Forcing Axioms and Cardinal Arithmetic

Boban Veličković

Equipe de Logique, Université de Paris 7, 2 Place Jussieu, 75251 Paris, France

Abstract

We survey some recent results on the impact of strong forcing axioms such as the Proper Forcing Axiom PFA and Martin's Maximum MM on cardinal arithmetic. We concentrate on three combinatorial principles which follow from strong forcing axioms: stationary set reflection, Moore's Mapping Reflection Principle MRP and the P-ideal dichotomy introduced by Abraham and Todorčević which play the key role in these results. We also discuss the structure of inner models of PFA and MM and present some open problems.

Key words: cardinal arithmetic, continuum problem, singular cardinal hypothesis, PFA, MM, stationary set reflection, MRP, P-ideal dichotomy
1991 MSC: Primary 03Exx

1 Introduction

Cardinal arithmetic has been one of the main fields of research in set theory since the foundational works by Cantor in the last quarter of the 19-th century [10]. The central question is the celebrated continuum problem which asks for the specific value of $2^{\aleph_0}$. A more general version of this problem is to determine all the rules which govern the exponential function $\kappa \mapsto 2^\kappa$ on infinite cardinals. Since the seminal work of Gödel [19] and Cohen [11] it has been known that the usual axioms ZFC of set theory do not decide the value of the continuum. Moreover, soon after Cohen introduced the method of forcing Easton [14] generalized Cohen's result and showed that the exponential function $\kappa \mapsto 2^\kappa$ on regular cardinals can be arbitrary modulo some mild restrictions. The situation for singular cardinals is much more subtle. Silver [33] showed that the Generalized Continuum Hypothesis (GCH) cannot first fail at a singular cardinal of uncountable cofinality. Subsequently, Shelah developed

Email address: boban@logique.jussieu.fr (Boban Veličković).

a rich and powerful theory of possible cofinalities (PCF theory) with important applications in cardinal arithmetic. One of the most striking of Shelah's results is that $\aleph_\omega^{\aleph_0} < \max\{\aleph_{\omega_4}, (2^{\aleph_0})^+\}$ holds in ZFC (see [31] or [2]).

In 1947 Gödel ([20] and [21]) speculated on the ontological status of set theory and conjectured correctly that ZFC would be too weak to settle the continuum problem. He concluded that it was necessary to seek new natural axioms that could give a satisfactory solution to the continuum problem as well as other natural problems arising in the field. Gödel stressed that these axioms should satisfy some form of maximality property which he opposed to the minimality condition satisfied by the constructible universe L (see [21], section 4, p.478, note 19). This became later known as Gödel's program. Over the years two types of new axioms satisfying Gödel's condition emerged: *large cardinal axioms* and *forcing axioms*. While large cardinals give rise to a very rich theory and decide many natural questions about the reals they have very little to say about the continuum problem. Forcing axioms on the other hand have strong influence on cardinal exponentiation and decide many natural questions about uncountable cardinals left open by ZFC. It should be pointed out that these two types of axioms are very closely intertwined. Typically one needs large cardinals to prove the consistency of strong forcing axioms.

One way to motivate forcing axioms is as generalizations of Baire's category theorem. Suppose $\mathcal{K}$ is a class of partial orderings and κ an infinite cardinal. Then $\mathrm{FA}(\mathcal{K}, \kappa)$ is the following statement.

> *Suppose $\mathcal{P}$ is in $\mathcal{K}$ and $\mathcal{D}$ is a family of κ dense subsets of $\mathcal{P}$. Then there is a filter G in $\mathcal{P}$ such that $G \cap D \neq \emptyset$, for all $D \in \mathcal{D}$.*

The study of these axioms was started by Martin and Solovay [26] who introduced Martin's axiom (MA) as an abstraction of Solovay and Tennenbaum's approach to solving Suslin's problem [34], a question about uncountable trees. MA is the statement that $\mathrm{FA}(\mathrm{ccc}, \kappa)$ holds for all $\kappa < \mathfrak{c}$, where ccc denotes the class of forcing notions having the countable chain condition. It was soon realized that MA together with the negation of the Continuum Hypothesis (CH) provides a rich structure theory for the reals. As the method of forcing was further developed generalizations of MA + $\neg$CH to larger classes of forcing notions were considered as well. One of the most successful of these axioms is the Proper Forcing Axiom (PFA) introduced by Baumgartner and Shelah in the early 1980s (see, for example, Baumgartner's survey paper [7]). PFA says that $\mathrm{FA}(\mathrm{Proper}, \aleph_1)$ holds, where Proper is the class of proper forcing notions. In the mid 1980s Foreman, Magidor and Shelah [16] formulated Martin's Maximum (MM) the provably strongest forcing axiom and showed that it is relatively consistent with ZFC. MM is the statement that $\mathrm{FA}(\mathcal{K}, \aleph_1)$ holds, where $\mathcal{K}$ is the class of forcing notions preserving stationary subsets of ω_1. It should be pointed out that while the consistency of MA + $\neg$CH does not require any large cardinals, the proofs of the consistency of PFA and MM use a supercompact cardinal.

In recent years, bounded versions of traditional forcing axioms have received a considerable amount of attention as they have many of the same consequences, yet require much smaller large cardinal assumptions. These statements were first considered by Goldstern and Shelah in [22] who showed that the Bounded Proper Forcing Axiom (BPFA) is equiconsistent with a relatively modest large cardinal axiom, the existence of a Σ_1-reflecting cardinal. An appealing formulation of bounded forcing axioms as principles of generic absoluteness was provided by Bagaria [5]. Namely, suppose $\mathcal{K}$ is a class of forcing notions. The bounded forcing axiom BFA($\mathcal{K}$) is the statement asserting that for every $\mathcal{P} \in \mathcal{K}$,

$$(H_{\aleph_2}, \in) \prec_{\Sigma_1} (V^{\mathcal{P}}, \in).$$

Here $H_{\aleph_2}$ denotes the collection of all sets whose transitive closure has size at most $\aleph_1$. Thus, BFA($\mathcal{K}$) states that for every Σ_0 formula $\psi(x, a)$ with parameter $a \in H_{\aleph_2}$, if some forcing notion from $\mathcal{K}$ introduces a witness x for $\psi(x, a)$, then such an x already exists. For example, $\mathrm{MA}_{\aleph_1}$ is BFA(ccc), BPFA is BFA(Proper) and BMM is BFA($\mathcal{K}$) where $\mathcal{K}$ is the class of forcing notions that preserve stationary subsets of ω_1.

Martin's Axiom does not decide the value of the continuum, but PFA and MM have strong influence on cardinal arithmetic. Thus, Foreman, Magidor and Shelah [16] showed that MM implies that $\mathfrak{c} = \aleph_2$. In fact, they showed that it implies that $\kappa^{\aleph_1} = \kappa$, for all regular $\kappa \geq \aleph_2$. As a consequence of this and Silver's theorem [33] it follows that MM implies the Singular Cardinal Hypothesis (SCH). By a more involved argument Todorčević and the author [38] showed that the weaker Proper Forcing Axiom (PFA) also implies that $\mathfrak{c} = 2^{\aleph_1} = \aleph_2$. In [42] Woodin identified a statement ψ_{AC} which follows from both Woodin's $\mathbb{P}_{\max}$-axiom $(*)$ and from MM, and implies that $\mathfrak{c} = 2^{\aleph_1} = \aleph_2$ and that there is a well-ordering of the reals definable with parameters in $(H_{\aleph_2}, \in)$. Moreover, Woodin showed that BMM together with the existence of a measurable cardinal implies that the continuum is $\aleph_2$. The assumption of the existence of a measurable cardinal was later eliminated by Todorčević [36] who deduced these consequences of ψ_{AC} from a statement he called θ_{AC} that he showed follows from BMM. Recently, Moore [28] introduced the Mapping Reflection Principle (MRP) and deduced it from PFA. Although MRP does not follow from BPFA, Moore [28] used a bounded version of MRP to show that BPFA implies a certain statement υ_{AC} which in turn implies that there is a well ordering of the reals, and in fact of $\mathcal{P}(\omega_1)$, of order type ω_2 which is Δ_2-definable in the structure $(H_{\aleph_2}, \in)$ with parameter a subset of ω_1. This result was later improved by Caicedo and the author [9] who showed that BPFA implies the existence of a Δ_1-definable well ordering of the reals. Finally, in 2005 Viale [40] resolved a long standing problem by showing that PFA implies SCH. In fact, he produced two proofs of this result. In one he deduced SCH from MRP and in the other he obtained the same conclusion from the P-ideal dichotomy PID introduced by Abraham and Todorčević [3]. It was known previously that PID follows from PFA.

In this paper we survey the recent results concerning the impact of forcing axioms on cardinal arithmetic. Rather than working with forcing axioms directly we will mostly concentrate on three combinatorial principles: stationary set reflection, the Mapping Reflection Principle and the P-ideal dichotomy. They are all consequences of MM and the last two follow from PFA as well. These principles express a certain form of reflection and can be used to get bounds on the value of the continuum and more generally $\kappa^{\aleph_1}$, for $\kappa > \aleph_1$. Therefore, most of the arguments we present will be purely combinatorial and the use of forcing axioms is just to prove the relevant reflection principle.

The paper is organized as follows. In §2 we discuss various versions of stationary set reflection and present a simplified proof of a result of Shelah [32] saying that stationary set reflection implies the Singular Cardinal Hypothesis. In §3 we turn our attention to Moore's Mapping Reflection Principle and its consequences. In §4 we discuss the P-ideal dichotomy PID introduced by Abraham and Todorčević [3] and present a result of Viale [40] saying that PID implies the Singular Cardinal Hypothesis SCH. §5 is devoted to presenting the main ideas of [9] where it is shown that the Bounded Proper Forcing Axiom implies that there is a well ordering of the reals Δ_1 definable with parameter a subset of ω_1. In §6 we consider inner models of strong forcing axioms and present some rigidity results due to Viale. Finally, in §7 we present some open questions and directions for further research. We point out that §2, §3 and §4 are independent of each other, §5 requires §3, while §6 requires all the previous sections.

Our notation is mostly standard. If κ is an infinite cardinal, then H_κ is the family of sets whose transitive closure has cardinality smaller than κ, i.e., $H_\kappa = \{ x \colon |\mathrm{tc}(x)| < \kappa \}$, where $\mathrm{tc}(\cdot)$ denotes transitive closure. We use $V^{\mathcal{P}}$ to denote the Boolean-valued extension of V by the forcing notion $\mathcal{P}$, equivalently, and abusing language, $V^{\mathcal{P}}$ denotes any extension $V[G]$ where G is $\mathcal{P}$-generic over V. For notation and concepts from PCF theory we refer the reader to [2], for all other notation not explicitly defined in this paper as well as an introduction to set theory and forcing we refer the reader to [23].

Ackowledgement The author would like to thank Andres Caicedo, Brice Minaud, and Matteo Viale for their help in the preparation of this paper.

2 Stationary set reflection

The purpose of this section is to present some basic results on reflection of stationary sets and present a proof of a recent result of Shelah [32] saying that stationary set reflection implies the Singular Cardinal Hypothesis. We start by recalling some definitions. Recall that an *algebra* on an infinite set I is just a function $F : I^{<\omega} \to I$. For a subset X of I the closure $cl_F(X)$ of X under F is the smallest subset Y of I containing X and such that $F[Y^{<\omega}] \subseteq Y$. A subset C of $\mathcal{P}(I)$ is *club* if there is an algebra F such that $C = \{X \subseteq I : \mathrm{cl}_F(X) = X\}$. A subset S of $\mathcal{P}(I)$ is *stationary* if $S \cap C \neq \emptyset$, for all club $C \subseteq \mathcal{P}(I)$. These notions generalize the well known notions of club and stationary subsets of a regular cardinal κ. In particular, we have a version of Fodor's theorem for stationary subsets of $\mathcal{P}(I)$. For some basic information concerning club and stationary subsets of $\mathcal{P}(I)$ see [23, chapter 8 and 38].

Example 2.1 *Given an infinite cardinal $\kappa \leq |I|$ the following set is stationary*

$$[I]^\kappa = \{X \subseteq I : |X| = \kappa\}.$$

In what follows we shall restrict our attention to the space $[I]^\omega$ and the notions of club and stationary will be interpreted as relativized to this space. In many applications I will be H_θ, for some fixed regular cardinal θ. In this case stationary sets are those $S \subseteq [H_\theta]^\omega$ such that for any model $(H_\theta, \in, \ldots)$ there is $M \in S$ with $M \prec (H_\theta, \in, \ldots)$. We shall be mostly concerned with the following reflection principle.

