REPRESENTATIVE DEFINABLE $C^r$ FUNCTIONS ON DEFINABLE $C^r$ GROUPS

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Abstract. Let $G$ be a compact affine definable $C^r$ group and let $r$ be $\infty$ or $\omega$. We prove that the representative definable $C^r$ functions on $G$ is dense in the space of continuous functions on $G$.

1. Introduction.

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \ldots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. Everything is considered in $\mathcal{M}$, every definable map is assumed to be continuous and the term “definable” is used throughout in the sense of “definable with parameters in $\mathcal{N}$” unless otherwise stated. We assume that $r$ denotes $\infty$ or $\omega$.

General references on o-minimal structures are [1], [2], also see [13]. Definable $C^r$ manifolds and definable $G$ sets in $\mathcal{M}$ are studied in [8], [7], [6].

Let $G$ be a definable $C^r$ group and $Def^r(G)$ denote the space of definable $C^r$ functions. Left translations in $G$ induce an action of $G$ defined by $f : G \to \mathbb{R} \mapsto L(g, f) = f(g^{-1}x) : G \to \mathbb{R}$. A function $f$ on $G$ is representative if the functions $\{L(g, f) | g \in G\}$ generate a finite dimensional subspace of $Def^r(G)$.

**Theorem 1.1.** Let $G$ be a compact affine definable $C^r$ group. Then the representative definable $C^r$ functions on $G$ is dense in the strong topology in the space of continuous functions on $G$.

Let $X$ be a definable $C^rG$ manifold. We say that the action of $G$ on $X$ is definably $C^r$ linearizable (resp. $C^r$ linearizable) if there exist a definable $C^r$ representation of $G$ whose representation space $\Omega$, a definable $C^rG$ submanifold $Y$ of $\Omega$ and a definable $C^rG$ diffeomorphism (resp. $C^r$ diffeomorphism) from $X$ to $Y$.

**Theorem 1.2.** Let $G$ be a compact affine definable $C^r$ group and $X$ a compact definable $C^rG$ manifold. Then the action is $C^r$ linearizable.

Remark that if $\mathcal{M} = \mathcal{R}$, then for any positive dimensional compact connected $C^\infty G$ manifold with non-transitive action, it admits uncountably many nonaffine definable $C^\infty G$ manifold structures ([10]). In Theorem 1.2, we cannot replace $C^r$ linearizable by definably $C^r$ linearizable.

Locally definable $C^r$ manifolds are defined in [9].

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2010 Mathematics Subject Classification. 57S15, 03C64.

Keywords and Phrases. O-minimal, definable $C^r$ groups, definable $C^rG$ manifolds.

Partially supported by Kakenhi (23540101)
Theorem 1.3. Let $G$ be a connected locally definable $C^r$ group and $(\tilde{G}, \pi)$ the universal cover of $G$. Then $\tilde{G}$ can be equipped uniquely with the structure of a locally definable $C^r$ group such that $\pi$ is a locally definable $C^r$ group homomorphism.

A locally Nash case of Theorem 1.3 is proved in [5].

2. Preliminaries and proof of results

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets. A continuous map $f : X \to Y$ is definable if the graph of $f$ $(\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m)$ is a definable set.

We say that a group $G$ is a definable group if $G$ is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.

A Hausdorff space $X$ is an $n$-dimensional definable $C^r$ manifold if there exist a finite open cover $\{U_i\}_{i=1}^k$ of $X$, finite open sets $\{V_i\}_{i=1}^k$ of $\mathbb{R}^n$, and a finite collection of homeomorphisms $\{\phi_i : U_i \to V_i\}_{i=1}^k$ such that for any $i, j$ with $U_i \cap U_j \neq \emptyset$, $\phi_i(U_i \cap U_j)$ is definable and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable $C^r$ diffeomorphism. A definable $C^r$ manifold $X$ is affine if $X$ is definably $C^r$ diffeomorphic to a definable $C^r$ submanifold of some $\mathbb{R}^n$.

A definable $C^r$ manifold (resp. an affine definable $C^r$ manifold) $G$ is a definable $C^r$ group (resp. an affine definable $C^r$ group) if $G$ is a group and the group operations $G \times G \to G, G \to G$ are definable $C^r$ maps.

A subgroup of a definable $C^r$ group is a definable subgroup of it if it is a definable $C^r$ submanifold of it. Note that every definable $C^r$ subgroup of a definable $C^r$ group is closed ([12]) and a closed subgroup of a definable $C^r$ group is not necessarily definable.

Let $G$ be a definable $C^r$ group. A group homomorphism from $G$ to some $O_n(\mathbb{R})$ is a definable $C^r$ representation if it is a definable $C^r$ map. A definable $C^r$ representation space of $G$ is $\mathbb{R}^n$ with the orthogonal action induced from a definable $C^r$ representation of $G$. A definable $C^rG$ submanifold means a $G$ invariant definable $C^r$ submanifold of some definable $C^r$ representation space of $G$.

