On dependent conjunction and implication

Matthieu Herrmann and Alain Prouté

Abstract

In the context of our work on a new proof assistant, we were led to give a theoretical model of conjunctions $E \wedge F$ and implications $E \Rightarrow F$ where F is meaningful only when E is true, a situation that is very often encountered in everyday mathematics, and which was already considered by several type theorists. We present a version of these concepts that is more oriented towards usual mathematics than towards type theories, by using an extension of Lawvere's definition of quantifiers. Also, our presentation stresses the importance of the principle of description in this phenomenon. We explain why this dependency is obtained through the use of a "hidden" variable, and more generally the links of these concepts with the vernacular language of mathematics, which is actually our main motivation. Despite its links with topos theory, this article is readable by non-specialists.

Introduction

Conjunction, also known as the "logical and", and widely denoted $\wedge$, is generally defined as a commutative operation. In other words, for any statements E and F, it is assumed (or proved) that $E \wedge F$ is equivalent to $F \wedge E$.

However, there are very elementary and simple examples showing that this is not always the case. For example, let A be a subset of the set $\mathbb{N}$ of the natural integers, and consider the statement $A \neq \emptyset \wedge \inf(A) = 0$, where $\inf(A)$ denotes the infimum (greatest lower bound) of A. This statement is clearly meaningful, should it be true or not. On the contrary, the statement $\inf(A) = 0 \wedge A \neq \emptyset$ is meaningless, because $\inf(A)$ is not well defined if we do not know that A is nonempty. Notice that $A \neq \emptyset \Rightarrow \inf(A) = 0$ is meaningful and $\inf(A) = 0 \Rightarrow A \neq \emptyset$ meaningless for the same reason.

It is hard to find a treatement of this problem in the literature. It seems that the first occurrence of a theoretic analysis of this fact can be found in a series of lectures given by P. Martin-Löf in 1980 (see [7]), within the setting of intuitionistic type theory. The idea is mainly based on the fact that statements can be identified with sets (or types) in view of the analogy between the two situations : "the element a is of type X" and "the expression p is a proof of the statement E", which is now widely popularized under the name of the "Curry-Howard correspondance", to which Martin-Löf actually also refers. Subsequent works have appeared modelling these "dependent logical connectives", such as T. Coquand and G. Huet in 1986 [1] (see also M. Hyland and A. M. Pitts [4]), or D. Pavlović [8] in 1990, among others.

We turn now to a vocabulary that is more familiar to mathematicians. We propose an explanation of dependent conjunction and implication, which is not only a modelling of them, but also shows how much their existence is natural. It is indeed exactly as natural as the existence of the quantifiers $\exists$ and $\forall$, simply because dependent conjunction and implication play the same role with respect to assumptions as quantifiers with respect to declarations. It seems to us that the best way to understand this is to use one of the most beautiful discoveries of William Lawvere, namely his definition of quantifiers by way of adjoint increasing maps.([1])

It appears that a dependent conjunction $E \wedge F$ (and similarly for an implication) should be denoted $(\zeta \vdash E) \wedge F$ if we want to be completely explicit, where $\zeta \vdash E$ is a kind of declaration of ζ (actually a named assumption) and where F contains free occurrences of the variable ζ. Of course, in usual mathematics, this variable ζ is not visible, and we also explain on the one hand how F can have such a free occurrence of ζ, and on the other hand why ζ remains invisible (hidden) in F, and consequently not explicitly declared before E.

The content of this article holds for both intuitionistic and classical mathematics, and is part of type theory and topos theory's more or less explicit folklore. We hope to add some originality in the way we have chosen to present it, and to make the subject accessible to a wider audience.

The authors want to thank John Newsome Crossley for his careful reading, his suggestions and many corrections in our use of the English language.

Contents

1 Prerequisites

The meaning of logical connectives was historically defined in two different ways : the "logical" one, which defines under what conditions a given statement is true (or false),

[1] A pair of adjoint *decreasing* maps (almost the same concept) is also known as a "Galois correspondence".

possibly using truth tables (in other words, how to compute a truth value for a statement), and the "operational" one, which defines the meaning of logical connectives in terms of proofs. The second manner was initiated around 1930 by the intuitionists (and also holds for classical logic([2])) through the so-called Brouwer-Heyting-Kolmogorov interpretation, and also by G. Gentzen in his work on natural deduction and the sequent calculus. This operational point of view was more recently very elegantly reformulated using the categorical concept of adjoint functors (in the present case, adjoint increasing maps), in particular within the setting of topos theory.

For example, (ordinary) conjunction can be defined as the right adjoint to the duplication (diagonal) operation. In order to see this, consider two preordered sets X and Y,([3]) and two maps $f : X \to Y$ and $g : Y \to X$. We say that "f is left adjoint to g" (or equivalently that "g is right adjoint to f") if :

$$\forall x \in X\ \forall y \in Y\ \ f(x) \leq y \Leftrightarrow x \leq g(y)$$

This fact is denoted $f \dashv g$.([4]) This definition has several elementary consequences. First of all, the inequality $x \leq g(f(x))$ is true for any x in X (just replace y by $f(x)$ in the above statement, and use the reflexivity of the preorder relation). Symmetrically, we have $f(g(y)) \leq y$ for any y in Y. These two inequalities are known as the "unit" and the "co-unit" of the adjunction $f \dashv g$. From this and the transitivity of preorder relations we can deduce that f and g are increasing functions, and that if we have $f \dashv g$ and $f \dashv h$, then g and h are "equivalent" in the sense that for any y in Y, we have $g(y) \leq h(y)$ and $h(y) \leq g(y)$ (so that they are actually equal if the preorder on X is an order). Of course we also have the symmetric fact that $f \dashv h$ and $g \dashv h$ entail that f and g are equivalent (in the same sense). Furthermore, if $f \dashv h$ and $g \dashv k$, and if $g \circ f$ is meaningful, then $h \circ k$ is also meaningful and we have $g \circ f \dashv h \circ k$. Finally, we shall also need the fact that left adjoints commute (up to equivalence) with least upper bounds, and that right adjoints commute (up to equivalence) with greatest lower bounds.([5]) We leave these facts as easy exercises for the reader.

Hence, if a map has an adjoint (either on the left or on the right), this adjoint is essentially (i.e. up to equivalence) unique. As a consequence, if a map is defined as a left or right adjoint to a given map, it is well defined (up to equivalence).([6]) This kind of definition is a particular case of a "universal problem".

