Brown's Natural Twisting Cochain and The Eilenberg-Mac Lane Transformation

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Abstract

We construct a natural twisting cochain which commutes with the Eilenberg-Mac Lane natural transformation.

Algebraic topology uses repeatly a small set of "classical" natural transformations. All these transformations may be constructed by the "method of acyclic models", and the method also shows that these transformations are unique up to homotopy. This last fact implies that most diagrams involving these transformations are homotopy commutative.

Sometimes, the commutativity up to homotopy is not enough for the purpose of certain proofs, and we need the exact commutativity of a diagram of natural tranformations. Since the 50's, this kind of result have been proved by direct (and most often tedious) computations.

In [4], a very simple trick is described, which enables to reduce most of these computations to very short ones, even to none. However, this method gave only already known results, and we were not able to apply it to the diagram we are concerned with in this paper, which says that "Brown's twisting cochain commutes with the Eilenberg-Mac Lane transformation" (see the section "Stating the Problem" below).

Finally, we give here a proof of the exact commutativity of this diagram, which does not use the method of [4].

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1 Prerequisites

A *reduced* simplicial set is a simplicial set with only one 0-simplex. The standard n-simplex Δ_n may be transformed into a reduced simplicial set by identifying all its vertices to a single point. The resulting simplicial set will be denoted by $\overline{\Delta}_n$. Since $\overline{\Delta}_n$ and Δ_n have the same simplexes in positive dimensions, we may describe the positive dimensional simplexes of $\overline{\Delta}_n$ by the sequence of the vertices of the corresponding simplex of Δ_n. The canonical map $\overline{e}_n : \Delta_n \longrightarrow \overline{\Delta}_n$ is the *fondamental simplex* of $\overline{\Delta}_n$. If X is reduced, any n-simplex of X may be seen as a simplicial map $\overline{\Delta}_n \longrightarrow X$.

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The complex of normalized[1] simplicial chains of X (over some fixed commutative ring Λ)[2] is denoted $C_*^{\mathrm{N}}(X)$. This is a differential graded coalgebra, with the Alexander-Whitney diagonal

$$C_*^{\mathrm{N}}(X) \xrightarrow{\Delta} C_*^{\mathrm{N}}(X) \otimes C_*^{\mathrm{N}}(X).$$

which is associative but not commutative (see [5]).

For any simplicial set S, we denote by $\varepsilon : C_*^{\mathrm{N}}(S) \longrightarrow \Lambda$ the usual augmentation. If S is either reduced or a simplicial group, we also have a coaugmentation $\eta : \Lambda \longrightarrow C_*^{\mathrm{N}}(S)$, which maps 1 either to the unique 0-simplex or to the neutral element in dimension 0.

ε is the co-unit of the coalgebra $C_*^{\mathrm{N}}(S)$, which means that

$$(1 \otimes \varepsilon)\Delta = (\varepsilon \otimes 1)\Delta = 1,$$

where 1 denotes identity.

Throughout this paper, we adopt *Koszul's convention*, which defines the tensor product of two (homogeneous) graded morphisms as:

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y),$$

which forces the following composition formula $(f \otimes g)(h \otimes k) = (-1)^{|g||h|} fh \otimes gk$, and which enables to write the differential of a tensor product of complexes in the simple form $\partial \otimes 1 + 1 \otimes \partial$.

We also define T by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$. This forces the relation $T(f \otimes g)T = (-1)^{|f||g|} g \otimes f$ for any graded morphisms f and g.

The Eilenberg-Mac Lane transformation is the unique (see [4]) natural transformation

$$C_*^{\mathrm{N}}(X) \otimes C_*^{\mathrm{N}}(Y) \xrightarrow{\nabla} C_*^{\mathrm{N}}(X \times Y).$$

It is associative and commutative and a morphism of differential coalgebras, so that

$$\partial \nabla = \nabla(\partial \otimes 1 + 1 \otimes \partial)$$
$$\Delta \nabla = (\nabla \otimes \nabla)(1 \otimes T \otimes 1)(\Delta \otimes \Delta).$$

If H is a simplicial group, $C_*^{\mathrm{N}}(H)$ is a Hopf algebra. Its product μ (the Pontriagin product) is given by

$$\mu = m_* \nabla : C_*^{\mathrm{N}}(H) \otimes C_*^{\mathrm{N}}(H) \longrightarrow C_*^{\mathrm{N}}(H)$$

where m is the multiplication of the group. So, we have the formula $\Delta\mu = (\mu \otimes \mu)(1 \otimes T \otimes 1)(\Delta \otimes \Delta)$.

