Counting in Pseudofinite Structures

Tingxiang Zou
PhD student at Institut Carmille Jordan, Université Lyon 1
Supervisor: Frank Wagner

December 20, 2018

1. Pseudofinite dimensions

2. Coarse dimension and pseudofinite H-structures

3. Coarse dimension and pseudofinite difference fields

Pseudofinite Dimensions ([Hrushovski, 2013])

Fix a pseudofinite structure $\mathcal{M} := \prod_{i \in I} M_i/\mathcal{U}$. Let $\mathbb{R}^* := \prod_{i \in I} \mathbb{R}/\mathcal{U}$.

Definition: Let C be a non-zero convex subgroup of $\mathbb{R}^*$ containing $\mathbb{R}$. Let

$$\delta_C(D) := \log |D| + C \quad ,$$

for definable set D. ($\mathbb{R}^*/C$ is an ordered $\mathbb{Q}$-vector space.)

General properties:

- $\delta_C(X) = 0$ for finite X;
- $\delta_C(X \cup Y) = \max\{\delta_C(X), \delta_C(Y)\}$;
- $\delta_C(X \times Y) = \delta_C(X) + \delta_C(Y)$;
- (subadditivity) $f : X \to Y$ definable function. If $\delta_C(f^{-1}(y)) \leq \alpha$ for all $y \in Y$ and $\delta_C(Y) \leq \beta$, then $\delta_C(X) \leq \alpha + \beta$.
- The definable subsets φ of X with $\delta_C(\varphi) < \delta_C(X)$ form an ideal.

Larsen-Pink inequality: $\delta_C(\Gamma \cap Z) \leq (\dim(Z)/\dim(G))\delta_C(\Gamma)$.
G simple group, Z constructable, Γ pseudofinite Zariski dense subgroup.

Fine Dimension

- Let C_{fin} be the convex hull of standard reals. Define $\delta_{\text{fin}} := \delta_{C_{\text{fin}}}$.

- Among all δ_C, the characteristic feature of δ_{fin} is that any dimension $\alpha \in \mathbb{R}^*/C_{\text{fin}}$ comes with a real-valued measure μ_α (up to a scalar multiple) such that
  - $\mu_\alpha(X) = 0$ iff $\delta_{\text{fin}}(X) < \alpha$;
  - $\mu_\alpha(X) = \infty$ iff $\delta_{\text{fin}}(X) > \alpha$;
  - if $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y) = \alpha$, then $\mu_\alpha(X) = \text{st}(|X|/|Y|)\mu_\alpha(Y)$.

Having a canonical definable set X with $\delta_{\text{fin}}(X) = \alpha$ in mind, we can define $\mu_\alpha(D) := \text{st}(|D|/|X|)$.

- For a definable set X with $\delta_{\text{fin}}(X) = \alpha$, the collection

$$\{D \subseteq X : \mu_\alpha(D) = 0\} = \{D \subseteq X : \delta_{\text{fin}}(D) < \delta_{\text{fin}}(X)\}$$

forms an S1-ideal.

Asymptotic Classes

- [Macpherson and Steinhorn, 2008]

A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is called a one-dimensional asymptotic class, if the following holds for every $\varphi(x;\bar{y})$:

- There is a positive constant C and a finite set $E \subseteq \mathbb{R}_{>0}$ such that for any $M \in \mathcal{C}$ and $\bar{b} \in M^{|\bar{y}|}$, either $|\varphi(M;\bar{b})| < C$ or there is $\mu \in E$ with
$$||\varphi(M;\bar{b})| - \mu|M|| < C \cdot |M|^{\frac{1}{2}}. \quad (*)$$
- For every $\mu \in E$ there is an $\mathcal{L}$-formula $\varphi_\mu(\bar{y})$ (over $\emptyset$) such that for any $M \in \mathcal{C}$ and $\bar{b} \in M^{|\bar{y}|}$, we have $M \models \varphi_\mu(\bar{b})$ iff $(*)$ holds.

- Examples: Finite fields; finite cyclic groups...

- Let $\mathcal{C}$ be a one-dimensional asymptotic class and M be an infinite ultraproduct of members of $\mathcal{C}$. Then $Th(M)$ is supersimple of SU-rank 1.

Multidimensional Asymptotic Classes

- Anscombe, Macpherson, Steinhorn and Wolf, 201?.

Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures and $\mathcal{R}$ be any set of functions $\mathcal{C} \to \mathbb{R}_{\geq 0}$ (*measuring functions*). $\mathcal{C}$ is called a $\mathcal{R}$-multidimensional asymptotic class ($\mathcal{R}$-m.a.c.), if the following holds for every $\varphi(x; \bar{y})$:

- There is a finite set $E \subseteq \mathcal{R}$ such that for any $M \in \mathcal{C}$ and $\bar{b} \in M^{|\bar{y}|}$, there is $h \in E$ with
$$||\varphi(M; \bar{b})| - h(M)| = o(h(M)). \quad (*)$$
- For every $h \in E$ there is an $\mathcal{L}$-formula $\varphi_h(\bar{y})$ (over $\emptyset$) such that for any $M \in \mathcal{C}$ and $\bar{b} \in M^{|\bar{y}|}$, we have $M \models \varphi_h(\bar{b})$ iff $(*)$ holds.

- Examples: one/N-dimensional asymptotic classes; finite vector spaces over finite fields in two-sorted language (measuring functions are polynomials in two variables over $\mathbb{Q}$)...

- Under certain assumptions of $\mathcal{R}$, infinite ultraproduct of members of $\mathcal{C}$ are simple. (There are examples of ultraproduct of m.a.c. being NSOP1 non-simple.)

Coarse Dimension

- Fix a pseudofinite set $X \subseteq \mathcal{M}^n$ with $\log|X| = \alpha$.
- Let $C_{<\alpha}$ be the maximal convex subgroup not containing α, and C_α be the smallest convex subgroup containing α, then $C_\alpha / C_{<\alpha} \leq \mathbb{R}^* / C_{<\alpha}$ can be mapped (by isomorphism) to $\mathbb{R}$. Or

$$\boldsymbol{\delta}_\alpha(D) := \mathrm{st}(\frac{\log|D|}{\alpha}) = \mathrm{st}(\frac{\log|D|}{\log|X|}).$$

We also write $\boldsymbol{\delta}_\alpha$ as $\boldsymbol{\delta}_X$ and call it the coarse dimension with respect to X.

- We say $\boldsymbol{\delta}_X$ is continuous if for any $\varphi(x, y)$ and $\alpha < \beta \in \mathbb{R}$, there is $\emptyset$-definable D such that
$$\{y : \boldsymbol{\delta}_X(\varphi(M^{|x|}, y)) \leq \alpha\} \subseteq D \subseteq \{y : \boldsymbol{\delta}_X(\varphi(M^{|x|}, y)) < \beta\}.$$
- For a tuple a and $A \subseteq M$, define
$$\boldsymbol{\delta}_X(a/A) := \inf \ \{\boldsymbol{\delta}_X(\varphi) : \varphi \in \mathrm{tp}(a/A)\}.$$
- (Hrushovski) If $\boldsymbol{\delta}_X$ is continuous, then $\boldsymbol{\delta}_X$ is additive:
$$\boldsymbol{\delta}_X(a, b/A) = \boldsymbol{\delta}_X(a/A, b) + \boldsymbol{\delta}_X(b/A).$$

Coarse Dimension

- Fix X, write $\boldsymbol{\delta}_X$ as $\boldsymbol{\delta}$.
- Define independence relation $a \mathop{\smile\!\!\!\!\!\!\,|}\nolimits^{\boldsymbol{\delta}}_A b$ for finite tuples a, b and small set A as: $\boldsymbol{\delta}(a/b, A) = \boldsymbol{\delta}(a/A)$.
- Additivity of $\boldsymbol{\delta}$ implies $\mathop{\smile\!\!\!\!\!\!\,|}\nolimits^{\boldsymbol{\delta}}$ is symmetric (for tules of finite coarse dimension).
- Extension: If $a \mathop{\smile\!\!\!\!\!\!\,|}\nolimits^{\boldsymbol{\delta}}_A b$, then for any c, there is $a \equiv_{A,b} a'$ with

$$a' \mathop{\smile\!\!\!\!\!\!\,|}\nolimits^{\boldsymbol{\delta}}_A b, c.$$

- In the expanded language, $\mathop{\smile\!\!\!\!\!\!\,|}\nolimits^{\boldsymbol{\delta}}$ satisfies in addition: invariance; transitivity; local character ...
- Let $\mathcal{M}$ be a field expansion. A a $\boldsymbol{\delta}$-closed subset (if $\boldsymbol{\delta}(b/A) = 0$, then $b \in A$). Let $\dim(a/A)$ be defined as transcendence degree.