Now, consider the meta-set $\mathcal{E}$ of all mathematical (closed) statements.([7]) This meta-set is

[2] This is just because all intuitionistic proof principles are also valid in classical logic.

[3] In other words, sets with a binary relation $\leq$ that is reflexive and transitive.

[4] Warning : the symbol $\dashv$ is not a reversed notation for $\vdash$. The two symbols have completely distinct uses (but they sometimes appear together in a single expression).

[5] In other words, if $g : Y \to X$ is a right adjoint and if $\inf_{i \in I}(y_i)$ exists, then $\inf_{i \in I}(g(y_i))$ exists and is equivalent to $g(\inf_{i \in I}(y_i))$.

[6] This does not imply that the adjoint exists. It is only "unambiguously defined (up to equivalence)" .

[7] Here, we say *meta-set* instead of *set*, because this is a set of expressions of the language, not a set of truth values.

preordered by the relation of deductibility, that we denote by $\leq$ in this article. In other words, $E \leq F$ means that F can be deduced from E (i.e. proved under hypothesis E). This is clearly a preorder.([8]) For any preordered set X, the product set $X \times X$ is also preordered by the relation defined by the condition that $(x, y) \leq (u, v)$ if and only if $x \leq u$ and $y \leq v$, and we have a "diagonal" (or "duplication") map $\Delta : X \to X \times X$ defined by $\Delta(x) = (x, x)$ (which is obviously increasing).

Now, we can define a map $\wedge : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ just by postulating that $\Delta \dashv \wedge$. The map $\wedge$ is well defined (up to provable equivalence of statements). It is easy to recognize that this map cannot be anything other than our usual conjunction. This is just a consequence of the fact that the usual conjunction indeed has the property $\Delta \dashv \wedge$, since we have :

$$E \leq F \wedge G \quad \text{iff} \quad E \leq F \text{ and } E \leq G \quad \text{iff} \quad \Delta(E) = (E, E) \leq (F, G)$$

This definition (which is well known in topos theory) of conjunction is one of the last known avatars of the Brouwer-Heyting-Kolmogorov interpretation (concerning conjunction). It is undoubtebly the most elegant thing we can do for defining (ordinary) conjunction.

Why is such a definition "operational" ? This is just because all the proof rules concerning the conjunction can be deduced from it, which is not the case if the conjunction is defined by the "logical" method, i.e. in terms of truth values. Indeed, the adjunction $\Delta \dashv \wedge$ means that $E \leq F \wedge G$ if and only if $E \leq F$ and $E \leq G$, in other words that it is equivalent to prove $F \wedge G$ under the hypothesis E, or to prove separately F and G under the same hypothesis. This is of course a rule that anyone uses everyday when a conjunction must be proved. Now, consider the co-unit of the adjunction, which writes :

$$(E \wedge F, E \wedge F) = \Delta(E \wedge F) \leq (E, F)$$

and which gives the rules $E \wedge F \leq E$ and $E \wedge F \leq F$ (which are generally taken as axioms in most presentations of the properties of conjunction, or are deduced from truth tables). The reader is urged to check that similarly we have $\vee \dashv \Delta$, and that the principle of "reasoning by disjunction of cases" is actually a consequence of this definition.([9])

Actually, $E \wedge F$ is nothing other than the greatest lower bound of E and F, as is the case of any concept defined as the right adjoint to a diagonal map (as for example gcd using the divisibility relation, or intersection using the inclusion of subsets relation). However, the notion of adjoint maps is much more general than the notion of greatest lower bound. Of course, commutativity of the (ordinary) conjunction is a consequence of the commutativity of the diagonal Δ.

Because we shall use it in another section below, we also recall the definition of the usual implication via adjunctions. Given a statement F, we consider the map $E \mapsto E \wedge F$ from $\mathcal{E}$

[8] But not an order, because we are very careful not to confuse statements (that are "signifiers", a syntactic notion) with truth values (the corresponding "signified", a semantic notion).

[9] It should be remarqued that there is no dependent disjunction in mathematics. Indeed, a disjunction $E \vee F$ should be true if E is false and F true. But in this case, F cannot depend on the truth of E.

to $\mathcal{E}$. By definition, the map $G \mapsto F \Rightarrow G$ is a right adjoint to the previous one. In other words, we have :

$$E \wedge F \leq G \quad \text{iff} \quad E \leq F \Rightarrow G$$

This means that in order to prove the implication $F \Rightarrow G$ under the hypothesis E it is enough to prove G under the hypothesis $E \wedge F$. This is of course what everyone does when an implication must be proved, and is known as the "auxiliary hypothesis method".([10])

The co-unit of this adjunction gives :

$$(F \Rightarrow G) \wedge F \leq G$$

which is the principle known as "modus ponens".

Another advantage of this method, is that it gives a clear and exciting definition of the quantifiers, as was revealed by Lawvere.

2 Lawvere's definition of the quantifiers

In order to express Lawvere's definition of the quantifiers, we must first discuss the notion of "context". In mathematical texts, it is often the case that we "declare" variables. For example, we can say "Let x be a real number.". Such a sentence is called a "declaration". In this article, a "generic" declaration will be denoted $(x \in X)$, where x is a symbol and where X is a set.

Within any part of a mathematical text, we are "working in a context", which is just the consequence of the declarations made so far.([11]) Let Γ denote an arbitrary context, and let $\Gamma(x \in X)$ denote the context obtained by declaring x as an element of X in the context Γ. In other words, $\Gamma(x \in X)$ is the context obtained by "enriching" Γ by a new declaration.([12]) We require that a variable cannot be declared twice in the same context. In other words, the context $\Gamma(x \in X)$ is invalid if x is already declared in Γ.

We also denote by $\mathcal{E}_\Gamma$ the meta-set of all statements that are meaningful in the context Γ, in other words, the free variables of which are declared in Γ. This meta-set $\mathcal{E}_\Gamma$ is still preordered by the deductibility relation, that we denote $\leq_\Gamma$. There is a canonical inclusion $\mathcal{J}_{x \in X} : \mathcal{E}_\Gamma \to \mathcal{E}_{\Gamma(x \in X)}$, since if a free variable in a statement E is declared in Γ, it is *a fortiori* declared in $\Gamma(x \in X)$. Notice the importance of the fact that a variable cannot be declared

[10]Purists will argue that the auxiliary hypothesis method consists in considering F as an *extra hypothesis*, not to replace the hypothesis E by the hypothesis $E \wedge F$. They are right. Passing from one point of view to the other one can of course be formalized, and we will need to do it in section 3. For the time being, we need to present these concepts as we do, because otherwise adjunctions are not as simply applicable.

