Low Distortion Embeddings of Infinite Metric Spaces

Stefan Geschke

July 9, 2007

K-embeddings

Definition

Let $K > 1$. An embedding $e : M \to N$ between metric spaces is a *K-embedding*, respectively *K-bi-Lipschitz*, if for all $x, y \in M$ with $x \neq y$ we have

$$\frac{1}{K} \leq \frac{d_N(e(x), e(y))}{d_M(x, y)} \leq K.$$

Definition

A metric space is *K-linear* if it K-embeds into the real line.

A metric space is *K-Hilbertian* if it K-embeds into a Hilbert space.

A metric space is *K-uniform* if it K-embeds into metric space where any two distinct points have the same distance.

K-embeddability and Ramsey theory

Theorem (Bourgain, Figiel, Milman)

Let $K > 1$. Then every metric space of size n has a K-Hilbertian subset of size $\Omega(\log n)$.

Compare this to the finite Ramsey Theorem:

Theorem (Ramsey)

For every coloring $c : [X]^2 \to \{0, 1\}$ of the two element subsets of an n-element set X there is a set $H \subseteq X$ of size $\Omega(\log n)$ that is homogeneous with respect to c, i.e., c is constant on $[H]^2$.

Just as the finite Ramsey Theorem, the theorem of Bourgain, Figiel and Milman has an infinite analog:

Theorem

Every infinite metric space has an infinite K-Hilbertian subset.

In fact, much more is true:

Theorem (Matoušek)

Every infinite metric space has an infinite subset that is either K-linear or K-uniform.

In the uncountable, we can prove an even stronger statement:

Theorem

Every uncountable Polish space has a non-empty perfect subset that is K-linear.

This theorem has a Ramsey-theoretic analog as well, a theorem of Galvin.

We need some definitions.

Definition

For a topological space X, $[X]^2$ carries the topology generated by all sets of the form

$$\{\{x, y\} : x \in U \wedge y \in V\},$$

where U and V are disjoint open subsets of X.

A *pair cover on* X is a finite collection $C = (K_1, \dots, K_n)$ such that

$$[X]^2 = K_1 \cup \dots \cup K_n.$$

A pair cover $C = (K_1, \ldots, K_n)$ on X is open (clopen; has the Baire property) if each K_i is open (clopen; has the Baire property).

$H \subseteq X$ is C-homogeneous of color $i \in \{1, \ldots, n\}$ if $[H]^2 \subseteq K_i$.

Theorem (Galvin)

Let X be an uncountable Polish space and suppose that $C = (K_1, \ldots, K_n)$ is a pair cover on X with the Baire property. Then X has a non-empty perfect C-homogeneous subset.

Covering by small sets

Definition

DOCA (Dual Open Coloring Axiom) is the statement that for every open pair cover $C = (K_1, \ldots, K_n)$ on a Polish space X there is a family $\mathcal{H}$ of C-homogeneous subsets of X such that $|\mathcal{H}| < 2^{\aleph_0}$ and $X = \bigcup \mathcal{H}$.

Theorem

DOCA is consistent with ZFC. In fact, DOCA follows from DOCA restricted to clopen pair covers on 2^ω, which is known to be consistent.

We have the following metric analog:

Theorem

It is consistent that for every $K > 1$ and every separable metric space X, X is the union of $< 2^{\aleph_0}$ K-linear sets.

The coverability of separable metric spaces by small families of K-linear sets is consistent with the negation of DOCA. On the other hand, at least for finite dimensional spaces the coverability by small families of K-linear sets follows from DOCA.

Theorem

Assume DOCA. Then for every $K > 1$ and every $n > 1$, $\mathbb{R}^n$ can be covered by $< 2^{\aleph_0}$ K-bi-Lipschitz copies of $\mathbb{R}$.

Compactness

We will need two compactness results related to K-embeddability.

Theorem (Matoušek)

For every $n > 1$ there is $K > 1$ such that every n-element metric space K-embeds into $\mathbb{R}$.

Theorem

Let $K > 1$. If M is a separable metric space and X a homogeneous metric space in which every closed bounded set is compact, then M K-embeds into X iff every finite subspace of M does.

K-embeddability and localization numbers

Definition

Let $n \geq 1$. A subset S of $(n+1)^\omega$ is *n-ary* if no $n+1$ elements of S pairwise split for the first time at the same coordinate.

The *localization number* $\mathfrak{l}_{n,n+1}$ is the least size of a family of n-ary sets that covers $(n+1)^\omega$.

It is easily checked that

$$2^{\aleph_0} = \mathfrak{l}_{1,2} \geq \mathfrak{l}_{2,3} \geq \mathfrak{l}_{3,4} \geq \dots$$

Theorem

Let $K > 1$ and $n \geq 2$ be such that there is some $(n+1)$-element subset of $\mathbb{R}^2$ that is not K-linear. Then for every $k > 1$ with $k < K$, at least $\mathfrak{l}_{n,n+1}$ k-linear sets are necessary to cover all of $\mathbb{R}^2$.

Theorem

Let $k > 1$ and $n \geq 2$ be such that every $(n+1)$-element subset of $\mathbb{R}^2$ is k-linear. If $K > k$, then there is a forcing extension of the set-theoretic universe where $\mathbb{R}^2$ can be covered by less than $\mathfrak{l}_{n,n+1}$ K-linear sets.

Corollary

There is an increasing sequence $(K_n)_{n\in\omega}$ of real numbers > 1 such that whenever $n < m$, there is a forcing extension of the universe where less K_m-linear sets are necessary to cover the plane than K_n-linear sets.

There are many ways to define cardinal invariants in terms of K-embeddability. One of the more interesting open questions is the following:

Problem

Given $K > k > 1$. Is it consistent that less K-linear sets are needed to cover the plane than k-linear sets?