ITERATION OF SEMIPROPER FORCING REVISITED

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Abstract. We present a method for iterating semiproper forcing which uses side conditions and is inspired by the technique recently introduced by Neeman.

Introduction

One of the most important issues in the theory of forcing is whether certain classes of nicely behaved forcing notions can be iterated while remaining in the same class. We are primarily interested in preserving certain cardinals, but often we also wish to preserve some particular objects or properties of the ground model. The theory of iterated forcing was developed largely by Shelah (see [12]), who isolated important classes of posets such as proper and semiproper forcing and developed suitable iteration techniques for them. In order to iterate proper forcing Shelah uses *countable support* iteration, initially developed by Laver [8], Baumgartner [1], and others. This theory is quite well understood and has been used in many applications, most important of which is the consistency of the Proper Forcing Axiom (PFA), obtained by Baumgartner [2] and Shelah [12]. In order to iterate semiproper forcing Shelah had to devise a new type of support, called the *revised countable support*, and show that the iteration of semiproper forcing notions using this support is also semiproper. The most spectacular application of this theory is the relative consistency of Martin's Maximum (MM), which was shown by Foreman, Magidor and Shelah [3]. However, the definition of revised countable support is quite intricate and, despite several attempts such as [10] and [4] to simplify it, it still remains somewhat of a mystery.

The starting point for our work is a new iteration technique recently introduced by Neeman [11] who used it to give an alternative proof of the consistency of PFA. Neeman's method uses finite supports for the working part together with side conditions which are finite $\in$-chains of elementary submodels of two types, countable and transitive. The side conditions also have to be closed under intersection. Neeman's primary motivation was to generalize this type of iteration in order to obtain generalizations of PFA to higher cardinals. However, even the basic properties of Neeman's technique can be used to obtain new results or give simpler and unified proofs of known theorems, see e.g. [13].

Key words and phrases. forcing, semiproper, side conditions.

Now, it is quite natural to ask if Neeman's method can be adapted to iterate more general classes of posets, such as semiproper forcing. However, when attempting to do that one encounters serious difficulties. The most important one comes from the fact that for non proper forcing the operations of taking a generic extension and intersection of two models do not commute. When attempting to resolve this difficulty we were lead to the notion of *virtual models* which turns out to be more suitable in this context. The basic idea of virtual models is that they encode the essential information relative to an initial segment of the iteration that is contained in some actual models which do not yet appear as side conditions at that stage of the iteration. The upshot of these considerations is that we now use only countable models and the operation of taking the intersection is replaced by a better behaved operation of projection. The added benefit is that we now get a true iteration, namely each stage of the iteration is a complete suborder of all the later stages, which was not the case of Neeman's iteration.

The paper is organized as follows. In §1 we introduce the notion of virtual models and establish some of their basic properties. In §2 we recall some facts about generic extensions and elementary submodels. Mostly we make definitions that will be relevant for the iteration of semiproper forcing. Finally, in §3 we present our iteration technique and show that if each iterand is semiproper then so is the resulting forcing notion. If one wishes, one can then use this method to reprove the consistency of MM or some other applications of revised countable support iteration. The hope is that this technique is quite flexible and will allow us to obtain results that cannot be proved by standard methods, in particular it may be possible to generalize it for higher cardinal versions of semiproper forcing. Finally, let us note that a related but somewhat different approach to iterating semiproper forcing was proposed by Gitik and Magidor [5]. In order to read this paper, only basic knowledge of iterated forcing is needed, such as the one presented in [6] or [7]. For all undefined terms we refer the reader to one of these two monographs.

1. Virtual models

In this section we present the collection of models we shall use as side conditions. We shall consider the language $\mathcal{L}$ obtained by adding a unary function symbol U to the standard language $\mathcal{L}_\epsilon$ of set theory. Let us say that a structure $\mathcal{A}$ of the form $(A, \in, U^{\mathcal{A}})$ is *admissible* if A is a transitive set, $U^{\mathcal{A}}$ is a function from ORD^A to A, and $\mathcal{A}$ satisfies ZFC in the expanded language $\mathcal{L}$. We shall often abuse notation and refer to the structure $(A, \in, U^{\mathcal{A}})$ simply by A. Suppose $\mathcal{A}$ is an admissible structure. If α is an ordinal in A, we let A_α denote $A \cap V_\alpha$. Finally, we let

$$E_{\mathcal{A}} = \{\alpha \in A : (A_\alpha, \in, U^{\mathcal{A}} \restriction \alpha) \prec (A, \in, U^{\mathcal{A}})\}.$$

Note that $E_{\mathcal{A}}$ is a closed, possibly empty, subset of ORD^A. For $\alpha \in A$ we let $\mathrm{next}_{\mathcal{A}}(\alpha)$ be the least ordinal in $E_{\mathcal{A}}$ above α, if such an ordinal exists. Otherwise, we leave $\mathrm{next}_{\mathcal{A}}(\alpha)$ undefined. We start with a simple technical lemma.

Lemma 1.1. *Suppose M is an elementary submodel of an admissible structure $\mathcal{A}$. Then* $\sup(E_{\mathcal{A}} \cap M) = \sup(E_{\mathcal{A}} \cap \sup(M \cap \mathrm{ORD}^A))$.

Proof. Suppose $\beta \in E_{\mathcal{A}}$ and $(M \cap \mathrm{ORD}^A) \setminus \beta$ is nonempty. Let γ be the least ordinal in $M \setminus \beta$. We show that A_γ is an elementary submodel of $\mathcal{A}$. Suppose otherwise, then by the Tarski-Vaught criterion, there is a tuple $\bar{x} \in A_\gamma$ and a formula $\varphi(y, \bar{x})$ such that $\mathcal{A} \models \exists y \varphi(y, \bar{x})$, but there is no $y \in A_\gamma$ such that $\mathcal{A} \models \exists y \varphi(y, \bar{x})$. Since $\gamma \in M$ and M is an elementary submodel of $\mathcal{A}$, there is such a tuple $\bar{x} \in A_\gamma \cap M$. Now, γ is the least ordinal in M above β, therefore $\bar{x} \in M \cap A_\beta$. Since A_β is an elementary submodel of $\mathcal{A}$, there is $y' \in A_\beta$ witnessing that $A_\beta \models \varphi(y', \bar{x})$ and so $A \models \varphi(y', \bar{x})$. Since $\beta \leq \gamma$, it follows that $y' \in A_\gamma$, a contradiction. □

Suppose M is an elementary submodel of an admissible structure $\mathcal{A}$ and X is a subset of A. Let

$$\mathrm{Hull}(M, X) = \{f(\bar{x}) : f \in M, \bar{x} \in X^{<\omega}, f \text{ is a function, and } \bar{x} \in \mathrm{dom}(f)\}.$$

Lemma 1.2. *Suppose $\mathcal{A}$ is an admissible structure, M is an elementary submodel of $\mathcal{A}$ and X is a subset of A. Let δ be* $\sup(M \cap \mathrm{ORD})$ *and suppose $X \cap A_\delta$ is nonempty. Then* $\mathrm{Hull}(M, X)$ *is the least elementary submodel of $\mathcal{A}$ containing M and $X \cap A_\delta$.*

Proof. For each $\gamma \in A$, let id_γ be the identity function on A_γ. Clearly, if $\gamma \in M$ then $\mathrm{id}_\gamma \in M$. Therefore, $X \cap A_\delta$ is a subset of $\mathrm{Hull}(M, X)$. Let $\gamma \in M$ be such that $X \cap A_\gamma$ is nonempty. For each $z \in M$, the constant function c_z defined on A_γ is in M, therefore M is a subset of $\mathrm{Hull}(M, X)$. The minimality of $\mathrm{Hull}(M, X)$ is clear from the definition. It remains to show that $\mathrm{Hull}(M, X)$ is an elementary submodel of A. We check the Tarski-Vaught criterion for $\mathrm{Hull}(M, X)$ and A. Let φ be a formula and $a_1, \ldots, a_n \in \mathrm{Hull}(M, X)$ such that $A \models \exists u \varphi(u, a_1, \ldots, a_n)$. Then we can find functions $f_1, \ldots, f_n \in M$ and tuples $\bar{x}_1, \ldots, \bar{x}_n \in X^{<\omega}$ such that $a_i = f_i(\bar{x}_i)$, for all i. If D_i is the domain of f_i, this implies that $\bar{x}_i \in D_i$. By regularity and the axiom of choice in A we can find a function g defined on $D_1 \times \ldots \times D_n$ such that for every $\bar{y}_1 \in D_1, \ldots, \bar{y}_n \in D_n$, if there is u such that $A \models \varphi(u, f_1(\bar{y}_1), \ldots, f_n(\bar{y}_n))$ then $g(\bar{y}_1, \ldots, \bar{y}_n)$ is such a u. Moreover, by elementarity of M, we may assume that $g \in M$. Let $a = g(\bar{x}_1, \ldots, \bar{x}_n)$. It follows that $a \in \mathrm{Hull}(M, X)$ and $A \models \varphi(a, a_1, \ldots, a_n)$. Therefore, $\mathrm{Hull}(M, X)$ is an elementary submodel of A. □

Now, let us fix an inaccessible cardinal κ and a function $U^{\mathcal{V}_\kappa} : \kappa \to V_\kappa$. Let $\mathcal{V}_\kappa = (V_\kappa, \in, U^{\mathcal{V}_\kappa})$. Clearly, $\mathcal{V}_\kappa$ is admissible. We shall abuse notation and write simply V_κ instead of $\mathcal{V}_\kappa$ and U instead of $U^{\mathcal{V}_\kappa}$. We shall also write E instead of $E_{\mathcal{V}_\kappa}$ and $\mathrm{next}(\alpha)$ instead of $\mathrm{next}_{\mathcal{V}_\kappa}(\alpha)$.