[11]And the "scope" of which we have not yet left.

[12]Formally, contexts are generated by the two rules : there is an "empty" context, and for any context Γ, $\Gamma(x \in X)$ is a context.

twice in the same context. Indeed, if a variable x is already declared as an element of some set Y in Γ, it can be the case that a statement E which is meaningful in Γ becomes meaningless in $\Gamma(x \in X)$. This phenomenon is analogous to the well known "variable capture".

Since $\mathcal{J}_{x\in X}$ is a canonical inclusion, any statement E is identical to $\mathcal{J}_{x\in X}(E)$ from a syntactic point of view. However, it is in many places very important to still see this operator $\mathcal{J}_{x\in X}$. This is why, instead of writing $\mathcal{J}_{x\in X}(E)$ as E, we write it in grey : $\mathcal{J}_{x\in X}(E)$.

Now, if a is *any* expression representing an element of X in the context Γ, and if $E \in \mathcal{E}_{\Gamma(x\in X)}$, the process of replacing all (free) occurrences of x in E by the expression a produces a statement interpretable (i.e. meaningful) in the context Γ, in other words, an element of $\mathcal{E}_\Gamma$, that we denote by $E[a/x]$ (read "E where a replaces x"). Hence, we have a map $[a/x] : \mathcal{E}_{\Gamma(x\in X)} \to \mathcal{E}_\Gamma$ (for each such a), and it is clear that this map is a retraction for $\mathcal{J}_{x\in X}$ (i.e. that $[a/x] \circ \mathcal{J}_{x\in X} = 1_{\mathcal{E}_\Gamma}$). Furthermore, this map is increasing (in other words, if $E \leq_{\Gamma(x\in X)} F$, then $E[a/x] \leq_\Gamma F[a/x]$) because the replacement can also be performed within proofs.([13])

Lawvere defines the quantifiers by postulating that :([14])([15])

$$\exists_{x\in X} \dashv \mathcal{J}_{x\in X} \dashv \forall_{x\in X}$$

Let us check that our "usual" quantifiers have this property, so that they are actually equivalent to Lawvere's quantifiers. In the case of the universal quantifier, Lawvere's definition writes :

$$\mathcal{J}_{x\in X}(E) \leq_{\Gamma(x\in X)} F \quad \text{iff} \quad E \leq_\Gamma \forall_{x\in X}\ F$$

What this says is that in order to prove $\forall_{x\in X}\ F$ under the hypothesis E, we can just declare

[13] We do not give any precise definition of the replacement. We assume only that it is a retraction for $\mathcal{J}_{x\in X}$ and that it is increasing. These two properties are of course easy consequences of the "usual" definition of the replacement, and of the definition of proofs (that we do not explicitly need here).

Important warning : The inclusion $\mathcal{J}_{x\in X} : \mathcal{E}_\Gamma \to \mathcal{E}_{\Gamma(x\in T)}$ allows to identify $\mathcal{E}_\Gamma$ to a subset of $\mathcal{E}_{\Gamma(x\in X)}$, but *not as a preordered subset*. Indeed, we can have $\mathcal{J}_{x\in X}(E) \leq_{\Gamma(x\in X)} \mathcal{J}_{x\in X}(F)$, and not have $E \leq_\Gamma F$, because the fact of declaring x in X can allow to prove things which are not provable in the context Γ. Think for example of what happens if you declare an element in the empty set.

[14] Maybe the word "defines" needs an explanation. The reality is that such adjunctions can be used to define the deductibility relation itself, and this provides a meaning for the logical connectives as a by-product. See [10] Chapitre 1, for a detailed explanation of this fact. Lawvere did not actually give the definition exactly in this form. His "definition" (or more accurately, his characterization) looks much more in the form of a pair of internal (in the sens of the "internal logic" of topos theory) adjoints to the "inverse image" arrow $\mathcal{P}(Y) \to \mathcal{P}(X)$ (for a given arrow $X \to Y$), where $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are seen as "internally ordered objects". There is also an "external" version of this using **Sub** (the subobject functor) instead of $\mathcal{P}$ (the power object functor). In some sens, Lawvere's actual definition uses subsets where we are using statements, but this is of course equivalent in view of the one-to-one correspondance between subobjects (or subsets) and characteristic arrows. See Lawvere [6].

[15] Writing the declaration after a quantifier as subparts is not usual practice. However, we want to stress the fact that the declaration $x \in X$ plays the role of a *parameter* for the function $E \mapsto \forall_{x\in X}\ E$. This is also coherent with the notation $\mathcal{J}_{x\in X}$.

x in X and prove F under this same hypothesis E (and that the converse is also true). This is clearly what we do everyday.

The co-unit of the adjunction writes :

$$\mathcal{J}_{x\in X}(\forall_{x\in X}\ F) \leq_{\Gamma(x\in X)} F$$

which is the most general case of particularization of a universally quantified statement. It says that if $\forall_{x\in X}\ F$ is true, then F is true, provided of course that we interpret this in the context $\Gamma(x \in X)$ (otherwise, F would anyway be meaningless). Applying the replacement map $[a/x]$ to both sides, we get :

$$\forall_{x\in X}\ F \leq_\Gamma F[a/x]$$

which is the "usual" particularization principle.

A symmetric analysis can be done for the existential quantifier. Precisely, the adjunction writes :

$$\exists_{x\in X}\ E \leq_\Gamma F \quad \text{iff} \quad E \leq_{\Gamma(x\in X)} \mathcal{J}_{x\in X}(F)$$

This shows how to use an existence hypothesis. Indeed, in order to prove F under the hypothesis $\exists_{x\in X}\ E$, we can first declare $x \in X$, and then prove F under the hypothesis E. This is again what everybody does instinctively. Applying the replacement $[a/x]$ to the unit of the adjunction yields :

$$E[a/x] \leq_\Gamma \exists_{x\in X}\ E$$

which is the usual "exhibition principle" used for proving an existence, which justs says that if you already know an element a in X satisfying property E, you have proved $\exists_{x\in X}\ E$.

3 Dependent conjunction and implication

We now arrive at the heart of our subject. Contexts are constructed not only via declarations, but also via "assumptions". For example, we can find the following sentence within a mathematical text : "Let x be a real number, and assume that $x > 0$.", i.e. a declaration followed by an assumption. Obviously, both are modifying the context in the intuitive sense, simply because they both provide new means for proving.