Definition 2.2 (Reflection Principle (RP)) *For every regular $\lambda \geq \aleph_2$ the following principle* RP(λ) *holds:*

> *If S is a stationary subset of $[\lambda]^\omega$ then there is $X \subseteq \lambda$ of cardinality $\aleph_1$ such that $S \cap [X]^\omega$ is stationary in $[X]^\omega$.*

RP follows from Martin's Maximum and has a number of interesting consequences. In particular, it implies that every forcing notion preserving stationary subsets of ω_1 is semiproper and this in turn implies that the nonstationary ideal NS_{ω_1} is precipitous, see [16]. Therefore RP has large cardinal strength. However, many applications of MM require an even stronger reflection principle for stationary sets. In order to state this principle we will need the following definition.

Definition 2.3 *A set $S \subseteq [\lambda]^\omega$ is* projective stationary *if for every stationary set $T \subseteq \omega_1$, the set the set $\{X \in S : X \cap \omega_1 \in T\}$ is stationary.*

Definition 2.4 (Strong Reflection Principle (SRP)) *If $\lambda \geq \aleph_2$ is regular then the following principle* SRP(λ) *holds:*

If S is projective stationary in $[H_\lambda]^\omega$ then there is continuous increasing $\in$-chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of H_λ such that $M_\alpha \in S$, for all α.

The following result is implicitly proved in [30].

Theorem 2.5 MM *implies* SRP. $\Box$

SRP is a very strong combinatorial principle and it implies many of the key consequences of MM, such as Strong Chang's Conjecture, the saturation of the nonstationary ideal NS_{ω_1} on ω_1, etc. (see [23], Chapter 37).

Proposition 2.6 SRP(λ) *implies* RP(λ)*, for every regular* $\lambda \geq \omega_2$.

PROOF: We show that SRP(λ) implies the following stronger version RP*(λ) of stationary set reflection:

If S is a stationary subset of $[H_\lambda]^\omega$ then there is a continuous increasing $\in$-chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of H_λ such that $\{\alpha < \omega_1 : M_\alpha \in S\}$ is stationary.

Let $S \subseteq [H_\lambda]^\omega$ be stationary. Since NS_{ω_1} is saturated by SRP($\aleph_2$) there is a stationary $A \subseteq \omega_1$ such that for every stationary $B \subseteq A$ the set $\{M \in S : M \cap \omega_1 \in B\}$ is stationary. Therefore, $T = \{M : M \in S \text{ or } M \cap \omega_1 \notin A\}$ is projective stationary. By SRP(λ), T contains a continuous increasing chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of H_λ. It follows that $M_\alpha \in S$, for all $\alpha \in A$. $\Box$

Concerning the impact of stationary reflection principles on cardinal arithmetic, Foreman, Magidor and Shelah's proof from [16] that MM implies the Singular Cardinal Hypothesis, SCH, can be factored through SRP. Todorčević (unpublished) showed that RP($\aleph_2$) implies $2^{\aleph_0} = \aleph_2$ and the author showed in [38] that RP* implies SCH. The same conclusion was later obtained in [17] form simultaneous reflection of 3 stationary sets in $[\lambda]^\omega$. In the remainder of this section we present a simplified proof of the main result of Shelah [32] saying that RP implies that SCH. We begin by recalling the following.

Definition 2.7 *The Singular Cardinal Hypothesis* (SCH) *states that, for any singular cardinal κ, if $2^{\mathrm{cof}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cof}(\kappa)} = \kappa^+$.*

In order to prove that RP implies SCH we shall need Lemma 2.9 in order to set off the induction in the main proof (Lemma 2.8 is used to prove Lemma 2.9). Then we state and prove the main result in Theorem 2.10. The fact that stationary set reflection implies SCH comes in Corollary 2.17 as a consequence of the main theorem. Finally, in Corollary 2.18, we show that the previous results still hold without any constraint on the size of the reflecting sets.

Lemma 2.8 RP($\aleph_2$) *implies that for every stationary subset S of $[\omega_2]^\omega$ there is $\alpha < \omega_2$ such that $S \cap [\alpha]^\omega$ is stationary in $[\alpha]^\omega$.*

PROOF: Let S be a stationary subset of $[\omega_2]^\omega$, and suppose that S does not reflect in any $\alpha \in \omega_2$. Then for each $\alpha \in \omega_2$ there is a club set $C(f_\alpha)$ in $[\alpha]^\omega$ containing the closure points in $[\alpha]^\omega$ of some function $f_\alpha : [\alpha]^{<\omega} \to \alpha$, and such that $S \cap [\alpha]^\omega \cap C(f_\alpha) = \emptyset$. Letting $f_\alpha(e) = \min(e)$ for $e \in [\omega_2]^{<\omega} \setminus [\alpha]^{<\omega}$, we can extend f_α to a function from $[\omega_2]^{<\omega}$ into ω_2. We denote this new function also by f_α.

By making simple definitions by cases, we can build two functions f and g from $[\omega_2]^{<\omega}$ into ω_2 such that for all $X \subseteq \omega_2$:

(1) if X is closed by f, then for all $\alpha \in X$, X is closed by f_α;
(2) if X is closed by g and $Card(X) = \aleph_1$, then either $X \in \omega_2$ or $\mathrm{ot}(X) = \omega_1$.

The definition of g, for instance, goes as follows.

(1) For $e \in [\omega_2]^n$ with $n > 2$, $g(e) = n - 3$. Thus, if $X \subseteq \omega_2$ is closed by g, then $\omega \subseteq X$.
(2) For $n \in \omega$ and $\xi \in \omega_1 - \omega$, $g(\{n, \xi\}) = h_\xi(n)$, where $h_\xi : \omega \to \xi$ is a fixed bijection. Thus, if $X \subseteq \omega_2$ is closed by g, then $X \cap \omega_1 \in \omega_1 + 1$.
(3) For $\xi \in \omega_1$ and $\alpha \in \omega_2 - \omega_1$, $g(\{\xi, \alpha\}) = i_\alpha(\xi)$, where $i_\alpha : \omega_1 \to \alpha$ is a fixed bijection. Thus, if $X \in [\omega_2]^{\aleph_1}$ is closed by g and $\omega_1 \subseteq X$, then $X \in \omega_2$.
(4) For $\alpha < \beta$ in $\omega_2 - \omega_1$, $g(\{\alpha, \beta\}) = i_\beta^{-1}(\alpha)$. Thus, if $X \in [\omega_2]^{\aleph_1}$ is closed by g and $\mathrm{ot}(X) > \omega_1$, then $X \cap \omega_1$ is unbounded, hence $\omega_1 \subseteq X$ by point 2, hence $X \in \omega_2$ by point 3.
(5) In all other cases $g(e)$ equals 0.

Let $C(f)$ and $C(g)$ be the respective club sets of closure points of f and g in $[\omega_2]^\omega$.

Finally, let $A \in [\omega_2]^{\aleph_1}$ such that $S \cap C(f) \cap C(g)$ reflects in A. A is closed by g, but $A \notin \omega_2$ by hypothesis, so $\mathrm{ot}(A) = \omega_1$. Then the set of proper initial segments of A is a club in $[A]^{\aleph_0}$, so there exists $\alpha \in A$ such that $A \cap \alpha \in S$. Since A is closed under f and $\alpha \in A$, by the choice of f we know that A is closed by f_α, hence, due to the definition of f_α, so is $A \cap \alpha$. On the other hand, since $S \cap [\alpha]^\omega \cap C(f_\alpha) = \emptyset$ and $A \cap \alpha \in S \cap [\alpha]^\omega$, $A \cap \alpha$ cannot be closed under f_α, a contradiction. □

Lemma 2.9 RP($\aleph_2$) *implies* $\aleph_2^{\aleph_0} = \aleph_2$.

PROOF: For each $\alpha \in \omega_2$, let us pick $\langle X_\xi^\alpha : \xi < \omega_1 \rangle$ a continuous increasing sequence of countable subsets of α such that $\bigcup\{X_\xi^\alpha : \xi < \omega_1\} = \alpha$. Let $C = \{X_\xi^\alpha : \xi < \omega_1, \alpha < \omega_2\}$. Notice that $[\omega_2]^\omega \setminus C$ cannot reflect in any $\alpha \in \omega_2$

by the choice of C, so by Lemma 2.8 it does not reflect at all, hence it is not stationary, hence C contains a club set. Since club sets in $[\omega_2]^\omega$ are of size $\aleph_2^{\aleph_0}$ (see [8], Theorem 3.2) and $Card(C) = \aleph_2$, we conclude that $\aleph_2^{\aleph_0} = \aleph_2$. $\square$

The following theorem is due to Shelah [32].

Theorem 2.10 RP *implies that* $\lambda^{\aleph_0} = \lambda$, *for any regular cardinal* $\lambda \geq \aleph_2$.

PROOF: Assume that the theorem does not hold, and let λ be the least counterexample. Basic cardinal arithmetic (along with Lemma 2.9) shows that λ is the successor of some κ of cofinality $\aleph_0$, and $\kappa^{\aleph_0} > \lambda$. Furthermore, Lemma 2.9 implies that $2^{\aleph_0} < \kappa$. Our goal is to show that RP(λ) does not hold. To that end we will need the following notion from PCF theory [31]. We borrow the terminology from [12].

Definition 2.11 *Given a sequence* $\langle \mu_\alpha : \alpha < \beta \rangle$ *of regular ordinals and an ideal* I *on* β, *an* I-scale *on* $\langle \mu_\alpha : \alpha < \beta \rangle$ *is a* $<_I$*-strictly increasing and cofinal sequence* $\langle f_\xi : \xi < \gamma \rangle$ *in* $\prod_{\alpha<\beta} \mu_\alpha$. *The scale* $\langle f_\xi : \xi < \gamma \rangle$ *is said to be* better *if and only if, for every ordinal* $\alpha < \gamma$ *with* $\mathrm{cof}(\alpha) > \beta$, *there exists a club set* $C \subseteq \alpha$, *and* $Z_i \in I$, *for each* $i \in C$, *such that if* $i, j \in C$ *and* $i < j$ *then* $f_i \restriction (\beta \setminus (Z_i \cup Z_j)) < f_j \restriction (\beta \setminus (Z_i \cup Z_j))$.

PCF theory shows that we can choose an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals in κ with limit κ so as to have a *better scale* $\langle f_\xi : \xi < \lambda \rangle$ on $\langle \kappa_n : n < \omega \rangle$, with respect to the ideal FIN of finite subsets of ω (see [31], II, Claim 1.5A). In fact, for the purpose of this proof, we shall only need the *better* scale property to hold for α of cofinality $\aleph_1$.

For $X \subseteq$ ORD, let $\delta(X) = \sup(X \cap \lambda)$, and $\chi(X)(n) = \sup(X \cap \kappa_n)$. Most of the proof will hinge on the comparison, for $X \in [\lambda]^\omega$, between $\chi(X)$ and $f_{\delta(X)}$. Let us then define:

$$E_X = \{n < \omega : \chi(X)(n) \leq f_{\delta(X)}(n)\}$$

Let $\{A_\xi : \xi < \omega_1\}$ be a family of pairwise almost-disjoint infinite subsets of ω, that is, for all $\xi \neq \zeta$ in ω_1, $A_\xi \cap A_\zeta$ is finite; and let $\phi : \mathcal{P}(\omega) \to \omega_1$ be a partial function defined by

$$\phi(E) = \min\{\xi < \omega_1 : Card(A_\xi \cap E) = \aleph_0\}$$

if such an ordinal ξ exists. Finally, let us consider the set:

$$\mathcal{S} = \{X \in [\lambda]^\omega : \phi(E_X) \text{ is defined and } \phi(E_X) \geq \mathrm{ot}(X), \\ X \text{ is closed by } x \mapsto f_x(n) \text{ for all } n\}$$

We are going to show that $\mathcal{S}$ is stationary, yet does not reflect in any $A \in [\lambda]^{\aleph_1}$.

Claim 2.12 *$\mathcal{S}$ does not reflect in any $A \in [\lambda]^{\aleph_1}$.*

PROOF: Let us assume to the contrary that $\mathcal{S}$ reflects in some $A \in [\lambda]^{\aleph_1}$. Let $\langle X_i : i < \omega_1 \rangle$ be a continuous cofinal increasing sequence of countable subsets of A, and let $R = \{i < \omega_1 : X_i \in \mathcal{S}\}$. Since $\{X_i : i < \omega_1\}$ is club in $[A]^\omega$, saying that $\mathcal{S}$ reflects in A is the same as saying that $\{X_i : i \in R\}$ is stationary in $[A]^\omega$, or that R is stationary in ω_1. First, we show that $\mathrm{cof}(\sup(A)) = \aleph_1$.

