1.8 Consider the expanded by externally definable sets model $M^*$. The property that new relations are externally definable preserves in elementary extensions of $M^*$, i.e. if $M^+_1$ is an elementary extension of $M^*$, and $P_{\phi(x,\bar{a})}$ is a new relational symbol, then there is an elementary extension $N_1$ of $M_1$ (in the initial language), such that $P_{\phi(x,\bar{a})}(M_1) = \phi(N_1, \bar{a}_1) \cap M^+_1$, where $\phi$ is defined with a new parameter $\bar{a}_1$ from $N_1$.

Proof. Let $N$ be an $|M|^+$-saturated elementary extension of $M$, and $M^+_1$ an elementary extension of $M$. By definition $P_{\phi(x,\bar{a})}(M) = \phi(N, \bar{a}) \cap M^k$ for some $\bar{a} \in N$. Consider $\theta(\bar{y}; \bar{x}) \equiv \phi(\bar{x}; \bar{y})$. Obviously, the formula $P_{\phi(x,\bar{a})}(\bar{x})$ is a definition of the $\theta(\bar{y}; \bar{x})$-type $p(\bar{y})$ of $\bar{a}$ over $M$. Let $p_1$ is a unique $\theta$-type over the model $M_1$, which extends $p$ with the same definition $P_{\phi(x,\bar{a})}(\bar{x})$. Let $N_1$ be $|M_1|^+$-saturated elementary extension of $M_1$, and $\bar{a}_1 \in N_1$ realize the type $p_1$. Then $P_{\phi(x,\bar{a})}(M_1) = \phi(N_1, \bar{a}_1) \cap M^k$. \hfill $\square$

Theorem 1.9 Let $T$ be an o-stable theory of a language $L$, and $M \prec N$ two models of $T$ such that $N$ is $|M|^+$-saturated. Then the elementary theory $T^*$ of the expansion $M^*$ of $M$ is o-stable.

Proof. Since $T^*$ does not have the independence property it is sufficient to prove that it has no the strict order property onside a cut. Assume the contrary, that $T^*$ is not o-stable and then has the strict order property inside a cut witnessed by some $L^*$-formula $\Theta(x, \bar{y})$ and a cut $\langle C, D \rangle$ in $M^+_1$. Clearly, $M^+_1$ is an elementary extension of $M^*$. Let $M^+_1$ be an expansion of $M_1$ by all extradefinable formulae. By Lemma 1.8 $M^+_1$ is an expansion of $M^*_1$. For simplicity of notation without loss of generality I may assume that $M_1 = M$.

Let $N^*$ be an $|M|^+$-saturated elementary extension of $M^*$. Then by the assumption above there is an infinite sequence $(\bar{b}_i : i \in I)$ such that $\Theta(N, \bar{b}_i) \cap \langle C, D \rangle(N) \subset \Theta(N, \bar{b}_j) \cap \langle C, D \rangle(N)$ for all $i < j$.

Note that the convex set $C$ is definable by some $L^*$-formula. Indeed let $\alpha \in N$ realize the cut $\langle C, D \rangle$. Then $(x < \alpha) \cap M$ defines $C$. Without lost of generality I may assume that for all $i < j$ there is $c_{i,j}$ such that in the expanded model $M^*$ the following holds:

$$\Theta(M^*, \bar{b}_i) \cap (c_{i,j}, \sup C) \subset \Theta(M^*, \bar{b}_j) \cap (c_{i,j}, \sup C)$$

Recall, that $\Theta(M^*, \bar{b}_i) = \theta(N, \bar{b}_i, \alpha) \cap M$ for some $\bar{a} \in N$ and some $L$-formula $\theta$. Let $\bar{C}$ be the convex hull of $C$ in $N$, and $\bar{D}$ the complement of $\bar{C}$ in $N$. Then for any cut $\langle \bar{C}, \bar{D} \rangle$ in $I$ the following type

$$\{\neg \theta(x, \bar{b}_i) : i \in C\} \cup \{\theta(x, \bar{b}_i) : i \in D\}$$

is consistent with the cut $\langle \bar{C}, \bar{D} \rangle$ in $N$. This contradicts o-stability of $T$. \hfill $\square$

There is an analog of Morley’s theorem for o-$\omega$-stable theories.
Lemma 1.10 Let $T$ be o-$\omega$-stable. Then for any $\lambda > \omega$ the theory $T$ is o-$\lambda$-stable.

Proof. The proof is similar to the proof of analogous Morley’s theorem. In this proof it is sufficient to replace the set of all one-types with the set of all one-types which are consistent with a given cut. \hfill \Box

Lemma 1.11 Let $T$ be o-$\lambda$-stable, $\mathcal{M} = (M, <, \ldots) \models T$, and $A$ a definable subset of $M$. Then $A$ with the full induced structure is o-$\lambda$-stable.

Proof. Let $\langle C, D \rangle$ be a cut in $A$. Note that $\sup C$ and $\inf D$ define cuts in $\mathcal{M}$ (possibly, these cuts are equal). If a formula $\varphi(x)$ is consistent with the cut $\langle C, D \rangle$ in $A$, then either there are as large as we want elements from $C$, satisfying $\varphi(x)$, or there are as small as we want elements from $D$, satisfying $\varphi(x)$. Then the formula $\varphi(x)$ is consistent either with the cut $\sup C$ in $\mathcal{M}$ or with the cut $\inf D$ in $\mathcal{M}$. Since for infinite cardinals the equality $\lambda + \lambda = \lambda$ holds, there are at most $\lambda$ 1-types over a subset $B$ of $A$ with $|B| \leq \lambda$, which are consistent with the cut $\langle C, D \rangle$ in $A$. \hfill \Box

Lemma 1.12 Let $T$ be o-$\lambda$-stable, $\mathcal{M} = (M, <, \ldots) \models T$, and let $E$ be a definable equivalence relation with convex classes. Then $M/E$ with the full induced structure is ordered and o-$\lambda$-stable.

Proof. If a cut in $M/E$ has too many extensions to complete types, then the corresponding cut in $M$ has the same extensions to complete types. \hfill \Box

Lemma 1.13 Let $T$ be o-$\lambda$-stable, $\mathcal{M} = (M, <, \ldots) \models T$, and $E$ is a definable equivalence relation with unbounded classes. Then $M/E$ with the full induced structure is $\lambda$-stable.

Proof. Consider a definable set in $M/E$: it is a union of $E$-classes. Then this set is consistent with the cut $+\infty$. Since $T$ is o-$\lambda$-stable, the elementary theory of $M/E$ is $\lambda$-stable. \hfill \Box

Observe that if $T$ is o-$\omega$-stable, then for each cut $\langle C, D \rangle$ I can introduce local Morley rank of a formula as well as of a type.

Definition 1.14 1. I say that Morley rank of a formula $\phi(x)$ inside a cut $\langle C, D \rangle$ is equal to or greater than 1 and write $\text{RM}_{\langle C, D \rangle}(\phi) \geq 1$ for this, if $\{\phi(x)\} \cup \langle C, D \rangle$ is consistent.

2. $\text{RM}_{\langle C, D \rangle}(\phi) \geq \alpha + 1$ if there are infinitely many pairwise inconsistent formulae $\psi_i(x)$ such that $\text{RM}_{\langle C, D \rangle}(\phi(x) \land \psi_i(x)) \geq \alpha$. 7
3. If $\alpha$ is a limit ordinal, then $RM_{(C,D)}(\phi) \geq \alpha$ if $RM_{(C,D)}(\phi) \geq \beta$ for all $\beta < \alpha$.

4. $RM_{(C,D)}(\phi) = \alpha$ if $RM_{(C,D)}(\phi) \geq \alpha$ and $RM_{(C,D)}(\phi) \not\geq \alpha + 1$.

In a similar way I can define local Morley degree of a formula. As usual I define Morley rank of a type. More generally, if $r$ is some rank of a type from stability theory, one may introduce a localization $r_{(C,D)}$ of this rank by replacing the set of all types with the set off all types consistent with the cut $(C, D)$.

2 Commutativity and general properties of ordered o-stable groups

Throughout this section $G$ is an ordered group with unit $e$ whose elementary theory is o-stable.

Let $H$ be a convex subgroup of $G$ (not necessary definable). In the following consideration I work in the cut $\sup H$. Similar to stability for any formula $\varphi(x, \bar{y})$ there is a natural number $n$ such that each chain $K_1 \cap H \subset K_2 \cap H \subset \ldots \subset K_m \cap H$ has length at most $n$, provided that $K_i$ is definable by $\varphi(x, \bar{a}_i)$ for some $\bar{a}_i$ and that each $K_i$ is not bounded in the convex subgroup $H$. I call this trivial chain condition for $H$. Here I use only the fact that $K_i$ are subsets of $G$. For two subsets $A$ and $B$ of $G$ denote

$$A \cap \cap B \triangleq \begin{cases} A \cap B & \text{if } A \cap B \text{ is not bounded in } B, \\ \{e\} & \text{otherwise.} \end{cases}$$

Note that here $A \cap \cap B$ is not necessary equal to $B \cap \cap A$. Using this notation I can rewrite the trivial chain condition as for any formula $\varphi(x, \bar{y})$ there is a natural number $n$ such that each chain $K_1 \cap \cap H \subset K_2 \cap \cap H \subset \ldots \subset K_m \cap \cap H$ has length at most $n$, provided that $K_i = \varphi(G, \bar{a}_i)$.

The proofs of the following three lemmata are similar to the proofs of the corresponding facts for stable groups. One can see any textbook on stable groups for details, for instance, [11]. The first lemma is a modification of Baldwin-Saxl condition for stable groups [2].

**Lemma 2.1 (Baldwin-Saxl condition)** For any formula $\varphi(x, \bar{y})$ and any convex subgroup $H$ there is a natural $n$ such that the intersection of a family of subgroups of the form $K_i \cap \cap H$, where $K_i = \varphi(G, \bar{a}_i)$, is the intersection of just $n$ of them. Consequently, subgroups which are finite or infinite intersections of $K_i$, form an almost uniform family in that sense that for any set of indices $I$ there are $\bar{b}_0, \ldots, \bar{b}_{n-1}$ such that $\bigcap_{i \in I}(K_i \cap \cap H) = \bigcap_{j < n}(K_j \cap \cap H)$. So I may apply the trivial chain condition.
Lemma 2.2 (O-superstable chain condition) In an o-superstable ordered group \( G \) for any convex subgroup \( H \) there is no infinite decreasing sequence \( K_0 \supset H \supset K_1 \supset H \supset \ldots \supset K_n \supset H \supset \ldots \), where \( K_i \) are definable subgroups of \( G \), such that \( |K_n \supset H : K_{n+1} \supset H| = \infty \) for each \( n \).

Lemma 2.3 (O-omega-stable chain condition) In an o-omega-stable ordered group \( G \) for any convex subgroup \( H \) there is no an infinite decreasing sequence \( K_0 \supset H \supset K_1 \supset H \supset \ldots \supset K_n \supset H \supset \ldots \), where \( K_i \) are definable subgroups of \( G \).

Lemma 2.4 Let \( H \) and \( K \) be definable subgroups of \( G \). Then \((H \cap K)^c = H^c \cap K^c\), i.e. if \( K^c \leq H^c \), then \( H \cap K \) is not bounded in \( K \).