Let's use the notation $\zeta \vdash E$ (read "ζ proves E") for a generic assumption, where E is a statement, and ζ the "name" of the assumption. As previously with declarations, we consider contexts of the form $\Gamma(\zeta \vdash E)$ obtained by enriching Γ by the "assumption" $\zeta \vdash E$,([16]) and we have a canonical inclusion $\mathcal{J}_{\zeta\vdash E} : \mathcal{E}_\Gamma \to \mathcal{E}_{\Gamma(\zeta\vdash E)}$.([17])

[16] Declarations and assumptions can be mixed in a single context.

[17] In general, assumptions are anonymous : "Assume E.", but can also be named such as in : "Assume E

Imitating Lawvere, we introduce, for any statement E, two "declarative operators",

$$F \mapsto (\zeta \vdash E) \wedge F \quad \text{and} \quad F \mapsto (\zeta \vdash E) \Rightarrow F$$

which are actually two maps from $\mathcal{E}_{\Gamma(\zeta\vdash E)}$ to $\mathcal{E}_\Gamma$, by postulating :

$$(\zeta \vdash E) \wedge \dashv \mathcal{J}_{\zeta\vdash E} \dashv (\zeta \vdash E) \Rightarrow$$

Before proving anything from this definition, we must capture the fact that E is true in the context $\Gamma(\zeta \vdash E)$ (and even a little more than this). Nothing up to here can entail this fact, because there is *a priori* no link between the deductibility relation and the concept represented by the sign $\vdash$, which actually has no precise meaning up to now, and is mainly just a syntactic gadget. In other words, we must state a principle that establishes the link between "deductibility" and "assumption". We propose the following *special rule* :

$$E \wedge F \leq_\Gamma G \quad \text{iff} \quad \mathcal{J}_{\zeta\vdash E}(F) \leq_{\Gamma(\zeta\vdash E)} \mathcal{J}_{\zeta\vdash E}(G)$$

(where $\wedge$ is ordinary conjunction) which says that it is the same thing to deduce G from $E \wedge F$ or to deduce G from F alone after having assumed E. It is actually the formalisation of the difference refered to in footnote 10.([18])

Now, let us examine the consequences of the above definition. Concerning the right adjoint (the "dependent implication") we have the equivalence :

$$\mathcal{J}_{\zeta\vdash E}(F) \leq_{\Gamma(\zeta\vdash E)} G \quad \text{iff} \quad F \leq_\Gamma (\zeta \vdash E) \Rightarrow G$$

This means that proving the implication $(\zeta \vdash E) \Rightarrow G$ under the hypothesis F is the same as proving G under the same hypothesis F, but after having assumed E. This is clearly the same thing as the usual "auxiliary hypothesis method",([19]) and this just shows that this method is compatible with the dependency of G on the truth of E.

(ζ).", where ζ is a symbol (a name, but in practice, a number is generally used) by which we can later refer to this assumption. The reason why assumptions are generally anonymous is a consequence of a fundamental principle of mathematics, called "proof-irrelevance", that we shall not discuss here. For our explanations, we need to be explicit, hence the notation $\zeta \vdash E$. Furthermore, if p is any proof of E in the context Γ, the replacement operation $[p/\zeta] : \mathcal{E}_{\Gamma(\zeta\vdash E)} \to \mathcal{E}_\Gamma$ is a retraction for $\mathcal{J}_{\zeta\vdash E}$ and is increasing. In this article, we do not give any precise definition of the notion of proof. This is not necessary for our purpose, and we will actually not use this kind of replacement.

[18] A similar rule exists in Gentzen's work (left $\wedge$-rule), which establishes the link between the conjunction and the coma in the left members of sequents. This coma may be considered as an external version of the conjunction, and indeed, concatanation of assumptions in a context is also a kind of "external" conjunction. From this special rule, we can derive the rule : $H \leq_{\Gamma(\zeta\vdash E)} \mathcal{J}_{\zeta\vdash E}(E)$ for any statement E meaningful in Γ and any statement H meaningful in $\Gamma(\zeta \vdash E)$. Indeed, it is enough to replace F by $\top$ (a greatest element in $\mathcal{E}_\Gamma$), and G by E in the special rule, and to use the facts that $E \wedge \top \simeq E$ and that $H \leq_{\Gamma(\zeta\vdash E)} \top \simeq \mathcal{J}_{\zeta\vdash E}(\top)$, because indeed, $\mathcal{J}_{\zeta\vdash E}$, as a right adjoint, preserves greatest elements which are the greatest lower bounds of the empty subset. This derived rule just says that if you assume E, then E *becomes* true.

[19] And here, we are much closer to the usual meaning of this method than in the introduction. See footnote 10.

Notice that the co-unit of this adjunction says :

$$\mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \Rightarrow G) \leq_{\Gamma(\zeta \vdash E)} G$$

which means that if you have the hypothesis $(\zeta \vdash E) \Rightarrow G$ in a context where E is assumed, you can deduce G. This is the most general dependent version of "modus ponens". There is no hope to have exactly the form $(E \Rightarrow G) \wedge E \leq G$ because the left hand side should be interpretable in a context not declaring ζ, where G is not meaningful (because it depends on ζ).([20])

Concerning the left adjoint ("dependent conjunction") we have the equivalence :

$$(\zeta \vdash E) \wedge F \leq_{\Gamma} G \quad \text{iff} \quad F \leq_{\Gamma(\zeta \vdash E)} \mathcal{J}_{\zeta \vdash E}(G)$$

which means that in order to use the hypothesis $(\zeta \vdash E) \wedge F$ in order to prove G, we can just use only the hypothesis F, but after having assumed E. This is again indeed what we do daily in a natural and instinctive way.([21])

The unit of this adjunction says :

$$F \leq_{\Gamma(\zeta \vdash E)} \mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \wedge F)$$

which means that $(\zeta \vdash E) \wedge F$ can be deduced from F, provided that E is assumed, which is again an everyday proof method (actually, the *canonical* way for proving a dependent conjunction).

Notice that the non dependent counterpart of this last fact is just the fact that in order to prove $E \wedge F$ under some hypothesis, we can prove E and then F under the same hypothesis. In the dependent situation this works similarly except that E must be proved first, because the fact that E is true is necessary to make F meaningful.