Let us assume towards contradiction that $\mathrm{cof}(\sup(A)) < \aleph_1$. Then there exists $\gamma \in \omega_1$ such that $\sup(X_\gamma) = \sup(A)$. Let $\delta = \sup(A)$. Now for any $n \in \omega$, let $\alpha(n) = \min\{\alpha \in R : \chi(X_\alpha)(n) > f_\delta(n)\}$, if such α exists, and $\alpha(n) = \min(R)$ otherwise. Letting $\alpha = \max\{\gamma, \sup_{n\in\omega} \alpha(n)\}$, for all $\beta \geq \alpha$, $E_{X_\alpha} = E_{X_\beta}$, and in particular $\phi(E_{X_\beta}) = \phi(E_{X_\alpha})$. However, since $X_\beta \in \mathcal{S}$, we also have $\phi(E_{X_\beta}) \geq \mathrm{ot}(X_\beta)$, and $\mathrm{ot}(X_\beta)$ converges to ω_1, so there is a contradiction.

Since $\mathrm{cof}(\sup(A)) = \aleph_1$, we may assume that $\delta(X_i) = \sup(X_i)$ is strictly increasing, trimming $\langle X_i : i < \omega_1 \rangle$ if necessary. Let $\delta_i = \delta(X_i)$, and let $\beta_i = \min(A \setminus \delta_i)$. Trimming $\langle X_i : i < \omega_1 \rangle$ two more times, we can ensure that:

$$\forall i < j \in R,\ (\beta_i < \delta_j) \wedge (\beta_i \in X_j) \tag{2.1}$$

Now, by applying the better scale property of $\langle f_\xi : \xi < \gamma \rangle$ to $\delta(A)$, there exists a club set $C \subseteq \omega_1$, along with a sequence $\langle n_i : i \in C \rangle$ of elements of ω such that for $i < j \in R \cap C$ and $n \geq n_i, n_j$, we have $f_{\delta_i}(n) < f_{\delta_j}(n)$. As $i \mapsto n_i$ yields a partition of $C \cap R$ into $\aleph_0$ subsets, one of them is stationary, let us rename it R. Thus, we may assume that there exists $k \in \omega$ such that for all $i < j$ in R, $f_{\delta_i} \restriction [k, \omega) < f_{\delta_j} \restriction [k, \omega)$.

Because of (2.1), we know that for i in R if $j = \min(R \setminus (i+1))$ then $f_{\delta_i} \leq_{FIN} f_{\beta_i} <_{FIN} f_{\delta_j}$, so there exists $m_i \in \omega$ such that for all $n \geq m_i$, $f_{\delta_i}(n) \leq f_{\beta_i}(n) < f_{\delta_j}(n)$. By the same reasoning as before, we can thin R so as to have $m_i = m$ a constant, and increase k so that $k \geq m$. As a result:

$$\forall i < j \in R,\ f_{\delta_i} \restriction [k, \omega) \leq f_{\beta_i} \restriction [k, \omega) < f_{\delta_j} \restriction [k, \omega) \tag{2.2}$$

Now let $f \in \prod_{n<\omega} \kappa_n$ be defined by $f(n) = \sup_{i \in R} f_{\beta_i}(n)$ if $n \geq k$ and 0 otherwise. Because of (2.1) and the closure properties of $\mathcal{S}$, for $i < j \in R$ and $n \in \omega$ we have $f_{\beta_i}(n) \in X_j$, so $f(n) \leq \sup(\bigcup_{i\in R}(X_i \cap \kappa_n)) = \chi_A(n)$. Let $B = \{n \in [k, \omega) : f(n) = \chi_A(n)\}$ and $\overline{B} = \{n \in [k, \omega) : f(n) < \chi_A(n)\} = [k, \omega) \setminus B$. We are going to prove that, for all i in some stationary subset of R, $f_{\delta_i} \restriction B \geq \chi(X_i) \restriction B$ and $f_{\delta_i} \restriction \overline{B} < \chi(X_i) \restriction \overline{B}$.

Let $n \in B$. Since $f(n) = \chi(A)(n)$ and $\mathrm{cof}(f(n)) = \aleph_1$, we can define a club set C_n in $\chi(A)(n)$ such that for all $i < j$ in C_n, $\chi(X_i)(n) < f_{\delta_j}(n)$, and also $f_{\delta_i}(n) \in X_j$. As a result, for l a limit point of C_n, we get $f_{\delta_l}(n) \geq$

$\bigcup_{i\in C_n\cap l} f_{\delta_i}(n) = \bigcup_{i\in C_n\cap l} \chi(X_i)(n) = \chi(X_l)(n)$. Let D_n be the club set of limit points of C_n, as $R \cap \bigcap\{D_n : n < \omega\}$ is stationary, we can rename it R again, and so we may assume that for all $i \in R$ we have $f_{\delta_i} \restriction B \geq \chi(X_i) \restriction B$.

Let $n \in \overline{B}$. Since $f(n) < \chi(A)(n)$, there exists $i(n) \in \omega_1$ such that $f(n) < \chi(X_{i(n)}(n))$. As $\sup_{n\in B}(i(n)) < \omega_1$, $R \setminus \sup_{n\in B}(i(n))$ is stationary and we can rename it again R. Thus, for all $i \in R$ we have $f_{\delta_i} \restriction \overline{B} < \chi(X_i) \restriction \overline{B}$.

We have shown that for $i \in R$, $\{n \in [k,\omega) : \chi(X_i)(n) \leq f_{\delta_i}(n)\} = B$, so $E_{X_i} =_{FIN} B$. In particular, $\phi(E_{X_i})$ remains constant on R. That is a contradiction, since $\phi(E_{X_i}) \geq \text{ot}(X_i)$ and $\text{ot}(X_i)$ converges to ω_1. □

Claim 2.13 *$\mathcal{S}$ is stationary.*

PROOF: Let C be a club in $[\lambda]^\omega$ and fix a function $f_C : [\lambda]^{<\omega} \to \lambda$ such that $C = \{X \in [\lambda]^\omega : cl_{f_C}(X) = X\}$. We need to find $X \in \mathcal{S}$ such that $f_C[X^{<\omega}] \subseteq X$. In order to build such a set $X \in \mathcal{S}$, the main issue is to control both E_X and $\text{ot}(X)$, so that $\phi(E_X) \geq \text{ot}(X)$. For that purpose, we consider a closed two-player game G_ε, for each choice of $\varepsilon \in \omega_1$. Player 1 sets up constraints that will later on allow us to control E_X and ensure that $\phi(E_X) \geq \varepsilon$; meanwhile, player 2 tries to meet these constraints, build the set X, bound $\chi(X)$, as well as prove that $\text{ot}(X) \leq \varepsilon$.

In the first part of the proof, we show that player 2 has a winning strategy for some $\varepsilon \in \omega_1$. In the second part, we show how player 1 should play against that strategy in order to obtain X as required. We begin by describing G_ε.

Let θ be a sufficiently large regular cardinal, say $\theta = (2^\lambda)^+$, and let us fix a well-ordering $\lhd$ of H_θ. For $X \subseteq \text{ORD}$, let $sk(X)$ be the Skolem hull of X in $(H_\theta, \in, \lhd)$, $sk_\lambda(X) = sk(X) \cap \lambda$, and $cl(X) = sk_\lambda(X \cup \{\langle f_\xi : \xi < \lambda\rangle, f_C\})$. Then we can find functions $t_n : \lambda^n \to \lambda$ such that if $X = \{\alpha_n : n < \omega\}$ is a subset of λ then $cl(X) = \{t_n(\alpha_0, \ldots, \alpha_n) : n < \omega\}$.

At stage n the game G_ε proceeds as follows.

(1) (a) If n is of the form $2l$: player 1 picks an ordinal $\xi_{2l} \in \kappa_l$. Player 2 then picks α_{2l} and γ_{2l} in κ_l such that $\xi_{2l} \leq \alpha_{2l} \leq \gamma_{2l}$.
 (b) If n is of the form $2l+1$: player 1 picks an ordinal $\xi_{2l+1} \in \lambda$. Player 2 then picks α_{2l+1} in λ such that $\xi_{2l+1} \leq \alpha_{2l+1}$.
(2) Player 2 chooses an ordinal ζ_n in ε.

Once the game is over, let $X = cl(\{\alpha_n : n \in \omega\})$ and $\tau_n = t_n(\alpha_0, \ldots, \alpha_n)$, for all n. Then $\{\tau_n : n < \omega\}$ constitutes an enumeration of X.

We say that player 2 wins the game iff:

(1) For all $n \in \omega$, $X \cap \kappa_n \subseteq \gamma_{2n}$.
(2) The mapping $g : X \to \varepsilon$ given by $\tau_n \overset{g}{\mapsto} \zeta_n$, for all n, is well-defined and order preserving. As such g witnesses $\mathrm{ot}(X) \leq \varepsilon$.

Fact 2.14 *There exists $\varepsilon \in \omega_1$ such that player 2 has a winning strategy for G_ε.*

PROOF: Let $\varepsilon \in \omega_1$. The first point is that the game G_ε is closed, because if player 2 loses, that loss is apparent after a finite number of moves. Indeed, if at the end of the game, for some $n \in \omega$, $X \cap \kappa_n \nsubseteq \gamma_n$, then some element τ_n of X witnesses it and the value τ_n can be computed at stage n of the game. The same goes for the second winning condition.

Since G_ε is closed, the Gale-Stewart theorem [18] guarantees that one of the two players has a winning strategy. Let us assume towards contradiction that player 1 has a winning strategy σ_ε, for all $\varepsilon \in \omega_1$. The crux of the matter here is that player 1's best interest is always to play ξ_n as high possible. In particular, if we modify σ_ε to increase player 1's answer ξ_n to some sequence of moves by player 2, we still get a winning strategy.

Assuming that player 1 follows the strategy σ_ε, for any given sequence s of moves by player 2 up to step n of the game, let $\sigma_\varepsilon(s)$ be the answer ξ_n dictated to player 1 by his strategy σ_ε (letting $\sigma_\varepsilon(s) = 0$ if s is not a possible sequence of moves for player 2 when player 1 applies σ_ε). We can define a new strategy σ for player 1 by $\sigma(s) = \sup\{\sigma_\varepsilon(s) : \varepsilon \in \omega_1\}$. Due to the remark above, σ is a winning strategy for player 1 for all the games G_ε.

Let us assume for a while that player 2 always plays $\alpha_n = \xi_n$, i.e. as the first part of his move he copies the move of player 1 and then chooses some γ_n (if n is even), and some $\zeta_n < \varepsilon$. We let player 1 follow his winning strategy σ. Thus, up to step $2n$, this subgame is determined by player 2's choices of the $\gamma_{2i} < \kappa_i$, for $i \leq n$, and the $\zeta_n < \varepsilon < \omega_1$. As a result, there are only κ_n possible sequences of moves up to step $2n+1$ (we may assume $\omega_1 < \kappa_0$), so the set of all possible plays ξ_{2n+2} by player 1 is bounded in κ_{n+1}. Thus, improving σ if necessary, we can assume that player 1's moves are independent of all the previous moves. Let $\langle \xi_n : n < \omega \rangle$ be the sequence of player 1's moves.

Let us now turn back to the regular games G_ϵ, and play as player 2 against strategy σ using the following strategy of our own. We are going to play $\alpha_n = \xi_n$ at every turn, so we know from the start the set $X = cl(\{\alpha_n : n < \omega\}) = cl(\{\xi_n : n < \omega\})$. Let $\varepsilon = \mathrm{ot}(X)$. We play in the game G_ε.

Since we know X we can fix an order preserving mapping $g : X \to \varepsilon$ in advance. At each turn, we play $\alpha_n = \xi_n$ and $\gamma_n = \sup(X \cap \kappa_n)$ in case n is even. We can also compute $\tau_n = t_n(\alpha_0, \ldots, \alpha_n)$ and we play $\zeta_n = g(\tau_n)$. By playing in this way player 2 clearly wins the game, so σ is not a winning strategy, which is a contradiction. $\square$

Let then $\varepsilon \in \omega_1$ be such that player 2 has a winning strategy τ for G_ε. If we play against τ, we know that we will get a set X such that $\mathrm{ot}(X) \leq \varepsilon$, $X \in C$, and X is closed by all the relevant functions, thus the last remaining point is to ensure $\phi(E_X) \geq \varepsilon$. Letting $A = A_\varepsilon$, we are going to achieve this by having $E_X =_{FIN} A$.