Proof. Let \( \varphi(x,a) \triangleq \exists y \in K(e \leq y \leq a \land x \in H \cdot y) \). Obviously if \( a < b \) then \( \varphi(G,a) \cap (h, \sup H) \subseteq \varphi(G, b) \cap (h, \sup H) \) for any \( h \in H \). Since \( G \) is o-stable, there is no strict order property inside any cut, and in particular in the cut defined by \( \sup H \). Then there is \( a_0 \in K \) such that for any \( b \in K \), which is bigger than \( a_0 \) there is \( h_b \in H \) such that \( \varphi(G,a) \cap (h_b, \sup H) = \varphi(G, b) \cap (h_b, \sup H) \).

Let \( k_1 > a_0 \) be an element of \( K \). Then \( H \cdot k_1 \subseteq \varphi(G, a_0) \) holds eventually in the cut \( \sup H \). Choose an arbitrary large element \( h_1 \in H \). Since \( h_1 k_1 \in \varphi(G, a_0) \) there are \( h_2 \in H \) and \( k_2 \in [e, a_0] \cap K \) such that \( h_1 k_1 = h_2 k_2 \). Rewrite this equality as \( k_1 k_2^{-1} = h_1^{-1} h_2 \). Obviously, \( k_1 k_2^{-1} \in K \cap H \). Observe that \( k_2 \) is bounded by \( a_0 \) and \( k_1 \) may be chosen arbitrary large. This implies that \( k_1 k_2^{-1} \) can be arbitrary large and consequently \( K \cap H \) is not bounded in \( K \). \( \square \)

Lemma 2.5 If \( H \) is a minimal definable subgroup unbounded in \( G \), then \( H \) is the least definable subgroup unbounded in \( G \).

Proof. Assume that \( K \) is a definable unbounded subgroup. By Lemma 2.4 the intersection \( K \cap H \) is not bounded in \( G \). Since \( H \) is minimal, \( K \cap H = H \). \( \square \)

Lemma 2.6 For any element \( a \in G \) there is a convex subgroup \( H_a \) of \( G \) containing \( a \), such that the center \( Z(H_a) \) of \( H_a \) is not bounded in \( H_a \).

Proof. Let \( a \) be a positive element of \( G \) and \( C(a) \) the centralizer of \( a \). Consider a formula \( \psi(x,a,b) \) which says that \( a \leq b \) and \( x \in (C(c))c \) for all \( c \in [a, b] \). Define a formula \( \theta(y,a) \) as \( \psi(y,a,y) \). Note that \( \theta(G, a) \) is not empty and \( \sup \theta(G, a) \) defines a convex subgroup which I denote by \( H_a \). Indeed, \( \theta(G, a) \) contains \( a \) and \( b^2 \) whenever \( b \in \theta(G, a) \).

By Lemma 2.4 for any \( b_1 < b_2 < \ldots < b_n < \sup H_a \) with \( b_1 > a \) the intersection of all \( C(b_i) \) is not bounded in \( H_a \). Since \( G \) is o-stable I may apply...
Baldwin-Saxl’s condition to the intersection of uniformly definable family of subgroups inside the cut defined by sup $H_a$. Then $\bigcap\{C(b) : b \in [a, \sup H_a]\}$ is an intersection of just $n$ of them and is not bounded in $H_a$ by Lemma 2.4. It is easy to see that this intersection is the center of $H_a$. Thus $Z(H_a)$ is not bounded in $H_a$. □

Lemma 2.7 If the center of an arbitrary densely ordered group is dense in the group, then this group is abelian.

Proof. First, observe that in an ordered group both functions $f_a(x) = ax$ and $g_a(x) = xa$ are continuous. Indeed, $ax_0\varepsilon^{-1} < ax < ax_0\varepsilon$ iff $x_0\varepsilon^{-1} < x < x_0\varepsilon$.

Let $a$ and $b$ be arbitrary elements. Since the center $Z$ is dense, there is a sequence $\{c_\alpha\}$ of elements from the center, which converges to $b$. Then

$$ab = a \cdot \lim c_\alpha = \lim ac_\alpha = \lim c_\alpha a = (\lim c_\alpha) \cdot a = ba$$

Thus the group is abelian. □

Theorem 2.8 An ordered o-stable group is abelian.

Proof. Without loss of generality I may assume that $G = H_a$ for some element $a$. Indeed, if $H_a$ is abelian for any element $a$, then $G$ is abelian.

Let $Z$ be the center of $G$ and $H$ the greatest subgroup of $Z$, which is convex in $G$. Obviously, $H$ is a definable normal subgroup of $G$.

Assume the contrary, that $G$ is not abelian, then $H$ is a bounded subgroup of $G$. Define sets $A$ and $B$ as follows:

$$B \triangleq \{g \in Z : g > H\}, \quad A \triangleq \{g \in G : H < g < B\}$$

Claim, that the set $A$ consists of cosets of $H$.

Claim 2.8.1 The index $|A : H|$ is finite.

Proof. Consider the following formula:

$$\varphi(x, b) \triangleq \exists y(e \leq y < b \land x \in yZ)$$

Let $a_1 < a_2 < \ldots < a_n$ be elements of $A$ such that $a_i \cdot H < a_{i+1}$. Then $a_{i+1} \cdot Z \not\subseteq \varphi(G, a_i)$. By Lemma 2.6 the center $Z$ is not bounded in $G$, which implies that any coset of $Z$ is consistent with the cut $+\infty$. Then $\varphi(G, a_i)$ is eventually in the cut $+\infty$ a strict subset of $\varphi(G, a_{i+1})$. Since there is no the strict order property inside any cut, there is a bound on the length of $a_1 < a_2 < \ldots < a_n$. Thus $|A : H|$ is finite. □
Claim 2.8.2 The set $A$ is empty.

Proof. Assume the contrary, that the set $A$ is not empty. Then in virtue of Claim 2.8.1 the quotient group $G/H$ is discretely ordered and there are an injective homomorphism $\tau : \mathbb{Z} \to G/H$, such that $\tau(1)$ is the least positive coset of $H$, and there is a natural number $n$ such that any representative of $\tau(n)$ in $G$ is central.

Observe that $\tau(\mathbb{Z})$ is a subgroup of the center $Z(G/H)$ of the quotient group $G/H$. Indeed, if $\tau(1) \notin Z(G/H)$ then there is an element $g \in G/H$ such that $\tau(1)g \neq g\tau(1)$, say, $\tau(1)g < g\tau(1)$. Since $\tau(1)$ is positive, $g \in (0, \tau(1))g \subset (0, g\tau(1))$. Eliminating $g$ I obtain that $\tau(0) = g^{-1}g < g^{-1}g\tau(1)$, which is contradictory, because $\tau(1)$ is the least positive element in $G/H$.

Let $b$ be a representative in $G$ of $\tau(1)$. Since $b$ is not central there is an element $c$ such that $bc \neq cb$. From the other hand $bH$ is central in $G/H$. Hence $[b, c] \in H$. By easy calculations

$$b^n c = b^{n-1}(bc) = b^{n-1}cb[b, c] = b^{n-2}c^2b[b, c] = \ldots = cb^n[b, c]^n$$

I obtain that $e = [b^n, c] = [b, c]^n$, because $b^n$ is central as a representative in $G$ of $\tau(n)$. This yields a contradiction, because any ordered group is torsion-free. □

The consequence of Claim 2.8.2 is that $Z(G)/H$ is dense in $G/H$. By Lemma 2.7 $G/H$ is abelian, because $Z(G)/H \leq Z(G/H)$.

Assume that there are elements $a$ and $b$ of $G$ such that $ab \neq ba$. Let $c = [a, b]$. Since $G/H$ is abelian, the element $c$ belongs to $H$. As in Claim 2.8.2 I obtain that $[a, b^n] = [a, b]^n = c^n$. Consider the following formula $\varphi(x, a, d) \triangleq d^{-1} < [a, x] < d$. Observe that $[a, f] = [a, b] = c$ for any element $f \in C(a) \cdot b$. Indeed, if $f = a_1b$ then $aa_1b = a_1ab = a_1bac$. Thus $\varphi(G, a, d)$ consists of cosets of $C(a)$. It is easy to see that $C(a) \cdot b^n \nsubseteq \varphi(G, a, c^n)$ and $C(a) \cdot b^n \subseteq \varphi(G, a, c^{n+1})$. Thus I obtain the strict order property witnessed by $\varphi(x; a, y)$ in the cut defined by sup $C(a)$. This contradicts o-stability of $G$. □

From now on I shall use additive notation for the group operation.

As a corollary of Lemma 1.13 I obtain the following.

Lemma 2.9 Let $K$ be a definable unbounded subgroup. Then $G/K$ with the full induced structure is stable.

Let $\Sigma$ be the family of all convex (not necessary definable) subgroups of $G$. Two convex subgroups $A$ and $B \in \Sigma$ are called a jump if $A$ is a subgroup of $B$ and there is no subgroup $C \in \Sigma$ such that $A < C < B$. 11
Lemma 2.10 For each natural \( n \geq 2 \) the number of jumps \( A < B \), such that \( B/A \) is not \( n \)-divisible, is finite.

Proof. Let \( \varphi_n(x,a) \triangleq \exists y (0 \leq y \leq a \land x \in y + nG) \). If for some \( n \) the number of jumps \( A < B \) such that \( B/A \) is not \( n \)-divisible is infinite, then \( \varphi(x;y) \) has the strict order property inside the cut \( +\infty \). Indeed, if \( b \in B \) is such that \( b + A \) is not divisible by \( n \) in the quotient group \( B/A \), then for any \( a \in A \) the coset \( b + nG \) is not a subset of \( \varphi(G,a) \).

Lemma 2.11 If the elementary theory of \( G \) is o-superstable, then \( |G : nG| < \infty \) for any positive integer \( n \).

Proof. The lemma follows from the o-superstable chain condition. Indeed, \( nG \) is not bounded in \( G \) and if \( |G : nG| = \infty \), then \( |n^qG : n^{q+1}G| = \infty \) for any positive integer \( q \).

Lemma 2.12 If the elementary theory of \( G \) is o-\( \omega \)-stable, then \( G \) as a pure ordered group is elementary equivalent to the ordered group of rationals.

Proof. Since an ordered o-stable group \( G \) is abelian, it is sufficient to prove that \( G \) is divisible. Because of the o-\( \omega \)-stable chain condition any chain of the form \( G \geq pG \geq p^2G \geq \ldots \geq p^kG \geq \ldots \) stabilizes in finite many steps. Since \( G \) is ordered, any chain \( G > pG > p^2G > \ldots > p^kG > \ldots \) cannot be finite. Thus \( G = pG \) and \( G \) is divisible.

Question 1 Characterize all pure ordered o-stable and o-superstable groups:

1. is an ordered group \( G \) o-stable iff \( G \) is abelian and for each natural \( n \geq 2 \) the number of jumps \( A < B \) such that \( B/A \) is not \( n \)-divisible is finite?

2. is an ordered group \( G \) o-superstable iff \( G \) is abelian and \( |G : nG| < \infty \) for each positive integer \( n \)?

3 Definable subsets of ordered o-stable groups

Throughout this section \( G \) is a sufficiently saturated ordered group whose elementary theory is o-stable. I say that a formula \( \varphi \) is eventually minimal inside a cut \( s \) if \( \varphi \) is consistent with \( s \) and for any formula \( \psi \) exactly one of \( \varphi \land \psi \) and \( \varphi \land \neg \psi \) is consistent with \( s \). The aim of this section is to investigate the eventual stabilizer

\[
K_\varphi(x) \triangleq \exists z \forall y [z < y \rightarrow (\varphi(y) \leftrightarrow \varphi(y + x))]
\]
of a formula \( \varphi \), which is eventually minimal in the cut \( +\infty \). Obviously, \( K_\varphi \triangleq K_\varphi(G) \) is a subgroup of \( G \). Observe that for any \( g \in G \) the formula \( \varphi(x + g) \) is also eventually minimal.