Definition 1.3. *Suppose $\alpha \in E$. We let $\mathscr{A}_\alpha$ denote the set of all admissible structures $\mathcal{A}$ that are elementary end extensions of $(V_\alpha, \in, U \restriction \alpha)$ and have the same cardinality as V_α.*

Note that if $A \in \mathscr{A}_\alpha$ and $\alpha \in A$ then $E_{\mathcal{A}} \cap (\alpha+1) = E \cap (\alpha+1)$. Of course, $E_{\mathcal{A}}$ may have some elements above α which are not in E. Another important point is that $\mathscr{A}_\alpha$ is uniformly definable in $\mathcal{V}_\kappa$ with parameter α. Our plan is to define an iteration $(\mathbb{P}_\alpha : \alpha \in E^*)$, where $E^* = E \cup \{\alpha+1 : \alpha \in E\}$. Once we have defined $\mathbb{P}_\alpha$, we will define $\mathbb{P}_{\alpha+1}$ to be essentially $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is the $\mathbb{P}_\alpha$-name for a semiproper forcing given by $U(\alpha)$. We will have that the initial segment of the iteration, $(\mathbb{P}_\gamma : \gamma \in E^* \cap (\alpha+1))$, is uniformly definable in V_κ with parameter α and $U \restriction \alpha$, for every $\alpha \in E$. Now, if $\alpha \in E$ and $\mathcal{A} \in \mathscr{A}_\alpha$, the same definition can be applied in $\mathcal{A}$ and we get an iteration $(\mathbb{P}^{\mathcal{A}}_\gamma : \gamma \in E_{\mathcal{A}})$. By elementarity, we will then have that $\mathbb{P}^{\mathcal{A}}_\gamma = \mathbb{P}_\gamma$, for every $\gamma \in E \cap \alpha$.

We are now ready to define the collection of models that we plan use as side conditions in our iteration.

Definition 1.4. *Suppose $\alpha \in E$. We let $\mathscr{C}_\alpha$ denote the collection of all countable submodels M of V_κ such that, if we let $A = \mathrm{Hull}(M, V_\alpha)$, then A is transitive and there exists a function $U^{\mathcal{A}}$ such that the structure $\mathcal{A} = (A, \in, U^{\mathcal{A}})$ belongs to $\mathscr{A}_\alpha$. We refer to the members of $\mathscr{C}_\alpha$ as the* α-models. *We write $\mathscr{C}_{<\alpha}$ for $\bigcup\{\mathscr{C}_\gamma : \gamma \in E \cap \alpha\}$ and $\mathscr{C}_{\leq\alpha}$ for $\mathscr{C}_{<\alpha} \cup \mathscr{C}_\alpha$. We write $\mathscr{C}_{\geq\alpha}$ for $\bigcup\{\mathscr{C}_\gamma : \gamma \in E \setminus \alpha\}$. Finally, we let $\mathscr{C}$ denote $\mathscr{C}_{<\kappa}$.*

Remark 1.5. Note that if $M \in \mathscr{C}_\alpha$ then $\sup(M \cap \mathrm{ORD}) \geq \alpha$. In general, M is not elementary in V_κ, in fact, this only happens if $M \subseteq V_\alpha$. Finally, note that we are not requiring that the function $U^{\mathcal{A}}$ be unique. However, we will have that $U^{\mathcal{A}} \restriction \alpha = U \restriction \alpha$, and that is all we care about.

We plan to use members of $\mathscr{C}_{\leq\alpha}$ as side conditions in the forcing $\mathbb{P}_\alpha$. This will guarantee that $\mathbb{P}_\alpha$ is small, i.e. has size less than $\mathrm{next}(\alpha)$. We will use α-models to control the working parts of conditions below α. Now, we need to ensure that $\mathbb{P}_\alpha$ is a complete suborder of $\mathbb{P}_\beta$, for $\alpha < \beta$. So, if M is a β-model appearing in some condition from $\mathbb{P}_\beta$, we need to find an α-model N which has the same impact as M on the iteration up to α. This motivates the following definition.

Definition 1.6. *Suppose $M, N \in \mathscr{C}$ and $\alpha \in E$. We say that M and N are α-*isomorphic *and write $M \cong_\alpha N$ if there is an isomorphism σ between* $\mathrm{Hull}(M, V_\alpha)$ *and* $\mathrm{Hull}(N, V_\alpha)$ *such that $\sigma[M] = N$. Of course, if such a σ exists, it is unique.*

Clearly, $\cong_\alpha$ is an equivalence relation, for every α. Note that if $M \in \mathscr{C}_\gamma$, for some $\gamma < \alpha$, then the only model α-isomorphic to M is M itself. Suppose $\alpha, \beta \in E$ and $\alpha \leq \beta$. It is easy to see that, if $M, N \in \mathscr{C}$ are β-isomorphic, then they are α-isomorphic. We will now see that, if $\alpha < \beta$, then for every β-model M there is a canonical representative of the $\cong_\alpha$-equivalence class of M which is an α-model.

Definition 1.7. *Suppose α and β are members of E with $\alpha \leq \beta$, and M be a β-model. Let $\overline{\mathrm{Hull}(M, V_\alpha)}$ be the transitive collapse of $\mathrm{Hull}(M, V_\alpha)$ and let π be the collapsing map. We define $M \restriction \alpha$ to be $\pi[M]$, i.e. the image of M under the collapsing map of $\mathrm{Hull}(M, V_\alpha)$.*

Note that $\overline{\mathrm{Hull}(M, V_\alpha)}$ belongs to $\mathscr{A}_\alpha$, so $M \restriction \alpha$ is an α-model which is α-isomorphic to M. Note also that if $\beta = \alpha$ then $M \restriction \alpha = M$, since $\mathrm{Hull}(M, V_\alpha)$ is already transitive. The following is straightforward.

Proposition 1.8. *Suppose $\alpha, \beta, \gamma \in E$ with $\alpha \leq \beta \leq \gamma$. Let $M \in \mathscr{C}_\gamma$. Then $(M \restriction \beta) \restriction \alpha = M \restriction \alpha$.* □

We also need to define a version of the membership relation, for every α in E.

Definition 1.9. *Suppose $M, N \in \mathscr{C}$. We let $M \in_\alpha N$ if there is $M' \in N$ such that $M' \cong_\alpha M$.*

Note that if $M \subseteq V_\alpha$ this simply means that $M \in N$. However, in general, we may have $M \in_\alpha N$ even if the rank of M is higher than the rank of N. Note that $\in_\alpha$ is transitive on members of $\mathscr{C}$. We will need the following simple fact.

Proposition 1.10. *Let $\alpha, \beta \in E$ with $\alpha \leq \beta$. Suppose M and N are β-models and $M \in_\beta N$. Then $M \restriction \alpha \in_\alpha N \restriction \alpha$.* □

2. The scaffolding

In this section we recall some elementary facts about iterated forcing and make some definitions that will be relevant for our iteration. Let us fix a transitive model $(A, \in, \ldots)$ of ZFC, possibly with some additional structure, and a forcing notion $\mathbb{Q} \in A$. Then $A^{\mathbb{Q}}$ denotes the class of all $\mathbb{Q}$-names, as defined in A. If G is an A-generic filter over $\mathbb{Q}$, we define the interpretation of $\mathbb{Q}$-names by G by $\in^*$-recursion as follows:

$$\tau_G = \{\sigma_G : \text{there is } p \in G \text{ such that } (p, \sigma) \in \tau\}.$$

The generic extension $A[G]$ is equal to $\{\tau_G : \tau \in A^{\mathbb{Q}}\}$. Fix now an elementary submodel M of A with $\mathbb{Q} \in M$. We do not assume that M belongs to A or even $A[G]$. We let

$$M[G] = \{\tau_G : \tau \text{ is a } \mathbb{Q}\text{-name and } \tau \in M\}.$$

Then $M[G]$ is an elementary submodel of $A[G]$. The following definition is non standard.

Definition 2.1. *Let $M(G)$ denote the trace of $M[G]$ on A, i.e. $M[G] \cap A$.*

By a result of Laver [9] and Woodin [14] the ground model A is definable in $A[G]$ from the parameter $\mathcal{P}^A(\lambda)$, the power set of λ as computed in A, where λ is the A cardinality of $\mathbb{Q}$. It follows that $M(G)$ is an elementary submodel of A, although, of course, it may not belong to A. Therefore it makes sense to define $M(G)[G]$. We will need the following easy facts, using the above notation.

Lemma 2.2. $M(G)[G] = M[G]$.