First, we need a countable $M \prec H_\theta$ such that $\chi(M) \leq_{FIN} f_{\delta(M)}$, and M contains all the relevant objects: $\langle f_\xi : \xi < \lambda \rangle$, $\langle \kappa_i : i < \omega \rangle$, τ, f_C. To obtain such an M we build a continuous $\in$-chain $\langle M_\zeta : \zeta < \omega_1 \rangle$ of countable elementary submodels of H_θ such that the aforementioned objects belong to M_0. Let $\delta_\zeta = \sup(M_\zeta \cap \lambda)$, for all ζ, and let $\delta = \sup\{\delta_\zeta : \zeta < \omega_1\}$. Then $\chi(M_\zeta) <_{FIN} f_{\delta_{\zeta+1}} < \chi(M_{\zeta+2})$, for all ζ. Since $\{\delta_\zeta : \zeta < \omega_1\}$ is a club in δ we can use the better scale property of $\langle f_\xi : \xi < \lambda \rangle$ at δ to find a club D in ω_1 and l_ζ, for each $\zeta \in D$, such that if $\zeta, \eta \in D$ and $n \geq l_\zeta, l_\eta$ then $f_{\delta_\zeta}(n) < f_{\delta_\eta}(n)$. Moreover, by increasing the l_ζ if necessary, we may assume that if $\zeta \in D$ and we let $\zeta^+ = \min(D \setminus (\zeta + 1))$ then $\chi(M_\zeta)(n) < f_{\delta_{\zeta^+}}(n)$, for all $n \geq l_\zeta$. Let $k_\zeta = \max(l_\zeta, l_{\zeta^+})$, for each $\zeta \in D$. The map $\zeta \mapsto k_\zeta$ yields a partition of D into countably many pieces. Therefore, there is k such that the set $E = \{\zeta \in D : k_\zeta = k\}$ is stationary in ω_1. Fix a limit point η of E, an increasing sequence $\langle \eta(i) : i < \omega \rangle$ of elements of E converging to η and let $M = M_\eta$. We now have that for every every i and every $n \geq k$,

$$f_{\delta_{\eta(i)}}(n) < \chi(M_{\eta(i+1)})(n) < f_{\delta_{\eta(i+2)}}(n) < f_{\delta_\eta}(n).$$

Since $\chi(M) = \sup\{\chi(M_{\eta(i)}) : i < \omega\}$ and $\delta(M) = \delta_\eta$ it follows that $\chi(M) \restriction [k, \omega) \leq f_{\delta(M)} \restriction [k, \omega)$, as required.

Now, for simplicity of notation let us assume that $\chi(M)(n) \leq f_{\delta(M)}(n)$, for all n, i.e. that $k = 0$.

Fact 2.15 $\delta(sk_\lambda(M \cup \kappa)) = \delta(M)$. *Similarly,* $\chi(sk_\lambda(M \cup \kappa_n))(n+1) = \chi(M)(n+1)$, *for all* $n < \omega$.

PROOF: Let $\alpha \in sk_\lambda(M \cup \kappa)$. Then here exists a Skolem term z_1 with parameters $x_1, \ldots, x_m$ in M and $y_1, \ldots, y_n$ in κ such that $\alpha = z_1(\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n)$. Now if we consider the Skolem function $z_2(\beta_1, \ldots, \beta_m)$ representing

$$\sup\{z_1(\beta_1, \ldots, \beta_m, x_1, \ldots, x_n) : x_1, \ldots, x_n \in \kappa\}$$

it is clear that $z_2(\beta_1, \ldots, \beta_m) \geq \alpha$. Then $z_2(\beta_1, \ldots, \beta_m) \in M$, since all the β_i are in M. Necessarily $z_2(\beta_1, \ldots, \beta_m) < \lambda$, for cofinality reasons. Since $\alpha \in sk_\lambda(M \cup \kappa)$ was arbitrary, it follows that $\delta(sk_\lambda(M \cup \kappa) \leq \delta(M)$. The converse also holds since $M \cap \lambda \subset sk_\lambda(M \cup \kappa)$, so we have an equality. We can apply the same reasoning to obtain the second equality. $\square$

Let $\langle m_i : i < \omega \rangle$ be an enumeration of $M \cap \lambda$. Now we play the game G_ε as player 1 against player 2's winning strategy τ as follows.

(1) At step $2l$:
   (a) if $l \in A$, we play $\xi_{2l} = 0$;
   (b) if $l \notin A$, we play $\xi_{2l} = f_{\delta_M}(l) + 1$.
(2) At step $2l+1$, we play $\xi_{2l+1} = m_l$.

At stage $2l$, player 2 played $\alpha_{2l}, \gamma_{2l}, \zeta_{2l}$ such that $\xi_{2l} \leq \alpha_{2l} \leq \gamma_{2l} < \kappa_l$, and at stage $2l+1$ player 2 played $\alpha_{2l+1}, \zeta_{2l+1}$ such that $\xi_{2l+1} \leq \alpha_{2l+1} < \lambda$. All the ζ_n are less than ε. As before let $\tau_n = t_n(\alpha_0, \ldots, \alpha_n)$ and let $X = cl(\{\alpha_n : n < \omega\}) = \{\tau_n : n < \omega\}$. Since player 2 played following his winning strategy we know that the map $\tau_n \mapsto \zeta_n$ witnesses that X is of order type at most ε and that $\chi(X)(l) \leq \gamma_{2l}$, for all l.

Fact 2.16 $E_X = A$.

PROOF: Since player 1's move at stage $2l+1$ is $\xi_{2l+1} = m_l$ and $\alpha_{2l+1} \geq \xi_{2l+1}$ it follows that $\delta(M) \leq \delta(X)$. Conversely, $X \subset sk_\lambda(M \cup \kappa)$, so Fact 2.15 yields $\delta_M = \delta_X$.

Let us look at step $2l$ of the game we have described. First, if $l \in A$, player 1 played $\xi_{2l} = 0$. Since all the moves of the game up to that point as well as the winning strategy τ belong to $M \cup \kappa_{l-1}$ the move played by player 2 at this stage belongs to $sk(M \cup \kappa_{l-1})$, as well. Therefore

$$\chi(X)(l) \leq \gamma_{2l} \leq \chi(sk_\lambda(M \cup \kappa_{l-1}))(l) = \chi(M)(l) \leq f_{\delta(M)}(l) = f_{\delta(X)}(l).$$

Therefore $l \in E_X$.

Second, if $l \notin A$, player 1 played $\xi_{2l} = f_{\delta(M)}(l) + 1$. Since $\xi_{2l} \leq \alpha_{2l} < \kappa_l$ and $\delta(X) = \delta(M)$ we have that $\chi(X)(l) > f_{\delta(X)}(l)$, i.e. $l \notin E_X$. $\square$

Corollary 2.17 RP *implies* SCH.

PROOF: Assume to the contrary that SCH does not hold and let κ be the first cardinal for which SCH fails. Silver's theorem [33] implies that $\mathrm{cof}(\kappa) = \aleph_0$, so we have $\kappa^{\aleph_0} > \kappa^+$. Theorem 2.10, on the other hand, states that $(\kappa^+)^{\aleph_0} = \kappa^+$, so there is a contradiction. $\square$

Corollary 2.18 *Assume for all regular $\lambda \geq \aleph_2$ every stationary subset of $[\lambda]^\omega$ reflects in some subset of λ of cardinality $< \lambda$. Then $\lambda^{\aleph_0} = \lambda$, for all regular $\lambda \geq \aleph_2$. In particular,* SCH *holds.*

PROOF: That is, Theorem 2.10 still holds if we allow reflection in any $A \in [\lambda]^{<\lambda}$. This stems from the fact that, in the proof of Theorem 2.10, the set $\mathcal{S}$ does not actually reflect in any $A \in [\lambda]^{<\lambda}$.

Assume towards contradiction that $\mathcal{S}$ reflects in some $A \subseteq \lambda$ of cardinality $< \lambda$. First, suppose that $Card(A) < \kappa$. Then we can collapse $Card(A)$ to ω_1 by the usual forcing with countable conditions $Coll(\omega_1, \kappa)$. Since this forcing does not add new ω-sequences and preserves stationary sets, using the same definition in the generic extension $V[G]$ we get the same set $\mathcal{S}$. Moreover, $\mathcal{S}$ still reflects in A, while $Card(A)$ becomes ω_1. However, this contradicts Theorem 2.10 applied in $V[G]$. As a side effect, however, since ordinals in λ that were formerly of cofinality greater than $\aleph_1$ may end up after the forcing with cofinality $\aleph_1$ and we still need to apply the *better* scale property to them, we now have to use the better scale property on all ordinals of cofinality greater than or equal to $\aleph_1$ and not just on those of cofinality $\aleph_1$, as was the case in Theorem 2.10.

Now suppose that $Card(A) = \kappa$. Let $\delta = \sup(A)$ and $\gamma = \mathrm{cof}(\delta)$. Since κ is singular, we have $\gamma < \kappa$. Recall that in Claim 2.12, we have shown that $\mathrm{cof}(\sup(A)) > \aleph_0$. The reasoning we used does not really depend on $Card(A)$ and still holds. Thus we can apply the better scale property of $\langle f_\xi : \xi < \lambda \rangle$ to δ in order to conclude that the set $\{n \in \omega : \mathrm{cof}(f_\delta(n)) = \gamma\}$ is cofinite. Let then $m \in \omega$ be such that $n \geq m$ implies $\mathrm{cof}(f_\delta(n)) = \gamma$.

Let $U_0 = \{n \geq m : \chi(A)(n) = f_\delta(n)\}$. For each $n \in U_0$ choose $B_n \subseteq A \cap [\kappa_{n-1}, \kappa_n)$ cofinal in $\chi(A)(n)$ and of cardinality γ. Let $U_1 = \{n \geq m : \chi(A)(n) > f_\delta(n)\}$. For each $n \in U_1$ let $\alpha_n = \min(A \setminus (f_\delta(n) + 1))$ and let $R = \{\alpha_n : n \in U_1\}$. Finally, let $U_3 = (\omega \setminus m) \setminus (U_0 \cup U_1)$. Since $\mathrm{cof}(\delta) > \aleph_0$, using the better scale property at δ we can find $\alpha < \delta$ such that $f_\alpha > \chi(A)(n)$, for all but finitely many elements of U_3. Let $B_\lambda \subseteq A \cap [\kappa, \delta)$ be cofinal in δ and of cardinality γ. Finally, let

$$B = \bigcup \{B_n : n \in U_0\} \cup B_\lambda \cup R \cup \{\alpha\}.$$

Increasing B if necessary, we may assume that B is closed by $x \mapsto f_x(n)$, for all n (because A satisfies this condition).

The construction of B ensures that the set

$$\begin{aligned} D = \{X \in [A]^\omega : R \cup \{\alpha\} \subseteq X, \delta(X) = \delta(X \cap B) \\ \wedge\, n \in U_0 \implies \chi(X)(n) = \chi(X \cap B)(n)\} \end{aligned}$$

is club in $[A]^\omega$. The construction of D, in turn, ensures that for each $X \in \mathcal{S} \cap D$, $E_X =_{FIN} E_{X \cap B}$ (recall that $E_X = \{n < \omega : \chi(X)(n) \leq f_{\delta(X)}(n)\}$), therefore $\phi(X \cap B) = \phi(X)$. On the other hand, obviously $\mathrm{ot}(X \cap B) \leq \mathrm{ot}(X)$. By the definition of $\mathcal{S}$ this means that if $X \in \mathcal{S} \cap D$ then $X \cap B \in \mathcal{S}$. As a result $\mathcal{S}$ reflects in B. Indeed, let C be a club set in $[B]^\omega$. Then $C_A = \{X \in [A]^\omega : X \cap B \in C\}$ is also a club. Let $X \in \mathcal{S} \cap D \cap C_A$. It follows that $X \cap B \in \mathcal{S} \cap C$. Therefore, $\mathcal{S}$ reflects in B, as well. However, B is of cardinality $\gamma < \kappa$ and by the previous case this give a contradiction. □

3 Mapping Reflection Principle

The Mapping Reflection Principle was introduced by Moore in [28] and used by him and others to obtain a number of interesting consequences. It can be thought of as a localized version of stationary set reflection, but is in fact independent of the statements RP and SRP discussed in the previous section. While PFA does not imply RP it does imply Mapping Reflection Principle and, indeed, proofs of some important consequences of PFA can be factored through MRP. In particular, MRP plays a key role in Moore's proof [27] that PFA implies the existence of a five element basis for the class of uncountable linear orderings. In this section we present this principle and discuss some of its consequences. We start by recalling the relevant definitions from [28].