The coaugmentation η is the unit of the Pontriagin product, which means that:

$$\mu(\eta \otimes 1) = \mu(1 \otimes \eta) = 1.$$

One can also easily verify that the Eilenberg-Mac Lane transformation is a morphism of Hopf algebras, so that we have $\mu(\nabla \otimes \nabla) = \nabla(\mu \otimes \mu)(1 \otimes T \otimes 1)$.

If X is any reduced simplicial set, and H any simplicial group, a *twisting function* $\tau : X \longrightarrow H$ is a map of degree -1 which satisfies the following identities (see [3] or [6] for more informations and some motivations):

$$\begin{array}{rcll} d_0(\tau(x)) & = & \tau(d_0(x))^{-1}\tau(d_1(x)) & \\ d_i(\tau(x)) & = & \tau(d_{i+1}(x)) & \text{for } i \geq 1 \\ \tau(s_0(x)) & = & 1 & \\ s_i(\tau(x)) & = & \tau(s_{i+1}(x)) & \text{for } i \geq 0 \end{array}$$

[1] This is the quotient of the ordinary simplicial chain complex by the subcomplex generated by the degenerated simplexes, so that any degenerated simplex represents 0 in $C_*^{\mathrm{N}}(X)$.

[2] For any Λ-module M, we identify $M \otimes \Lambda$ and $\Lambda \otimes M$ with M.

The cartesian product of two twisting functions is easily seen to be a twisting function.

An initial object in the category of all twisting function from some fixed reduced simplicial set X to any simplicial group H (with simplicial group morphisms as morphisms) is denoted:

$$X \xrightarrow{\tau_X} G(X)$$

(we sometimes write τ instead of τ_X) and has been first described by D. Kan in [2]. The simplicial group $G(X)$ is free and generated by the $\tau(x)$ for all simplexes x not in the image of s_0. Geometrically, it represents the *loop space* of X. For our convenience, we will denote $\tau(x)^{-1}$ by $\sigma(x)$. Notice that G is a functor, so that $G(x)$ makes sens for any morphism x of reduced simplicial sets.

$\overline{\Delta}_0$ and $G(\overline{\Delta}_0)$ are one point simplicial sets. For such a set (say $*$), we identify $X \times *$ with X, and $C^{\mathrm{N}}_*(*)$ with Λ. Then, the following particular instance of the Eilenberg-Mac Lane transformation

$$C^{\mathrm{N}}_*(X) \otimes C^{\mathrm{N}}_*(*) \xrightarrow{\nabla} C^{\mathrm{N}}_*(X \times *)$$

is the identity.

If X and Y are two reduced simplicial sets, there is a unique simplical group morphism:

$$G(X \times Y) \xrightarrow{\varphi} G(X) \times G(Y)$$

such that $\varphi\tau_{X\times Y} = \tau_X \times \tau_Y$, so that for any x and y, we have $\varphi(\tau(x,y)) = (\tau(x), \tau(y))$. φ is clearly onto, because if x or y is not in the image of s_0, so is (x,y). Notice that if Y is $\overline{\Delta}_0$, φ reduces to the identity morphism of $G(X)$.

A *twisting cochain* is a map of degree -1, $C \xrightarrow{t} A$, where C is a differential graded coalgebra, and A a differential graded algebra, such that

$$\begin{aligned} \partial t + t\partial + \mu(t \otimes t)\Delta &= 0 \quad \text{(Brown's relation)}, \\ \varepsilon t &= 0, \\ t\eta &= 0. \end{aligned}$$

The *cartesian sum* of two twisting cochains t and t' is defined by

$$t * t' = t \otimes \eta\varepsilon + \eta\varepsilon \otimes t',$$

and is a twisting cochain. We will also use this definition even if t and t' are not twisting cochains. The reader may verify the following relation which will be used in the last section.

$$(\mu \otimes \mu)(1 \otimes T \otimes 1)((t * t) \otimes (t * t))(1 \otimes T \otimes 1)(\Delta \otimes \Delta) = (\mu(t \otimes t)\Delta) * (\mu(t \otimes t)\Delta)$$