[20] However, if we have a proof p of E in the context Γ, applying $[p/\zeta]$ to both sides, we get $(\zeta \vdash E) \Rightarrow G \leq_{\Gamma} G[p/\zeta]$, which looks like another variation on modus ponens. We also have the following inequality suggested to us by Paul-André Melliès :

$$\mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \wedge \mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \Rightarrow G)) \leq_{\Gamma(\zeta \vdash E)} G$$

Indeed, we first have $(\zeta \vdash E) \wedge \mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \Rightarrow G) \leq_{\Gamma} (\zeta \vdash E) \Rightarrow G$, by the co-unit of the adjunction $(\zeta \vdash E) \wedge \dashv \mathcal{J}_{\zeta \vdash E}$, and since $\mathcal{J}_{\zeta \vdash E}$ is increasing, we have :

$$\mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \wedge \mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \Rightarrow G)) \leq_{\Gamma(\zeta \vdash E)} \mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \Rightarrow G)$$

Now, we also have $\mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \Rightarrow G) \leq_{\Gamma(\zeta \vdash E)} G$, by the co-unit of the other adjunction. Actually, this is a particular case of the fact that if F, G and H are functors such that $F \dashv G \dashv H$, then we have the natural transformation $\varepsilon \circ G\varepsilon H$ from $GFGH$ to the identity functor (where the two ε are the co-units of the two adjunctions).

[21] Notice the similarity between this equivalence and the special rule. In some sense, the special rule says that the ordinary conjunction also has this property.

Now, the co-unit of the adjunction says:

$$(\zeta \vdash E) \wedge \mathcal{J}_{\zeta \vdash E}(F) \leq_\Gamma F$$

which is analogous to $E \wedge F \leq_\Gamma F$ for ordinary conjunction. However, in the above inequality F cannot depend on ζ since it is meaningful in Γ. Removing this restriction on dependency is more problematic and discussed in another section below.

We would also like to have the analog of $E \wedge F \leq_\Gamma E$, i.e. the inequality $(\zeta \vdash E) \wedge F \leq_\Gamma E$. It is equivalent to $F \leq_{\Gamma(\zeta \vdash E)} \mathcal{J}_{\zeta \vdash E}(E)$, which is a consequence of the special rule as explained in footnote 18.

4 Comparison with ordinary conjunction and implication

A natural question is that of the equivalence of $(\zeta \vdash E) \wedge \mathcal{J}_{\zeta \vdash E}(F)$ and ordinary conjunction $E \wedge F$, and similarly for implication. Here, F is meaningful in the same context as E, so that $\mathcal{J}_{\zeta \vdash E}(F)$ does not "actually depend" on the truth of E.([22])

In order to avoid any confusion, we recall that a dependent conjunction is always denoted $(\zeta \vdash E) \wedge F$ even if F does not actually depend on the truth of E (i.e., even if F is in the image of $\mathcal{J}_{\zeta \vdash E}$). On the contrary, of course, ordinary conjunction is denoted $E \wedge F$. The same rule applies to the two kinds of implication.

For conjunction, we have successively :

$$\begin{array}{rlll} E \wedge F & \leq_\Gamma & E \wedge F & \text{reflexivity} \\ \mathcal{J}_{\zeta \vdash E}(F) & \leq_{\Gamma(\zeta \vdash E)} & \mathcal{J}_{\zeta \vdash E}(E \wedge F) & \text{(special rule)} \\ (\zeta \vdash E) \wedge \mathcal{J}_{\zeta \vdash E}(F) & \leq_\Gamma & E \wedge F & ((\zeta \vdash E) \wedge \dashv \mathcal{J}_{\zeta \vdash E}) \end{array}$$

For the converse inequality, we have :

$$\begin{array}{rlll} \mathcal{J}_{\zeta \vdash E}(F) & \leq_{\Gamma(\zeta \vdash E)} & \mathcal{J}_{\zeta \vdash E}((\zeta \vdash E) \wedge \mathcal{J}_{\zeta \vdash E}(F))) & \text{(unit of the adjunction)} \\ E \wedge F & \leq_\Gamma & (\zeta \vdash E) \wedge \mathcal{J}_{\zeta \vdash E}(F) & \text{(special rule)} \end{array}$$

For implication, we have successively :

$$\begin{array}{rlll} (E \Rightarrow F) \wedge E & \leq_\Gamma & F & \text{(modus ponens)} \\ \mathcal{J}_{\zeta \vdash E}(E \Rightarrow F) & \leq_{\Gamma(\zeta \vdash E)} & \mathcal{J}_{\zeta \vdash E}(F) & \text{(special rule)} \\ E \Rightarrow F & \leq_\Gamma & (\zeta \vdash E) \Rightarrow \mathcal{J}_{\zeta \vdash E}(F) & (\mathcal{J}_{\zeta \vdash E} \dashv (\zeta \vdash E) \Rightarrow) \end{array}$$

[22] Be careful not to confuse the notion of "non-dependent" which applies to ordinary conjunction and implication, and the notion of "not actually dependent", which applies to dependent conjunctions and implications of the forms $(\zeta \vdash E) \wedge \mathcal{J}_{\zeta \vdash E}(F)$ and $(\zeta \vdash E) \Rightarrow \mathcal{J}_{\zeta \vdash E}(F)$.

and for the converse inequality :

$$\begin{array}{rlll} \mathcal{J}_{\zeta\vdash E}((\zeta \vdash E) \Rightarrow \mathcal{J}_{\zeta\vdash E}(F)) & \leq_{\Gamma(\zeta\vdash E)} & \mathcal{J}_{\zeta\vdash E}(F) & \text{(co-unit of the adjunction)} \\ ((\zeta \vdash E) \Rightarrow \mathcal{J}_{\zeta\vdash E}(F)) \wedge E & \leq_{\Gamma} & F & \text{(special rule)} \\ (\zeta \vdash E) \Rightarrow \mathcal{J}_{\zeta\vdash E}(F) & \leq_{\Gamma} & E \Rightarrow F & (A \mapsto A \wedge E \dashv F \mapsto E \Rightarrow F) \end{array}$$

Hence, the dependent conjunction and implication are equivalent to ordinary conjunction and implication when their second operand does not "actually depend" on the truth of the first one. This is of course an important and reassuring fact.