Definition 3.1 *Let θ be a regular cardinal, let X be uncountable, and let M be a countable elementary submodel of H_θ such that $[X]^\omega \in M$. A subset Σ of $[X]^\omega$ is* M-stationary *iff for all $E \in M$ such that $E \subseteq [X]^\omega$ is club, $\Sigma \cap E \cap M \neq \emptyset$.*

Recall that the Ellentuck topology on $[X]^\omega$ is obtained by declaring a set open iff it is the union of sets of the form

$$[x, N] = \{Y \in [X]^\omega \colon x \subset Y \subseteq N\}$$

where $N \in [X]^\omega$ and $x \subset N$ is finite.

Definition 3.2 Σ *is an* open stationary set mapping *if and only if there is an uncountable set $X = X_\Sigma$ and a regular cardinal $\theta = \theta_\Sigma$ such that $[X]^\omega \in H_\theta$,* $\mathrm{dom}\,(\Sigma)$ *is a club in $[H_\theta]^\omega$ consisting of elementary submodels and for all $M \in \mathrm{dom}\,(\Sigma)$, $X \in M$ and $\Sigma(M) \subseteq [X]^\omega$ is open in the Ellentuck topology on $[X]^\omega$ and M-stationary.*

The Mapping Reflection Principle MRP is the following statement.

> *If Σ is an open stationary set mapping, there is a continuous $\in$-chain $\vec{N} = \langle N_\xi \colon \xi < \omega_1 \rangle$ of elements in the domain of Σ such that for all limit ordinals $\xi < \omega_1$ there is $\nu < \xi$ such that $N_\eta \cap X \in \Sigma(N_\xi)$ for all η such that $\nu < \eta < \xi$.*

If $\langle N_\xi \colon \xi < \omega_1 \rangle$ satisfies the conclusion of MRP for Σ then it is said to be a *reflecting sequence* for Σ. The following was proved in [28].

Theorem 3.3 PFA *implies* MRP. $\square$

Moore also showed in [28] that MRP implies the failure of the principle $\square(\kappa)$, for all regular $\kappa > \aleph_1$. Thus, MRP has considerable large cardinal strength. Moreover, Sharon [29] showed that MRP implies the failure of $\square_{\kappa,\omega}$, for all regular $\kappa \geq \aleph_1$, and that MRP together with $\mathrm{MA}_{\aleph_1}$ implies the failure of

$\square_{\kappa,\omega_1}$, for all regular $\kappa \geq \aleph_1$. On the other hand, MRP is *not* equivalent to PFA, and in fact it does not even imply MA. This is a consequence of the fact that the posets used to force the individual instances of MRP are proper and do not add reals and therefore they are ω^ω-bounding and the fact that ω^ω-bounding is preserved by countable support iteration of proper forcing notions, see [1].

Concerning the impact of MRP on cardinal arithmetic, Moore also showed in [28] that MRP implies $2^{\aleph_0} = \aleph_2$. In fact, for this result one only needs a bounded version of MRP and thus obtains that even BPFA implies $2^{\aleph_0} = \aleph_2$. Somewhat later, Viale [40] showed that MRP and therefore also PFA implies SCH, and thus resolved a long standing open problem. We now sketch Moore's proof that MRP implies $2^{\aleph_0} = \aleph_2$. We begin with the following definition.

Definition 3.4 *A sequence $\vec{C} = \langle C_\xi : \xi < \omega_1$ and $\lim(\xi)\rangle$ is called a C-*sequence *if C_ξ is an unbounded subset of ξ of order type ω, for all limit $\xi < \omega_1$.*

Fix a C-sequence $\vec{C}$. For $N \subseteq M$ countable sets of ordinals such that $\mathrm{ot}(M)$ is a limit ordinal and $\sup(N) < \sup(M)$, set

$$w(N, M) = |\sup(N) \cap \pi^{-1}[C_\alpha]|$$

where $\alpha = \mathrm{ot}(M)$ and $\pi : M \to \alpha$ is the transitive collapse of M.

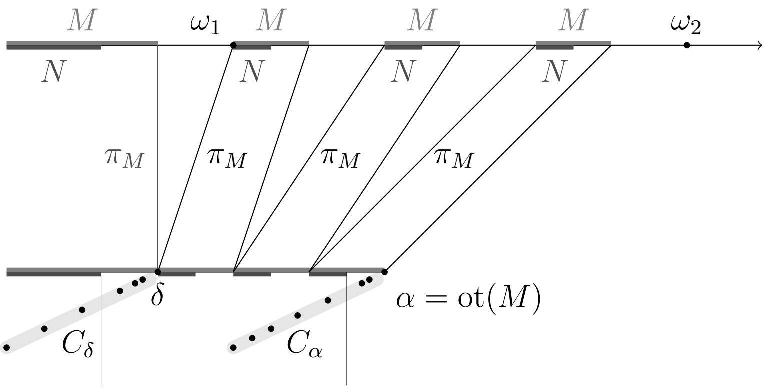

Fig. 1. $w(N \cap \omega_1, M \cap \omega_1) = 3$, $w(N, M) = 5$

Definition 3.5 *Let $A \subseteq \omega_1$. Then $\upsilon_{\mathrm{AC}}(A)$ holds iff there is an uncountable $\delta < \omega_2$ and an increasing continuous sequence $\langle N_\xi : \xi < \omega_1\rangle$ of countable subset of δ whose union is δ such that for all limit ordinals $\nu < \omega_1$ there is a $\nu_0 < \nu$ such that for all $\xi \in (\nu_0, \nu)$,*

$$N_\nu \cap \omega_1 \in A \textit{ iff } w(N_\xi \cap \omega_1, N_\nu \cap \omega_1) < w(N_\xi, N_\nu).$$

The principle υ_{AC} states that $\upsilon_{\mathrm{AC}}(A)$ holds for all $A \subseteq \omega_1$.

Theorem 3.6 MRP *implies that* υ_{AC} *holds. This also follows from* BPFA*.*

PROOF: It suffices to prove the following.

Lemma 3.7 *Let* M *be a countable elementary submodel* $H_{(2^{\aleph_1})^+}$*. Then* $\Sigma_<(M)$ *and* $\Sigma_\geq(M)$ *are open in the Ellentuck topology on* $[\omega_2]^\omega$ *and* M*-stationary, where*

$$\Sigma_<(M) = \{\, N \in [M \cap \omega_2]^\omega \colon w(N \cap \omega_1, M \cap \omega_1) < w(N, M) \,\}$$

and

$$\Sigma_\geq(M) = \{\, N \in [M \cap \omega_2]^\omega \colon w(N \cap \omega_1, M \cap \omega_1) \geq w(N, M) \,\}.$$

To see that the theorem follows from the lemma, let $A \subseteq \omega_1$ and notice that Lemma 3.7 implies immediately that Σ_A is open stationary where, for a countable elementary submodel M of $H_{(2^{\aleph_1})^+}$ we let

$$\Sigma_A(M) = \begin{cases} \Sigma_<(M) \text{ if } M \cap \omega_1 \in A \\ \Sigma_\geq(M) \text{ if } M \cap \omega_1 \notin A. \end{cases}$$

Let $\langle N_\xi \colon \xi < \omega_1 \rangle$ be a reflecting sequence for Σ_A and let $\delta = \omega_2 \cap \bigcup\{N_\xi : \xi < \omega_1\}$. Then δ and $\langle N_\xi \cap \omega_2 \colon \xi < \omega_1 \rangle$ witness $\upsilon_{\mathrm{AC}}(A)$.

To see that BPFA suffices, notice that for each $A \subseteq \omega_1$, $\upsilon_{\mathrm{AC}}(A)$ is a Σ_1 statement in the parameters A and $\vec{C}$. By the proof of Theorem 3.3 there is a proper poset $\mathcal{P}$ that forces $\upsilon_{\mathrm{AC}}(A)$. It follows that $\upsilon_{\mathrm{AC}}(A)$ holds under BPFA.

PROOF of Lemma 3.7: To see that $\Sigma_<(M)$ is open in the Ellentuck topology let $N \in \Sigma_<(M)$. Let $\beta \in N$ be such that

$$\sup(N) \cap \pi^{-1}[C_{ot(M \cap \omega_2)}] \subset \beta,$$

and let $\gamma \in N \cap \omega_1$ be such that $\sup(N \cap \omega_1) \cap \pi^{-1}[C_{ot(M \cap \omega_1)}] \subset \gamma$, where $\pi : M \cap \omega_2 \to ot(M \cap \omega_2)$ is the transitive collapse of $M \cap \omega_2$. β and γ exist because $N \in M$ is countable so $\sup(N) < \sup(M \cap \omega_2)$ and $\sup(N \cap \omega_1) < \sup(M \cap \omega_1)$, and each C_α has order type ω. Then $[\{\beta, \gamma\}, N] \subseteq \Sigma_<(M)$. Exactly the same argument works for $\Sigma_\geq(M)$.

To see that $\Sigma_<(M)$ is M-stationary, let $E \in M$ be club in $[\omega_2]^\omega$. By the pigeonhole principle there is $\gamma < \omega_1$ such that

$$\{\sup(N) : N \in E \text{ and } N \cap \omega_1 \subseteq \gamma\}$$

is unbounded in ω_2. By elementarity of M, there is $\gamma < M \cap \omega_1$ such that $\{\sup(N) : N \in E \cap M \text{ and } N \cap \omega_1 \subseteq \gamma\}$ is unbounded in $M \cap \omega_2$, so we can find $N \in E \cap M$ such that $N \cap \omega_1 \subseteq \gamma$ and

$$w(N, M \cap \omega_2) = |\sup(N) \cap \pi^{-1}[C_{ot(M\cap\omega_2)}]| > |C_{M\cap\omega_1} \cap \gamma|.$$

As before, N is countable and belongs to M, so $N \subset M$ and $\sup(N) < \sup(M)$. It follows that $N \in E \cap \Sigma_<(M) \cap M$.

Finally, we show that $\Sigma_\geq(M)$ is M-stationary. Let $E \in M$ be club in $[\omega_2]^\omega$. Let $\delta \in M$ be such that $\delta < \omega_2$, δ is uncountable, and $E \cap [\delta]^\omega$ is club in $[\delta]^\omega$. Now let $N \in M$ be such that $N \in E \cap [\delta]^\omega$,

$$|\sup(N \cap \omega_1) \cap C_{M\cap\omega_1}| \geq |\delta \cap \pi^{-1}[C_{ot(M\cap\omega_2)}]|$$

and $\delta \cap \pi^{-1}[C_{ot(M\cap\omega_2)}] \subseteq N$. Then $N \in E \cap \Sigma_\geq(M) \cap M$. This completes the proof of Lemma 3.7 and Theorem 3.6. $\Box$

The following theorem was also proved in [28].

Theorem 3.8 υ_{AC} *implies that* $L(\mathcal{P}(\omega_1))$ *satisfies the Axiom of Choice and that* $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.

PROOF: For $A \subseteq \omega_1$ let $[A]$ denote the equivalence class of A in the quotient $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ and let δ_A denote the smallest uncountable ordinal such that some sequence $\vec{N}_A = \langle N^A_\xi \colon \xi < \omega_1\rangle$ and δ_A witness $\upsilon_{\mathrm{AC}}(A)$. Notice that δ_A depends only the equivalence class $[A]$ of A. The assignment $[A] \mapsto \delta_A$ is in fact an injection of $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ into ω_2, since $\delta_A = \delta_B$ implies that for any witnessing sequences $\vec{N}_A$ and $\vec{N}_B$ as above, the set of $\xi < \omega_1$ such that $N^A_\xi = N^B_\xi$ is club in ω_1 and therefore for any ξ limit point of this club, $\xi \in A$ iff $\xi \in B$.

It follows that $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ is well-orderable in $L(\mathcal{P}(\omega_1))$ in order type ω_2. But this implies that $\mathcal{P}(\omega_1)$ itself is well-orderable, and therefore $L(\mathcal{P}(\omega_1)) \models \mathrm{AC}$. In effect, fix a sequence $\langle S_\alpha \colon \alpha < \omega_1\rangle$ of disjoint stationary subsets of ω_1 and notice that the map $A \mapsto [\bigcup_{\alpha\in A} S_\alpha]$ is an injection of $\mathcal{P}(\omega_1)$ into $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. In particular, $2^{\aleph_1} = \aleph_2$.