E.H. Brown ([1]) constructed for any reduced simplicial set X a twisting cochain

$$C^{\mathrm{N}}_*(X) \xrightarrow{t_X} C^{\mathrm{N}}_*(G(X))$$

which is natural with respect to X. This is used to prove the "twisted" Eilenberg-Zilber theorem, which generalises the Eilenberg-Zilber theorem from cartesian products to fiber bundles (see for example [3]). To get the right geometrical meaning, Brown's twisting cochain must be defined in dimension 1 ($|x| = 1$) by:

$$t_X(x) = 1 - \sigma(x).$$

See [3] for more details.

Brown's construction relies on some choices (it chooses repeatly an antecedent of some element by a non injective map). So, we cannot speak of *the* Brown's twisting cochain, but of *a* Brown's twisting cochain.

2 Stating the Problem

We can now state the problem to be solved. It consists in the existence of at least one Brown's twisting cochain such that the following diagram is commutative:

$$\begin{array}{ccc} C_*^{\mathrm{N}}(G(X)) \otimes C_*^{\mathrm{N}}(G(Y)) & \xrightarrow{\nabla} & C_*^{\mathrm{N}}(G(X) \times G(Y)) \\ & & \uparrow \varphi_* \\ \uparrow t_X * t_Y & & C_*^{\mathrm{N}}(G(X \times Y)) \\ & & \uparrow t_{X\times Y} \\ C_*^{\mathrm{N}}(X) \otimes C_*^{\mathrm{N}}(Y) & \xrightarrow[\nabla]{} & C_*^{\mathrm{N}}(X \times Y) \end{array}$$

We will exhibit such a Brown's twisting cochain in a constructive way. In order to do this we will reexamine Brown's construction, but right now we make some remarks about this diagram.

Since the functor $(X, Y) \mapsto C_*^{\mathrm{N}}(X) \otimes C_*^{\mathrm{N}}(Y)$ is generated by the models $((\overline{\Delta}_p, \overline{\Delta}_q), \overline{e}_p \otimes \overline{e}_q)$, it is enough to verify the formula

$$\varphi_* t \nabla(\overline{e}_p \otimes \overline{e}_q) = \nabla(t * t)(\overline{e}_p \otimes \overline{e}_q).$$

If $q = 0$ (or similarly if $p = 0$), this is trivialy verified, because if we put $Y = \overline{\Delta}_0$, our diagram reduces to

$$\begin{array}{ccc} C_*^{\mathrm{N}}(G(X)) & \xrightarrow{\mathrm{id}} & C_*^{\mathrm{N}}(G(X)) \\ & & \uparrow \mathrm{id} \\ \uparrow t_X & & C_*^{\mathrm{N}}(G(X)) \\ & & \uparrow t_X \\ C_*^{\mathrm{N}}(X) & \xrightarrow[\mathrm{id}]{} & C_*^{\mathrm{N}}(X) \end{array}$$

Now, by definition of $t * t$, $(t * t)(\overline{e}_p \otimes \overline{e}_q)$ is null as soon as $pq \neq 0$, so that (somewhat curiously) the only thing we have to prove is

$$\varphi_* t \nabla(\overline{e}_p \otimes \overline{e}_q) = 0$$

for $pq \neq 0$. This will occupy the rest of this paper, and will be proved by induction on $p + q$. More precisely, it is enough to show for $p + q \geq 2$ and $pq \neq 0$, that if $\varphi_* t \nabla(x \otimes y) = \nabla(t * t)(x \otimes y)$ for all tensors such that $|x \otimes y| < p + q$, then $\varphi_* t \nabla(\overline{e}_p \otimes \overline{e}_q) = 0$.

3 Constructing some Contractions

If C_* is a chain complex, we will call *contraction* a map $h : C_* \longrightarrow C_*$ of degree $+1$, such that $\partial h(x) + h\partial(x) = x$, whenever x has degree at least one (so, this is not a "regular" contraction, because we do not ask for $\partial h(x) + \eta\varepsilon(x) = x$ when $|x| = 0$). Notice that if x is a cycle and $|x| \geq 1$, then $\partial h(x) = x$.