Note that for any definable convex set \( A \) each of \( \inf A \) and \( \sup A \) defines a convex subgroup \( H^-_A \) and \( H^+_A \), respectively, in the following way:

\[
H^-_A \triangleq \{ g \in G : a - |g| \in A \text{ for any } a \in A \}
\]

\[
H^+_A \triangleq \{ g \in G : a + |g| \in A \text{ for any } a \in A \}
\]

Observe that the above restriction that I shall investigate the eventual stabilizer in the cut \( +\infty \) is not essential. Indeed, let \( \langle C, D \rangle \) be a cut, such that \( H^+_C \) is not trivial. Since the expansion by a unary predicate naming a convex set preserves o-stability, I may assume that this convex subgroup is definable and that in fact \( \sup C = \sup H^+_C \). So I can investigate the eventual stabilizer at the cut \( +\infty \) in \( H^+_C \), which is obviously equal to the eventual stabilizer at the cut \( \langle C, D \rangle \) in the whole group \( G \).

Now I prove some technical lemmata.

**Lemma 3.1** For any definable subset \( A \) the number of infinite convex components whose infimums belong to \( G \) is finite.

**Proof.** Assume the contrary. Let \( E(x, y) \) be an equivalence relation on \( A \) whose classes are convex components of \( A \). Since \( G \) is supposed to be sufficiently saturated I may find countably many of these classes \( \{[a_i] : i < \omega \} \) such that \( [a_i] < [a_j] \) for all \( i < j \) and that there is a positive element \( b \) such that \( nb \) is less than the length of convex components \( [a_i]_E \) for any positive integer \( n \) (in the discrete case I may suppose that \( \mathbb{Z} < G \) and that \( b > \mathbb{Z} \)).

Let a cut \( \langle C, D \rangle \) in \( G \) be defined as \( C = (-\infty, \sup \bigcup_i [a_i]) \). Then

\[
\inf[a]_E < \inf[a]_E + b < \sup[a]_E
\]

The convex set \( (\inf[a]_E, \inf[a]_E + b) \) is a proper subset of \( (\inf[a]_E, \inf[a]_E + 2b) \). Since there are infinitely many possibilities for \( b \) the following formula has the strict order property inside the cut \( \langle C, D \rangle \):

\[
\theta(x, y) \triangleq \exists z(z \in A \land x \in (\inf[z]_E, \inf[z]_E + y))
\]

This contradicts o-stability of \( G \). \( \Box \)

**Lemma 3.2** Let \( G \) be densely ordered. Then for any infinite definable set there is an interval where this set is dense. By other words, no infinite nowhere dense subset is definable.

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Proof. Assume that an infinite definable set $A$ is not dense in any interval. Let

$$B = \{ a \in A : (a, c) \cap A = \emptyset \text{ for some } c > a \}$$

By Lemma 3.1 the set $B$ is finite. Thus without loss of generality I may assume that $B$ is empty. Also I can suppose that for any $a \in A$ and $c < a$ there is an element $a_1 \in A$ with $c < a_1 < a$.

Since the set $A$ is infinite I can find a monotone (say, decreasing) sequence $\langle a_n : n < \omega \rangle$ of elements of $A$. Consider the cut $\langle C, D \rangle$ defined by the infimum of this sequence. Observe that for any $a_n$ and any $\epsilon > 0$ there is some $a \in A \cap (a_n, a_n + \epsilon)$ such that $(a_n, a)$ contains an interval in the complement of $A$. Hence in every neighbourhood of $a_n$ there are convex components of the complement of $A$ of arbitrarily small length.

Let a formula $\phi(x, y)$ says that $x$ belongs to a convex component of the complement of $A$, which has length less than $y$. Since $G$ does not have the strict order property, for any positive $g \in G$ there is a natural number $n_g$ such that for any $n > n_g$ any convex component inside $(a_{n+1}, a_n)$ of the complement of $A$ has length less than $g$. Let $k > n_g$. Consider an arbitrary convex component of the complement of $A$ in $(a_{k+1}, a_k)$. Let $h$ be smaller than the length of this convex component. Then $n_g < n_h$. Since $G$ is assumed to be uncountable, I obtain a contradiction. \hfill $\square$

As a corollary of this lemma I obtain that each definable subset consists of dense components and of finitely many isolated points. The following lemma also is immediate from Lemma 3.2.

**Lemma 3.3** Any definable subgroup of a dense o-stable ordered group $G$ is densely ordered. In particular, the elementary theory of the ordered group of reals $(\mathbb{R}, <, +, 0, \mathbb{Z})$ with named subgroup of integers is not o-stable.

**Lemma 3.4** Let $G$ be an ordered group with dense order whose elementary theory is o-stable. Then there is no definable function $f : A \to B$ such that the set $A$ is definable and dense, $B \subseteq G$, $\langle C, D \rangle$ is a cut in $B$ and there are infinitely many sequences $\langle a_{n,i} : i \in I_n \rangle_{n<\omega}$ of elements of $A$ converging to $\alpha_n$ with $\alpha_n \neq \alpha_k$ for all $n < k < \omega$ such that $\lim_{i \to \infty} f(a_{n,i}) = \sup C$ for any $n < \omega$, i.e. that $\{ f(x) : |\alpha_n - x| < |\alpha_n - a_{n,i}| \}$ is consistent with $\langle C, D \rangle$ for any $n$ and $i$.

Proof. Without loss of generality I may assume that $\alpha_n < \alpha_k$ for $n < k$ and that for each $n$ the sequence $\langle a_{n,i} : i \in I_n \rangle$ is strictly monotone, say, strictly increasing. Let $\langle C, D \rangle$ be a cut in $G$ defined as $\sup C = \sup \{ \alpha_n : n \in \omega \}$. By compactness I may assume that there are $c_1 \in C$ and $d_1 \in D$ such that for each $n$ there is $i_{n,1}$ such that $f(a_{n,i_{n,1}}) \in (c_1, d_1)$. Assume that I have found
\[ c_k \in C \text{ and } d_k \in D. \] Then again by compactness I can find \( c_{k+1} \in C \) and \( d_{k+1} \in D \) such that for each \( n \) \( f(a_{n,i,n,k}) \not\in (c_{k+1},d_{k+1}) \) and there is \( i_{n,k+1} \) such that \( f(a_{n,i,n,k+1}) \in (c_{k+1},d_{k+1}) \). Then the following formula

\[ \varphi(x; y, z) \triangleq y < f(x) < z \]

has the strict order property inside the cut \( \langle C, D \rangle \), contradicting \( \sigma \)-stability of \( G \). \qed

**Lemma 3.5** Let \( \varphi \) be such that for any \( g \in G \) either \( \varphi(G) \cong (\varphi(G) + g) \) or \( \varphi(G) \cap (\varphi(G) + g) \cong \emptyset \). Then \( K_\varphi \) is unbounded.

**Proof.** Assume the contrary, that \( K_\varphi \) is bounded. Let \( H_1 \) be the convex hull of \( K_\varphi \). If \( G/H_1 \) is dense, then let \( H \triangleq H_1 \). If \( G/H_1 \) is discrete, then let \( g \) be a representative of the least positive coset of \( H_1 \). In this case let \( H \triangleq \bigcup_{n \in \mathbb{Z}} (ng + H_1) \). Since \( G \) is sufficiently saturated, \( G/H \) is dense.

Then for any \( g \not\in H \) the intersection \( \varphi(G) \cap (\varphi(G) + g) \) is eventually empty.

Define a formula

\[ \psi(x, y, z) \triangleq \exists t (y < t < z \land \varphi(x - t)) \]

**Claim 3.5.1** There are \( a < b \) such that \( b - a > \sup H \) and for any \( c \) and \( d \) with \( a < c < d < b \) and \( d - c > \sup H \) it holds that \( \psi(x, a, b) \) and \( \psi(x, c, d) \) are eventually equal at the cut \( +\infty \).

**Proof.** Since there is no the strict order property inside a cut. \qed

Fix some element \( c \in (a, b) \) and some \( e \in (\sup(c + H), b) \). Then

**Claim 3.5.2** Eventually \( \varphi(G) + c \subseteq \psi(G, e, b) \)

**Proof.** Eventually \( \psi(G, c, b) \subseteq \psi(G, e, b) \cup (\varphi(G) + c) \subseteq \psi(G, a, b) = \psi(G, e, b) \). \qed

**Claim 3.5.3** For any \( d \in (e, b) \) the intersection \( (\varphi(G) + c) \cap (\varphi(G) + d) \) is eventually empty at the cut \( +\infty \).

**Proof.** It follows from the assumption that for any \( g \not\in H \) the intersection \( \varphi(G) \cap (\varphi(G) + g) \) is eventually empty. \qed

Now I define a function \( f : (e, b) \to G \) as

\[ f(x) = \sup[(\varphi(G) + c) \cap (\varphi(G) + x)] \] 

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By Claims 3.5.1–3.5.3 the function $f$ is not bounded in any subinterval $(b_1, b_2)$ of $(c, b)$ with $b_1 + H < b_2$. So I may find infinitely many disjoint intervals $(b_{n,1}, b_{n,2})$ such that $f$ is not bounded in each $(b_{n,1}, b_{n,2})$. In each interval I can find a sequence $\langle a_{n,i} : i \in I_n \rangle$ such that $\{f(a_{n,i}) : i \in I_n \}$ is unbounded. This contradicts to Lemma 3.4. Thus $K_\varphi$ contains an element which does not belong to $H$. \hfill \square

Note that a formula $\varphi$ such that for any $g \in G$ either $\varphi(G) \equiv (\varphi(G) + g)$ or $\varphi(G) \cap (\varphi(G) + g) \not\equiv \emptyset$ does exist in any o-stable group, because the $\psi(x + y)$-rank of any formula $\psi(x)$ is finite at $+\infty$.

**Lemma 3.6** Let $\varphi(x)$ be an unbounded formula such that the eventual stabilizer $K_\varphi$ is not bounded. Then $\varphi(G) \cap (g + K_\varphi) \not\supseteq \emptyset$ for some $g$. If in addition $\varphi(x)$ is eventually minimal, then eventually $\varphi(G) \subseteq g + K_\varphi$.

**Proof.** Take an arbitrary $a \in K_\varphi$. By the definition of $K_\varphi$ there is some $b$ such that for any $c$, if $b < c$ and $b < a + c$ then $\varphi(c) \not\leftrightarrow \varphi(c + a)$. Let $c \in \varphi(G)$ be bigger than $b$. Then $c + n \cdot a \in \varphi(G)$ for all natural $n$, and so $\varphi(G) \cap (c + K_\varphi)$ is infinite and is dense in some interval by Lemma 3.2.

I show that there is an element $c$, such that $\varphi(G) \cap (c + K_\varphi) \not\supseteq \emptyset$. Let

$$
\psi_1(y, d) \equiv \exists u \exists v (d < u < v \land [\varphi(G) \cap (y + K_\varphi) \text{ is dense in } (u, v)])
$$

$$
\psi(x, d) \equiv \exists y (\psi_1(y, d) \land x \in (y + K_\varphi))
$$

Since there is no the strict order property inside the cut $+\infty$, there is $d_1$ such that $\psi(G, d_1) \not\equiv \psi(G, d_2)$ for any $d_2 > d_1$. Also $\psi(G, d_1)$ is not eventually empty, because $K_\varphi$ is unbounded and the above mentioned element $c$ can be chosen arbitrary large. Let $c \in \varphi(G)$ be bigger than $d_1$. Then $\varphi(G) \cap (c + K_\varphi) \not\supseteq \emptyset$.

