Perfect Trees and Large Cardinals

$\kappa$ is measurable iff there is $j : V \to M$ with critical point $\kappa$

$\kappa$ is $\lambda$-hypermeasurable iff in addition $H(\lambda) \subseteq M$

$\kappa$ is $\lambda$-supercompact iff in addition $M^\lambda \subseteq M$

(Measurable $=$ $\kappa^+$-hypermeasurable $=$ $\kappa$-supercompact.)

Question: Suppose $\kappa$ is a large cardinal and $G$ is $P$-generic over $V$. Is $\kappa$ still a large cardinal in $V[G]$?
Lifting method (Silver):

Given $j : V \to M$ and $P$-generic $G$ over $V$.

Let $P^*$ be $j(P)$.

Find $P^*$-generic $G^*$ over $M$ s.t. $j[G] \subseteq G^*$.

Then $j : V \to M$ lifts to $j^* : V[G] \to M[G^*]$.

If $G^*$ belongs to $V[G]$ then $j^*$ is $V[G]$-definable, so $\kappa$ is still measurable (and maybe more) in $V[G]$. 
Singular cardinal hypothesis

SCH: The GCH holds at singular, strong limit cardinals

Prikry: Con(GCH fails at a measurable) \rightarrow Con(\text{not SCH})

Silver: Con(\kappa \text{ is } \kappa^{++}-\text{supercompact}) \rightarrow Con(\text{GCH fails at a measurable})

Easy fact: GCH fails at measurable \kappa \rightarrow GCH fails at measure-one \alpha < \kappa.

So for Silver’s theorem, must violate GCH not only at \kappa, but also below \kappa.
Silver’s strategy: Iterated Cohen forcing

Cohen(\(\alpha, \alpha^{++}\)) = \alpha^{++}\)-product of \(\alpha\)-Cohen forcing (with supports of size < \(\alpha\))

\(P_0\) is trivial
\(P_{\alpha+1} = P_\alpha * \text{Cohen}(\alpha, \alpha^{++}), \ \alpha \text{ inaccessible}\)
\(P_{\alpha+1} = P_\alpha, \ \text{otherwise}\)
Inverse limits at singular ordinals, direct limits otherwise

\(P = \text{Direct limit of } P_\alpha, \ \alpha \in \text{Ord}.\)

\(P\) preserves cofinalities and forces not GCH at each inaccessible.
Assume GCH in \( V \).
Let \( j : V \to M \) witness \( \kappa^{++} \)-supercompactness.
Let \( G \) be \( P \)-generic.
Want generic \( G^* \) for \( P^* = j(P), j[G] \subseteq G^* \).

Write \( P^* = P^* (\langle j(\kappa) \rangle) * P^* (j(\kappa)) * P^* (\rangle j(\kappa)) \).

1. (Below \( j(\kappa) \)) Easy to build generic \( G^* (\langle j(\kappa) \rangle) \) containing \( j[G(\langle \kappa \rangle)] = G(\langle \kappa \rangle) \).

2. (At \( j(\kappa) \), key step) Using supercompactness, the conditions in \( j[G(\kappa)] \subseteq P^* (j(\kappa)) \) have a common lower bound (master condition) \( p \).
   Choose \( G^* (j(\kappa)) \) to include \( p \).

3. (Above \( j(\kappa) \)) Using distributivity of \( P(\rangle \kappa) \), easy to show that \( j[G(\rangle \kappa)] \) generates a generic \( G^* (\rangle j(\kappa)) \).

So \( G^* = G^* (\langle j(\kappa) \rangle) * G^* (j(\kappa)) * G^* (\rangle j(\kappa)) \) contains \( j[G] \), as desired.
Woodin: Can replace $\kappa^{++}$-supercompactness with $\kappa^{++}$-hyperstrength in the Silver strategy.

Subtle argument:

Derived measure: Use both $j : V \to M$ and its derived measure embedding $j_0 : V \to M_0$.

Leaving the universe: Force a generic $G_0^*(j_0(\kappa))$ over $V[G]$. $\kappa$ is measurable in $V[G][G_0^*(j_0(\kappa))]$.

Generic modification: Use $G_0^*(j_0(\kappa))$ to obtain a generic $G^*(j(\kappa))$ for $P^*(j(\kappa))$, which must be modified to get the desired generic $G^*(j(\kappa))$.

A new strategy: Iterated Sacks forcing

Let $\alpha$ be inaccessible.