To show that $2^{\aleph_0} = 2^{\aleph_1}$, we refute the weak diamond principle of Devlin and Shelah, see [13]. Suppose then that υ_{AC} holds and let $F : 2^{<\omega_1} \to 2$ be given by $F(f) = 0$ iff f codes in some reasonable fashion an increasing continuous $\langle N_\alpha : \alpha < \beta\rangle$ with β limit such that if $N = \bigcup_{\alpha<\beta} N_\alpha$, then there is a $\nu < \beta$ such that for all $\alpha \in (\nu, \beta)$, $w(N_\alpha, N) < w(N_\alpha \cap \omega_1, N \cap \omega_1)$. Let $g \in 2^{\omega_1}$ witness weak diamond for F, let $S = g^{-1}(1)$ and let $f \in 2^{\omega_1}$ code $\langle N_\alpha : \alpha < \omega_1\rangle$ witnessing $\upsilon_{\mathrm{AC}}(S)$. There is a club $C \subseteq \omega_1$ such that $f \upharpoonright \nu$ codes $\langle N_\alpha : \alpha < \nu\rangle$, for all $\nu \in C$. Let $A = \{\nu : F(f \upharpoonright \nu) = g(\nu)\}$, so A is stationary. If $A \cap S$ is stationary, let $\nu \in A \cap S \cap C$. Then $F(f \upharpoonright \nu) = 0$, contradicting that $g(\nu) = 1$. It follows that $A \setminus S$ is stationary. Let $\nu \in C \cap (A \setminus S)$. Then $F(f \upharpoonright \nu) = 1$, contradicting that $g(\nu) = 0$. This contradiction implies that weak diamond fails and therefore $2^{\aleph_0} = 2^{\aleph_1}$. $\Box$

Remark This well-ordering is $\Delta_2(\vec{C})$, since being stationary is Π_1 and we need this to state that two sets are assigned different ordinals under this well-ordering. In §5 we show that this can be improved and that MRP and BPFA imply that there is a $\Delta_1(\vec{C})$ well-ordering of $\mathcal{P}(\omega_1)$, where $\vec{C}$ is a given C-sequence.

Our next goal is to present a result of Viale [40] saying that MRP implies SCH.

Theorem 3.9 MRP *implies that* $\kappa^{\aleph_0} = \kappa$*, for all regular cardinals* $\kappa > \aleph_1$.

We will need the following definition.

Definition 3.10 *Let* κ *be an infinite cardinal. A* covering matrix *for* κ^+ *is a sequence* $\mathcal{C} = \{K(n, \beta) : n < \omega, \beta < \kappa^+\}$ *such that*

(1) $|K(n, \beta)| < \kappa$*, for all* n *and* β*,*
(2) $K(n, \beta)$ *is a closed subset of* β *in the order topology, for all* n*,*
(3) $K(n, \beta) \subseteq K(m, \beta)$*, for all* β *and* $n < m$*,*
(4) $\beta = \bigcup\{K(n, \beta) : n < \omega\}$*, for all* β*,*
(5) for all $\alpha < \beta$ *and* n *there is* m *such that* $K(n, \alpha) \subseteq K(m, \beta)$*.*

Lemma 3.11 *Assume* κ *is a singular cardinal of cofinality* ω*. Then there exists a covering matrix for* κ^+*.*

PROOF: Fix a sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals converging to κ and for each $\xi < \kappa^+$ a surjection $\varphi_\xi : \kappa \to \xi$. We define a sequence $\{K(n, \beta) : n < \omega\}$ by induction on $\beta < \kappa^+$. In order to guarantee condition (1) we also arrange that $|K(n, \beta)| \leq \kappa_n$, for all β and n. Assume we have defined $K(n, \xi)$, for all n and $\xi < \beta$. We let

$$K(n, \beta) = \overline{\varphi_\beta[\kappa_n] \cup \bigcup\{K(n, \xi) : \xi \in \varphi_\beta[\kappa_n]\}} \cap \beta,$$

where the closure is taken in the order topology. It is easy to see that this sequences satisfies the required conditions. □

PROOF of Theorem 3.9: We prove the theorem by induction on κ. The only problem occurs at successors of singular cardinals of cofinality ω. So, assume $\mathrm{cof}(\kappa) = \omega$ and fix a covering matrix for κ^+, say $\{K(n, \beta) : n < \omega, \beta < \kappa^+\}$. By the inductive hypothesis we have that $|[K(n, \beta)]^\omega| < \kappa$, for all n and β. We also fix a C-sequence $\langle C_\xi : \xi < \omega_1 \rangle$. For a countable subset X of κ^+ let

$$\alpha_X = \sup(X \cap \omega_1) \text{ and } \delta_X = \sup(X).$$

If $\alpha < \gamma < \omega_1$ let $\mathrm{ht}_\gamma(\alpha) = |C_\gamma \cap \alpha|$. If $\xi < \beta < \kappa^+$ let

$$w(\xi, \beta) = \min\{n : \xi \in K(n, \beta)\}.$$

Now, fix a sufficiently large regular cardinal θ and let M be a countable elementary submodel of H_θ containing all the relevant information. Fix an ordinal $\beta_M \geq \delta_M$ such that for all $\gamma < \kappa^+$ and all $n < \omega$ there is m such that $K(n, \gamma) \cap M \subseteq K(m, \beta_M)$. To see that such an ordinal β_M exists note that $\{K(n, \gamma) \cap M : n < \omega, \gamma < \kappa^+\}$ is of cardinality at most $2^{\aleph_0}$ and $\mathrm{cof}(\kappa^+) > 2^{\aleph_0}$, and use property (5) of the covering matrix. Let $\Sigma(M)$ be the set of all $X \subseteq M \cap \kappa^+$ such that

$$\alpha_X < \alpha_M, \delta_X < \delta_M \text{ and } \mathrm{ht}_{\alpha_M}(\alpha_X) < w(\delta_X, \beta_M).$$

Claim 3.12 *$\Sigma(M)$ is open in the Ellentuck topology.*

PROOF: If $X \in \Sigma(M)$ we have to find a finite $x \subseteq X$ such that $[x, X] \subseteq \Sigma(M)$. First note that if $\delta_X \in X$ then $[\{\delta_X\}, X] \subseteq \Sigma(M)$. So, assume now that X does not have a maximal element and let $n = w(\delta_X, \beta_M)$. By definition of w and the fact that the $K(n, \beta_M)$ are closed in the order topology it follows that $K(n-1, \beta_M) \cap \delta_X$ is bounded in δ_X. So, we can find $\gamma < \delta_X$ such that $[\gamma, \delta_X) \cap K(n-1, \beta_M) = \emptyset$. It follows now that $[\{\gamma\}, X] \subseteq \Sigma(M)$, as desired. $\Box$

In general we cannot show that $\Sigma(M)$ is M-stationary, but we do have the following.

Claim 3.13 *Assume $\kappa^{\aleph_0} > \kappa^+$. Then $\Sigma(M)$ is M-stationary.*

PROOF: Suppose $f : [\kappa^+]^{<\omega} \to \kappa^+$ belongs to M. We need to find $Y \in M \cap \Sigma(M)$ which is closed under f. First, find a countable elementary submodel N of $H_{\kappa^{++}}$ containing f and all other relevant objects and such that $N \in M$. Let $m = \mathrm{ht}_{\alpha_M}(\alpha_N)$. Let E be the set of $\gamma < \kappa^+$ of cofinality ω which are closed under f. Then E is of cardinality κ^+ and so, by our assumption $|[E]^{\aleph_0}| > \kappa^+$. By the inductive hypothesis, $\mu^{\aleph_0} < \kappa$, for all $\mu < \kappa$. Therefore $[E]^{\aleph_0}$ is not a subset of

$$\bigcup\{[K(n, \beta)]^{\aleph_0} : n < \omega, \beta < \kappa^+\}.$$

By elementarity of N there exists $X \in [E]^{\aleph_0} \cap N$ such that $X \nsubseteq K(n, \beta)$, for all n and β. In particular, $X \nsubseteq K(m, \beta_M)$. Since $X \in N$ and X is countable it follows that $X \subseteq N$. We conclude that there exists $\gamma \in E \cap N$ such that $w(\gamma, \beta_M) > m$. Since γ is of cofinality ω and closed under f, we can pick a countable subset Y of γ which is cofinal in γ, closed under f and belongs to N. Then we have that $\delta_Y = \gamma$. Since $Y \subseteq N$ we have that $\alpha_Y \leq \alpha_N$. Therefore

$$\mathrm{ht}_{\alpha_M}(\alpha_Y) \leq \mathrm{ht}_{\alpha_M}(\alpha_N) = m < w(\delta_Y, \beta_M).$$

It follows that $Y \in M \cap \Sigma(M)$, as desired. $\square$

Now apply MRP to Σ and let $\langle N_\xi : \xi < \omega_1 \rangle$ be a reflecting sequence. Let $\delta_\xi = \sup(N_\xi \cap \kappa^+)$, for all $\xi < \omega_1$. Then $C = \{\delta_\xi : \xi < \omega_1\}$ is a club in $\delta = \sup C$. Since δ is of cofinality ω_1 there is n such that $K(n, \delta)$ is unbounded in δ. Since $K(n, \delta)$ is also closed it follows that there exists a club D in ω_1 such that $\{\delta_\xi : \xi \in D\} \subseteq K(n, \delta)$. Let ν be any limit point of D and let $M = N_\nu$. By the definition of β_M and properties (3) and (5) of the covering matrix there exists m such that $K(n, \delta) \cap M \subseteq K(m, \beta_M)$. It follows that $w(\delta_\xi, \beta_M) \leq m$, for all $\xi \in D \cap \nu$. On the other hand, $\mathrm{ht}_{\alpha_M}(\alpha_{N_\xi})$ converges to ω, as ξ converges to ν. It follows that $\mathrm{ht}_{\alpha_M}(\alpha_{N_\xi}) > w(\delta_\xi, \beta_M)$, for eventually all $\xi \in D \cap \nu$, which contradicts the fact that $\langle N_\xi : \xi < \omega_1 \rangle$ is a reflecting sequence for Σ. This completes the proof of Theorem 3.9. $\square$

Using the same argument as in Corollary 2.17 we have the following.

Corollary 3.14 MRP *implies* SCH. $\square$

4 P-ideal dichotomy

In this section we introduce a combinatorial principle which captures some of the essential features of PFA yet is compatible with CH. A restricted version of this principle was introduced and studied by Abraham and Todorčević in [3] and the principle was later extended and generalized by Todorčević in [35].

We start with some basic definitions concerning P-ideals of countable sets. We fix an infinite set X. For $A, B \subseteq X$ let us write $A \subseteq_* B$ if $A \setminus B$ is finite. We say that A and B are *orthogonal* and write $A \perp B$ if and only if $A \cap B$ is finite.

Definition 4.1 *Let X be a set. Suppose $\mathcal{I} \subseteq [X]^{\leq \aleph_0}$ is an ideal containing all finite sets. We say that $\mathcal{I}$ is a* P-ideal of countable sets *if for every sequence $\{X_n : n < \omega\} \subseteq \mathcal{I}$ there is $X \in \mathcal{I}$ such that $X_n \subseteq_* X$, for all n.*

Definition 4.2 *The* P-ideal dichotomy PID *is the statement that for every* P*-ideal $\mathcal{I}$ of countable sets on an uncountable set X, either*

(1) there is an uncountable $Y \subseteq X$ such that $[Y]^{\leq \aleph_0} \subseteq \mathcal{I}$, or
(2) we can write $X = \bigcup\{X_n : n < \omega\}$, where $X_n \perp \mathcal{I}$, for all n.

The following was shown for P-ideals of countable sets on ω_1 in [3] and for general P-ideals of countable sets in [35].

Theorem 4.3 PFA *implies* PID. $\square$

The following was also shown in [35].

Theorem 4.4 *Assume there is a supercompact cardinal κ. Then there is generic extension satisfying* GCH *in which* PID *holds.* $\square$

It was shown in [3] that PID refutes some standard consequences of $\diamondsuit$ and therefore these statements do not follow from CH. For instance, PID implies that there are no ω_1-Souslin trees and that all (ω_1, ω_1^*)-gaps are Hausdorff. In [35] it was shown, among other things, that PID implies that $\square(\kappa)$ fails, for all regular $\kappa > \aleph_1$. Thus, PID has considerable large cardinal strength. For many more applications of PID see [6], [39] and [15].

We will primarily be interested in the consequences of PID to cardinal arithmetic. Before we start we make some definitions which will be useful in the future. Given a subset $\mathcal{A}$ of $\mathcal{P}(X)$ let

$$\mathcal{A}^\perp = \{B \subseteq X : B \perp A, \text{ for all } A \in \mathcal{A}\}.$$

Notice that $\mathcal{A}^\perp$ is always an ideal, but not necessarily a P-ideal.