If $\varphi(x)$ is eventually minimal, then obviously eventually $\varphi(G) \subseteq g + K_\varphi$. \hfill \square

**Lemma 3.7** Let $K_\varphi$ be the eventual stabilizer of an arbitrary formula $\varphi(x)$. Assume that $K_\varphi$ contains a positive element. Then for any $g \in K_\varphi$ there is a non-zero convex in $K_\varphi$ subgroup $H$ of $K_\varphi$ such that $g \in H$ and eventually $\varphi(G)$ is equal to $\varphi(G) + H$ (here, $H$ need not be definable).

**Proof.** Let $a \in K_\varphi$ be positive. Then by the definition of $K_\varphi$ there is $c$ such that $\varphi(G) \cap (c, \infty) = (\varphi(G) + a) \cap (c, \infty)$. I define a function $f$ as

$$
f(x) \equiv \inf \{c \in G : \varphi(G) \cap (c, \infty) = (\varphi(G) + x) \cap (c, \infty)\}$$

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By Lemma 3.4 in any interval \((a_1, b_1)\) with \(|g| < a_1\) there is a subinterval \([a_2, b_2]\) such that the function \(f\) is bounded on \([a_2, b_2)\cap K_\varphi\) by some \(c\). Then \(f\) is bounded on \([-b_2, -a_2] \cap K_\varphi\) with a bound \(c + b_2\). Moreover, \(f\) is bounded on the intersection \([a_2 - b_2, b_2 - a_2]\cap K_\varphi\). Indeed, let \(h \in [0, b_2 - a_2]\). Then

\[
x \in \varphi(G) \iff x + (a_2 + h) \in \varphi(G) \iff (x + a_2 + h) - a_2 \in \varphi(G) \iff x + h \in \varphi(G)
\]

Since \(f(2x) \leq f(x) + x\) the function \(f\) is bounded on \([2(a_2 - b_2), 2(b_2 - a_2)]\cap K_\varphi\) as well as on \([n(a_2 - b_2), n(b_2 - a_2)]\cap K_\varphi\) for each natural \(n\).

Let \(H = \bigcup_{n \in \mathbb{N}}[n(a_2 - b_2), n(b_2 - a_2)]\cap K_\varphi\). By compactness \(f\) is bounded on \(H\). Then eventually \(\varphi(G)\) is equal to \(\varphi(G) + H\). Since \(|g| < a_1 \leq a_2\), the subgroup \(H\) contains \(g\).

I say that \(G\) has **boundedly many definable convex subgroups** if there is a cardinal \(\lambda\) such that in any group which is elementary equivalent to \(G\) the number of convex definable subgroups does not exceed \(\lambda\). Otherwise I say that \(G\) has **unboundedly many definable convex subgroups**. I say that a convex set \(A\) is **coset-infinite**, if it is not a finite union of cosets of definable subgroups.

**Lemma 3.8** If for some formula \(\varphi(x)\) the number of coset-infinite convex components of \(\varphi(G)\) is infinite, then \(G\) has unboundedly many definable convex subgroups.

**Proof.** Assume the contrary, that such a formula \(\varphi(x)\) does exist and \(G\) has boundedly many definable convex subgroups. Recall that for any convex \(A\) each of \(\inf A\) and \(\sup A\) defines a convex subgroup \(H^-_A\) and \(H^+_A\), respectively.

Since \(G\) has boundedly many definable convex subgroups, so convex components of \(\varphi(G)\) define finitely many convex subgroups. Thus I can suppose that each convex component of \(\varphi(G)\) is not a coset of a convex subgroup, is coset-infinite and \(H^-_A = H^-_B\) and \(H^+_A = H^+_B\) for any convex components \(A\) and \(B\). Without loss of generality I may assume that \(H^-_A \leq H^+_A\). Since \(G/H^-_A\) with the full induced structure is also o-stable, I can suppose that \(H^-_A = \{0\}\). Then \(\inf A \in G\). The lemma follows from Lemma 3.1.

**Theorem 3.9** Let \(A\) be a definable subset of \(G\) such that both \(A\) and its complement \(B = G \setminus A\) are unbounded and the eventual stabilizer of \(A\) is equal to \(G\). Then \(G\) has unboundedly many definable convex subgroups.

**Proof.** Assume that \(G\) contains boundedly many definable convex subgroups. Since \(G\) is supposed to be sufficiently saturated, there is an element \(g\) which does not belong to any definable convex proper subgroup of \(G\). By Lemma 3.7
there is a convex subgroup $H$ of $G$, which contains $g$ and satisfies the eventual equality $A + H \cong A$. Thus $A$ contains convex components of length at least $g$. The complement $B$ of $A$ satisfies the same condition. So there are infinitely many convex components of $A$ of length at least $g$. Then these convex components are coset-infinite. This contradicts Lemma 3.8.

**Lemma 3.10** Assume that $G$ has boundedly many definable convex subgroups. Then for any formula $\varphi(x, \bar{y})$ there is a natural number $k_\varphi$ such that for any $\bar{a}$ the number of infinite dense components of $\varphi(G, \bar{a})$ which are not cosets of subgroups is at most $k_\varphi$.

**Proof.** Let $A$ be the topological closure of $\varphi(G, \bar{a})$ in the topology induced by the ordering. Then the intersection of a convex component of $A$ with $\varphi(G, \bar{a})$ gives a dense component of $\varphi(G, \bar{a})$. Thus the lemma follows from Lemma 3.1. \qed

Recall that $G$ is of non-valuational type if it contains no non-trivial definable convex subgroup.

**Lemma 3.11** Let $G$ be of non-valuational type. Then any equivalence relation in $G$ has at most finitely many infinite convex classes.

**Proof.** The lemma directly follows from Lemma 3.1, because there is no definable convex non-trivial subgroup. \qed

**Lemma 3.12** Let $G$ be of non-valuational type with dense order. Let $\phi(x)$ be a formula and $\langle C, D \rangle$ a cut. Then $\phi(G)$ is either eventually dense in $C$ or eventually empty in $C$. (The same holds for $D$ with the inverse ordering)

**Proof.** Let $E_\phi(x, y)$ be an equivalence relation with convex classes such that each class is a maximal convex set which either is infinite and has an empty intersection with $\phi(G)$, or is not necessary infinite and $\phi(G)$ is dense in this class. By Lemma 3.11 $E_\phi$ has at most finitely many convex classes, that is sufficient for us. \qed

I shall prove that if $G$ has boundedly many convex definable subgroups, then an eventually minimal formula is eventually equal to a coset of its eventual stabilizer, which is the least definable unbounded subgroup. Also I shall construct an example of a group with unboundedly many definable convex subgroups, such that an eventually minimal formula is eventually a proper subset of its eventual stabilizer.
**Theorem 3.13** Let $G$ have boundedly many convex definable subgroups, $A$ be a definable unbounded subset, and $K_A$ its eventual unbounded stabilizer. Then there is $g \in G$ such that $A \cap (g, +\infty)$ is equal to $(A \cap (g, +\infty)) + K_A \cap (g, +\infty)$.

**Proof.** If $A$ is eventually equal to an interval $(g, +\infty)$, then obviously the theorem holds. So assume that the compliment of $A$ is unbounded, too. By theorem 3.9 $K_A$ is a proper subgroup of $G$.

Let $g \in G$. Then $A \cap (g, +\infty)$ is equal to $(A \cap (g, +\infty)) + K_A \cap (g, +\infty)$ iff for any $h \in A \cap (g, +\infty)$ the intersection $(h + K_A) \cap (g, +\infty)$ is a subset of $A$. Assume the contrary, that there is no $g \in G$ such that $A \cap (g, +\infty)$ is equal to $(A \cap (g, +\infty)) + K_A \cap (g, +\infty)$. Then for each $g$ there is a coset $h + K_A$ for some $h \in A$ such that $(h + K_A) \cap (g, +\infty) \not\subseteq A$. Let $A_h = A \cap (h + K_A)$. Then $A_h - h \subseteq K_A$ and $K_A$ is the stabilizer of $A_h - h$. Since $K_A$ with the full induced structure is o-stable, and $G$ (and then $K_A$) has boundedly many definable convex subgroups, either $A_h - h$ is bounded or its compliment is bounded by Theorem 3.9. Then there is $g_1 > g$ such that either $A \cap (g_1, +\infty) \cap (h + K_A) = \emptyset$ or $(h + K_A) \cap (g_1, +\infty) \subseteq A$.

Now I construct a sequence $g_0 = g < h_0 = h < g_1 < h_1 < g_2 < h_2 < \ldots$ such that $h_n$ relates to $g_n$ as $h_0$ relates to $g_0$. Then $(h_n + K_A) \cap (g_n, +\infty) \not\subseteq A$ and either $A \cap (g_{n+1}, +\infty) \cap (h_n + K_A) = \emptyset$ or $(h_n + K_A) \cap (g_{n+1}, +\infty) \subseteq A$. Since all of these holds for any $g$, at least one property holds infinitely many times, say $(h + K_A) \cap (g_1, +\infty) \subseteq A$ (the other case is).

Now I define a formula with the strict order property in the cut $+\infty$, for a contradiction. Let $\theta(x, g)$ say that $x$ belongs to a coset $h + K_A$ for some $h$ such that $(h + K_A) \cap (g, +\infty) \not\subseteq A$ and $(h + K_A) \cap (g_1, +\infty) \subseteq A$ for some $g_1 > g$.

**Theorem 3.14** Let $G$ have boundedly many convex definable subgroups, and $A$ an eventually minimal definable subset. Then there is the least definable unbounded subgroup $K$. Moreover, the set $A$ is eventually equal to a coset of $K$. In particular, $K$ is divisible.

**Proof.** By Lemma 3.6 $A$ is eventually a subset of $g + K_A$ for some $g$. For simplicity of notation I assume that $g = 0$. Then $A$ is eventually a subset of $K_A$. By Theorem 3.13 the set $A$ is eventually equal to $A + K_A$, which means that $A$ is eventually equal to $K_A$.

**Theorem 3.15** Let $G$ be an o-stable ordered group with boundedly many definable convex subgroups. Assume that $G$ is not weakly o-minimal, that is there is a definable subset $A$ consisting of infinitely many convex components and a
non-rational cut \langle C, D \rangle such that both \( A \) and the complement of \( A \) are consistent with this cut. Then there is an externally definable unbounded proper subgroup \( K \) of \( H_C^+ \) (where \( H_C^+ \) is the stabilizer of the set \( C \)). If in addition the cut \( \langle C, D \rangle \) is definable then \( K \) is definable.

Proof. Let \( A \) be a definable subset consisting of infinitely many convex components. Since \( G \) is sufficiently saturated there are convex components \( A_i \) of \( A \) such that \( A_i < A_j \) for \( i < j < \omega \). Let \( C = \sup \bigcup_{i<\omega} A_i \). Note that \( \sup C \) defines a subgroup \( H_C^+ \), which is not zero-group, because \( G \) is sufficiently saturated. Thus \( \sup C = \sup (H_C^+ + g) \) for some \( g \in G \). For simplicity of notations I assume that \( g = 0 \). Obviously, \( H_C^+ \) is at least externally definable as a convex subset. By Theorem 1.9 the elementary theory of \( (G, H_C^+) \) is o-stable and by Lemma 1.11 the elementary theory of the full induced structure of \( H_C^+ \) is o-stable. So without loss of generality I may assume that \( A \) is an unbounded subset of \( G \).