$$a \underset{A}{\overset{\boldsymbol{\delta}}{\mathop{\smile\!\!\!\!\!\!\,|}}} b \quad \Longrightarrow \dim(a, b/A) = \dim(a/A) + \dim(b/A).$$

H-structures

- [Berenstein and Vassiliev, 2016]

Let T be a geometric theory in the language $\mathcal{L}$ ($\mathrm{acl}_{\mathcal{L}}$ forms a pre-geometry and elimination of $\exists^\infty$). Let $\mathcal{L}_H$ be the expansion of $\mathcal{L}$ with a predicate H.

Let $M \models T$, and $H(M)$ be an $acl_{\mathcal{L}}$-independent subset of M. We say $(M, H(M))$ is an *H-structure of T* if: for any finite subset $A_0 \subseteq M$, any $A \subseteq acl_{\mathcal{L}}(A_0)$, and any non-algebraic type $q \in S_1(A)$,

- there is $a \in H(M)$ such that $a \models q$;
- there is $a \in M$ such that $a \models q$ and $a \notin acl_{\mathcal{L}}(A \cup H(M))$.

- Let T be a geometric theory, then H-structures of T exist, and all of them are elementary equivalent. We call the common theory T^H.
- If T is strongly minimal/supersimple/superrosy of rank 1 and $\mathrm{acl}_{\mathcal{L}}$ is non-trivial, then T^H is ω-stable/supersimple/superrosy of rank ω.

Pseudofinite H-structures

Fact: Theories of ultraproducts of one-dimensional asymptotic classes are supersimple of SU-rank 1, hence geometric.

Theorem[Zou, 2018]

Let $\mathcal{C}$ be a one-dimensional asymptotic class in a countable language $\mathcal{L}$. Let $M := \prod_{i \in I} M_i/\mathcal{U}$ be an infinite ultraproduct of members in $\mathcal{C}$.

Then there exists $(M, H) := \prod_{i \in I}(M_i, H_i)/\mathcal{U}$ with each $H_i \subseteq M_i$ such that (M, H) is an H-structure.

Remark: $|H_i| << |M_i|$ (in fact $|H_i| \approx C_i \log M_i$, with $\lim_{i \in I} C_i = \infty$); in particular $\boldsymbol{\delta}_M(H) = 0$.

Questions: What is the relation between SU-rank and pseudofinite dimensions? What is the class $\{(M_n, H_n) : n \in \mathbb{N}\}$?

Expectation: It is a multidimensional asymptotic class.

Coarse dimension and H-structures

Let $\mathcal{M} = (M, H) = \prod_{i \in I}(M_i, H_i)/\mathcal{U}$ be a pseudofinite H-structure arising from an one-dimensional asymptotic class of SU-rank ω.

Theorem (Coarse dimension)

For any $\mathcal{L}_H$-formula $\varphi(\bar{x}, \bar{y})$ and any $\bar{b} \in M^{|\bar{y}|}$,

$$\boldsymbol{\delta}_M(\varphi(M^{|\bar{x}|}, \bar{b})) \in \{0, \cdots, |\bar{x}|\}.$$

Moreover, each $\{\bar{y} \in M^{|\bar{y}|} : \boldsymbol{\delta}_M(\varphi(M^{|\bar{x}|}, \bar{y})) = i\}$ is $\emptyset$-definable.
In particular, $\boldsymbol{\delta}_M$ is continuous and additive in $\mathcal{L}_H$.

Theorem (Coarse dimension and SU-rank)

Let $\bar{a}$ be a tuple. Suppose $SU_H(\bar{a}/A) = \omega \cdot k + n$. Define $\dim_{rk}(\bar{a}/A) := k$.
For any tuple $\bar{a}$ in (M, H) and any countable subset $A \subseteq M$,

$$\dim_{rk}(\bar{a}/A) = \boldsymbol{\delta}_M(\bar{a}/A).$$

m.a.c. ?