$\alpha$-Sacks: $\alpha$-closed, binary trees of height $\alpha$, with CUB-many splitting levels.

In the Silver strategy, replace Cohen($\alpha, \alpha^{++}$) by Sacks($\alpha, \alpha^{++}$), the $\alpha^{++}$-product of $\alpha$-Sacks (with supports of size $\alpha$).
Assume GCH in $V$.
Let $j : V \to M$ witness $\kappa^{++}$-hypermeasurability.
Let $G$ be generic for $P = \text{iterated Sacks}(\alpha, \alpha^{++})$.
Let $P^* = j(P)$.
We want a $P^*$-generic $G^*$ s.t. $j[G] \subseteq G^*$.

The construction of $G^*$ is now easy.
Do not need the derived measure, leaving the universe or generic modification.

$\alpha$-Sacks has a weak form of $\alpha^+$-closure called $\alpha$-fusion:

Write $S \leq^i T$ iff $S \leq T$ and $S$ has the same $i$-th splitting level as $T$. Then any sequence $T_0 \geq_0 T_1 \geq_1 T_2 \geq_2 \cdots$ of length $\alpha$ has a lower bound.

$\alpha$-Sacks is $\alpha$-closed and $\alpha^{++}$-cc.
$\alpha$-fusion implies that $\alpha^+$ is preserved.
If $G$ is $\alpha$-Sacks generic then $G = \{T \mid f \in [T]\}$ for some unique $f : \alpha \to 2$. We also say that $f$ is $\alpha$-Sacks generic.

**Tuning fork lemma** (F - Katie Thompson)
Suppose $j : V \to M$ with critical point $\kappa$ and $G$ is $\kappa$-Sacks generic. Then the intersection of the trees in $j[G]$ consists of exactly two $f_0, f_1 : j(\kappa) \to 2$, which agree below $\kappa$ and disagree at $\kappa$. Moreover each $f_i$ is $j(\kappa)$-Sacks generic over $M$.

Reason: The splitting levels of $j(T)$, $T \in G$, form CUB subsets $j(C)$ of $j(\kappa)$. The intersection of the $j(C)$’s is $\{\kappa\}$. (We assume that $j$ is given by an extender ultrapower.)

There is a version of the Tuning Fork Lemma for Sacks$(\kappa, \kappa^{++})$, giving:
Theorem 1. (F - Thompson) Assume GCH. Suppose $j : V \to M$ witnesses that $\kappa$ is $\kappa^{++}$-hypermeasurable and $G$ is generic for the iteration of Sacks($\alpha, \alpha^{++}$), $\alpha$ inaccessible. Then $j$ lifts to $j^* : V[G] \to M[G^*]$, witnessing the failure of GCH at the measurable cardinal $\kappa$.

Using a result of Gitik, we also get:

$\text{Con}(\rho(\kappa) = \kappa^{++}) \leftrightarrow \text{Con}(\text{GCH fails at a measurable})$
The Tree Property and Large Cardinals

\(\kappa\)-Aronszajn tree = \(\kappa\)-tree with no \(\kappa\)-branch

TP(\(\kappa\)): There is no \(\kappa\)-Aronszajn tree.

GCH holds at \(\kappa\) \(\rightarrow\) TP(\(\kappa^{++}\)) fails

Question: What is the consistency strength of TP(\(\kappa^{++}\)), \(\kappa\) measurable?

**Lemma** (F - Natasha Dobrinen) Assume GCH, \(\kappa\) is regular, \(\lambda\) is weakly compact, \(\kappa < \lambda\) and \(G\) is generic for Sacksi(t)(\(\kappa, \lambda\)) = the \(\lambda\)-iteration of \(\kappa\)-Sacks (with supports of size \(\kappa\)). Then in \(V[G]\), \(\lambda = \kappa^{++}\) and TP(\(\kappa^{++}\)) holds.

Using a version of the Tuning Fork Lemma, we get:
Theorem 2. (F - Dobrinen) Assume GCH and $j : V \to M$ witnesses that $\kappa$ is $\lambda$-hypermeasurable, where $\lambda$ is weakly compact and greater than $\kappa$. Let $G$ be generic for the iteration of Sacksit($\alpha, \lambda_\alpha$), $\alpha$ an inaccessible limit of weakly compacts, $\lambda_\alpha$ the least weakly compact above $\alpha$. Then in $V[G]$, $\kappa$ is measurable and $\text{TP}(\kappa^{++})$ holds.