Let \( \varphi(x; y) = x \in (y + A) \). Since \( G \) is o-stable the \((2, \varphi)\)-rank of \( A \) inside the cut \(+\infty\) is finite. So there are \( a_i \in G \) for \( i < n \) and \( \tau \in ^n 2 \) such that the formula

\[
\theta(x) \triangleq \bigwedge_{i<n} \varphi^{\tau(i)}(x; a_i)
\]

satisfies the requirements of Lemma 3.5. Then \( \mathcal{K}_\theta \) is an externally definable unbounded subgroup of \( G \).

Obviously, if \( C \) is definable, there is no need to expand the language and then \( \mathcal{K}_\theta \) is definable. \( \Box \)

Now I shall give an example of an o-\( \omega \)-stable ordered group with a proper definable subset, whose eventual stabilizer is equal to the group.

Let \( G \) be the direct power \( \mathbb{R}^\mathbb{Q} \) with lexicographical ordering, and a subgroup \( H = \mathbb{Q}^\mathbb{Q} \). Let \( E \) be an equivalence relation with \( \neg E(a, -a) \) for any \( a \neq 0 \), whose classes are archimedean classes of \( G \). Then each positive element \( a \) defines a convex subgroups \( K_a = (-\inf E(G, a), \inf E(G, a)) \). Let \( B_a = E(G, a) \cap (H + K_a) \) and a unary predicate \( P \) names the union \( \bigcup_{a \in G} B_a \). It will be shown that \( (G, <, +, -0, P, E) \) is a group I am looking for. Obviously for any element \( g \in G \) it holds that

\[
P(G) \cap (\sup E(G, g), +\infty) = (P(G) + g) \cap (\sup E(G, g), +\infty)
\]

Thus the eventual stabilizer \( \mathcal{K}_P \) of \( P \) is equal to \( G \).

In [14] it has been proved that \( \text{Th}(G, <, +, -0, E) \) is weakly o-minimal and admits quantifier elimination. Below I prove that the elementary theory of the expansion of this structure by \( P \) also admits quantifier elimination and consequently is o-\( \omega \)-stable.
Theorem 3.16 The elementary theory $T$ of $(G, <, +, -, 0, P, E)$ admits quantifier elimination.

Proof. First, note that any term in variables $x_1, \ldots, x_n$ is equal to $k_1x_1 + \cdots + k_nx_n$, where $k_i \in \mathbb{Z}$. Second I consider how I can reduce $E(t_1, t_2)$, where $t_1, t_2$ are terms, to a more simple form. Let $x$ be a variable, and $u, v$ terms. Note, that $0 < x < y$ implies $E(x+y, y)$, since $y < y + x = 2y$, $E(y, 2y)$ and $E$-classes are convex. If $0 < -x < y \land \neg E(x, y)$, then $y/2 < y + x < y$, and consequently $E(x+y, y)$ holds.

I shall use the following notations:

$$
E_1(x, y) \triangleq E(x, y) \land \neg E(\lfloor x \rfloor, \lfloor x - y \rfloor)
$$

$$
\inf[y] < x \triangleq y < x \lor E(x, y)
$$

$$
\sup[y] < x \triangleq y < x \land \neg E(x, y)
$$

$$
\inf[y]_1 < x \triangleq y < x \lor E_1(x, y)
$$

$$
\sup[y]_1 < x \triangleq y < x \land \neg E_1(x, y)
$$

$$
E_g(x, y) \triangleq (\inf[-\lfloor g \rfloor] < x, y < \sup[\lfloor g \rfloor]) \lor E(\lfloor x - y \rfloor, \lfloor g \rfloor)
$$

$$
E_g^1 \triangleq E_g(x, y) \land \neg E_g(\lfloor x \rfloor, \lfloor x - y \rfloor)
$$

Other notations $x < \inf[y]$, $x < \sup[y]$, $x < \inf[y]_1$, $x < \sup[y]_1$, $x < \inf[y]_g$, $x < \sup[y]_g$ (respectively $E_g$, $x < \inf[y]_g^1$, $x < \sup[y]_g^1$) (respectively $E_g^1$) are defined similarly. Obviously, $E(x, y)$ is equivalent to $\inf[y] < x < \sup[y]$ and similar equivalences hold for other equivalence relations.

In a natural way I can define addition on the set of cuts in $G$. Let $\langle C_i, D_i \rangle$ be cuts, for $i = 1, 2, 3$. Then $\langle C_3, D_3 \rangle = \langle C_1, D_1 \rangle + \langle C_2, D_2 \rangle$ iff for any $c_3 \in C_3$ and $d_3 \in D_3$ there are $c_i \in C_i$ and $d_i \in D_i$ for $i = 1, 2$ such that $c_3 = c_1 + c_2$ and $d_3 = d_1 + d_2$. Then $\sup[a]_g = \sup[g] + a$, and $\sup[a]_g^1 = \sup[g]_1 + a$. Similar equalities hold for the infimum.

Note that

$$
E(x+y, z) \iff [E(x+y, y) \land E(y, z)] \lor [E(x+y, x) \land E(x, z)] \lor
\neg E(x+y, y) \land \neg E(x+y, x) \land E(x, -y) \land E(x+y, z)]
$$

and

$$
E(x+y, x) \iff \inf[y] < x < \sup[y] \lor |x| > \sup[|y|] \lor
(\inf[y] < -x < \sup[y] \land -x \notin (\inf[y]_1, \sup[y]_1))
$$

$$
E(x+y, y) \iff \inf[y] < x < \sup[y] \lor |x| < \inf[|y|] \lor
(\inf[y] < -x < \sup[y] \land -x \notin (\inf[y]_1, \sup[y]_1))
$$

$$
E(x+y, -x) \iff \inf[y] < -x < \sup[y] \land -x \notin (\inf[y]_1, \sup[y]_1))
$$
Now consider $E(x + y, x + z)$. Applying a reduction for $E(x + y, z)$ twice I obtain that it is equivalent to

$$\left( [E(x + y, y) \land E(x + z, z) \land E(y, z)] \lor [E(x + y, y) \land E(x + z, x) \land E(y, x)] \right) \lor$$

$$\left( [E(x + y, y) \land \neg E(x + z, z) \land \neg E(x + z, x) \land E(y, x + z)] \lor [E(x + y, x) \land E(x + z, x)] \right) \lor$$

$$\left( [\neg E(x + y, x) \land \neg E(x + y, y) \land E(x + z, x) \land E(x + y, z)] \lor [\neg E(x + y, x) \land \neg E(x + y, y) \land E(x + z, x) \land E(x + y, x)] \right) \lor$$

$$\left( [\neg E(x + y, x) \land \neg E(x + y, y) \land \neg E(x + z, z) \land \neg E(x + z, x) \land E(x + y, x + z)] \right).$$

Note that the third, fifth and sixth disjuncts are inconsistent. Consider the third disjunct. I have that $E(x, -z)$ and then $\forall t(E([x], t) \to |y| < t)$. Consequently $E(x + y, x) \land \neg E(x + y, y)$, this is inconsistent with $E(x + y, y)$. The fifth and sixth disjuncts are considered analogously.

Consider the last disjunct: $\neg E(x + y, x) \land \neg E(x + y, y) \land \neg E(x + z, z) \land \neg E(x + z, x) \land E(x + y, x + z)]$. Let $E(x + y, g)$ hold. Then $E_g(y, -x), E_g(-x, z), E_g(-x, z)$, and $E_g(y, z)$. So the last disjunct is equivalent to

$$-x = y = z \lor$$

$$(\text{sgn}(x + y) = \text{sgn}(x + z) \land [(y > z \land \inf[y - z] < -x < \sup[y - z]) \lor$$

$$(y \leq z \land \inf[z - y] < -x < \sup[z - y])])$$

In a similar way it is possible to show that $E(x + y, -x + z)$ is a boolean combination of convex sets of the same forms. Indeed, it is equivalent to

$$\left( [E(x + y, y) \land E(-x + z, z) \land E(y, z)] \lor [E(x + y, y) \land E(-x + z, -x) \land E(-y, y)] \right) \lor$$

$$\left( [E(x + y, y) \land \neg E(-x + z, z) \land \neg E(-x + z, -x) \land E(x, z) \land E(y, -x + z)] \lor [E(x + y, x) \land E(-x + z, x)] \right) \lor$$

$$\left( [\neg E(x + y, x) \land \neg E(x + y, y) \land \neg E(-x + z, z) \land \neg E(-x + z, -x) \land E(x + y, -x + z)] \right).$$

Consider the last disjunct. Let $E(x + y, g)$ hold. Then $E_g(y, -x), E_g(x, z), E_g(-x, -z)$ and $E_g(y, -z)$. So the last disjunct is equivalent to

$$x = -y = z \lor$$

$$(\text{sgn}(x + y) = \text{sgn}(-x + z) \land [(y > -z \land \inf[y + z] < -x < \sup[y + z]) \lor$$

$$(y \leq -z \land \inf[z + y] < -x < \sup[z + y])])$$
Thus $E(x + y, \pm x + z)$ is a boolean combination of convex sets of the form $(a_1, a_2)$, where either $a_i \in G \cup \{\pm \infty\}$ or $a_i = \sup\{b_i\}$, or $a_i = \inf\{b_i\}$, or $a_i = \sup\{b_i\}_1$, or $a_i = \inf\{b_i\}_1$.

Note that $\sup\{x\} < y$ iff $x < \inf\{y\}$.

Thus, to prove the theorem it is enough to consider a formula $\exists x \bigwedge_i \varphi_i(x, \bar{y})$, where every $\varphi_i$ (or $\neg \varphi_i$) is one of the following forms:

1. $mx = u$,
2. $mx < u$, or $mx > u$,
3. $mx < \sup\{v\}$, or $mx > \sup\{v\}$,
4. $mx < \inf\{v\}$, or $mx > \inf\{v\}$,
5. $mx < \sup\{v\}_1$, or $mx > \sup\{v\}_1$,
6. $mx < \inf\{v\}_1$, or $mx > \inf\{v\}_1$,
7. $P(mx + u)$,
8. $\neg P(mx + u)$.

Since $x \neq y \iff x > y \lor x < y$, $x \neq y \iff x > y \lor x = y$, and $mx \neq \sup\{v\} \iff mx > \sup\{v\}$ I assume that each $\varphi_i$ of the forms (1)–(6) occurs in the positive form.

Note that $m(\sup\{nx + u\} + v) = \sup\{nx + u\} + mv$ and that $\sup\{nx + u\} = \sup\{mnx + mu\}$. Observe also that $P(nx + u) \iff P(mnx + mu)$. Thus by multiplying each equality and inequality by some number I can obtain that the $m$’s in all conjuncts are equal. Since the group is divisible, for simplicity of notation I may assume that $m = 1$. If some $\varphi_i$ is of the form $x = u$, then replacing each occurrence of $x$ with $u$ in each conjuncts I obtain an equivalent quantifier-free formula.

Now I suppose that there is no conjunct of the first form. Moreover, I may assume that that at most one of the $\varphi_i$ is of the form (2)–(6). (where $<$ and $>$ count separately). Indeed, $x < u \land x < v$ is equivalent to $[x < u \land u \leq v] \lor [x < v \land v < u]$. A similar argument works for $x > u \land x > v$. Note that bound$_1[u_1] < bound_2[u_2]$ is definable by a quantifier-free formula, where bound$_i \in \{\inf, \sup\}$. Then a formula of the form $x < bound_1[v_1] \land x < bound_2[v_2]$ is equivalent to $(x < bound_1[v_1] \land bound_1[v_1] \leq bound_2[v_2]) \lor (x < bound_2[v_2] \land bound_2[v_2] \leq bound_1[v_1])$.