Proof. Since $M \subseteq M(G)$, it suffices to show $M(G)[G] \subseteq M[G]$. Let $\tau \in M$ be a $\mathbb{Q}$-name for an element of $A^{\mathbb{Q}}$. Working in M we can find a maximal antichain $\mathcal{X}$ in $\mathbb{Q}$ and, for every $q \in \mathcal{X}$, a $\mathbb{Q}$-name σ_q such that $q \Vdash \tau = \check{\sigma}_q$. By the Maximality Lemma we can find in M a single $\mathbb{Q}$-name σ such that $q \Vdash \sigma_q = \sigma$, for all $q \in A$. It follows that $(\tau_G)_G = \sigma_G$. □

Corollary 2.3. *Suppose N is another elementary submodel of A with $M \subseteq N \subseteq M(G)$. Then $N[G] = M[G]$.* □

We now generalize Definition 2.1 to a situation where we have not one forcing, but an iteration of forcing notions. Suppose $X \subseteq \mathrm{ORD} \cap A$ and $(\mathbb{P}_\xi : \xi \in X)$ is an increasing chain of posets in A such that $\mathbb{P}_\xi$ is a complete suborder of $\mathbb{P}_\eta$, for all $\xi, \eta \in X$ with $\xi < \eta$. Suppose moreover that $(\mathbb{P}_\xi : \xi \in X \cap (\alpha + 1))$ is uniformly definable in A with parameter α. Suppose $\delta \in X$ and G_δ is an A-generic filter over $\mathbb{P}_\delta$ and let G_α be $G_\delta \cap \mathbb{P}_\alpha$, for $\alpha \in X \cap (\delta + 1)$. It will be convenient to write $G_{<\alpha}$ for the union of the G_ξ, for $\xi \in X \cap \alpha$. So, if $X \cap \alpha$ has a largest element, say β, then $G_{<\alpha}$ is just G_β. Fix, as before, an elementary submodel M of A. We define, by induction on $\alpha \in X \cap (\delta + 1)$, what it means for α to be attainable from M by $G_{<\alpha}$ and we construct a model $M(G_\alpha)$. The sequence $(M(G_\alpha) : \alpha \in X \cap (\delta + 1))$ will form an increasing chain of elementary submodels of A and each of these models will contain M as a submodel. We let $M(G_{<\alpha})$ denote the union of the $M(G_\xi)$, for $\xi \in X \cap \alpha$. If α is the least point of X then $M(G_{<\alpha})$ is just M. By induction and the theorem on elementary chain of models, $M(G_{<\alpha})$ will also be an elementary submodel of A.

Definition 2.4. *Suppose $\alpha \in X \cap (\delta + 1)$. We say that α is* attainable *from M by $G_{<\alpha}$ if $\alpha \in M(G_{<\alpha})$. If this is the case we let $M[G_\alpha] = M(G_{<\alpha})[G_\alpha]$ and we let $M(G_\alpha) = M[G_\alpha] \cap A$. Otherwise we let $M(G_\alpha)$ be equal to $M(G_{<\alpha})$.*

The idea is that we start with M. If α is the least element of X we ask if $\alpha \in M$. If so, then $\mathbb{P}_\alpha \in M$ as well and we can define the model $M[G_\alpha]$. We then let $M(G_\alpha)$ be the trace of $M[G_\alpha]$ on the ground model, i.e. A. In general, we ask if α appears in the union of the previous models, i.e. $M(G_{<\alpha})$. If so, then we know that $\mathbb{P}_\alpha$ belongs to $M(G_{<\alpha})$, as well. Therefore, we can define the model $M(G_{<\alpha})[G_\alpha]$. We then let $M(G_\alpha)$ be the trace of this model on A. Otherwise we let $M(G_\alpha)$ be $M(G_{<\alpha})$. Clearly, in this way, we obtain an increasing chain of elementary submodels of A. Note also that the model $M(G_\alpha)$ only depends on M and G_α and does not depend on the future generic filters G_β, for $\beta > \alpha$.

We now define what it means for a condition to make an ordinal attainable from M. Before that we need to make some definitions. First of all, our transitive model A will belong to $\mathscr{A}_\delta$, for some $\delta \in E$. We will have a sequence of posets $(\mathbb{P}_\alpha : \alpha \in E^* \cap \delta)$ such that $\mathbb{P}_\xi$ is a complete suborder of $\mathbb{P}_\eta$, for all $\xi < \eta$. Moreover, for every $\alpha \in E^* \cap \delta$ the chain $(\mathbb{P}_\xi : \xi \in E^* \cap (\alpha + 1))$ will be uniformly definable in $(A, \in, U^{\mathcal{A}})$ with parameter α. M will be an elementary submodel of A in V. Notice that if $\alpha \in E^* \cap \delta$ then a filter G_α is V-generic over $\mathbb{P}_\alpha$ iff it is A-generic over $\mathbb{P}_\alpha$. Therefore, we can perform the construction of $M(G_\alpha)$ and $M[G_\alpha]$ in $V[G_\alpha]$. We let $M(\dot{G}_\alpha)$ denote the canonical $\mathbb{P}_\alpha$-name for $M(G_\alpha)$ and we let $M[\dot{G}_\alpha]$ denote the canonical $\mathbb{P}_\alpha$-name for $M[G_\alpha]$, provided this model makes sense, i.e. if α is attainable from M by $G_{<\alpha}$. Finally, for every condition p we let $\xi(p)$ be the least $\xi \in E^* \cap \delta$ such that $p \in \mathbb{P}_\xi$.

Definition 2.5. *Suppose p is a condition and $\alpha \leq \delta$. We say that p* makes α attainable from M *if there is $\xi \leq \min(\xi(p), \alpha)$ and $q \in \mathbb{P}_\xi$ such that $p \leq q$ and $q \Vdash_{\mathbb{P}_\xi} \alpha \in M(\dot{G}_\xi)$.*

In our situation there will be canonical projections $p \mapsto p \restriction \xi$, for $\xi \in E^* \cap \delta$. We will have that $p \leq p \restriction \xi$, for every p belonging to $\mathbb{P}_\alpha$, for some $\alpha \in E^* \cap \delta$. Moreover, if $q \in \mathbb{P}_\xi$ then $p \leq q$ iff $p \restriction \xi \leq q$, and q and p are compatible iff q and $p \restriction \xi$ are compatible. The point then is that, if p makes some $\alpha \in E^* \cap \delta$ attainable from M and G_α is V-generic over $\mathbb{P}_\alpha$ such that $p \restriction \alpha \in G_\alpha$, then we can define the model $M[G_\alpha]$.

We now review some definitions involving semiproper forcing and place them in our context. Suppose $\mathbb{Q}$ is a forcing notion and let θ be a sufficiently large regular cardinal. Let M be a countable elementary submodel of $H(\theta)$ such that $\mathbb{Q} \in M$. We say that a condition $q \in \mathbb{Q}$ is $(M, \mathbb{Q})$-*semigeneric* if:

$$q \Vdash_{\mathbb{Q}} M[\dot{G}] \cap \omega_1 = M \cap \omega_1,$$

where $\dot{G}$ is the canonical name for the V-generic filter over $\mathbb{Q}$. Of course, here we can also write $M(\dot{G})$ instead of $M[\dot{G}]$. We say that $\mathbb{Q}$ is *semiproper* if for every such M and every $p \in M \cap \mathbb{Q}$ there is $q \leq p$ which is $(M, \mathbb{Q})$-semigeneric. The important point is that semiproper forcing notions preserve ω_1, indeed, they preserve stationary subsets of ω_1. Suppose now $\mathbb{Q} \in V_\alpha$, for some $\alpha \in E$, and let M be an α-model such that $\mathbb{Q} \in M$. Note that a condition q is $(M, \mathbb{Q})$-semigeneric iff it is $(M \cap V_\alpha, \mathbb{Q})$-semigeneric, so we can reformulate semiproperness in terms of the existence of semigeneric conditions for all α-models containing $\mathbb{Q}$. We will often use without mentioning the following fact.

Proposition 2.6. *Suppose $\alpha, \beta \in E$ with $\alpha < \beta$. Let $\mathbb{Q} \in V_\alpha$ be a forcing notion and M a β-model such that $\mathbb{Q} \in M$. Let G be a V-generic filter over $\mathbb{Q}$. Then $M(G) \cong_\alpha (M \restriction \alpha)(G)$.* □

3. The iteration

We now describe our iteration. Let us fix a function $U : \kappa \to V_\kappa$. We refer to U as the bookkeeping device. For instance, if κ is supercompact U could be the Laver function for κ. Let $E^* = E \cup \{\alpha + 1 : \alpha \in E\}$. Our plan is to define an increasing sequence of semiproper posets $(\mathbb{P}_\alpha : \alpha \in E^*)$ such that

(*i*) $\mathbb{P}_\alpha$ is of size $<$ next(α) and is uniformly definable in V_κ, with parameters α and $U \restriction \alpha$, for every $\alpha \in E$,

(*ii*) if $\alpha \in E$ and $U(\alpha)$ is a $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}_\alpha$ for a semiproper forcing notion then $\mathbb{P}_{\alpha+1}$ is isomorphic to $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$,

(*iii*) if $\alpha, \beta \in E^*$ and $\alpha < \beta$ then $\mathbb{P}_\alpha$ is a complete suborder of $\mathbb{P}_\beta$,

(*iv*) if α is a limit of E and is either inaccessible or of cofinality ω_1 then $\mathbb{P}_\alpha$ is equivalent to the direct limit of $(\mathbb{P}_\gamma : \gamma \in E^* \cap \alpha)$.

We then let $\mathbb{P}_\kappa$ be the direct limit of $(\mathbb{P}_\alpha : \alpha \in E^*)$. Condition (*iv*) is not really needed for the construction itself. It comes in later in order to prove the κ-c.c. of $\mathbb{P}_\kappa$. It is also important in $\diamondsuit$-like arguments. For instance, if U is a Laver function and we wish our iteration to capture all semiproper forcing notions in $V^{\mathbb{P}_\kappa}$. We will say a few more words about this point later.

Before we start our construction let us make some preliminary remarks and definitions. Suppose $\alpha \in E^*$ and we have defined the $\mathbb{P}_\xi$, for $\xi \in E^* \cap (\alpha + 1)$, satisfying conditions (*i*)-(*iii*). Suppose $\delta \in E \setminus \alpha$ and $M \in \mathscr{C}_{\geq\delta}$. Recall that if r is a condition then $\xi(r)$ denotes the least $\xi \in E^*$ such that $r \in \mathbb{P}_\xi$.