The upper bound given by Theorem 2 is nearly optimal:

$$\text{Con}(\kappa \text{ is weakly compact hypermeasurable}) \rightarrow \text{Con}(\text{TP}(\kappa^{++}), \kappa \text{ measurable}) \rightarrow \text{Con}(\kappa \text{ is } < \text{ weakly compact hypermeasurable})$$
Easton’s theorem and large cardinals

Easton: Con(GCH fails at all regulars)

Question: What is the consistency strength of GCH fails at all regulars and there is a measurable cardinal?

We saw:
Con(κ++-hypermeasurable) → Con(GCH fails at a measurable)

The same proof yields:
Con(κ++-hypermeasurable) → Con(GCH fails at all regulars except at α+, α++ when α is inaccessible)

Using Sacks(α, α++) at inaccessibles and Cohen(α, α++) elsewhere, one gets:
Theorem 3. (F - Radek Honzík) Assume GCH. There is a forcing $P$ such that if $G$ is $P$-generic then GCH fails at all regulars in $V[G]$. Moreover, if $\kappa$ is $\kappa^{++}$-hypermeasurable in $V$, then $\kappa$ remains measurable in $V[G]$.

One can also replace $\kappa^{++}$-hypermeasurable by $o(\kappa) = \kappa^{++}$, the optimal hypothesis.
Global Domination

So far: Large cardinal preservation

Now: Internal consistency

φ is internally consistent iff φ holds in an inner model (assuming large cardinals).

ICon(φ) = φ is internally consistent.

Consistency result:
Con(ZFC + large cardinals) → Con(ZFC + φ)

Internal consistency result:
ICon(ZFC + large cardinals) → ICon(ZFC + φ)
Examples:

(a) (Easton) $\text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + \text{GCH fails at all regulars})$
(b) (F - Ondřejovič) $\text{ICon}(\text{ZFC} + 0\# \text{ exists}) \rightarrow \text{ICon}(\text{ZFC} + \text{GCH fails at all regulars})$

(F - Dobrinen)
(a) $\text{Con}(\text{ZFC} + \text{proper class of } \omega_1\text{-Erdős cards}) \rightarrow \text{Con}(\text{ZFC} + \text{Global costat of ground model})$
(b) $\text{ICon}(\text{ZFC} + \omega_1\text{-Erdős hyperstrong with a sufficiently large measurable above}) \rightarrow \text{ICon}(\text{ZFC} + \text{Global costat of ground model})$

(a) $\text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + \text{no } L\text{-inaccessible})$
(b) $\sim \text{ICon}(\text{ZFC} + \text{no } L\text{-inaccessible})$

*Internal consistency strength:* What large cardinals are needed to prove $\text{ICon}(\varphi)$?
An application of perfect trees to internal consistency strength:

\[ d(\kappa) = \text{dominating number for } f : \kappa \to \kappa \]

\[ \kappa < d(\kappa) \leq 2^\kappa \]

*Global Domination:* \( d(\kappa) < 2^\kappa \) for all \( \kappa \).

Cummings-Shelah: \( \text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + \text{Global Domination}) \)

Proof uses \( \text{Cohen}(\alpha, \alpha^{++}) \ast \text{Hechlerit}(\alpha, \alpha^+) \) for all regular \( \alpha \) and gives:

\[ \text{ICon}(\text{ZFC} + \kappa^+-\text{supercompact} + \text{measurable above}) \rightarrow \text{ICon}(\text{ZFC} + \text{Global Domination}) \]

Replacing \( \text{Cohen}(\alpha, \alpha^{++}) \ast \text{Hechlerit}(\alpha, \alpha^+) \) with \( \text{Sacks}(\alpha, \alpha^{++}) \) for inaccessible \( \alpha \) gives:
(F - Thompson)
ICon(ZFC + 0# exists) →
ICon(ZFC + Global Domination except at \(\alpha^+\), \(\alpha\) inaccessible)

And with Cohen(\(\alpha^+, \alpha^{+++}\)) followed by an interlacing of Hechlerit(\(\alpha^+, \alpha^{++}\)) with Sacksit(\(\alpha, \alpha^{++}\)) for inaccessible \(\alpha\), we get:

**Theorem 4.** (F - Thompson)
ICon(ZFC + 0# exists) →
ICon(ZFC + Global Domination)

**Conclusion**

For large cardinal preservation and internal consistency, Sacks is better than Cohen!