We now construct three contractions, $\overline{h}$, $\overline{k}$ and $\overline{H}$, respectively on $C_*^{\mathrm{N}}(G(\overline{\Delta}_n))$, $C_*^{\mathrm{N}}(G(\overline{\Delta}_p \times \overline{\Delta}_q))$ and $C_*^{\mathrm{N}}(G(\overline{\Delta}_p) \times G(\overline{\Delta}_q))$.

Let h_i be the (ordinary) group morphism $h_i : G(\overline{\Delta}_n)_i \longrightarrow G(\overline{\Delta}_n)_{i+1}$ defined on generators by

$$h_i(\tau(x)) = \tau(x.n) \qquad\qquad (|x| = i+1)$$

where $x.n$ denotes the simplex obtained by adding to x the last vertex of Δ_n ($x.n$ is the *cone* with base x and vertex n). $k_i : G(\overline{\Delta}_p \times \overline{\Delta}_q)_i \longrightarrow G(\overline{\Delta}_p \times \overline{\Delta}_q)_{i+1}$ is defined similarly as $k_i(\tau(x,y)) = \tau(x.p, y.q) = \tau((x,y).(p,q))$.

The following relations are easily verified on generators of the free groups, and extend to any element of these groups, since all operators involved are group morphisms:

$$\begin{aligned} d_j h_i &= h_{i-1} d_j \quad (\text{for } 0 \le j \le i) \\ d_{i+1} h_i &= 1 \\ h_i h_{i-1} &= s_i h_{i-1} \end{aligned}$$

The same relations work also for the k_i's.

Now define $\overline{h}$ on $C_*^{\mathrm{N}}(G(\overline{\Delta}_n))$ and $\overline{k}$ on $C_*^{\mathrm{N}}(G(\overline{\Delta}_p \times \overline{\Delta}_q))$ by:

$$\overline{h}_i = (-1)^{i+1} h_i, \qquad\qquad \overline{k}_i = (-1)^{i+1} k_i.$$

Then, for $i \ge 1$,

$$\partial_{i+1}\overline{h}_i + \overline{h}_{i-1}\partial_i = \sum_{j=0}^{i+1} (-1)^{j+i+1} d_j h_i + \sum_{j=0}^{i} (-1)^{j+i} h_{i-1} d_j = d_{i+1} h_i = 1,$$

so that $\overline{h}$ (and similarly $\overline{k}$) is a contraction. Furthermore, the relation $h_i h_{i-1} = s_i h_{i-1}$ shows that $\overline{h}\,\overline{h} = 0$.

We define $H_i : G(\overline{\Delta}_p)_i \times G(\overline{\Delta}_q)_i \longrightarrow G(\overline{\Delta}_p)_{i+1} \times G(\overline{\Delta}_q)_{i+1}$ by $H_i(\tau(x), \tau(y)) = (\tau(x.p), \tau(y.q))$, and $\overline{H}_i = (-1)^{i+1} H_i$. Obviously $\overline{H}\varphi_* = \varphi_* \overline{k}$, and $\overline{H}$ is also a contraction since φ_* is onto.

Let $x : \overline{\Delta}_{p+q} \longrightarrow \overline{\Delta}_p \times \overline{\Delta}_q$ be a *non degenerated* $(p+q)$-simplex (so that the last vertex of this simplex is (p,q)). Then, we have $kG(x) = G(x)h$, as is easily verified by testing on the generators of the free group $G(\overline{\Delta}_{p+q})$, and consequently $\overline{k}G(x)_* = G(x)_*\overline{h}$.

4 Brown's Construction

Since the functor $X \mapsto C_*^{\mathrm{N}}(X)$ is *projective* on models $(\overline{\Delta}_n, \overline{e}_n)$, it is enough to define $t_X(x)$ for $x = \overline{e}_n$ (and $X = \overline{\Delta}_n$) and for each n. Then, if x is any n-simplex in some reduced simplicial set X, we have

$$t_X(x) = G(x)_*(t_{\overline{\Delta}_n}(\overline{e}_n)).$$

Since the acyclicity of $C_*^{\mathrm{N}}(G(\overline{\Delta}_n))$ begins in dimension 1, and since t_X is of degree -1, t_X must be constructed "by hand" up to dimension 2. The acyclicity of $C_*^{\mathrm{N}}(G(\overline{\Delta}_n))$ may be used only for the construction in dimensions 3 or more. Of course t_X must be null in dimension 0.