Then $\exists x \bigwedge_i \varphi_i(x, \bar{y})$ is equivalent to a formula saying that there is an element $x$, which belongs to a convex set and satisfies $\bigwedge_j P(x + u_j)$ and $\bigwedge_k \neg P(x + u_k)$.
Consider $P(x + u)$. It is equivalent to $(|x| < \inf|u| \land P(u)) \lor (|u| < \inf|x| \land P(x)) \lor (E(|x|, |u|) \land P(u + x))$. Taking into account that

$$E(x, u) \land P(u + x) \land E(x, v) \land P(v + x)$$

is equivalent to $E(x, u) \land P(u + x) \land E(u, v) \land P(u - v)$, I can suppose that at most one of $\varphi_i$ is of the form $P(x + u)$.

Since $\neg P(x + u)$ is equivalent to $(|x| < \inf|u| \land \neg P(u)) \lor (|u| < \inf|x| \land \neg P(x)) \lor (E(|x|, |u|) \land \neg P(u + x))$ it is possible to assume that if at most one of $\varphi_i$ is of the form $P(x + u)$ then there is no conjunct of the form $\neg P(x + u)$.

Hence $\exists x \land_i \varphi_i(x, \bar{y})$ is equivalent to a formula saying that there is $x$, which belongs to a convex set $U$, which has non-empty intersection either with $P(x + u)$ or has no intersection with finitely many $P(x + u)'s. Let $a \in U$. If the length of $U$ is bigger than the length of $[a]_1$, then $U$ has a non-empty intersection with $P(G) + g$ for any $g$ with $|g| < \sup[a]$ as well as with the complement of a finite union of $(P(G) + g)'s$ with similar $g$'s. If the length of $U$ is less than or equal to the length of $[a]_1$, then $U$ has a non-empty intersection with $P(G) + g$ iff $a \in P(G) + g$. So, it is sufficient to show that all of these is expressible by a quantifier-free formula.

Case 1. Assume that one of $\varphi_i$ is of the form $P(x + g)$. Then I may suppose that there is no conjunct of the form $\neg P(x + g)$. If the considered convex set $U$ is unbounded then obviously there is an element which satisfy $P(x + g)$ from $U$. So, assume that both bounds in the formula do exist. Adding to bounds $\neg g$ I obtain an equivalent formula $\exists x(\inf U - g < x < \sup U - g \land P(x))$. So for simplicity of notation I assume that $g = 0$.

Case 1.1. bound$_1U = \text{bound}_2|b|$. Then any non-empty convex set $U$ has length bigger than $[b]_1$ and $\exists x(\inf U < x < \sup U \land P(x))$ is equivalent to $\inf U < \sup U$.

Case 1.2. bound$_1U = \text{bound}_2|b|$, say $\sup U = \sup[b]$. Then the length of $U$ is bigger that $[b]_1$ iff $\inf U < \inf[b]_1$. Otherwise either $\inf U = \inf[b]_1$ or $\inf[b]_1 < \inf U = c < \sup[b]_1$. In this case $\exists x(\inf U < x < \sup U \land P(x))$ is equivalent to $P(b)$.

Case 1.3. bound$_1U = \text{bound}_2|b| + g$ or bound$_1U = \text{bound}_2|b| + g$ for some $g$ with $|g| > \sup[|b|]$, say $\sup U = \sup[b] + g$. Then $\exists x(\inf U < x < \sup U \land P(x))$ is equivalent to $\inf U < \sup U \land \exists x(\inf U < x < \sup[b]_1 \land P(x))$ (case 1.2).

Case 1.4. $U = (b, c)$. Then $\exists x(b < x < c \land P(x))$ is equivalent to $b < c \land (\neg E_1(b, c) \lor P(b))$.

Case 2. No conjunct is of the form $P(x + g)$ and there are finitely many conjuncts of the form $\neg P(x + g)$. Note that $\neg P(x + g) \land E(|x|, |g|) \land \neg P(x + h) \land E(|x|, |h|)$ implies $E(|g|, |h|)$, thus I may assume that the considered convex set $U$ is a subset of $[g]$ and the absolute values of all $g's$ are in the same $E$-class. Consider $U$. Assume that $\sup U = \sup[b]$. If $E_1(\neg g, b)$, then $U \cap \neg P(x + g)$ is
equivalent to \((\inf U, \min\{\sup [b], b - \inf [b + [g]]\}) \land \neg P(x + g)\). Similar facts hold in the cases \(\inf U = \inf [b], \sup U = b, \inf U = b, \) bound \(U = b + \text{bound}[g], \) bound \(U = b + \text{bound}[g]\). So I may assume that if one of these conditions holds, then \(\neg E_1(b, -g)\).

Case 2.1. bound \(U = \text{bound}_2[b]\). Then any non-empty convex set \(U\) has length bigger than \(b\) and \(\exists x(\inf U < x < \sup U \land \bigwedge_i \neg P(x + g_i))\) is equivalent to \(\inf U < \sup U\).

Case 2.2. bound \(U = \text{bound}_2[b], \) say \(\sup U = \sup [b]\). Then the length of \(U\) is bigger than \(b\) iff \(\inf U < \inf [b]\). Otherwise either \(\inf U = \inf [b]\) or \(\inf [b] < \inf U = c < \sup [b]\). In this case \(\exists x(\inf U < x < \sup U \land \bigwedge_i \neg P(x + g_i))\) is equivalent to \(\bigwedge_i \neg P(b + g_i)\).

Case 2.3. bound \(U = \text{bound}_2[b] + g\) or bound \(U = \text{bound}_2[b] + g\) for some \(g\) with \(|g| < \sup[|b|]\), say \(\sup U = \sup [b] + g\). Then \(\exists x(\inf U < x < \sup U \land \bigwedge_i \neg P(x + g_i))\) is equivalent to \(\inf U < \sup U \land \exists x(\inf U < x < \sup [b] + g_i)\) (case 2.2).

Case 2.4. \(U = (b, c)\). Then \(\exists x(b < x < c \land \bigwedge_i \neg P(x + g_i))\) is equivalent to \(b < c \land \neg E_1(b, c) \lor \bigwedge_i \neg P(b + g_i) \lor \bigwedge_i \neg P(c + g_i))\). □

**Theorem 3.17** The elementary theory \(T\) of \((G, <, +, -, 0, P, E)\) is \(\omega\)-stable. More precisely, each cut over a model has finitely many extensions up to complete type over the model.

**Proof.** By the previous theorem \((G, <, +, -, 0, E)\) is weakly \(\omega\)-minimal, thus each cut has at most 2 completions. Let \(\langle C, D\rangle\) be a cut and \(p(x)\) a type over \(G\) extending \(\langle C, D\rangle\). For simplicity I may assume that \(x > 0 \in p(x)\).

Let \(\langle C, D\rangle\) be a cut and \(K_C\) be the stabilizer of \(C\).

Note that if \(0 < |x| < \inf |a|\), then \(P(x + a)\) is equivalent to \(P(a)\). If \(0 < |a| < \inf |x|\), then \(P(x + a)\) is equivalent to \(P(x)\). The formula \(E(x, a) \land P(x + a) \land E(|x|, |b|) \land P(x + b) \land \neg E_1(x, -b)\) is equivalent to \(E(x, a) \land P(x + a) \land E(a, b) \land P(b - a) \land \neg E(c, -b)\) for some \(c\) such that \(E_1(x, c) \in p\). If there is no \(c\) such that \(E_1(x, c) \in p\), then \(\langle C/E_1, D_1\rangle\) is an irrational cut in \(G/E_1\). Then \(\neg E_1(x, -b)\) holds for any \(b \in G\) and the above formula is equivalent in \(Th(G) \cup \{c < x < d : c \in C, d \in D\}\) to \(E(x, a) \land P(x + a) \land E(a, b) \land P(b - a)\).

Let \(\langle C/E_1, D_1\rangle\) be an irrational cut in \(G/E_1\). If \(E(x, a) \in p\) then there are extensions of the cut of two kinds: \(\{P(x + b), E(x, a)\}\) for some \(b \in [a]\) defines a complete type over \(G\) and \(\{\neg P(x + b), E(x, a) : b \in G\}\) defines a complete type. If \(E(x, a) \notin p\) for any \(p\) then \(\langle C/E, D\rangle\) is an irrational cut in \(G/E\). Then there are two extension of the cut: with \(P(x)\) or with \(\neg P(x)\).

So, assume that \(E_1(x, a) \in p\) for some \(a \in G\).

Case 1. There is \(b \in E_1(G, a)\) such that \(b \in D\) and \(\inf\{b - c : c \in C\}\) is a minimal or \(b \in C\) and \(\inf\{d - b : d \in D\}\) is a minimal. Then I can consider the
following cut, which has the same number of extensions and been considered above: \(⟨C − b, D − b⟩\).

Case 2. There is no such \(b ∈ E_1(G, a)\) as in Case 1. Then \(K_C = \{0\}\). Assume that \(P(x + c) ∈ p\) and \(|x + c| < \inf|c|\). Then there is \(c'\) such that \(|x + c'| < \inf|x + c|\). By the definition of \(P\) it holds that \(P(x + c) ⇔ P(x + c - (x + c'))\). So I may replace \(P(x + c)\) with \(P(c - c')\). Thus this cut has at most two extensions: with \(P(x + c)\) or with \(¬P(x + c)\). □

4 Ordered o-stable fields

Let \(R = (R, <, +, ·, 0, 1, \ldots)\) be an ordered o-stable field.

**Lemma 4.1** Any definable subgroup of the additive group of \(R\) is convex.

**Proof.** Assume the contrary, that \(H\) is a definable subgroup of \((R, +)\), which is not convex. Let \(K\) be the maximal convex subgroup of \(H\), and for some \(b > a > 0\) an interval \((a, b)\) contains infinitely many cosets of \(K\). Then infinitely many of these cosets are not subset of \(H\). Let \(h ∈ H\) and \(c ∈ R\) be such that \(0 < h < a\) and \(ch ∈ (a, b) \setminus H\). Then \(c > 1\) and \(\sup H ≤ cH\).

By Lemmata 2.4 and 3.3 the intersection \(H ∩ cH\) is not bounded in \(H\). By Baldwin-Saxl condition (Lemma 2.1) there are \(c_0, \ldots, c_{n−1}\) such that

\[H \cap \bigcap_{i<n} \{cH : c ∈ (ah^{-1}, bh^{-1})\} = H \cap \bigcap_{i<n} c_iH\]

Hence \(H_1 \triangleq H \cap \bigcap_{c ∈ (ah^{-1}, bh^{-1})} \{cH\}\) is not bounded in \(H\). By the trivial chain condition inside the cut \(\sup H\) I may assume that \(H_1\) is minimal. Let \(c ≠ c_i\) for \(i < n\). Then eventually in the cut defined by \(\sup H\) the equality \(H_1 ∩ cH_1 = H_1\) holds.