Let $G$ be a definable $C^r$ group. A definable $C^rG$ manifold is a pair $(X, \phi)$ consisting of a definable $C^r$ manifold $X$ and a definable $C^r$ action $\phi : G \times X \to X$ on $X$ of $G$. For abbreviation, we write $X$ instead of $(X, \phi)$. A definable $C^rG$ manifold is affine if it is definably $C^rG$ diffeomorphic to a definable $C^rG$ submanifold of some definable $C^r$ representation space of $G$.

Proof of Theorem 1.1. Since $G$ is compact and affine, there exists a definable $C^rG$ diffeomorphism $f$ from $G$ to a definable $C^rG$ submanifold $G'$ of some definable $C^r$ representation space $\Omega$ of $G$.

Let $r : G \to \mathbb{R}$ be a continuous function. Applying Polynomial Approximation Theorem to $r \circ f^{-1} : G' \to \mathbb{R}$, we have a polynomial function $q : G' \to \mathbb{R}$ approximating $r \circ f^{-1}$. Since $f$ is equivariant and $G$ acts orthogonally on $\Omega$ and by P107 [11], $q \circ f : G \to \mathbb{R}$ is a representative on $G$ which is a definable $C^r$ function approximating $r$.

By a way similar to the proof of results of [10], we have the following result.

Theorem 2.1. Let $G$ be a compact affine definable $C^r$ group and $X$ a compact $C^\infty G$ manifold. Then $X$ is $C^\infty G$ diffeomorphic to a definable $C^rG$ submanifold $Y$ of some representation space of $G$. 
Proof of Theorem 1.2. We only have to prove the case where \( r = \omega \). By Theorem 2.1, there exist a representation space \( \Omega \) of a definable \( C^r \) representation of \( G \), a definable \( C^r G \) submanifold \( Y \) of \( \Omega \) and a \( C^\infty G \) diffeomorphism \( f : X \to Y \). By [P 233, 4], any Whitney neighborhood of a \( C^\infty G \) map to a representation space contains a \( C^\omega G \) map. Thus we can approximate \( f \) by a \( C^\omega G \) map \( h : X \to \Omega \). Therefore we have a required \( C^\omega G \) imbedding.

A Hausdorff space \( X \) is an \( n \)-dimensional locally definable \( C^r \) manifold if there exist a countable open cover \( \{ U_i \}_{i=1}^\infty \) of \( X \), countably many open sets \( \{ V_j \}_{j=1}^\infty \) of \( \mathbb{R}^n \), and a countable collection of homeomorphisms \( \{ \phi_i : U_i \to V_i \}_{i=1}^\infty \) such that for any \( i, j \) with \( U_i \cap U_j \neq \emptyset \), \( \phi_i(U_i \cap U_j) \) is definable and \( \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \) is a definable \( C^r \) diffeomorphism. We call the \( (U_i, \phi_i) \)'s the definable charts of \( X \).

Note that locally definable \( (C^0) \) manifolds are considered in [3]. Let \( X, Y \) be locally definable \( C^r \) manifolds with definable charts \( (U_i, \phi_i)_{i \in I}, (W_j, \psi_j)_{j \in J} \) respectively. A continuous map \( f : X \to Y \) is a locally definable \( C^r \) map if for every finite subset \( I' \) of \( I \), there exists a finite subset \( J' \) of \( J \) such that \( f(\cup_{i \in I'} U_i) \subset \cup_{j \in J'} W_j \) and that \( f|_{\cup_{i \in I'} U_i : \cup_{i \in I'} U_i \to \cup_{j \in J'} W_j} \) is a definable \( C^r \) map.

A bijective locally definable \( C^r \) map \( f \) between locally definable \( C^r \) manifolds is a locally definable \( C^r \) diffeomorphism if \( f^{-1} \) is a locally definable \( C^r \) map.

A locally definable \( C^r \) manifold \( X \) is affine if \( X \) is locally definably \( C^r \) diffeomorphic to a locally definable \( C^r \) submanifold of some \( \mathbb{R}^n \). Note that for any positive integer \( s \), a locally definable \( C^r \) manifold is locally definably \( C^s \) imbeddable into some \( \mathbb{R}^l \) (1.3 [9]).

A locally definable \( C^r \) manifold \( (\text{resp. an affine locally definable } C^r \text{ manifold}) \) \( G \) is a locally definable \( C^r \) group \( (\text{resp. an affine locally definable } C^r \text{ group}) \) if \( G \) is a group and the group operations \( G \times G \to G, G \to G \) are locally definable \( C^r \) maps.

Proof of Theorem 1.3. By the construction of the universal cover \( \tilde{G} \) of \( G \), \( \tilde{G} \) is a \( C^r \) group whose charts are countable and \( \pi \) is a \( C^r \) map. Since \( G \) is a locally definable \( C^r \) group, every transition function is definable. □

References


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