In dimension 1, t_X must be defined as explained at the end of the "Prerequisites" section. In dimension 2, t_X may be defined by the formula given by Brown:

$$t_X(x) = -\sigma(x) s_0 \sigma(d_0(x)).$$

This formula ensures that Brown's relation is satisfied up to dimension 2 (see [3]). We will now omit the subscript in the notation t_X.

For dimension 3 and higher, we procede by induction on the dimension. Brown's relation requires the following

$$\partial t(\overline{e}_n) = -t\partial(\overline{e}_n) - \mu(t \otimes t)\Delta(\overline{e}_n).$$

Since $\partial(\overline{e}_n)$ is of dimension $n-1$, since $\Delta(\overline{e}_n)$ is a sum of tensors $\alpha \otimes \beta$ such that $|\alpha| + |\beta| = n$, and since t is null in dimension 0, we see that the right hand side of the above equation is already well defined.

In fact, this right hand side is a cycle as may be easily verified (see [3]), and the acyclicity of $C_*^{\mathrm{N}}(G(\overline{\Delta}_n))$ in appropriate dimensions shows that it is a boundary, hence the construction of t.

But we need to be more precise and avoid choices. So we make use of the contraction $\overline{h}$, and define:

$$t(\overline{e}_n) = \overline{h}(-t\partial(\overline{e}_n) - \mu(t \otimes t)\Delta(\overline{e}_n)),$$

so that Brown's relation is still satisfied.

We will also need the relation $\overline{h}t(\overline{e}_n) = 0$ for $n \geq 1$. This is verified by direct computation for $n = 1$ and $n = 2$ (remember that degenerated simplexes represent 0 in $C_*^{\mathrm{N}}(X)$), and follows from $\overline{h}\,\overline{h} = 0$ for $n \geq 3$.

5 A Commutative Diagram

We begin by proving that if

$$\varphi_* t\nabla(x \otimes y) = \nabla(t * t)(x \otimes y)$$

for any tensor $x \otimes y$ such that $|x \otimes y| < n$, then

$$\partial\varphi_* t\nabla(x \otimes y) = \partial\nabla(t * t)(x \otimes y)$$

for any tensor $x \otimes y$ such that $|x \otimes y| \leq n$.

Indeed, let x and y be such that $|x \otimes y| = n$, then

$$\begin{aligned}
\partial\varphi_* t\nabla(x \otimes y) &= \varphi_*\partial t\nabla(x \otimes y) \\
&= -\varphi_* t\partial\nabla(x \otimes y) - \varphi_*\mu(t \otimes t)\Delta\nabla(x \otimes y) \\
&= -\varphi_* t\nabla(\partial \otimes 1 + 1 \otimes \partial)(x \otimes y) \\
&\qquad -\mu(\varphi_* \otimes \varphi_*)(t \otimes t)(\nabla \otimes \nabla)(1 \otimes T \otimes 1)(\Delta \otimes \Delta)(x \otimes y) \\
&= -\varphi_* t\nabla(\partial \otimes 1 + 1 \otimes \partial)(x \otimes y) \\
&\qquad -\mu((\varphi_* t\nabla) \otimes (\varphi_* t\nabla))(1 \otimes T \otimes 1)(\Delta \otimes \Delta)(x \otimes y).
\end{aligned}$$

In this last expression, $\varphi_* t\nabla$ is applied only to tensors of degree at most n, and when applied to a tensor of degree n, it is killed by an application of t to something of degree 0. So, we may use the hypothesis, so that

$$\begin{aligned}
\partial\varphi_* t\nabla(x \otimes y) &= -\nabla(t * t)(\partial \otimes 1 + 1 \otimes \partial)(x \otimes y) \\
&\qquad -\mu(\nabla \otimes \nabla)((t * t) \otimes (t * t))(1 \otimes T \otimes 1)(\Delta \otimes \Delta)(x \otimes y) \\
&= -\nabla(t\partial * t\partial)(x \otimes y) \\
&\qquad -\nabla(\mu \otimes \mu)(1 \otimes T \otimes 1)((t * t) \otimes (t * t))(1 \otimes T \otimes 1)(\Delta \otimes \Delta)(x \otimes y) \\
&= -\nabla(t\partial * t\partial)(x \otimes y) \\
&\qquad -\nabla((\mu(t \otimes t)\Delta) * (\mu(t \otimes t)\Delta))(x \otimes y) \\
&= \nabla(\partial t * \partial t)(x \otimes y) \\
&= \partial\nabla(t * t)(x \otimes y).
\end{aligned}$$