Let \(h ∈ H_1\) and \(g < h\) be not in \(H_1\). Let \(c = hg^{-1}\). Then \(c > 1\) and \(\sup cH_1 ≥ \sup H_1\). So, the intersection \(H_1 ∩ cH_1\) is unbounded in \(H_1\) and since \(H_1\) is minimal this intersection is equal to \(H_1\). By the choice of \(c\) it holds that \(cg = h ∈ H_1 ⊆ cH_1\). Then \(cg = ch_1\) for some \(h_1 ∈ H_1\) and \(g ∈ H_1\), giving a contradiction. □

In [8] it has been shown that each weakly o-minimal ordered field is real closed.

**Theorem 4.2** Let \(R\) be an o-stable ordered field with boundedly many convex definable subgroups of the additive group. Then \(R\) is weakly o-minimal and real closed. In particular, if \(R\) is a strongly o-stable ordered field, then it is o-minimal.
Proof. Assume the contrary, that $\mathcal{R}$ is not weakly o-minimal. Then there is a definable subset $A$ with infinitely many convex components. So I may assume that there are convex components $[a_i]$ for $i < \omega_0$ such that $[a_i] < [a_j]$ iff $i < j$.

Let $\langle C, D \rangle$ be a cut defined by $\sup \bigcup_{i<\omega_0} [a_i]$. Then $\sup C = \sup (H + a)$ for some convex definable subgroup $H$. Since the $\varphi$-rank of $A$ inside $\langle C, D \rangle$ is finite, for $\varphi(x; y) \equiv x \in (y + A)$ there is a definable subset $B$ of $A$ which at the cut $\langle C, D \rangle$ has the following property: if $B \cap (B + g)$ is not eventually an empty set, then these sets are eventually equal.

Suppose that $H$ is a non-zero subgroup. Then the eventual stabilizer $K_B$ of $B - a$ is not bounded in $H$. By Lemma 4.1 $K_B$ is convex. Then $K_B = H$. By Theorem 3.13 eventually $B$ equals to $B + K_B = B + H$, which is eventually equal to $H$. That gives a contradiction.

Thus $H$ is a zero subgroup. Let $b_1 < c_1 < b_2 < c_2 < \ldots < b_n < c_n < \ldots$ be an infinite sequence of elements such that $b_n \in B$ and $c_n \notin B$. Let $\alpha$ be an infinitesimal relatively $b_{n+1} - c_n$ and $c_n - b_n$ for all $n$. Let $C = \sup_{n<\omega} b_n$.

Then $\sup C = \sup_{n<\omega} b_n = \sup_{n<\omega} b_n + \alpha = \sup C + \alpha$. Thus I find another cut whose eventual stabilizer now is not a zero group. So I may repeat the above consideration.

Note that if $\mathcal{R}$ is strongly o-stable, then the supremum of any definable bounded subset is in $R$. Thus any definable convex set is an interval. Consequently, $\mathcal{R}$ is o-minimal. \hfill $\Box$

Due to this theorem it is quite natural to ask the following

**Question 2** Is there an ordered o-stable field which is not weakly o-minimal? Is an ordered o-stable (o-superstable, or o-$\omega$-stable) field real closed?

## 5 On ordered non-valuational o-omega-stable groups

Throughout this section $G$ is an ordered group whose elementary theory is non-valuational o-$\omega$-stable.

Simple examples of non-valuational (or even strongly) o-$\omega$-stable ordered groups which are not weakly o-minimal are the following: $(\mathbb{R}, <, +, \alpha \cdot \mathbb{Q})$, where $\alpha$ is some real number. Other examples will be considered in Subsection 5.1.

Note that by Lemma 2.4 the elementary theory of $(\mathbb{R}, <, +, \mathbb{Q}, \alpha \cdot \mathbb{Q})$ is not o-stable for any irrational $\alpha$.

**Lemma 5.1 (Descending chain condition)** There is no infinite descending chain of definable subgroups in an ordered non-valuational o-$\omega$-stable group.
Proof. Since there is no bounded non-zero subgroups, all non-zero definable subgroups are consistent with the cut $+\infty$. Now I can apply the $o$-omega-stable chain condition. □

Note that since an $o$-$\omega$-stable group is divisible, the ordering of $G$ is dense. The following theorem immediately follows from Theorem 3.14:

Theorem 5.2 Any formula with minimal Morley rank and degree in the cut $+\infty$ eventually is a coset of the least definable non-trivial subgroup of $G$. In particular, for any two definable eventually minimal subsets $A$ and $B$ there is an element $g$ such that eventually $A + g$ equals $B$.

Immediately I obtain the following

Corollary 5.3 Any formula of Morley rank 1 in the cut $+\infty$ eventually is a finite union of cosets of the least definable non-trivial subgroup of $G$.

Since any type consistent with the cut $+\infty$ is definable, I can speak of the eventual stabilizer of a type. Let $p \in S_1(G)$ be consistent with the cut $+\infty$, contain $\varphi(x)$ and have the same Morley rank and degree. Let

$$Fix_{+\infty}(p) = \{ g \in G : g + \alpha \models p \text{ for some (}\sim \text{ any) } \alpha \models p \}$$

It is easy to check that if $\varphi(x)$ has minimal positive Morley rank and degree then $Fix_{+\infty}(p) = K_\varphi$.

Lemma 5.4 The Morley rank of the eventual stabilizer of a type in the cut $+\infty$ is not bigger than the Morley rank of the type in this cut.

Proof. If the Morley rank of a type in the cut $+\infty$ is 1, then obviously, the Morley rank of its eventual stabilizer is 1. If the Morley rank of a type in the cut $+\infty$ is bigger than 1, then by Corollary 5.3 I can consider the quotient group $G'$ of $G$ modulo the least definable non-trivial group. By Lemma 1.13 $G'$ is a $\omega$-stable. Note that the preimage of the stabilizer of the quotient type in $G'$ will be the eventual stabilizer of the type in $G$. Since for $\omega$-stable group the analogous fact holds, the lemma follows. □

5.1 Examples of ordered strongly $o$-$\omega$-stable groups

Consider a divisible subgroup $K = (K,+,\ldots) = (K,L_K)$ of the group of reals, possibly with extra structure, which has zero intersection with the group of rationals $\mathbb{Q}$, such that the elementary theory $T_K$ of $K = (K,+,\ldots)$ admits quantifier elimination. Let $G = \mathbb{Q} + K$. 28
I construct the following structure: \( G = (G, <, +, -, Q, \ldots) \), where \( Q \) stands for unary predicate naming the set \( Q \) of rational numbers. For each relation \( R^n \) of \( K \) I define a relation with the same name \( R \) on \( G \) by

\[
G |\!| = R^n(a_1, \ldots, a_n) \iff K |\!| = R(a_1 + Q, \ldots, a_n + Q)
\]

By other words I define the structure of \( K \) on the quotient group \( G/Q \).

Theorem 5.5 The elementary theory \( T \) of \( G \) admits quantifier elimination.

Proof. Let \( \varphi(x, \bar{y}) \) be a conjunction of formulae of the following forms and its negation:

1. \( nx = t \);
2. \( nx < t \);
3. \( t < nx \);
4. \( Q(nx + t) \);
5. \( \psi(n_1 x + t_1, \ldots, n_k x + t_k) \), where \( \psi(z_1, \ldots, z_k) \) is a formula of \( L_K \);

here \( t \) and \( t_i \) are terms in \( \bar{y} \).

I am going to eliminate existential quantifier in the formula: \( \exists x \varphi(x, \bar{y}) \).

Since formulae of the form (1) – (4) are stable under multiplication by a positive natural number, I may assume that \( n \)'s in subformulae of \( \varphi \) of these form are the same. Obviously, \( nx = t_1 \land nx = t_2 \) is equivalent to \( nx = t_1 \land t_1 = t_2 \), and \( \neg nx = t \) is equivalent to \( nx < t \lor nx > t \). Thus I may assume that the formula \( \varphi \) contains at most one positive occurrence of a formula of the form (1). Also I may assume that the formula \( \varphi \) contains at most one positive occurrence of a formula of the form (2) and of the form (3). Indeed, \( nx < t_1 \land nx < t_2 \) is equivalent to \( (nx < t_1 \land t_1 < t_2) \lor (nx < t_2 \land t_2 \leq t_1) \).

Consider \( Q(nx + t_1) \land Q(nx + t_2) \). It is equivalent to \( Q(nx + t_1) \land Q(t_2 - t_1) \). Hence, I may assume that the formula \( \varphi \) contains at most one subformula of the form (4).

Assume that \( \varphi \) contains a subformula of the form (1). Then I can replace \( nx \) with \( t \) in subformulae of the form (2) – (4). Consequently, \( \exists x \varphi(x, \bar{y}) \) is equivalent to \( \exists x(nx = t \land \psi(n_1 x + t_1, \ldots, n_k x + t_k)) \land \theta(\bar{y}) \). For simplicity of notation I omit \( \theta \). By the construction of \( G \) the formula \( \psi(n_1 x + t_1, \ldots, n_k x + t_k) \) is equivalent to \( \psi(n_1 x + t_1 + q_1, \ldots, n_k x + t_k + q_n) \) for arbitrary rational \( q_i \). Then \( \exists x(nx = t \land \psi(n_1 x + t_1, \ldots, n_k x + t_k)) \) is equivalent to \( \exists x(nx = t + q \land \psi(n_1 x + t_1, \ldots, n_k x + t_k)) \) for arbitrary \( q \), which in turn is equivalent to \( \exists x(Q(nx - t) \land \psi(n_1 x + t_1, \ldots, n_k x + t_k)) \), which can be considered as a formula
\[\exists x(nx = t \land \psi(n_1x + t_1, \ldots, n_kx + t_k))\] of the language \(\mathcal{L}_K\) in \(T_K\). Since by our assumption \(T_K\) admits quantifier elimination, I can omit the existential quantifier.

Thus I may assume that \(\varphi\) does not contain a subformula of the form (1). Then \(\varphi(x, \bar{y})\) has the following general form:

\[v_1 < nx < v_2 \land Q(nx, v_3) \land \bigwedge_i \neg Q(nx, w_i) \land \psi(n_1x + t_1, \ldots, n_kx + t_k)\]

By the construction of \(G\) I may replace \(x\) with \(x + q\) for any rational \(q\). So the subformula \(v_1 < x < v_2\) can be replaced with \(v_1 < v_2\). Again by the construction of \(G\) I may replace considering a formula

\[\exists x \left( Q(nx, v_3) \land \bigwedge_i \neg Q(nx, w_i) \land \psi(n_1x + t_1, \ldots, n_kx + t_k) \right)\]

with considering the following formula of \(K\)

\[\exists x \left( nx = v_3 \land \bigwedge_i nx \neq w_i \land \psi(n_1x + t_1, \ldots, n_kx + t_k) \right)\]

Since \(T_K\) admits quantifier elimination, I may eliminate the existential quantifier.

Clearly, the other cases are similar. So, \(T\) admits quantifier elimination. \(\square\)

Now let \(\mathcal{K}\) be an arbitrary abelian torsion-free omega-stable group with cardinality at most continuum. Note that I mean really totally transcendental, not o-\(\omega\)-stable group. Then \(\mathcal{K}\) as a pure group is a direct product of copies of the set \(\mathbb{Q}\) of rationals and the number of copies is equal to the cardinality of \(K\). Let \(\{a_\alpha : \alpha < |\mathcal{K}|\}\) be a linearly independent over \(\mathbb{Q}\) subset of the set of reals \(\mathbb{R}\). Then there is an isomorphism \(\tau\) of \(K\) as a pure group onto a group of the form \(\sum_{\alpha < |\mathcal{K}|} a_\alpha \cdot \mathbb{Q}\), which can be ordered by the natural ordering of reals.