Possibly surprising may be the fact that usual conjunction comes as a right adjoint (and consequently is a "multiplicative" connective), whereas dependent conjunction comes as a left adjoint (and hence is an "additive" connective). In other words, the meaning of the usual conjunction is defined in terms of its behaviour when it is in the position of the conclusion, whereas the meaning of dependent conjunction is defined in terms of its behaviour when it is in the position of the hypothesis, as we saw above. A fact which can demystify this point for the reader is the similar fact that an indexed disjoint union of sets, such as

$$\coprod_{x\in X} Y$$

(a notion of an additive nature, which can actually be defined through the use of a left adjoint) can be identified with the cartesian product $X \times Y$ (a notion of a multiplicative nature) when Y does not actually depend on x, which is after all just a more elaborate example than $2 + 2 + 2 = 3 \times 2$.

However, we can temper the mysterious aspect of this by remarking that ordinary conjunction as a right adjoint is a binary operation, whereas dependant conjunction is in no way a binary operation. It looks much like a declarative operation, i.e. it constructs expressions containing a declaration and a "body" which is the scope of this declaration. There are many such declarative operations in mathematics, for example :

$$\forall_{x\in X}\ E \qquad \{x \in X \mid E\} \qquad (x \in X) \mapsto E \qquad \coprod_{x\in X} E \qquad \text{etc}\ldots$$

(where E is the body). So, the dependent conjunction is a quite different operation from ordinary conjunction. In a conclusion, we may be troubled by the fact that the two conjunctions have the same name (which is nevertheless justified by what we proved above).

The fact that conjunction should preferably be considered as a left adjoint (or an additive connective) rather than a right adjoint (or a multiplicative connective) could have been (but actually was not) suggested in the first half of the twentieth century. Indeed, the Brouwer-Heyting-Kolmogorov interpretation, which is a definition of the meaning of the logical connectives in terms of proofs, says that :([23])

[23]For a discussion of this interpertation stressing the fact that it hides a confusion between the signifier and the signified, see [10], pages 25-26. This confusion disappears if one replaces "proof" by what proofs represent.

- a proof of $E \wedge F$ is a *pair* (p, q) where p is a proof of E and q a proof of F,
- a proof of $E \vee F$ is a *pair* (i, p) where i is either 0 or 1, and p a proof of E if $i = 0$ and a proof of F if $i = 1$,
- a proof of $E \Rightarrow F$ is a *method*([24]) for producing a proof of F from a proof of E,
- a proof of $\forall_{x \in X}\ E$ is a *method* for producing a proof of E from any x in X,
- a proof of $\exists_{x \in X}\ E$ is a *pair* (x, p) where x is an element of X, and p a proof of E.

As one can see, additive connectives are those for which a proof is a pair, whereas multiplicative connectives are those for which a proof is a method, *provided that the conjunction is considered as an additive connective*. However, it should be remarked that pairs which are proofs of $E \vee F$ and $\exists_{x \in X}\ E$ are clearly "dependent pairs", in the sense that the statement proved by the second component of the pair depends on the *value* of the first component, and it is certain that, despite the fact that they were necessarily aware of this dependency, the early intuitionists always considered pairs that are proofs of a conjunction as "ordinary pairs". In the work of Gentzen, there is also no mention of a dependent conjunction. The reason for this could be that for some reason, these mathematicians did not take into account in their models of reasoning the fact that the definition of a mathematical object can depend on the truth of a statement.

5 From types to sets

The definition given in section 3 of dependent conjunction and implication provides a clear explanation of another phenomenon that we now consider.

The authors of this article were working on a compiler for a typed system similar to those familiar to topos theorists, such as the type theory one can find in Lambek and Scott [5], but with a type constructor W (taking a statement as its unique operand) with the meaning that data of type $W(E)$ are "warrantors" of E, i.e. objects that "warrant" the truth of E.([25]) At the "low level" of this system, the declaration following a quantifier has the form $x : T$ where T is a type (not a set !), and where the symbol : plays a role similar to $\in$, but means "is of type" instead of "belongs to". In this system, sets are defined as data whose type has the form $\mathcal{P}(T)$, where the type constructor $\mathcal{P}$ is of course a syntactic version of the power object functor of topos theory. More precisely, a datum of type $\mathcal{P}(T)$ is called a "set hosted by T". We also say that T is the "host" of X.

At the "high level" of the system, types are invisible and only sets are manipulated. In particular, the quantifiers have the form $\forall_{x \in X}\ E$ and $\exists_{x \in X}\ E$ where X is a set (not a type !).

[24] By "method", you can understand "algorithm".
[25] More on this notion in section 6.

The compiler of the system must translate this high level formulation into a low level one. The formulas for this transformation are :

$$\begin{array}{rcl} \forall_{x \in X}\ E & := & \forall_{x:T}\ (x \in X \Rightarrow E) \\ \exists_{x \in X}\ E & := & \exists_{x:T}\ (x \in X \wedge E) \end{array}$$

where T is the host of X, and where $x \in X$ in the right hand members is not a declaration but a statement (whereas it is a "high level" declaration in the left hand members).

Why these definitions should be written like this is intuitively obvious. However, it is not so easy to give an explanation of why they *must* be written like this. This can be done as follows.

First of all, we need a definition for the high level declaration $x \in X$. It is defined as the low level declaration $x : T$ (where T is the host of X), accompanied by the assumption $x \in X$.([26]) In other words, the high level context enrichment by $(x \in X)$ must be translated into the low level "two step" context enrichment as $(x : T)(\zeta \vdash x \in X)$.

Now, since the adjoint of a composition is the (reversed) composition of the adjoints, we have :([27])

$$\exists_{x:T} \circ (\zeta \vdash x \in X) \wedge \dashv \mathcal{J}_{\zeta \vdash x \in X} \circ \mathcal{J}_{x:T} \dashv \forall_{x:T} \circ (\zeta \vdash x \in X) \Rightarrow$$

which is just the wanted explanation.

Notice that this explanation is made possible by the fact (among other things) that dependent conjunction is defined as a *left* adjoint. Also notice that in the expressions $\forall_{x:T}\ (x \in X \Rightarrow E)$ and $\exists_{x:T}\ (x \in X \wedge E)$, the implication and the conjunction are obviously dependent, since E can be meaningless if x does not belong to X.