Definition 3.1. *Suppose $r \in \mathbb{P}_\alpha$, $\delta \in E \setminus \alpha$, and $M \in \mathscr{C}_{\geq\delta}$. We say that r* makes M active at δ *if there is some $\xi \leq \xi(r)$ which is made attainable from M by r such that*

$$r \Vdash_{\mathbb{P}_{\xi(r)}} M[\dot{G}_\xi] \cap E \cap \delta \text{ is cofinal in } E \cap \delta.$$

Notice that if δ is a successor point of E, say $\delta = \text{next}(\gamma)$, for $\gamma \in E$, this simply means that r makes γ attainable from M. If δ is a limit point of E, by Lemma 1.1, this means that r forces that $M[\dot{G}_\xi] \cap \delta$ is cofinal in δ.

Now, suppose $\mathcal{M}$ is a finite subset of $\mathscr{C}$. We let

$$\mathcal{M}^{r,\delta} = \{M \restriction \delta : M \in \mathcal{M} \cap \mathscr{C}_{\geq\delta} \text{ and } r \text{ makes } M \text{ active at } \delta\}.$$

Notice that if $s \leq r$ then $\mathcal{M}^{r,\delta}$ is a subset of $\mathcal{M}^{s,\delta}$. Let $\dot{\mathcal{M}}^{\alpha,\delta}$ be the $\mathbb{P}_\alpha$-name for the union of the $\mathcal{M}^{r,\delta}$, for r in the generic filter G_α over $\mathbb{P}_\alpha$. Since $\mathcal{M}$ is finite there is always going to be a condition $r \in G_\alpha$ which *decides* $\dot{\mathcal{M}}^{\alpha,\delta}$, i.e. such that $\mathcal{M}^{s,\delta} = \mathcal{M}^{r,\delta}$, for all $s \leq r$ with $s \in \mathbb{P}_\alpha$. If $\delta = \text{next}(\alpha)$ we will write $\dot{\mathcal{M}}^\delta$ instead of $\dot{\mathcal{M}}^{\alpha,\delta}$. We will one more definition.

Definition 3.2. *Let $\mathcal{M}$ is a finite collection of δ-models and let G_α be V-generic over $\mathbb{P}_\alpha$. We say that $\mathcal{M}$ is a* weak $\in_\delta$-chain at α *if, for every $M, N \in \mathcal{M}$, we have:*

(1) *if* $M \cap \omega_1 = N \cap \omega_1$ *then* $M = N$,
(2) *if* $M \cap \omega_1 < N \cap \omega_1$ *then* $M \in_\delta N(G_\alpha)$.

This definition is of course made in the generic extension $V[G_\alpha]$ and depends on our choice of the generic filter G_α. Notice that if a condition $p \in \mathbb{P}_\alpha$ forces $\mathcal{M}$ to be a weak $\in_\delta$-chain at α, then it forces it to be a weak $\in_\delta$-chain at α', for any $\alpha' \in E^*$ such that $\alpha \leq \alpha' \leq \delta$.

We now outline our construction. We will define the $\mathbb{P}_\alpha$ by induction on α. Once we have defined $\mathbb{P}_\alpha$, for $\alpha \in E$, we define a $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}_\alpha$. If $U(\alpha)$ is a $\mathbb{P}_\alpha$-name for a semiproper forcing notion, we let $\dot{\mathbb{Q}}_\alpha$ be equal to $U(\alpha)$, otherwise we let $\dot{\mathbb{Q}}_\alpha$ be the canonical $\mathbb{P}_\alpha$-name for the trivial forcing notion $\mathbb{1}$. If $\alpha \in E$, let us say that p is an *α-pair* if it is of the form $(\mathcal{M}_p, w_p)$, where $\mathcal{M}_p$ is a finite collection of models from $\mathscr{C}_{\leq\alpha}$ and w_p is a finite function with $\mathrm{dom}(w_p)$ a subset of $E \cap \alpha$. The forcing notion $\mathbb{P}_\alpha$ will consist of certain α-pairs. Once we have defined $\mathbb{P}_\alpha$ we define $\mathbb{P}_{\alpha+1}$ as the set of all pairs $(\mathcal{M}_p, w_p)$, where $\mathcal{M}_p$ is a finite subset of $\mathscr{C}_{\leq\alpha}$ and w_p is a finite function with $\mathrm{dom}(w_p)$ a subset of $E \cap (\alpha+1)$ such that $(\mathcal{M}_p, w_p \restriction \alpha) \in \mathbb{P}_\alpha$ and, if $\alpha \in \mathrm{dom}(w_p)$, then $w_p(\alpha)$ is a canonical $\mathbb{P}_\alpha$-name for a condition in $\dot{\mathbb{Q}}_\alpha$. The order on $\mathbb{P}_{\alpha+1}$ is defined by letting $q \leq p$ if $(\mathcal{M}_q, w_q \restriction \alpha) \leq \mathcal{M}_p, w_p \restriction \alpha)$ and, if $\alpha \in \mathrm{dom}(w_p)$ then $\alpha \in \mathrm{dom}(w_q)$ and $(\mathcal{M}_q, w_q \restriction \alpha) \Vdash_{\mathbb{P}_\alpha} w_q(\alpha) \leq w_p(\alpha)$. Thus, $\mathbb{P}_{\alpha+1}$ will be canonically isomorphic to $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. If p is an α-pair and $\gamma \in E \cap \alpha$ we will let $p \restriction \gamma$ denote the pair $(\mathcal{M}_p \restriction \gamma, w_p \restriction \gamma)$ where $\mathcal{M}_p$ is the collection $\{M \restriction \gamma : M \in \mathcal{M}_p\}$ and $w_p \restriction \gamma$ is simply the restriction of w_p to $\mathrm{dom}(w_p) \cap \gamma$. Similarly, we will let $p \restriction (\gamma+1)$ denote $(\mathcal{M}_p \restriction \gamma, w_p \restriction \gamma+1)$. It will be immediate from the construction that if $p \in \mathbb{P}_\alpha$ then $p \restriction \gamma \in \mathbb{P}_\gamma$, for all $\gamma \in E^* \cap \alpha$. In fact, the map $p \mapsto p \restriction \gamma$ will be a canonical projection of $\mathbb{P}_\alpha$ to $\mathbb{P}_\gamma$. In order for a pair p to be in $\mathbb{P}_\alpha$ there will be two types of requirements. First, if $\gamma \in \mathrm{dom}(w_p)$ we will require that $w_p(\gamma)$ be forced by $p \restriction \gamma$ to be $(M[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)$-semigeneric, for all $M \in \mathcal{M}_p$ for which this makes sense. The second is that for each $\delta \in E \cap (\alpha+1)$, if $\gamma \in E^* \cap \delta$ is sufficiently large, then $p \restriction \gamma$ forces $\dot{\mathcal{M}}_p^{\gamma,\delta}$ to be a weak $\in_\delta$-chain at γ. Of course, if δ is a successor point of E, say $\delta = \mathrm{next}(\gamma)$, we may simply say that $p \restriction (\gamma+1)$ forces $\dot{\mathcal{M}}_p^{\delta}$ to be a weak $\in_\delta$-chain at $\gamma+1$. We will need the second condition in order to be able to extend w_p to any $\gamma \in E \cap \alpha$ and also to prove the semiproperness of the iteration.

With these remarks in mind we are ready to define the posets $\mathbb{P}_\alpha$. Of course, formally the definition is by induction on α.

Definition 3.3. *Suppose* $\alpha \in E$. *We say that a pair* p *of the form* $(\mathcal{M}_p, w_p)$ *belongs to* $\mathbb{P}_\alpha$ *if* $\mathcal{M}_p$ *is a finite subset of* $\mathscr{C}_{\leq\alpha}$, w_p *is a finite function with domain included in* $E \cap \alpha$ *and:*

(1) *for every* $\delta \in E \cap (\alpha+1)$ *there exists* $\bar{\delta} \in E \cap \delta$ *such that, if* $\gamma \in E^*$ *and* $\bar{\delta} < \gamma < \delta$*, then*

$$p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \dot{\mathcal{M}}_p^{\gamma,\delta} \text{ is a weak } \in_\delta\text{-chain at } \gamma.$$

(2) *If* $\gamma \in \mathrm{dom}(w_p)$ *and* $\delta = \mathrm{next}(\gamma)$ *then*

$$p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} w_p(\gamma) \text{ is } (M[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)\text{-semigeneric, for all } M \in \dot{\mathcal{M}}_p^\delta.$$

The order is defined as follows: $q \leq p$ *if for every* $\gamma \in E \cap (\alpha+1)$ *and a* γ*-model* $M \in \mathcal{M}_p$ *there is* $N \in \mathcal{M}_q$ *such that* $M = N \restriction \gamma$, $\mathrm{dom}(w_p) \subseteq \mathrm{dom}(w_q)$*, and, for every* $\gamma \in \mathrm{dom}(w_p)$*,*

$$q \restriction \gamma \Vdash_{\mathbb{P}_\gamma} w_q(\gamma) \leq w_p(\gamma).$$