Let \(G\) be constructed from \(\tau(\mathcal{K})\) as above. The previous theorem implies the following:

**Theorem 5.6** Let the elementary theory \(T_K\) of \(\mathcal{K}\) is \(\omega\)-stable. Then the theory \(T\) is strongly \(\alpha\)-\(\omega\)-stable. Moreover if in \(\mathcal{K}\) a formula \(\varphi(x)\) has the Morley rank \(n\), then for any cut \(\langle C, D \rangle\) of \(G\) the Morley rank of the corresponding formula \(\varphi'(x)\) inside the cut \(\langle C, D \rangle\) equals \(n + 1\).
6 Unary definable functions and predicates

Throughout this subsection $G$ is an ordered group whose elementary theory is non-valuational $\omega$-stable and is not weakly $\omega$-minimal. I assume also that there is a least non-trivial definable subgroup $G_1$ of $G$. Note that if $G$ is $\omega$-stable and non-valuational, then $G_1$ does exists. One more assumption is that the full induced structure on $G_1$ is a weakly $\omega$-minimal structure. Note that it cannot be $\omega$-minimal because for any $g \not\in G_1$ the intersection $(-\infty, g) \cap G_1$ is not an interval in $G_1$. Observe that since the group $G$ is of non-valuational type, the group $G_1$ is also of non-valuational type. The condition that $G_1$ is weakly $\omega$-minimal is not a corollary of the fact that $G_1$ is the least definable unbounded subgroup. Consider the following example: $(\mathbb{R}, <, +, Q)$, where the predicate $Q$ names the set of rationals. The elementary theory of this group is $\omega$-stable, then the elementary theory of $(\mathbb{R}, <, +, P)$ is $\omega$-stable, where $P(\mathbb{R}) = \mathbb{Q} \cap (0, 1)$ is the set of rationals from the interval $(0, 1)$. Obviously, the subgroup of rationals is not definable in $(\mathbb{R}, <, +, P)$, moreover, there is no definable non-trivial subgroup. So, here $G_1 = \mathbb{R}$. Nevertheless this group is not weakly $\omega$-minimal.

Weak $\omega$-minimality of $G_1$ implies that for any cut $s = \langle C, D \rangle$ and any definable subset $A$ eventually in the cut $s$ the set $A$ consists of cosets of $G_1$. Indeed $A \cap (g + G_1)$ is a finite union of convex sets. Hence either eventually in $s$ the intersection $A \cap (g + G_1)$ is equal to $g + G_1$ or is empty. Let

$$B \triangleq A \setminus (\bigcup \{g + G_1 : A \cap (g + G_1) \text{ eventually in } s \text{ equals } g + G_1\})$$

Obviously, the intersection $B \cap (g + G_1)$ is empty for any $g$. It is sufficient now to show that eventually in $s$ the set $B$ is empty. If not I define function

$$f(g, c) \triangleq \sup B \cap (g + G_1) \cap (-\infty, c)$$

where $g$ runs over $G$ and $c$ over $C$. Since the formula

$$\varphi(x, a, c) \triangleq x \in \bigcup \{g + G_1 : f(g, c) < a\}$$

has not the strict order property inside any cut the image of $f$ is finite. It means that $B$ is bounded in $C$, that is eventually in $s$ it is empty.

I shall use the following fact, proved by R. Wencel:

**Fact 6.1** [17] Let $M$ be an ordered non-valuational weakly $\omega$-minimal group. Then any definable unary function is piecewise strictly monotone and continuous, where pieces are convex.

Obviously that this theorem is applicable to any coset of $G_1$:  

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Corollary 6.2 The restriction of a definable function to any coset of $G_1$ is piecewise strictly monotone and continuous.

Theorem 6.3 Let $G$ be an ordered non-valuational o-stable group with a non-valuational weakly o-minimal minimal non-trivial definable subgroup Any definable unary function is piecewise monotone and continuous, i.e. there is a finite definable partition of the group $G$ such that the restriction of the function to any element of the partition is strictly monotone and continuous. Note, however, that the partition need not consist of convex sets.

Proof. Let $f$ be a definable unary function in $G$. Since any coset of $G_1$ is non-valuational weakly o-minimal, the restriction of $f$ onto any coset of $G_1$ is piecewise monotone and continuous. Let $E(x, y)$ be an equivalence relation which says that $x - y \in G_1$ and the function $f$ is monotone and continuous on the set $(x, y) \cup (y, x) \cap (G_1 + x)$. Since $Th(G_1)$ is non-valuational weakly o-minimal there is a natural number $n$ such that there are at most $n$ equivalence classes of $E$ on any coset of $G_1$. Observe that $E$-classes are convex in cosets of $G_1$.

I say that two cosets $[a]$ and $[b]$ of $G_1$ are of the same type if they have the same number of $E$-classes, the restriction of $f$ to the first $E$-class in $[a]$ is strictly increasing iff the restriction of $F$ to the first class in $[b]$ is strictly increasing, the same holds if the restriction of $f$ is strictly decreasing or constant and the same holds for the second $E$-class, the third one, . . . , and the last one. Clearly there are finitely many types of cosets of $G_1$ and these types are definable. So, for simplicity I may assume that all cosets of $G_1$ are of the same types.

I say that a coset $[a]$ of $G_1$ is less than a coset $[b]$ of $G_1$ if the supremum in $G$ of the first $E$-class in $[a]$ is less than the supremum in $G$ of the first $E$-class in $[b]$.

Consider a formula $\varphi(x; y)$ which says that the supremum of the first $E$-class in $x + G_1$ is less than $y$. Since the cut $+\infty$ has no the strict order property, the set of supremums of the first $E$-classes of all coset of $G_1$ is finite. So, I may assume that this set consists of a unique elements, as well as the set of supremums of the second $E$-classes, of the third $E$-classes and so on. Moreover I may assume that there is only one $E$-class in each coset of $G_1$.

Since $G_1$ is dense in $G$ and the restriction of $f$ to any coset of $G_1$ is strictly monotone and continuous, I may construct an extension $f_\alpha$ of the restriction of $f$ on a coset $[a]$ of $G_1$ to the whole group $G$ as follows. Let $a_i \rightarrow g$, where $a_i \in [a]$. Then $f_\alpha(g) = \lim f(a_i)$.

If the restrictions of $f$ and $f_\alpha$ to a coset $[b]$ of $G_1$ are not equal, there is a first convex set in $[b]$ where for any element $c$ either $f(c) < f_\alpha(c)$, or $f(c) > f_\alpha(c)$. If the first inequality holds I say that the coset $[a]$ is bigger than the coset
Since there is no strict order property inside the cut $+\infty$, there is a finite definable partition of $G$ into $A_0, \ldots, A_{m-1}$ such that for any $i < m$ and for any $a, b \in A_i$ it holds that the restriction of $f$ on the coset $[b]$ of $G_1$ is equal to the restriction of $f_a$ on the coset $[b]$.

Again I may assume that $m = 1$. Then $f$ is continuous on $G$.

Assume that the restriction of $f$ on any coset of $G_1$ is strictly increasing. I prove that $f$ is strictly increasing on $G$. It follows from the above that $f$ is increasing. Assume that there are $a$ and $b$ such that $f(a) = f(b)$. Since $G_1$ is dense in $G$ there is an element $c \in a + G_1$ such that $a < c < b$. Then $f(a) = f(c)$, that contradicts to our assumption that the restriction of $f$ on any coset of $G_1$ is strictly increasing.

If $f$ is strictly decreasing or constant in any coset of $G_1$ then the proof is similar. \hfill $\Box$

**Theorem 6.4** In an ordered non-valuational o-stable group with a non-valuational weakly o-minimal minimal non-trivial definable subgroup the algebraic closure satisfies the exchange principle: if $c \in \text{acl}(Ab) \setminus \text{acl}(A)$ then $b \in \text{acl}(Ac) \setminus \text{acl}(A)$.

**Proof.** Let $c \in \text{acl}(Ab) \setminus \text{acl}(A)$. Since in any ordered structure the algebraic closure coincides with the definable closure, there is an $A$-definable function $f$ such that $f(b) = c$. By Theorem 6.3 there is a definable subset $B$ containing the element $b$ such that the restriction of $f$ on $B$ is continuous and is either strictly increasing, or strictly decreasing. It cannot be constant because then $c \in \text{acl}(A)$. Obviously, the restriction of $f$ on $B$ is a bijection between $B$ and $C = f(B)$. Then $f^{-1}$ can be defined and $b = f^{-1}(c)$. The last implies that $b \in \text{acl}(Ac)$. \hfill $\Box$

In [5] the following was proved. Let $T$ be a stable one-based theory and let $G$ be a group interpreted in $T$. The induced structure on $G$ is of the following kind: $G$ is abelian-by-finite, there are subgroups $H_i$, definable over acl($\emptyset$), such that in $G$ every formula is equivalent to a boolean combination of cosets mod the $H_i$’s. Here I can apply this result to o-stable groups.

**Theorem 6.5** Let $G$ be an ordered non-valuational o-stable group with a non-valuational weakly o-minimal minimal non-trivial definable subgroup $G_1$. Let the stable quotient group $G/G_1$ with the full induced structure be one-based. Then any definable subset of $G$ is a finite union of cosets of definable subgroups intersected with definable convex sets.

**Proof.** Let $A$ be a definable subset of $G$, and $\langle C, D \rangle$ a cut in $G$ such that $A$ is not eventually empty at this cut. First I consider the case when this cut is
Then eventually $A$ consists of cosets of $G_1$ and I can define the eventual quotient $A/G_1$ of the set $A$ by the following way: $g + G_1 \in A/G_1$ iff eventually $g + G_1$ is a subset of $A$. Since $G/G_1$ is one-based, $A/G_1$ is a finite union of cosets of definable subgroups of $G/G_1$. Let $B$ be the lift of the eventual quotient $A/G_1$ to $G$. So for each $g \in G$ eventually $B \cap (g + G_1)$ is equal to $A \cap (g + G_1)$. Let

$$f(g) = \inf\{a \in G : B \cap (g + G_1) \cap (a, \infty) = A \cap (g + G_1) \cap (a, \infty)\}$$

Since there is no the strict order property inside the cut $+\infty$ witnessed by the formula $\varphi(x, a) \equiv x \in \bigcup_{f(g)<a} g + G_1$ the image of $f$ must be finite. So there is a definable convex unbounded set $U$ such that $A \cap U = B \cap U$.

Now assume that $\langle C, D \rangle$ is of the form $a^-$. Since eventually in $a^-$ the set $A$ consists of cosets of $G_1$ I can repeat the above arguments.

Thus in some neighbourhood of a point in $G$ set $A$ equals the finite union of cosets of definable groups of $G$.

Let $E$ be an equivalence relation which says that each equivalence class is a maximal convex set such that for any $a$ and $b$ from this set eventually in the cut $a^-$ the set $A$ is equal to $A + a - b$, that is the eventual in $a^-$ quotient $A/G_1$ is equal to the eventual in $b^-$ quotient $A/G_1$. Since the group $G$ is non-valuational, the equivalence relation may contain at most finitely many infinite $E$-classes. Note that since $G$ is dense each finite $E$-class consists of one element. If there are infinitely many one-element $E$-classes then there is an open interval consisting of one-element $E$-classes. Let $a$ be in this interval. Then by the above arguments there is $b < a$ such that $A \cap (b, a) = B \cap (b, a)$, implying that $(b, a)$ is a subset of an $E$-class. Thus $E$ has finitely many equivalence classes and the theorem is proved. \[\square\]

References


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