We can apply the same method to similar situations. We give just one example. Denote by $\mathcal{T}(T)_\Gamma$ the meta-set of all terms of type T meaningful in the context Γ.([28]) The meta-set $\mathcal{T}(\mathcal{P}(T))_\Gamma$ of all expressions representing sets hosted by T is preordered by the relation of inclusion, denoted $\subset$.([29]) We use the more precise notation $\subset_\Gamma$ in order to stress the fact that the inclusion is valid in the context Γ (which entails that both operands of $\subset_\Gamma$ are meaningful in the context Γ). We have the two "low level" maps :

$$\mathcal{E}_{\Gamma(x:T)} \underset{A \mapsto x \in A}{\overset{E \mapsto \{x:T \ \mid \ E\}}{\rightleftarrows}} \mathcal{T}(\mathcal{P}(T))_\Gamma$$

[26] This is just because X is to be considered intuitively as a "part" of T.

[27] Recall that the notations $(\zeta \vdash x \in X) \wedge$ and $(\zeta \vdash x \in X) \Rightarrow$ are abbreviations for $E \mapsto (\zeta \vdash x \in X) \wedge E$ and $E \mapsto (\zeta \vdash x \in X) \Rightarrow E$.

[28] $\mathcal{E}_\Gamma$ could be considered as the special case $\mathcal{T}(\Omega)_\Gamma$, where the object Ω is the subobject classifier.

[29] Indeed, the sets hosted by T are ordered by inclusion, whereas the expressions which represent them are only preordered by inclusion, since several distinct expressions can represent the same set. In logic, we should never confuse a signifier with the corresponding signified, as we were taught by Gottlob Frege [2].

and similarly, two "high level" maps by replacing $x : T$ by the declaration $x \in X$, and $\mathcal{P}(T)$ by $\mathcal{P}(X)$. Notice that $E \mapsto \{x : T \mid E\}$ is left adjoint to $A \mapsto x \in A$, since :

$$\{x : T \mid E\} \subset_\Gamma A \quad \text{iff} \quad E \leq_{\Gamma(x:T)} x \in A$$

and similarly for the high level maps. Now, we have :

$$(E \mapsto \{x : T \mid E\}) \circ ((\zeta \vdash x \in X) \wedge) \dashv \mathcal{J}_{\zeta \vdash x \in X} \circ (A \mapsto x \in A)$$

again by the composition of adjoints. Since, $\mathcal{J}_{\zeta \vdash x \in X}(x \in A)$ can also be written $x \in A$, we see that the high level comprehension $\{x \in X \mid E\}$ must be translated at the low level into $\{x : T \mid x \in X \wedge E\}$.

The reader can check that $E \mapsto \{x : T \mid E\}$ is also right adjoint to $A \mapsto x \in A$ at the low level, but that the similar fact is not true at the high level. The reason is that types have no subtype whereas sets have subsets. Consequently, the argument does (hopefully) not apply for proving that the translation of $\{x \in X \mid E\}$ could also be $\{x : T \mid x \in X \Rightarrow E\}$.

6 How an invisible variable can occur free in an expression

The declaration of ζ in $(\zeta \vdash E) \wedge F$ and in $(\zeta \vdash E) \Rightarrow F$ would be useless if ζ had no free occurrence (should it be hidden) in F. Of course, we know that in usual mathematics, this ζ is not explicitly declared and not explicitly written in the expression F (which is why the phenomenon is somewhat mysterious).

Returning to our example $A \neq \emptyset \wedge \inf(A) = 0$ in the introduction, we want to write it as follows :

$$(\zeta \vdash A \neq \emptyset) \wedge \inf[\zeta](A) = 0$$

in order to make everything explicit, and in particular the fact that inf *needs* ζ. The purpose of this section is to explain how a free occurrence of ζ can be present (even if invisible) in the statement $\inf(A) = 0$.

In everyday mathematics, there is a principle that we apply without being in general conscious of the fact that it is one of the key bricks of the system. This is the "principle of description", which (informally) says that if you have proved the *existence* and *uniqueness* of a mathematical object, then this object is well defined. In topos theory, this principle is a theorem that can be stated as follows. Given two objects X and Y in a topos, and an arrow $\varphi : X \times Y \to \Omega$,([30]) such that the statement of the internal language $\forall_{x \in X}\ \exists!_{y \in Y}\ \varphi(x, y)$([31])

[30]Recall that Ω is the subobject classifier, which plays the role of an "object of truth values", similar to the usual set of booleans $\{\text{true}, \text{false}\}$, but possibly much more complicated than the booleans (and which is not necessarily a set).

[31]Where as usual, $\exists!$ means "there exists a unique".

is true (in the empty context), then there is one and only one arrow $f : X \to Y$ such that the statement of the internal language $\forall_{x \in X}\ \varphi(x, f(x))$ is true.([32]) The importance of the role played in mathematics by this principle is discussed in [9].

If we want to make everything explicit (so that mathematics look like a programming language), we introduce a so-called "description operator", taking as unique operand a proof([33]) p of a statement of the form $\exists!_{x \in X}\ E$, and producing an element of X (satisfying the statement E). This operator exists in every topos.([34])

Now, it is clear that the definition of the greatest lower bound of a nonempty subset of $\mathbb{N}$ looks like this : "*the unique integer such that ...*", in other word, it uses the principle of description, or, if we want everything to be explicit, the description operator. This is how the variable ζ has a free occurrence in $\inf(A)$, since the expression defining $\inf(A)$ necessarily has a subterm of the form $\delta(p)$ (where δ is the description operator), where p is a proof of a statement of the form $\exists!_{x:T}\ E$, and has at least one free occurrence of ζ, since it must use ζ as an hypothesis.

Now the question is: "why is such a variable invisible" ? Before answering this question, we need to state the fact that the description operator is the *only mecanism* by which a mathematical object can be constructed from a proof. This is justified by the axiomatization of the structure of elementary topos given in [9]. Notice that the axiom of choice does not construct any mathematical object from a proof since it just states the existence of a choice function but does not provide any expression representing this function.

The description operator δ maps a proof of a statement of the form $\exists!_{x \in X}\ E$ onto the unique x in X such that E, from which it results that if p and q are two proofs of $\exists!_{x \in X}\ E$, $\delta(p)$ and $\delta(q)$ are *both the unique x in X such that E*, so that necessarily, we have $\delta(p) = \delta(q)$, and this *independently of proof-irrelevance*, since it just results from the uniqueness part in the statement $\exists!_{x \in X}\ E$. In other words, $\delta(p) = \delta(q)$ holds even if $p \neq q$.