Once we have $\mathbb{P}_\alpha$, we define $\mathbb{P}_{\alpha+1}$, for $\alpha \in E$, as described above. Clearly, the order relation $\leq$ is transitive on $\mathbb{P}_\alpha$, for $\alpha \in E^*$. Suppose $\alpha, \beta \in E^*$ and $\alpha < \beta$. It follows from the definition that $\mathbb{P}_\alpha$ is contained in $\mathbb{P}_\beta$ and the order relation on $\mathbb{P}_\alpha$ is the restriction of the order relation on $\mathbb{P}_\beta$. Suppose $p \in \mathbb{P}_\beta$, $q \in \mathbb{P}_\alpha$ and $q \leq p \restriction \alpha$. Then we can define a condition r as follows. Set $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$ and let $w_r = w_q \cup w_p \restriction [\alpha, \beta)$. It is straightforward to check that $r \in \mathbb{P}_\beta$ and is a greatest lower bound of p and q. We will denote r by $p \wedge q$. It follows that the map $p \mapsto p \restriction \alpha$ is a projection from $\mathbb{P}_\beta$ to $\mathbb{P}_\alpha$. It is easy to see that the set of conditions p in $\mathbb{P}_\alpha$ such that $p \restriction \gamma$ fixes $\mathcal{M}_p$ at γ, for every $\gamma \in \mathrm{dom}(w_p)$, is dense in $\mathbb{P}_\alpha$. It will be convenient to write $\mathbb{P}_{<\alpha}$ for the union of the $\mathbb{P}_\xi$, for $\xi \in E^* \cap \alpha$. Also, if $p \in \mathbb{P}_{<\kappa}$ and $\xi \in E^*$ we write $\mathbb{P}_\xi \restriction p$ for the set of all $q \in \mathbb{P}_\xi$ extending $p \restriction \xi$. Finally, we write $\mathbb{P}_{<\alpha} \restriction p$ for the union of the $\mathbb{P}_\xi \restriction p$, for $\xi \in E^* \cap \alpha$.

Lemma 3.4. *Suppose* $\alpha \in E$ *and* p *is a condition in* $\mathbb{P}_\alpha$*. Let* M *be a* β*-model, for some* $\beta > \alpha$*, and suppose that* $p \in M$*. Then there is a condition* $p^M \leq p$ *such that* $M \restriction \alpha \in \mathcal{M}_{p^M}$*.*

Proof. Let $\mathcal{M}_{p^M} = \mathcal{M}_p \cup \{M \restriction \alpha\}$. Let $\gamma \in \mathrm{dom}(w_p)$. By Proposition 1.10, we have that if $N \in \mathcal{M}_p$ then $N \restriction \gamma \in_\gamma M \restriction \gamma$, for every $\gamma \in E \cap (\alpha+1)$. We now define w_{p^M}. We will have $\mathrm{dom}(w_{p^M}) = \mathrm{dom}(w_p)$. Suppose $\gamma \in \mathrm{dom}(w_p)$ and note that γ as well as $w_p(\gamma)$ are in M. Since $U \restriction (\gamma+1) \in M$, it follows that $\mathbb{P}_\gamma$ and $\dot{\mathbb{Q}}_\gamma$ also belong to M. Now $\dot{\mathbb{Q}}_\gamma$ is forced to be semiproper and $w_p(\gamma) \in M$, so we can find a $\mathbb{P}_\gamma$-name $w_{p^M}(\gamma)$ for a condition in $\dot{\mathbb{Q}}_\gamma$ which is forced by $p \restriction \gamma$ to extend $w_p(\gamma)$ and be $(M[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)$-semigeneric. If we let $M' = M \restriction \mathrm{next}(\gamma)$, then clearly $p \restriction \gamma$ also forces that $w_{p^M}(\gamma)$ is $(M'[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)$-semigeneric. Finally, we let $p^M = (\mathcal{M}_{p^M}, w_{p^M})$. It is straightforward to check that w_{p^M} is as required. □

Lemma 3.5. *Let* $\gamma \in E$ *and* $\delta = \mathrm{next}(\gamma)$*. Suppose* $p \in \mathbb{P}_\delta$ *and* $\gamma \notin \mathrm{dom}(w_p)$*. Suppose also* $M \in \mathcal{M}_p$ *and* $\dot{w}$ *is a* $\mathbb{P}_\gamma$*-name for a condition in* $\dot{\mathbb{Q}}_\gamma$*. Suppose that* $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} M \in \dot{\mathcal{M}}_p^\delta$ *and* $\dot{w} \in M[\dot{G}_\gamma]$*. Then there is a* $\mathbb{P}_\gamma$*-name* $\dot{w}^*$ *for a condition in* $\dot{\mathbb{Q}}_\gamma$ *such that* $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \dot{w}^* \leq \dot{w}$ *and* $\dot{w}^*$ *is* $(N[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)$*-semigeneric, for all* $N \in \dot{\mathcal{M}}_p^\delta$*, such that* $M \cap \omega_1 \leq N \cap \omega_1$*.*

Proof. Let G_γ be any V-generic filter over $\mathbb{P}_\gamma$ containing $p \restriction \gamma$. Let $\mathcal{M}$ be the interpretation of $\dot{\mathcal{M}}_p^\delta$ by G_γ and let $\{M_0, \dots, M_{n-1}\}$ be the enumeration of $\mathcal{M}$ in the increasing order of the intersections with ω_1. Then $M_i \in_\delta M_j[G_\gamma]$, for all $i < j$. Let $\mathbb{Q}_\gamma$ be the interpretation of $\dot{\mathbb{Q}}_\gamma$ by G_γ. Then $\mathbb{Q}_\gamma$ is a semiproper forcing notion and is definable from $U \restriction (\gamma+1)$. Therefore it belongs to $M_i[G_\gamma]$, for all i. Suppose $M = M_k$ and let w be the interpretation of $\dot{w}$ by G_γ. Then $w \in M_k[G_\gamma]$. Let λ be the cardinality of $\mathbb{Q}_\gamma$ and assume, without loss of generality, that the domain of $\mathbb{Q}_\gamma$ is λ. Let $N_i = M_i[G_\gamma] \cap H(2^{\lambda^+})$, for all i. Note that a condition is $(M_i[G_\gamma], \mathbb{Q})$-semigeneric iff it is $(N_i[G_\gamma], \mathbb{Q})$-semigeneric. Let $\mathcal{N} = \{N_i : k \leq i < n\}$. Then $\mathcal{N}$ is an actual $\in$-chain of countable models containing $\mathbb{Q}_\gamma$. We let $w_k = w$ and use the definition of semiproperness to build a decreasing chain of conditions $w_k \geq w_{k+1} \geq \ldots \geq w_n$, such that $w_i \in N_i$ and w_{i+1} is $(N_i[G_\gamma], \mathbb{Q}_\gamma)$-semigeneric, for $k \leq i < n$. Let $w^* = w_n$ and pick a canonical $\mathbb{P}_\gamma$-name $\dot{w}^*$ for w^*. Since G_γ was an arbitrary generic filter over $\mathbb{P}_\gamma$ containing $p \restriction \gamma$, we have that $\dot{w}^*$ is as required. □

We are now ready to state the main technical result.

Theorem 3.6. *Suppose $\gamma \in E$ and $p \in \mathbb{P}_\gamma$. Let M be an α-model, for some $\alpha > \gamma$, and suppose that $M \restriction \gamma \in \mathcal{M}_p$. Then $p \Vdash_{\mathbb{P}_\gamma} M(\dot{G}_\gamma) \cap \omega_1 = M \cap \omega_1$.*

Proof. We show this by induction on γ. The basic idea is that the only ground model objects that are added to M by G_γ are added by $G_{\mathbb{Q}_\xi}$, for some $\xi \in E \cap \gamma$, where $G_{\mathbb{Q}_\xi}$ is the generic over $\mathbb{Q}_\xi$ induced by G_γ. We proceed by cases according to whether γ is a successor or a limit point of E.

Case 1. Let us first assume that γ is a successor point of E, say $\gamma = \text{next}(\beta)$, for $\beta \in E$. Let us pick an arbitrary V-generic filter G_β over $\mathbb{P}_\beta$ containing $p \restriction \beta$ and show that $M(G_\gamma) \cap \omega_1 = M(G_\beta) \cap \omega_1$, for any V-generic filter G_γ over $\mathbb{P}_\gamma$ which extends G_β and contains p. We may assume that γ and hence also β belongs to $M(G_\beta)$, otherwise we have that $M(G_\gamma) = M(G_\beta)$. Also, we may assume that $\beta \in \text{dom}(w_p)$. Since $M \restriction \gamma \in \text{val}_{G_\beta}(\dot{\mathcal{M}}_p^\gamma)$ we have that $\text{val}_{G_\beta}(w_p(\beta))$ is $(M[G_\beta], \mathbb{Q}_\beta)$-semigeneric, where $\mathbb{Q}_\beta$ is $\text{val}_{G_\beta}(\dot{\mathbb{Q}}_\beta)$. Hence, if $G_{\mathbb{Q}_\beta}$ is $V[G_\beta]$-generic over $\mathbb{Q}_\beta$ containing $\text{val}_{G_\beta}(w_p(\beta))$ and $G_{\beta+1}$ is the associated V-generic over $\mathbb{P}_{\beta+1}$, then $M(G_{\beta+1}) \cap \omega_1 = M(G_\beta) \cap \omega_1$ and so, by the inductive hypothesis, $M(G_{\beta+1}) \cap \omega_1 = M \cap \omega_1$. Let $\mathbb{R}_\gamma$ be the quotient forcing $\mathbb{P}_\gamma / G_\beta$. We will be done once we establish the following.

Claim 3.7. $p \Vdash_{\mathbb{R}_\gamma} M(\dot{G}_\gamma) = M(\dot{G}_{\beta+1})$.