Now, let $p+q \geq 2$, $pq \neq 0$, and suppose that $\varphi_* t\nabla(x \otimes y) = \nabla(t * t)(x \otimes y)$ for all tensors $x \otimes y$ such that $|x \otimes y| < p+q$. We must prove that $\varphi_* t\nabla(\overline{e}_p \otimes \overline{e}_q) = 0$. From the preceding computation, we have

$$\overline{H}\partial\varphi_* t\nabla(\overline{e}_p \otimes \overline{e}_q) = \overline{H}\partial\nabla(t * t)(\overline{e}_p \otimes \overline{e}_q) = 0.$$

Since $\overline{H}\partial = 1 - \partial\overline{H}$, it is enough to prove that $\overline{H}\varphi_* t\nabla(\overline{e}_p \otimes \overline{e}_q) = 0$, or

$$\varphi_* \overline{k} t\nabla(\overline{e}_p \otimes \overline{e}_q) = 0.$$

$\nabla(\overline{e}_p \otimes \overline{e}_q)$ is a linear combination of non degenerated $(p+q)$-simplexes of $\overline{\Delta}_p \times \overline{\Delta}_q$, let's say:

$$\nabla(\overline{e}_p \otimes \overline{e}_q) = \sum_i \varepsilon_i x_i.$$

So, it is enough to prove that $\overline{k}t(x_i) = 0$. Now, by the construction of t, we have $t(x_i) = G(x_i)_* t(\overline{e}_{p+q})$, and since x_i is non degenerated, we have $\overline{k}G(x_i)_* = G(x_i)_* \overline{h}$. So, we have done, because $\overline{h}t(\overline{e}_{p+q}) = 0$.

6 Comparison with Szczarba's Cochain

Our twisting cochain turns out to be equal to the one defined by R.H. Szczarba in [7], so that our result may be rephrased as *"Szczarba's cochain commutes with the Eilenberg-Mac Lane transformation"*.

Of course, in dimension 1, Szczarba's cochain is identical to our's, because this is a matter of definition. Our cochain is characterized by

$$t(\overline{e}_n) = \overline{h}\partial t(\overline{e}_n) \qquad \text{for } n \geq 2.$$

Indeed, this relation is satisfied for $n \geq 3$ by definition, and may be verified for $n = 2$ by direct computation. The relation $\overline{h}\partial + \partial\overline{h} = 1$ shows that it is the only one such that $\overline{h}t(\overline{e}_n)$ is a cycle, for $n \geq 2$. So, it is also characterized by the seemingly stronger relation $\overline{h}t(\overline{e}_n) = 0$. We will see that this last relation is satisfied by Szczarba's cochain ϕ.

$\phi(\overline{e}_n)$ is defined in [7] as a linear combination of words, in which each letter is of the form

$$w = D^n_{j,i}\sigma(d_0^j(\overline{e}_n))$$

(see THEOREM 2.1 of [7]), where $D^n_{j,i}$ is an inductively defined operator involving degeneracy and face operators.

We claim that w is of the form $\sigma(y)$, where y is a *right justified* simplex, that is to say a simplex of $\overline{\Delta}_n$ such that the corresponding simplex of Δ_n contains n, the *last vertex* of Δ_n. This forces $\overline{h}\phi(\overline{e}_n)) = 0$ by the very definition of $\overline{h}$.

y is of the form

$$D^{n'}_{j,i} d_0^j(\overline{e}_n)$$

where $D^{n'}_{j,i}$ is the *derived* operator of $D^n_{j,i}$. It is obvious that if some operator D transforms right justified simplexes into right justified simplexes, then so does the derived operator. So, the only problem may come from the case

$$D^{n+1}_{j,i+k(n-1)!} = D^{n'}_{j,i} s_0 d_{k-j} \qquad \text{for } 0 \leq j < k \leq n-1$$

in Szczarba's inductive definition of the operators $D^n_{j,i}$, because of the presence of the face operator d_{k-j}. However, this face may be the last one only if $k-j = n-j$ ($n-j$ is the initial dimension of $D^{n+1}_{j,i+k(n-1)!}$), which is impossible.

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