- Class $\{(M_i, H_i) : i \in I\}$

Moreover, for any $\mathcal{L}_H$-formula $\varphi(x, \bar{y})$, there is a finite set $E \subseteq \mathbb{R}_{>0}$ and $C > 0$ such that for any $i \in I$ and $\bar{b} \in (M_i)^{|\bar{y}|}$,

- either $|\varphi(M_i, \bar{b})| \leq C|H_i|^C$;
- or there is $\mu \in E$, with

$$||\varphi(M_i, \bar{b})| - \mu|M_i|| = o(\mu|M_i|).$$

And there is an definable partition of the parameter space for these situations.

Remark:

- To show m.a.c., remains to show that small sets are "proportional" (e.g. polynomial) to the size of H. (Joint project with Dario Garcia and Alexander Berenstein)
- Actually the above fact means the measure μ_α associated with the fine dimension $\alpha := \delta_{\text{fin}}(M)$ is well-behaved.
- Key lemma: formula $\forall \bar{h} \in H^n \varphi(x, \bar{h}, \bar{y})$ with $\varphi(x, \bar{h}, \bar{y})$ an $\mathcal{L}$-formula is a finite conjunction (up to a small set) of instances of $\varphi(x, \bar{h}, \bar{y})$ and can be done uniformly.

Difference Fields

- A Difference Field is a field $(F, +, \cdot, 0, 1)$ together with a field automorphism σ which is surjective (hence $\sigma^{-1}, \sigma^{-2}, \cdots$ exist).
- $\mathcal{L}_\sigma$ the language of difference rings is the language of rings augmented by a uniry function symbol σ.
- Example: ACFA (Chatzidakis, Hrushovski,...)
  - The model companion of difference fields.
  - Supersimple of SU-rank ω.
  - For an element $SU(a/A) < \omega$ iff $\deg_\sigma(a/A_\sigma) < \infty$ iff a is not transformally transcendental over A_σ.
  - Let $K_q := (\mathbb{F}_p^{alg}, \mathrm{Frob}_q)$ where q is a power of the prime p.

    (Hrushovski) Any non-trivial ultraproduct along the class $\{K_q : q \text{ a power of some prime}\}$ is a model of ACFA.
- What can we say about pseudofinite difference fields ?

Coarse dimension in pseudofinite difference fields

Theorem

There is a function $f : \mathbb{N} \to \mathbb{N}$ and let

$$\mathcal{S} := \left\{ \prod_{p \in \mathbb{P}} (\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p)/\mathcal{U} : k_p \geq f(p),\ \mathcal{U} \text{ non-principal ultrafilter} \right\}.$$

For any $(F, \mathrm{Frob}) \in \mathcal{S}$, the pseudofinite coarse dimension with respect to F, $\boldsymbol{\delta}_F$, of any $\mathcal{L}_\sigma$-definable set is integer-valued.

Moreover, $\boldsymbol{\delta}_F$ is continuous in $\mathcal{L}_\sigma$. In particular, $\boldsymbol{\delta}_F$ is additive on definable sets (without expanding the language).

Remark: The statement also holds for $char(F) = p > 0$, i.e., for $\prod_{i \in I} (\mathbb{F}_{p^{k_i}}, \mathrm{Frob}_{p^{t_i}})/\mathcal{U}$ provided $k_i >> t_i$ for almost all i.

Proof

Statement: Let $(F, \mathrm{Frob}) := \prod_{p \in \mathbb{P}} (\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p)/\mathcal{U} \in \mathcal{S}$. Then $\boldsymbol{\delta}_F$ is integer-valued.

Fix an $\mathcal{L}_\sigma$-formula $\varphi(x, a)$ with $a = (a_p)_{p \in \mathbb{P}}/\mathcal{U}$

$\Downarrow$

For $p \in \mathbb{P}$, translate $\varphi(x, y)$ to $\varphi_p(x, y) \in \mathcal{L}_{ring}$ replacing t by t^p.

$\Downarrow$ (Chatzidakis, van den Dries and Macintyre)

$\varphi_p(x, a_p)$ has a dimension and measure (d_p, μ_p) with $d_p \leq |x|$.