As a consequence, since any (free) occurrence of ζ in the statement F must live within a proof which is itself within a δ, we see that F is *semantically independent* of ζ. In other words, replacing ζ in F by any proof of E, if it can actually change the (completely explicitely written) expression F, cannot change its meaning, since it states the *same* properties about the *same* mathematical objects.

Here, we see that the dependency of F on ζ in $(\zeta \vdash E) \wedge F$ and $(\zeta \vdash E) \Rightarrow F$ is somehow a *weak dependency*, since F does not depend on which precise value ζ may have, but only on the fact that ζ exists. This is nevertheless a dependency.

[32]It is easier to see this theorem as a version of the informal explanation above if you consider that X represents a context (instead of the domain of a function), or alternatively by putting $X = \mathbf{1}$.

[33]Or more accurately a warrantor as explained in section 5.

[34]It is essentially the arrow denoted $\sharp$ in [9].

7 More wanted properties

For ordinary conjunction, we have $E \wedge F \leq_\Gamma F$. Remember that we did not prove a similar fact for the dependent conjunction.

If E and F are two statements in the same context Γ, we have $E \wedge F \leq_\Gamma E \Rightarrow F$. Indeed, by definition of $\Rightarrow$, this is equivalent to $(E \wedge F) \wedge E \leq_\Gamma F$, which is a consequence of $(E \wedge F) \wedge E \leq_\Gamma F \wedge E \leq_\Gamma F$.

A similar *dependent* inequality would write $(\zeta \vdash E) \wedge F \leq_\Gamma (\zeta \vdash E) \Rightarrow F$, and seems to be true since under the hypothesis $(\zeta \vdash E) \wedge F$, we intuitively have that E should be true (so that F is well defined) and that F should also be true, from which it should result that $(\zeta \vdash E) \Rightarrow F$ should be true. At least, anyone would accept this reasonning even in the dependent case.

It is easy to see that such an inequality cannot be a consequence of the definitions of $(\zeta \vdash E) \wedge$ and $(\zeta \vdash E) \Rightarrow$ by adjunctions. Indeed, if it were the case, we would also have $\exists_{x \in X} E \leq_\Gamma \forall_{x \in X} E$, which is clearly false.([35])

Now, consider the statement $(\zeta \vdash E) \wedge F$. A proof of this statement should be a pair (p, q), where p is a proof of E and q a proof of $F[p/\zeta]$. Because ζ is a proof of E in the context $\Gamma(\zeta \vdash E)$, the two statements F and $\mathscr{J}_{\zeta \vdash E}(F[p/\zeta])$ (in the context $\Gamma(\zeta \vdash E)$) are *the same statement* in the sense that they differ only by the fact that $\mathscr{J}_{\zeta \vdash E}(F[p/\zeta])$ is obtained from F by replacing a term in F by an equal term, since these terms differ only by a proof necessarily contained within a description operator. Consequently, $\mathscr{J}_{\zeta \vdash E}(q)$ must be a proof of F in the context $\Gamma(\zeta \vdash E)$, so that $(\zeta \vdash E) \mapsto \mathscr{J}_{\zeta \vdash E}(q)$ must be a proof of $(\zeta \vdash E) \Rightarrow F$ in the context Γ.

In other words, the inequality $(\zeta \vdash E) \wedge F \leq_\Gamma (\zeta \vdash E) \Rightarrow F$ is a consequence of the very definition of the description operator, but this does not mean that it is provable from our presentation of connectives by way of adjunctions, even in the presence of the special rule. Indeed, this could become false if we introduce another mecanism than δ for making terms depend on proofs. The fact that δ is *the only way* of making terms and statements depend on proofs is of course essential here.

Now, because of the adjunctions, we also have the two desirable inequalities $\mathscr{J}_{\zeta \vdash E}((\zeta \vdash E) \wedge F) \leq_{\Gamma(\zeta \vdash E)} F$ and $F \leq_{\Gamma(\zeta \vdash E)} \mathscr{J}_{\zeta \vdash E}((\zeta \vdash E) \Rightarrow F)$.

References

[1] **T. Coquand, G. Huet** : *Constructions : a higher order proof system for mechanizing mathematics.* EUROCAL 85, Linz, LNCS 203 (Springer) 1986.

[35] Except precisely when X has at most one element, i.e. if all elements of X are equal. This is of course an allusion to proof-irrelevance, but we recall that proof-irrelevance is not needed in this article.

[2] **G. Frege** : *Sinn und Bedeutung.* Zeitschrift für Philosophie und philosophische Kritik (100), 1892.

[3] **J. van Heijenoort (ed.)** : *From Frege to Gödel.* Harvard University Press, Cambridge, 1967.

[4] **M. Hyland, A. M. Pitts** : *The theory of constructions : categorical semantics and topos-theoretic models.* Categories in Computer Sciences and Logic. Contemporary Maths. (AMS) 1987.

[5] **J. Lambek, P. J. Scott** : *Introduction to higher order categorical logic.* Cambridge University Press, Cambridge 1986.

[6] **F. W. Lawvere** : *Adjointness in Foundations.* Reprints in Theory and Applications of Categories, No. 16, 2006, pp. 1–16.

[7] **P. Martin-Löf** : *Intuitionnistic type theory.* Studies in Proof Theory, Bibliopolis, Naples, 1984.

[8] **D. Pavlović** : *Predicates and Fibrations.* PhD Thesis, Univ. Utrecht, 1990. (`ftp://ftp.kestrel.edu/pub/papers/pavlovic/proefschrift-dusko.pdf`)

[9] **A. Prouté** : *On the Role of Description.* J. of Pure and Applied Algebra 158 (2001), 295-307.

[10] **A. Prouté** : *Introduction à la logique catégorique.* `http://www.logique.jussieu.fr/~alp/cours_2010.pdf`

[11] **A. Prouté** : `http://www.logique.jussieu.fr/~alp/luminy_05_2007.pdf`

M. Herrmann :
1. Institut Mathématique de Jussieu (IMJ), Projet Logique, Université Paris-Diderot. France
2. BioEmergences, INAF - CNRS, ISC - Paris Ile de France, Avenue de la Terrasse, 91190 Gif Sur Yvette, France – funded on France BioImaging Agence Nationale de la Recherche ANR-10-INBS-04-05
3. ISC-PIF, Paris, France

A. Prouté :
1. Institut Mathématique de Jussieu (IMJ), Projet Logique, Université Paris-Diderot. France

`matthieu.herrmann@math.univ-paris-diderot.fr`
`alp@math.univ-paris-diderot.fr`