Definition 4.5 *An ideal $\mathcal{J}$ on a set X is* locally countably generated *if $\mathcal{J} \restriction Y$ is countably generated, for every countable $Y \subseteq X$.*

Lemma 4.6 *Assume an ideal $\mathcal{J}$ on a set X is locally countably generated, then $\mathcal{J}^\perp \cap [X]^{\leq\aleph_0}$ is a* P*-ideal.*

PROOF: Let $\mathcal{I} = \mathcal{J}^\perp \cap [X]^{\aleph_0}$. Assume $Y_n \in \mathcal{I}$, for all n. Let $Y^* = \bigcup_n Y_n$. Fix a generating family $\{Z_n : n < \omega\}$ for $\mathcal{J} \restriction Y^*$. We may assume $Z_n \subseteq Z_{n+1}$, for all n. Let $Y = \bigcup\{Y_n \setminus Z_n : n < \omega\}$. It follows that $Y \in \mathcal{I}$ and $Y_n \subseteq_* Y$, for all n. Therefore $\mathcal{I}$ is a P-ideal. $\square$

Lemma 4.7 *Assume $\mathcal{J}$ is a locally countably generated ideal on a set X. Let $\mathcal{I} = \mathcal{J}^\perp \cap [X]^{\leq\aleph_0}$. Then $\mathcal{I}^\perp \cap [X]^{\aleph_0} \subseteq \mathcal{J}$.*

PROOF: Let $Y \in \mathcal{I}^\perp$ be countable. Let $\{Z_n : n < \omega\}$ be a generating family for $\mathcal{J} \restriction Y$. We may assume $Z_n \subseteq Z_{n+1}$, for all n. We need to show that $Y \subseteq Z_n$, for some n. Assume otherwise and pick distinct points $x_n \in Y \setminus Z_n$, for $n < \omega$. Then $U = \{x_n : n < \omega\}$ is an infinite subset of Y and is orthogonal to all the Z_n, so $U \in \mathcal{I}$. But this contradicts the fact that $Y \in \mathcal{I}^\perp$. $\square$

We now turn to consequences of PID to cardinal arithmetic. As a warm up let us first prove the following result from [3].

Theorem 4.8 PID *implies there are no ω_1-Suslin tree.*

PROOF: Assume $(T, \leq)$ is a Souslin tree. For each $t \in T$ let $\hat{t} = \{s \in T : s <_T t\}$. Let $\mathcal{J}$ be the ideal generated by $\{\hat{t} : t \in T\}$. Then it is easy to see that $\mathcal{J}$

is locally countably generated and so $\mathcal{I} = \mathcal{J}^\perp \cap [T]^{\leq\aleph_0}$ is a P-ideal. Consider the two alternatives of PID.

Suppose first that we can write $T = \bigcup\{X_n : n < \omega\}$, where $X_n \in \mathcal{I}^\perp$, for all n. Fix any n such that X_n is uncountable. By Lemma 4.7 we have that $[X_n]^{\leq\aleph_0} \subseteq \mathcal{J}$. Therefore X_n has no infinite antichains, and so by a standard argument has an uncountable chain, a contradiction.

Suppose now that there is an uncountable $X \subseteq T$ such that $[X]^{\leq\aleph_0} \subseteq \mathcal{I}$. Then $\hat{t} \cap X$ is finite, for every $t \in T$. It follows that X, considered as a tree with the inherited ordering has at most $\leq \omega$ levels. Therefore X contains an uncountable antichain, again a contradiction. $\square$

It is an open question if PID bounds the value of the continuum, in particular, if it implies that $2^{\aleph_0} \leq \aleph_2$. However it follows from some results of Todorčević [35] that PID implies that the bounding number $\mathfrak{b}$ is at most $\aleph_2$. In order to discuss this result let us recall the following.

Definition 4.9 *Let S be a set. A* pregap *on S is a pair $(\mathcal{A}, \mathcal{B})$ of two orthogonal families of countable subsets of S.*

A pregap $(\mathcal{A}, \mathcal{B})$ is a gap *iff there does not exist $x \subseteq S$ such that $a \subseteq_* x$, for all $a \in \mathcal{A}$, and $b \perp x$, for all $b \in \mathcal{B}$.*

Furthermore, two families $\{a_\xi : \xi < \omega_1\}$ and $\{b_\xi : \xi < \omega_1\}$ form a Hausdorff gap *if the sets $\{\xi < \alpha : a_\alpha \cap b_\xi \subseteq a_\alpha[n]\}$ are finite for all $\alpha < \omega_1$ and $n < \omega$, where $a_\alpha[n]$ denotes the set of the first n elements of a_α.*

Notice that a Hausdorff gap is a gap. Indeed, it is easily seen that for every $x \subseteq \bigcup_{\xi<\omega_1}(a_\xi \cup b_\xi)$, either $\{\xi < \omega_1 : a_\xi \perp x\}$ or $\{\xi < \omega_1 : b_\xi \subseteq_* x\}$ is countable.

For a pregap $(\mathcal{A}, \mathcal{B})$ we consider the ideal $\mathcal{I}_{\mathcal{A},\mathcal{B}}$ of countable subsets B of $\mathcal{B}$ for which there is $a \in \mathcal{A}$ such that the set $B(a, n) = \{b \in B : a \cap b \subseteq a[n]\}$ is finite, for all n. The purpose of this condition is to capture the Hausdorff property, in the following sense.

Lemma 4.10 *If there is an uncountable $X \subseteq \mathcal{B}$ such that $[X]^\omega \subseteq \mathcal{I}_{\mathcal{A},\mathcal{B}}$ then $(\mathcal{A}, \mathcal{B})$ contains a Hausdorff subgap.*

PROOF: Simply pick a sequence $\{b_\xi : \xi < \omega_1\}$ of distinct elements of X and $\{a_\xi : \xi < \omega_1\}$ such that a_α witnesses $\{b_\xi : \xi < \alpha\} \in \mathcal{I}_{\mathcal{A},\mathcal{B}}$, for all α. $\square$

Lemma 4.11 *If $\mathcal{A}$ is σ-directed, then $\mathcal{I}_{\mathcal{A},\mathcal{B}}$ is a* P*-ideal.*

PROOF: Let $\{B_i : i < \omega\}$ be a family of elements of $\mathcal{I}_{\mathcal{A},\mathcal{B}}$, and let $a_i \in \mathcal{A}$ witness that $B_i \in \mathcal{I}_{\mathcal{A},\mathcal{B}}$, for all i. Since $\mathcal{A}$ is σ-directed, we can choose $a \in \mathcal{A}$ such that $a_i \subseteq_* a$, for all i. Notice that $B_i(a, n)$ is finite, for all i and n, since as soon as $m \geq n$ is large enough that $a_i \setminus a$ and $a[n] \cap a_i$ are included in $a_i[m]$, we get $B_i(a,n) \subseteq B_i(a_i,m)$. Finally, letting $B = \bigcup\{B_n \setminus B_n(a,n) : n < \omega\}$, it is clear that $B_n \subseteq_* B$, for all $n < \omega$, and a witnesses $B \in \mathcal{I}_{\mathcal{A},\mathcal{B}}$. $\square$

Thus, as long as $\mathcal{A}$ is σ-directed we can apply the P-ideal dichotomy to $\mathcal{I}_{\mathcal{A},\mathcal{B}}$. We have seen in Lemma 4.10 how the first alternative of the dichotomy translates in terms of gaps and we are now left to examine the second alternative.

Lemma 4.12 *Let $X \subseteq \mathcal{B}$ be orthogonal to $\mathcal{I}_{\mathcal{A},\mathcal{B}}$. Then $\bigcup X$ is orthogonal to $\mathcal{A}$.*

PROOF: Assume to the contrary that there exists $a \in A$ such that $a \cap (\bigcup X)$ is infinite. Then for each $n < \omega$ we can choose $b_n \in X$ such that $b_n \cap a \not\subseteq a[n]$. The family of all such b_n's is an infinite subset of X belonging to $\mathcal{I}_{\mathcal{A},\mathcal{B}}$, which is a contradiction. $\square$

This concludes the proof of the following theorem from [35].

Theorem 4.13 *Assume* PID *and let $(\mathcal{A},\mathcal{B})$ be a pregap on some set S such that $\mathcal{A}$ is σ-directed under $\subseteq_*$. Then one of the following holds.*

(1) $(\mathcal{A},\mathcal{B})$ contains a Hausdorff subgap.
(2) There exists a countable family $\{S_n : n < \omega\}$ of subsets of S such that S_n is orthogonal to $\mathcal{A}$, for all n, and every element of $\mathcal{B}$ is included in S_n, for some n. $\square$

Given two regular cardinals κ and λ recall that a (κ,λ)-*gap* is a gap $(\mathcal{A},\mathcal{B})$ such that $\mathcal{A}$ is well ordered by $\subseteq_*$ in order type κ and $\mathcal{B}$ is well ordered by $\subseteq_*$ in order type κ. We now have the following.

Corollary 4.14 *Assume* PID *and let κ and λ be two uncountable regular cardinals such that there exists a (κ,λ)-gap. Then $\kappa = \lambda = \aleph_1$.*

PROOF: Assume otherwise and let $(\mathcal{A},\mathcal{B})$ be a (κ,λ)-gap on a set S. Since $\mathcal{A}$ is well ordered by $\subseteq_*$ in order type κ and κ is regular and uncountable, it follows that $\mathcal{A}$ is σ-directed by $\subseteq_*$. Notice that the second alternative of Theorem 4.13 cannot hold. Namely, suppose $\{S_n : n < \omega\}$ is a family of subsets of S such that S_n is orthogonal to $\mathcal{A}$, for all n, and for every $b \in \mathcal{B}$ there is n such that $b \subseteq_* S_n$. Since λ is also regular and uncountable there is n such that $\{b \in \mathcal{B} : b \subseteq_* S_n\}$ is cofinal in $\mathcal{B}$ under $\subseteq_*$. It follows that $S_n \perp a$, for all $a \in \mathcal{A}$, and $b \subseteq_* S_n$, for all $b \in \mathcal{B}$, a contradiction. Therefore, $(\mathcal{A},\mathcal{B})$ has a Hausdorff subgap $(\mathcal{A}^*,\mathcal{B}^*)$. But then $\mathcal{A}^*$ is a cofinal subset of $\mathcal{A}$ and $\mathcal{B}^*$ is a cofinal subset of $\mathcal{B}^*$. Since both $\mathcal{A}^*$ and $\mathcal{B}^*$ have cardinality $\aleph_1$ it follows that $\kappa = \lambda = \aleph_1$. $\square$

Recall that the *bounding number* $\mathfrak{b}$ is the least cardinal κ such that there is a subset $\mathcal{F}$ of ω^ω of cardinality κ which is unbounded under eventual dominance $\leq_*$. We now have the following.

Corollary 4.15 PID *implies that* $\mathfrak{b} \leq \aleph_2$.

PROOF: Considering the structure $(\omega^\omega, \leq_*)$ one can define a similar notion of a (κ, λ^*)-gap and show, assuming PID, that if κ and λ are uncountable and regular and there is a (κ, λ^*)-gap in $(\omega^\omega, \leq_*)$ then $\kappa = \lambda = \aleph_1$. Now, assuming $\mathfrak{b} > \aleph_2$ one can build an (ω_2, λ^*)-gap in $(\omega^\omega, \leq_*)$ for some regular $\lambda \geq \aleph_1$, see for instance [23, Theorem 29.8, pp.559]. This is a contradiction. □

We now turn to the following result of Viale [41].

Theorem 4.16 *Assume* PID. *Then* $\kappa^{\aleph_0} = \kappa$*, for all regular* $\kappa \geq 2^{\aleph_0}$.

PROOF: We prove this by induction on κ. The only problem occurs at successors of singular cardinals of countable cofinality. Thus, assume κ is singular of countable cofinality and the theorem holds for all regular $\mu < \kappa$. By Lemma 3.11 we can fix a covering matrix for κ^+, say $\mathcal{C} = \{K(n,\beta) : n < \omega, \beta < \kappa^+\}$. Let $\mathcal{J}$ be the ideal generated by $\mathcal{C}$ and let $\mathcal{I} = \mathcal{J}^\perp \cap [\kappa^+]^{\leq \aleph_0}$.