Proof. Let $\tau \in M[G_\beta]$ be an $\mathbb{R}_\gamma$-name for an element of the ground model and let D be the set of conditions deciding the value of τ. Then D is a dense open subset of $\mathbb{R}_\gamma$. It suffices to show that for every $q \in \mathbb{R}_\gamma$ with $q \leq p$ there is a condition $r \in D$ and $s \leq q, r$ such that $s \restriction (\beta + 1) \Vdash_{\mathbb{Q}_\beta} r \in M(\dot{G}_{\beta+1})$. Then

$M(\dot{G}_{\beta+1})$ will be forced by s to contain the value of τ, as decided by r. Consider now one such $q \leq p$ and assume, without loss of generality, that $q \in D$. Let $w = \mathrm{val}_{G_\beta}(w_q(\beta))$ and let $\mathcal{M} = \mathrm{val}_{G_\beta}(\dot{\mathcal{M}}^\gamma_q)$. Since q is a condition, we know that $q \restriction (\beta+1)$ forces $\mathcal{M}$ to be a weak $\in_\gamma$-chain at $\beta+1$. Hence, for every $Q, R \in \mathcal{M}$, if $Q \cap \omega_1 = R \cap \omega_1$ then $Q = R$ and if $Q \cap \omega_1 < R \cap \omega_1$ then $w \Vdash_{\mathbb{Q}_\beta} Q \in_\gamma R(\dot{G}_{\beta+1})$. Let $\{M_0, \dots, M_{n-1}\}$ be the enumeration of the members of $\mathcal{M}$ in the order given by their intersections with ω_1. Suppose $M \restriction \gamma = M_k$ and let $\mathcal{N} = \{M_0, \dots, M_{k-1}\}$. We claim that $w \Vdash_{\mathbb{Q}_\beta} M_i \in M(\dot{G}_{\beta+1})$, for all $i < k$, and hence $w \Vdash_{\mathbb{Q}_\beta} \mathcal{N} \in M(\dot{G}_{\beta+1})$. To see that, pick an arbitrary $V[G_\beta]$-generic filter $G_{\mathbb{Q}_\beta}$ over $\mathbb{Q}_\beta$ containing w and let $G_{\beta+1}$ be the associated V-generic filter over $\mathbb{P}_{\beta+1}$. For each $i < k$ we know that $M_i \in_\gamma (M \restriction \gamma)(G_{\beta+1})$, so there is $M_i^\# \in M(G_{\beta+1})$ such that $M_i^\# \restriction \gamma = M_i$. However, $\gamma \in M(G_{\beta+1})$, so we can compute $M_i^\# \restriction \gamma$ inside $M(G_{\beta+1})$. It follows that $M_i \in M(G_{\beta+1})$, for all $i < k$. So, now we can find a $\mathbb{Q}_\beta$-name $\dot{\mathcal{N}} \in M[G_\beta]$ for a finite collection of γ-models such that $w \Vdash_{\mathbb{Q}_\beta} \dot{\mathcal{N}} = \check{\mathcal{N}}$. We define D^* to be the set of all $u \in \mathbb{Q}_\beta$ such that

(∗) there is $r \in D$ such that $\mathrm{val}_{G_\beta}(w_r(\beta)) = u$ and $u \Vdash_{\mathbb{Q}_\beta} \dot{\mathcal{N}} \subseteq \mathcal{M}_r$.

Clearly, D^* is an open subset of $\mathbb{Q}_\beta$. It may not be dense, so we let $(D^*)^\perp$ be the collection of all $u \in \mathbb{Q}_\beta$ that are incompatible with all members of D^* and we let $D^{**} = D^* \cup (D^*)^\perp$. Then D^{**} is a dense subset of $\mathbb{Q}_\beta$. Working in $M[G_\beta]$, fix a maximal antichain A contained in D^{**} and, for each $u \in A \cap D^*$, a condition $r_u \in D$ witnessing (∗) for u. Let $\dot{r} = \{(u, \check{r}_u) : u \in A \cap D^*\}$. Then $\dot{r}$ is a $\mathbb{Q}_\beta$-name which is forced to be either the empty set or an element of D. Moreover, $\dot{r} \in M[G_\beta]$. Let u be an element of A which is compatible with w and let w^* be a common extension of w and u. Let us show that $u \in D^*$. To see this, it suffices to show that $u \notin (D^*)^\perp$. Otherwise w^* would also be in $(D^*)^\perp$, but w^* extends w which is in D^* as witnessed by q itself, a contradiction. Notice that $w^* \Vdash_{\mathbb{Q}_\beta} \dot{r} = \check{r}_u$. To simplify notation, let $r = r_u$. We claim that q and r are compatible in $\mathbb{R}_\gamma$. To see this, first pick a $\mathbb{P}_\beta$-name $\dot{w}^*$ for w and let t be a common extension of q and r in G_β which forces all the relevant facts about q, r and $\dot{w}^*$. In particular, t decides $\dot{\mathcal{M}}^\gamma_q$ and $\dot{\mathcal{M}}^\gamma_r$. Now, let $\mathcal{M}_s = \mathcal{M}_t \cup \mathcal{M}_q \cup \mathcal{M}_r$, $w_s = w_t \cup \{(\beta, \dot{w}^*)\}$ and let $s = (\mathcal{M}_s, w_s)$. We check that $s \in \mathbb{P}_\gamma$. Since $s \restriction \beta = t$ which is in G_β and extends $q \restriction \beta$ and $r \restriction \beta$ this implies that $s \in \mathbb{R}_\gamma$. It suffices to verify that Definition 3.3 holds for s at γ. Now, t forces in $\mathbb{P}_\beta$ that $\dot{\mathcal{M}}^\gamma_s = \dot{\mathcal{M}}^\gamma_q \cup \dot{\mathcal{M}}^\gamma_r$ and it also forces that $\dot{w}^*$ is below $w_q(\beta)$ and below $w_r(\beta)$. Hence it forces that $\dot{w}^*$ is $(N[\dot{G}_\beta], \dot{\mathbb{Q}}_\beta)$-semigeneric, for all $N \in \dot{\mathcal{M}}^\gamma_s$. This verifies condition (2) of Definition 3.3. To see that condition (1) holds take $Q \in \mathcal{M}^{t,\gamma}_q$ and $R \in \mathcal{M}^{t,\gamma}_r$. If $Q \cap \omega_1 < M \cap \omega_1$ then Q belongs to $\mathcal{N}$ and $s \restriction (\beta+1) \Vdash_{\mathbb{P}_{\beta+1}} \mathcal{N} \subseteq \mathcal{M}_r$, so condition (1) holds for Q and R since $r \in \mathbb{P}_\gamma$. If $Q \cap \omega_1 = M \cap \omega_1$ then $Q \cong_\gamma M$ and we know that $t \Vdash_{\mathbb{P}_\beta} \mathcal{M}_r \subseteq M(\dot{G}_\beta)$. Hence $t \Vdash_{\mathbb{P}_\beta} R \in_\gamma Q(\dot{G}_\beta)$.

Finally, if $M \cap \omega_1 < Q \cap \omega_1$ then $s \restriction (\beta + 1) \Vdash_{\mathbb{P}_{\beta+1}} M \in_\gamma Q(\dot{G}_{\beta+1})$ and hence $s \restriction (\beta + 1) \Vdash_{\mathbb{P}_{\beta+1}} R \in_\gamma Q(\dot{G}_{\beta+1})$, by the transitivity of $\in_\gamma$. Therefore, we have condition (1) in this case as well. Finally, it is clear from the definition that s extends q and r. This establishes Claim 3.7 and completes the proof in **Case 1**. □

Case 2. Assume now that γ is a limit point of E. Before we start we need to establish a technical fact.

Claim 3.8. *Suppose $\delta \leq \gamma$ is a limit point of E, $q \in \mathbb{P}_\delta$ and $N \in \mathcal{M}_q$. Then there is an ordinal $\bar{\delta} \leq \delta$ which is also a limit point of E and $t \in \mathbb{P}_{<\bar{\delta}} \restriction q$ such that t makes N active at $\bar{\delta}$ and no condition in $\mathbb{P}_{<\delta} \restriction (q \wedge t)$ makes N active at any ordinal $\eta \in E \cap (\bar{\delta}, \delta]$.*

Proof. If $N \cap \mathrm{ORD}$ is contained in δ then we let $\bar{\delta}$ be $\sup(N \cap \delta)$. By Lemma 1.1, $\bar{\delta}$ is also a limit member of E. Now, if $\beta \in E$ and G_β is V-generic over $\mathbb{P}_\beta$ then $\sup(N(G_\beta) \cap \mathrm{ORD}) = \bar{\delta}$, hence N cannot be made active at any ordinal $\eta \in E \cap (\bar{\delta}, \delta]$, and we can let t be any condition in $\mathbb{P}_{<\bar{\delta}} \restriction q$. Suppose now that N has some ordinal $\geq \delta$ and let δ^* be the least ordinal $\geq \delta$ that can belong to $N(G_\beta)$, for some $\beta \in E \cap \delta$ and some V-generic G_β over $\mathbb{P}_\beta$ such that $q \restriction \beta \in G_\beta$. By taking the least possible β, we may assume that $\beta \in N(G_{<\beta})$ and hence we can form the model $N[G_\beta]$. Let λ be the cofinality of δ^* in $N[G_\beta]$. If $\lambda < \delta^*$, i.e. δ^* is singular in $N[G_\beta]$, then $\lambda < \delta$, hence $\lambda \in N[G_\beta] \cap \delta$. Now, by increasing β and going to a further generic extension, we may assume that $\lambda < \beta$. Since $\mathbb{P}_\beta$ collapses all cardinals $< \beta$ to have cardinality at most ω_1, it follows that we may assume that the cofinality of δ^* in $N[G_\beta]$ is either ω, ω_1, or δ^*. Suppose δ^* is regular in $N[G_\beta]$. Then, since δ is strong limit and δ^* is the least ordinal of $N[G_\beta] \setminus \delta$, it follows that δ^* is also strong limit and hence inaccessible in $N[G_\beta]$. Now, let $\bar{\delta} = \sup(N[G_\beta] \cap \delta)$. By elementarity of $N(G_\beta)$ in $\mathrm{Hull}(N, V_\delta)$, $\bar{\delta}$ is a limit point of E. Now, let $t \in G_\beta$ be a condition that makes β attainable from N, forces that $\delta^* \in N[\dot{G}_\beta]$ and decides whether the cofinality of δ^* in $N[\dot{G}_\beta]$ is ω, ω_1, or δ^*. Since $q \restriction \beta \in G_\beta$, we may also assume that $t \leq q \restriction \beta$. We claim that these $\bar{\delta}$ and t work. It suffices to show the following.