$\Downarrow$

$$|\varphi_p(\mathbb{F}_{p^{k_p}}, a_p)| \approx \mu_p \cdot (p^{k_p})^{d_p} \pm C_p \cdot (p^{k_p})^{d_p - 1/2}$$

$\Downarrow$

$\mathcal{U}$ will choose one d among all d_p (as $d_p \leq |x|$)

$\Downarrow$

$\lim_{p \in \mathcal{U}} \frac{\log |\varphi_p(\mathbb{F}_{p^{k_p}}, a_p)|}{\log p^{k_p}} = d$ provided $k_p >> \mu_p, C_p$ for almost all p.

Small sets are wild !

Small fixed field

Let $(F, \text{Frob}) \in \mathcal{S}$ and
$\text{Fix}(F) := \{x \in F : \sigma(x) = x\} = \prod_{p \in \mathbb{P}} \mathbb{F}_p / \mathcal{U}$.

- $|\text{Fix}(F)| \leq \log |F|$.
- (Folklore) If $Char(F) \neq 2$, there is an $\mathcal{L}_{ring}$-formula $\varphi(x, y)$ such that for any pseudofintie subset $A \subseteq Fix(F)$ there is $b \in F$

$$\varphi(F, b) \cap Fix(F) = A.$$

Corollary

$Th(F, \text{Frob})$ has TP2 and strict order property.

$\boldsymbol{\delta}_F$ and $\dim_{rk}$

- Goal: Relate $\boldsymbol{\delta}_F(a/A)$ to something we know.
- $(F, \text{Frob}) \subseteq \prod_{p \in \mathbb{P}} (\mathbb{F}_p^{alg}, \text{Frob}_p)/\mathcal{U} := (\tilde{F}, \text{Frob}) \models ACFA$.
- For a and A in $(\tilde{F}, \text{Frob})$, we have $SU(a/A) = \omega \cdot k + n$, define an integer-valued additive dimension $\dim_{rk}(a/A) := k$.
- $\dim_{rk}(a/A)$ is the transformal transcendence degree of a over A_σ.
- Let a and A in (F, Frob), then $\boldsymbol{\delta}_F(a/A) \leq \dim_{rk}(a/A)$.

Theorem

Let $\varphi(x) := \exists y \psi(x, y)$ be an $\mathcal{L}_\sigma$-formula such that $\psi(x, y)$ is quantifier-free with parameters in a finite set $A \subseteq F$.

- Then $\dim_{rk}(a/A) \leq \boldsymbol{\delta}_F(\varphi(F^{|x|}))$ for any $(F, \text{Frob}) \models \varphi(a)$.
- In particular, if $(F, \text{Frob}) \models \varphi(a)$ and $\boldsymbol{\delta}_F(\varphi(F^{|x|})) = \boldsymbol{\delta}_F(a/A)$, then $\dim_{rk}(a/A) = \boldsymbol{\delta}_F(a/A)$.

Proof

$\varphi(x,y)$ existential and y parameter.

$\Downarrow$

ONS if $\boldsymbol{\delta}_F(\varphi((F,\mathrm{Frob})^{|x|},b))=0$ then $\deg_\sigma(\varphi(x,b)/\{b\}_\sigma)<\infty$.

$\Downarrow$

If $\boldsymbol{\delta}_F(\varphi(x,b))=0$ then $|\varphi_p(\mathbb{F}_p^{alg},b_p)|<\infty$.

$\Downarrow$

$\varphi(x,b_p)$ has finitely many solutions in $(\mathbb{F}_p^{alg},\mathrm{Frob}_p)$ for almost all p.

$\Downarrow$ (Ryten, Tomašić)

Estimate number of solutions in $(\mathbb{F}_p^{alg},\mathrm{Frob}_p)$ uniform in p.

$\Downarrow$

$\deg_\sigma(\varphi(x,b)/\{b\}_\sigma)<d$ for some d.

Berenstein, A. and Vassiliev, E. (2016).
Geometric structures with a dense independent subset.
Selecta Mathematica, 22(1):191–225.

Hrushovski, E. (2013).
On pseudo-finite dimensions.
Notre Dame Journal of Formal Logic, 54(3-4):463–495.

Macpherson, D. and Steinhorn, C. (2008).
One-dimensional asymptotic classes of finite structures.
Transactions of the American Mathematical Society, 360(1):411–448.

Zou, T. (2018).
Pseudofinite h-structures and groups definable in supersimple h-structures.
https://arxiv.org/abs/1806.01398.