Subclaim. $(q \wedge t) \restriction \eta \Vdash_{\mathbb{P}_\eta} \sup(N(\dot{G}_\eta) \cap \delta) = \bar{\delta}$*, for all* $\eta \in E \cap [\beta, \delta)$

Proof. To see this, note that if the cofinality of δ^* in $N[G_\beta]$ is ω then $N[G_\beta]$ contains a cofinal ω-sequence in δ^*. It follows that $\delta^* = \delta$ and $N[G_\beta] \cap \delta$ is cofinal in δ, hence $\bar{\delta} = \delta$. Therefore, in this case, $t \Vdash_{\mathbb{P}_\eta} \sup(N(\dot{G}_\eta) \cap \delta) = \delta$, for all $\eta \in E \cap [\beta, \delta)$. Suppose now δ^* is inaccessible in $N[G_\beta]$. Then, for every $\eta \in N[G_\beta] \cap E \cap \delta$ which is bigger than β, the forcing notion $\mathbb{P}_\eta / G_\beta$ is smaller than δ^*, hence if G_η is V-generic over $\mathbb{P}_\eta$ extending G_β then $\sup(N[G_\eta] \cap \delta^*) = \sup(N[G_\beta] \cap \delta^*) = \bar{\delta}$. Moreover, δ^* remains inaccessible in $N[G_\eta]$ and so the same properties holds for $N[G_\eta]$ in place of $N[G_\beta]$. Finally, suppose δ^* has cofinality ω_1 in $N[G_\beta]$. Suppose $\eta \in E \cap \delta$ with $\eta > \beta$ and G_η is a V-generic filter over $\mathbb{P}_\eta$ extending G_β and such that $(q \wedge t) \restriction \eta \in G_\eta$. By our inductive assumption, we have that $N(G_\eta) \cap \omega_1 = N \cap \omega_1$. Therefore, we also have that $N(G_\eta) \cap \omega_1 = N(G_\beta) \cap \omega_1$. Since δ^* has cofinality ω_1 in $N[G_\beta]$, it follows that we also have $\sup(N(G_\eta) \cap \delta^*) = \sup(N(G_\beta) \cap \delta^*) = \bar{\delta}$. This completes the proof of the subclaim and Claim 3.8. $\square$

$\square$

Back to the proof of Theorem 3.6 in the case γ is a limit point of E. First, note that if p has no extension that makes γ attainable from M then the statement we are trying to prove follows by induction. So, by strengthening p we may assume that it makes γ attainable from M. By applying Claim 3.8 for $\delta = \gamma$ and strengthening p again we may fix $\bar{\gamma} \leq \gamma$ which is a limit of E and $\beta \in E \cap \bar{\gamma}$ such that $p \restriction \beta$ makes M active at $\bar{\gamma}$ and no condition $q \in \mathbb{P}_{<\gamma} \restriction p$ makes M active at any ordinal in $E \cap (\bar{\gamma}, \gamma]$. We may also assume that $p \restriction \beta$ makes β and γ attainable from M. We plan to show

$$p \Vdash_{\mathbb{P}_\gamma} M(\dot{G}_\gamma) = \bigcup \{M(\dot{G}_\eta) : \eta \in E \cap \bar{\gamma}\}.$$

By the inductive assumption this implies the statement we wish to prove. Let us pick an arbitrary V-generic filter G_β over $\mathbb{P}_\beta$ such that $p \restriction \beta \in G_\beta$ and let us work in $V[G_\beta]$ for a while. Since $p \restriction \beta$ makes β and γ attainable from M we can form the model $M[G_\beta]$ and we know that $\gamma \in M[G_\beta]$. Moreover, since $p \restriction \beta$ makes M active at $\bar{\gamma}$ we know that $E \cap M(G_\beta) \cap \bar{\gamma}$ is cofinal in $\bar{\gamma}$. Fix a $\mathbb{P}_\gamma/G_\beta$-name $\tau \in M[G_\beta]$ for an element of V and let D be the set of conditions in $\mathbb{P}_\gamma/G_\beta$ deciding the value of τ. Then D is a dense open subset of $\mathbb{P}_\gamma/G_\beta$ and belongs to $M[G_\beta]$. It suffices to show that for every $q \in \mathbb{P}_\gamma/G_\beta$ with $q \leq p$ there is $r \in D$ and $s \in \mathbb{P}_\gamma/G_\beta$ such that $s \leq q, r$ and $s \restriction \eta \Vdash_{\mathbb{P}_\eta/G_\beta} r \in M(\dot{G}_\eta)$, for some $\eta \in E \cap \bar{\gamma}$. Then $M(\dot{G}_\eta)$ will be forced by s to contain the value of τ, as decided by r. Consider now one such $q \leq p$ and assume, without loss of generality, that $q \in D$. By applying iteratively Claim 3.8 for each $N \in \mathcal{M}_q$ and a genericity argument, we can find an ordinal $\eta \in E \cap M(G_\beta) \cap \bar{\gamma}$ and $t \in \mathbb{P}_\eta/G_\beta$ such that $t \leq q \restriction \eta$ and such that, for every $N \in \mathcal{M}_q$, either t makes N active at $\bar{\gamma}$ or no extension of $q \wedge t$ in $\mathbb{P}_{\bar{\gamma}}/G_\beta$ makes N active at any ordinal in $E \cap (\eta, \bar{\gamma}]$. By increasing η, we may also assume that $\mathrm{dom}(w_q) \cap \bar{\gamma}$ is a subset of η. Let $\mathcal{M} = \mathcal{M}_q^{t,\bar{\gamma}}$. Since $q \restriction \beta$ already makes M active at $\bar{\gamma}$ we have that $M \restriction \bar{\gamma} \in \mathcal{M}$. By condition (1) of Definition 3.3 and increasing η again if necessary, we may assume that $(q \wedge t) \restriction \eta$ forces $\mathcal{M}$ to be a weak $\in_{\bar{\gamma}}$-chain at η. Let $\{M_0, \dots, M_{n-1}\}$ be the enumeration of the members of $\mathcal{M}$ given by their intersections with ω_1. Suppose $M \restriction \bar{\gamma} = M_k$. Now take any V-generic filter G_η over $\mathbb{P}_\eta$ extending G_β and containing $(q \wedge t) \restriction \eta$. Note that, since $\eta \in M(G_\beta)$ we can form the model $M[G_\eta]$. For each $i < k$ we know that $M_i \in_{\bar\gamma} (M \restriction \bar\gamma)(G_\eta)$, so there is $M_i^\# \in M(G_\eta)$ such that $M_i^\# \restriction \bar\gamma = M_i$. Let $\bar M_i$ be $M_i^\# \restriction \gamma$. Since $\gamma \in M(G_\eta)$, we can compute $\bar M_i$ inside $M(G_\eta)$, for all $i < k$. Let $\mathcal{N} = \{\bar M_i : i < k\}$. It follows that $\mathcal{N} \in M(G_\eta)$. Now, note that $\bar M_i \restriction \bar\gamma = M_i$, for all $i < k$. Moreover, since $q \restriction \eta \Vdash_{\mathbb{P}_\eta} \bar M_i \in_\gamma M(\dot G_\eta)$ and no condition compatible with q can make M active at any ordinal in $E \cap (\bar\gamma, \gamma]$ then the same must be true for $\bar M_i$, for all $i < k$. Therefore, if we let $\mathcal{M}_{\bar q} = \mathcal{M}_q \cup \mathcal{N}$ and $w_{\bar q} = w_q$, then $\bar q$ is a condition. In fact, it is equivalent to q and $\bar q \restriction \bar\gamma = q \restriction \bar\gamma$. Let $\bar D = D \cap G_\eta$. Then $\bar D$ is a dense open subset of $\mathbb{P}_\gamma / G_\eta$ and belongs to $M[G_\eta]$. Now, recall that M is an α-model and $\alpha > \gamma$. Then $M[G_\eta]$ is an elementary submodel of $\mathrm{Hull}(M, V_\alpha)[G_\eta]$. Since $\alpha > \gamma$, we know that $\bar q \in \mathrm{Hull}(M, V_\alpha)[G_\eta]$. Since $\bar q \restriction \eta$, we know that $\bar q \in G_\eta$. Now, by elementarity of $M[G_\eta]$ in $\mathrm{Hull}(M, V_\alpha)[G_\eta]$, we can find $r \in \bar D \cap M(G_\eta)$ such that $r \restriction \eta \in G_\eta$ and $\mathcal{N} \subseteq \mathcal{M}_r$.

Claim 3.9. *q and r are compatible.*

Proof. We need to define a common extension s of q and r. Let us first fix some $u \in G_\eta$ extending $(q \wedge t) \restriction \eta$ and $r \restriction \eta$ such that $u \Vdash_{\mathbb{P}_\eta} r \in M(\dot G_\eta)$. Notice that $\mathrm{dom}(w_q) \cap [\eta, \bar\gamma) = \emptyset$, while $w_r \in M[G_\eta]$. We know that $\sup(M[G_\eta] \cap \gamma) = \bar\gamma$, so $\mathrm{dom}(w_r) \subseteq \bar\gamma$. We first define a function w on $\mathrm{dom}(w_r) \setminus \eta$. So, suppose $\xi \in \mathrm{dom}(w_r) \setminus \eta$ and let $\nu = \mathrm{next}(\xi)$. Note that $u \Vdash_{\mathbb{P}_\eta} \check w_r \in M(\dot G_\eta)$. So, if we let $q^* = q \wedge u$ we can apply Lemma 3.5 to find a $\mathbb{P}_\xi$-name $w(\xi)$ for a condition in $\dot{\mathbb{Q}}_\xi$ such that $q^* \restriction \xi$ forces that $w(\xi)$ is stronger than $w_r(\xi)$ and is $(N[\dot G_\xi], \dot{\mathbb{Q}}_\xi)$-semigeneric, for all N in $\dot{\mathcal{M}}_q^\nu$ with $M \cap \omega_1 \leq N \cap \omega_1$. Now, let $\mathcal{M}_s = \mathcal{M}_u \cup \mathcal{M}_q \cup \mathcal{M}_r$ and $w_s = w_u \cup w \cup w_q \restriction [\bar\gamma, \gamma)$. We claim that $s = (\mathcal{M}_s, w_s)$ belongs to $\mathbb{P}_\gamma$ and is a common extension of q and r. We show that $s \restriction \xi$ is a condition, by induction on $\xi \in E \cap (\gamma + 1)$. Note that $s \restriction \eta = u$ which is a condition in $\mathbb{P}_\eta$. This starts the induction. Suppose first that $\xi \in E \cap (\eta, \bar\gamma + 1)$ and we have verified that $s \restriction \zeta$ is a condition, for all $\zeta \in E \cap \xi$. If $\xi = \mathrm{next}(\zeta)$ for some $\zeta \in \mathrm{dom}(w_r)$ we first check condition (2) of Definition 3.3. To begin, note that $r \restriction \zeta$ forces that $w_r(\zeta)$ is $(N[\dot G_\zeta], \dot{\mathbb{Q}}_\zeta)$-semigeneric, for all $N \in \dot{\mathcal{M}}_r^\xi$ and $q^* \restriction \zeta$ forces that $w(\zeta)$ extends $w_r(\zeta)$ and is $(N[\dot G_\zeta], \dot{\mathbb{Q}}_\zeta)$-semigeneric, for all $N \in \dot{\mathcal{M}}_q^\xi$ with $M \cap \omega_1 \leq N \cap \omega_1$. Also, $q^* \restriction \zeta$ forces that if some $N \in \mathcal{M}_q$ is active at ξ then $N \cong_\xi \bar N$, for some $\bar N \in \mathcal{N}$. Since $\mathcal{N}$ is contained in $\mathcal{M}_r$, it follows that $s \restriction \zeta$ forces in $\mathbb{P}_\zeta$ that $w(\xi)$ is $(N[\dot G_\zeta], \dot{\mathbb{Q}}_\zeta)$-semigeneric, for all $N \in \dot{\mathcal{M}}_s^\zeta$. Let us now check condition (1) of Definition 3.3 for s at ξ. Let $\zeta_0 \in E \cap \xi$ witness condition (1) for $r \restriction \xi$ and $\zeta_1 \in E \cap \xi$ witness (1) for $q^* \restriction \xi$. Let ζ be the maximum of ζ_0, ζ_1 and η. We check that that for every $\rho \in E^* \cap (\zeta, \xi)$ we have that $s \restriction \rho$ forces in $\mathbb{P}_\rho$ that $\dot{\mathcal{M}}_s^{\rho,\xi}$ is a weak $\in_\xi$-chain at ξ. To see this pick an arbitrary $z \in \mathbb{P}_\rho$ with $z \leq s \restriction \rho$ such that z decides $\dot{\mathcal{M}}_s^{\rho,\xi}$. Let $\mathcal{M}^*$ be the set of all $N \in \mathcal{M}_q^{z,\xi}$ such that $N \cap \omega_1 < M \cap \omega_1$. Note that $\mathcal{M}^* \subseteq \mathcal{N} \restriction \xi$ and $\mathcal{N} \subseteq \mathcal{M}_r$, hence $\mathcal{M}^* \subseteq \mathcal{M}_r^{z,\xi}$. Therefore, $\mathcal{M}_s^{z,\xi} = \mathcal{M}_r^{z,\xi} \cup (\mathcal{M}_q^{z,\xi} \setminus \mathcal{M}^*)$. On the other hand, u

forces in $\mathbb{P}_\eta$ that $\mathcal{M}_r \in M(\dot{G}_\eta)$, hence it forces that $N \restriction \xi \in_\xi M(\dot{G}_\eta) \restriction \xi$, for all $N \in \mathcal{M}_r$. Finally, z forces that $\mathcal{M}_q^{z,\xi} \setminus \mathcal{M}^*$ is a a weak $\in_\xi$-chain at ρ, since it is compatible with q which is a condition. Now, suppose $\xi \in E \cap (\bar{\gamma}, \gamma]$. Since $u \Vdash_{\mathbb{P}_\eta} r \in M(\dot{G}_\eta)$, and no condition compatible with q^* can make M active at any ordinal in $E \cap (\bar{\gamma}, \gamma]$, then no condition compatible with q^* can make any $N \in \mathcal{M}_r$ active at any ordinal in $E \cap (\bar{\gamma}, \gamma]$. Therefore, the fact that q^* satisfies Definition 3.3 implies that so does s at ordinals in $E \cap (\bar{\gamma}, \gamma]$. This completes the verification that $s \in \mathbb{P}_\gamma$. Clearly, s is then a common extension of q and r, as required. □

This completes the proof of Theorem 3.6. □

Definition 3.10. *For $\alpha \in E^*$, let $\mathbb{P}^*_\alpha$ be the set of all conditions $p \in \mathbb{P}_\alpha$ such that there are no γ-models in $\mathcal{M}_p$, for any γ which is either inaccessible or has cofinality ω_1.*

Lemma 3.11. *$\mathbb{P}^*_\alpha$ and $\mathbb{P}_\alpha$ are equivalent forcing notions, for all $\alpha \in E^*$.*

Proof. We may assume that $\alpha \in E$. Let $\bar{\mathbb{P}}_\alpha$ be the set of all conditions $p \in \mathbb{P}_\alpha$ such that there is $\beta \in E$ with $\beta > \alpha$ and a β-model M_p such that $M_p \restriction \alpha \in \mathcal{M}_p$, and α and $\mathcal{M}_p \setminus \{M_p \restriction \alpha\}$ belong to M_p. Note that, by Lemma 3.4, $\bar{\mathbb{P}}_\alpha$ is a dense subset of $\mathbb{P}_\alpha$. Now, for each $\gamma \in E \cap (\alpha + 1) \cap M_p$ which is either inaccessible or has cofinality ω_1, let $\gamma_p = \sup(\gamma \cap M_p)$. Then γ_p has countable cofinality and, by Lemma 1.1, we know that $\gamma_p \in E$. Note that, for every such γ, the model $M_p \restriction \alpha$ cannot be made active at any ordinal $\eta \in E \cap (\gamma_p, \gamma]$ by a condition compatible with p. For γ inaccessible this follows from the fact that $\mathbb{P}_\xi$ has size $< \text{next}(\xi)$, for all $\xi \in E$. For γ of cofinality ω_1 this follows from Theorem 3.6. Since all the models in $\mathcal{M}_p$ other than $M_p \restriction \alpha$ are elements of M_p, it follows that no model in $\mathcal{M}_p$ can be made active at any such ordinal by any condition compatible with p. Now define a map $\pi_\alpha : \bar{\mathbb{P}}_\alpha \to \mathbb{P}^*_\alpha$ by letting $\pi_\alpha(p) = (\mathcal{M}^*_p, w_p)$, where $\mathcal{M}^*_p$ is obtained by replacing any γ-model $N \in \mathcal{M}_p$, for γ either inaccessible or of cofinality ω_1, by $N \restriction \gamma_p$. By the above argument $\pi_\alpha(p)$ and p are equivalent conditions in $\mathbb{P}_\alpha$, i.e. any condition $q \in \mathbb{P}_\alpha$ is compatible with $\pi_\alpha(p)$ iff it is compatible with p. It follows that $\mathbb{P}_\alpha$ and $\mathbb{P}^*_\alpha$ are equivalent as forcing notions. □

Now, note that if $\alpha \in E$ is either inaccessible or has cofinality ω_1 then $\mathbb{P}^*_\alpha$ is the direct limit of the $\mathbb{P}^*_\xi$, for $\xi \in E \cap \alpha$. Therefore, if we let $\mathbb{P}^*_\kappa$ be the union of the $\mathbb{P}^*_\alpha$, for $\alpha \in E$, then by a standard argument $\mathbb{P}^*_\kappa$ satisfies the κ-chain condition. By putting all this together we get the final conclusion.

Theorem 3.12. *$\mathbb{P}^*_\kappa$ is semiproper and satisfies the κ-chain condition.* □

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