Claim 4.17 $\mathcal{J}$ *is locally countably generated and thus* $\mathcal{I}$ *is a* P*-ideal.*

PROOF: Let Y be a countable subset of κ^+. For each $\alpha < \kappa^+$ let $\mathcal{J}_\alpha$ be the ideal generated by $\{K(n,\alpha) : n < \omega\}$. By property 4. of the covering matrix $\mathcal{J}_\alpha \subseteq \mathcal{J}_\beta$, for $\alpha < \beta$. Since $|\mathcal{P}(Y)| = 2^{\aleph_0} < \kappa^+$ there is $\beta < \kappa^+$ such that

$$\mathcal{J}_\gamma \restriction Y = \mathcal{J}_\beta \restriction Y$$

for all $\gamma \geq \beta$. So, $\mathcal{J} \restriction Y$ is generated by $\{K(n,\beta) : n < \omega\}$. □

Claim 4.18 *There is no uncountable set* $X \subseteq \kappa^+$ *such that* $[X]^{\leq \aleph_0} \subseteq \mathcal{I}$.

PROOF: Suppose X is of size $\aleph_1$ and let $\alpha = \sup(X)$. Then for some n $K(n,\alpha) \cap X$ is uncountable. Let Z be an infinite subset of $K(n,\alpha)$. Then $Z \notin \mathcal{I}$. □

By the P-ideal dichotomy one gets a decomposition $\kappa^+ = \bigcup_n X_n$ such that $X_n \perp \mathcal{I}$, for all n. By Lemma 4.7 we have the following.

Claim 4.19 $[X_n]^{\aleph_0} \subseteq \bigcup\{[K(m,\beta)]^{\aleph_0} : m < \omega, \beta < \kappa^+\}$. □

Since there is n such that $|X_n| = \kappa^+$ and the $K(m,\beta)$ are all of size $< \kappa$ it follows that

$$(\kappa^+)^{\aleph_0} = \kappa^+ \cdot \sup_{\lambda < \kappa} \lambda^{\aleph_0} = \kappa^+.$$

This completes the proof of Theorem 4.16. □

As in Corollary 2.17 we have the following.

Corollary 4.20 PID *implies* SCH. □

5 Definable well-orderings of the reals

In this section we show that BPFA implies the existence of a definable well ordering of the reals of optimal complexity. This result comes from the paper of Caicedo and the author [9] and refines Theorem 3.8 due to Moore [28]. The interest of this result is that it suggests that there is a structure theory for models of strong forcing axioms. We will say more about this in §6.

The idea is to design a robust coding of reals by triples of ordinals smaller than ω_2 which guarantees that if M is an inner model, BPFA holds in both M and V, and $\aleph_2^M = \aleph_2$, then $\mathcal{P}(\omega_1) \subset M$. This allows us to show that BPFA implies the existence of a well-ordering of the reals which is Δ_1 with parameter a subset of ω_1. Obviously, under $\neg$CH there cannot be a Δ_0-definable well ordering of the reals with parameter a subset of ω_1. Also, a well ordering of the reals cannot be Δ_1 with real parameter r, since then, by a result of Mansfield (see [23, Theorem 25.39]), $\mathbb{R} \subseteq L[r]$, which satisfies CH.

We start by describing the coding machinery which is reminiscent of Moore's coding from §3. We fix a C-sequence $\vec{C} = \langle C_\xi \colon \xi < \omega_1 \text{ and } \lim(\xi)\rangle$. Given x, y, z sets of natural numbers, define an equivalence relation $\sim_x$ on $\omega \setminus x$ by setting $n \sim_x m$ (for $n \leq m$) iff $[n, m] \cap x = \emptyset$. Thus the equivalence classes of $\sim_x$ are simply the intervals between the consecutive members of x. Let $(I_k)_{k\leq t}$ ($t \leq \omega$) be the natural enumeration of those equivalence classes which intersect both y and z. Let the *oscillation* of x, y, z be the function $o(x, y, z) : t \to 2$ defined by letting for all $k < t$,

$$o(x, y, z)(k) = 0 \text{ iff } \min(I_k \cap y) \leq \min(I_k \cap z).$$

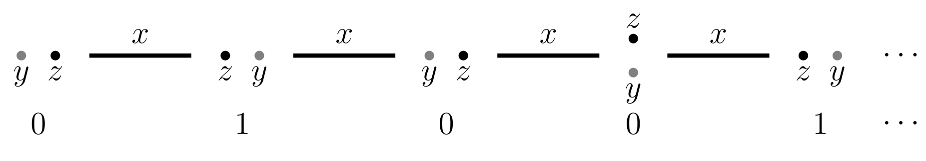

Fig. 2. $o(x, y, z) = 01001\ldots$

Let $\omega_1 < \beta < \gamma < \delta$ be fixed limit ordinals and suppose $N \subseteq M \subseteq \delta$ are countable sets of ordinals. Assume that $\{\omega_1, \beta, \gamma\} \subset N$, that $\sup(\xi \cap N) < \sup(\xi \cap M)$ and $\sup(\xi \cap M)$ is a limit ordinal, for every $\xi \in \{\omega_1, \beta, \gamma, \delta\}$. Then the pair (N, M) codes a finite binary sequence as follows. Take the transitive collapse $\bar{M}$ of M and let π be the collapsing map. Let $\alpha_M = \pi(\omega_1), \beta_M = \pi(\beta), \gamma_M = \pi(\gamma), \delta_M = \bar{M}$, each of these is a countable limit ordinal. Let the *height* of $\alpha_N = \sup(\pi[\omega_1 \cap N])$ in α_M be the integer

$$n = n(N, M) = |\alpha_N \cap C_{\alpha_M}|.$$

Define three sets x, y and z of integers by

$$x = \{\, |\pi(\xi) \cap C_{\beta_M}| \colon \xi \in \beta \cap N \,\}$$

and similarly for y and z with γ and δ, respectively, instead of β.

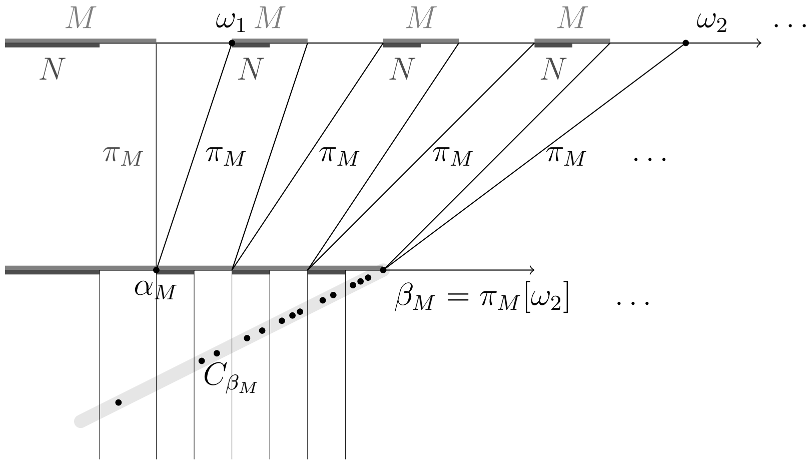

Fig. 3. $x = \{0, 1, 3, 4, 5, 8, 9, 10\}$

Notice that x, y, z are finite by our assumption on N and M. Now, we look at the oscillation of $x \setminus n$, $y \setminus n$ and $z \setminus n$, which is a binary sequence, and if its length is at least n then we let

$$s_{\beta\gamma\delta}(N, M) = o(x \setminus n, y \setminus n, z \setminus n) \restriction n.$$

Otherwise we let $s_{\beta\gamma\delta}(N, M) = *$. We similarly write

$$s_{\beta\gamma\delta}(N, M) \restriction l = *$$

if $l > n(N, M)$.

Remark Notice that there is a finite $T \subset N$ such that for any $S \subset N$, if $T \subset S$ then $s_{\beta\gamma\delta}(S, M) = s_{\beta\gamma\delta}(N, M)$. In effect, it suffices that T contains $\{\omega_1, \beta, \gamma\}$, $\pi^{-1}[\alpha_N \cap C_{\alpha_M}]$, one point of N for each interval in $\beta \cap M$ determined by $\pi^{-1}[C_{\beta_M}]$ that N meets, and similarly for γ and δ.

Finally, we say that the triple (β, γ, δ) *codes* a real r iff there is a continuous increasing sequence $\langle N_\xi \colon \xi < \omega_1 \rangle$ of countable sets whose union is δ such that for every countable limit ordinal ξ there is $\nu < \xi$ such that

$$r = \bigcup_{\nu<\eta<\xi} s_{\beta\gamma\delta}(N_\eta, N_\xi).$$

It was shown in [9] that BPFA implies the following two facts:

(1) Given ordinals $\omega_1 < \beta < \gamma < \delta < \omega_2$ of cofinality ω_1 there is an increasing continuous sequence $\langle N_\xi \colon \xi < \omega_1\rangle$ of countable sets whose union is δ such that for every limit ordinal $\xi < \omega_1$ and every integer n there is $\nu < \xi$ and $s^n_\xi \in \{0,1\}^n \cup \{*\}$ such that $s_{\beta\gamma\delta}(N_\eta, N_\xi) \restriction n = s^n_\xi$ for every η such that $\nu < \eta < \xi$.

(2) For each real r there are ordinals $\omega_1 < \beta < \gamma < \delta < \omega_2$ of cofinality ω_1 such that the triple (β, γ, δ) codes r.

We will not prove these two facts here and refer the interested to [9] for details.

Remark Notice that in (1) we are not claiming that for a fixed n the values of s^n_ξ cohere as ξ varies. In fact, this is not necessarily true. However, by the way we have defined our coding it follows that for a fixed limit ordinal ξ the values of s^n_ξ (other than $*$) cohere as n varies. Also, if $s^n_\xi = *$ then $s^m_\xi = *$ for all $m \geq n$.

We now show how facts (1) and (2) are used to obtain the desired consequences.

Theorem 5.1 *Assume M is an inner model,* BPFA *holds in both M and V, and $\omega_2^M = \omega_2$. Then $\mathcal{P}(\omega_1) \subset M$.*

PROOF: Assume that M is an inner model of V, that BPFA holds in both M and V and $\omega_1^M = \omega_1$. Fix a C-sequence in M to carry out the codings just described. Suppose $\omega_1 < \beta < \gamma < \delta < \omega_2^M$ are ordinals of cofinality ω_1 and the triple (β, γ, δ) codes in V a real r. Let $\langle N_\xi \colon \xi < \omega_1\rangle$ be a continuous increasing sequence of countable sets with union δ witnessing this. In M there is a sequence $\langle P_\xi \colon \xi < \omega_1\rangle$ witnessing (1) for (β, γ, δ). There is a club $C \subseteq \omega_1$ in V such that $N_\xi = P_\xi$, for every $\xi \in C$. Then it follows that for any ξ which is a limit point of C, $r = \bigcup_n s^n_\xi$, as computed in M relative to the sequence $\langle P_\xi \colon \xi < \omega_1\rangle$. It follows that $r \in M$. If, moreover, $\omega_2 = \omega_2^M$, then any real is coded in V by some triple of ordinals less than ω_2^M and thus all reals are in M. But then $\mathcal{P}(\omega_1) \subset M$, since, given any ω_1-sequence in M of almost disjoint reals, BPFA, and in fact $\mathrm{MA}_{\aleph_1}$, allows us to code any subset of ω_1 via this sequence and a real. $\square$

Since BPFA implies $\mathrm{MA}_{\aleph_1}$ this also implies that BPFA implies that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.

Theorem 5.2 BPFA *implies that there is a* Δ_1 *well-ordering of* $\mathcal{P}(\omega_1)$ *with parameter a subset of* ω_1. *The length of the well-ordering is* ω_2.

PROOF: Fix as parameter a C-sequence $\vec{C} = \langle C_\xi \colon \xi < \omega_1 \text{ and } \lim(\xi)\rangle$. Let $\mathrm{T} = \mathrm{T}_{\vec{C}}$ be the theory

$$\text{"ZFC}-\text{Power set}+\text{MA}_{\aleph_1}+\text{ (1) and (2) hold with respect to } \vec{C} \quad +\forall x\,(|x| \leq \aleph_1)\text{"}.$$

Notice that any transitive model M of T which contains $\vec{C}$ is uniquely determined by $\mathrm{ORD} \cap \mathrm{M}$. In effect, notice that since $\vec{C} \in M$, M computes ω_1 correctly. Suppose a real r is coded by some triple (β, γ, δ) of ordinals in M. Then, arguing as in the proof of Theorem 5.1, we see that $r \in M$. Notice that we are not claiming that M knows that (β, γ, δ) codes r, just that $r \in M$. Since M also satisfies (2) it follows that the reals in M are precisely the reals coded by some triple of ordinals which belong to M. Since $\mathrm{MA}_{\aleph_1}$ holds in M it follows that $\mathcal{P}(\omega_1)^M$ is completely determined as well. Namely, from $\vec{C}$ we can define a canonical ω_1-sequence $\vec{r}$ of almost disjoint reals, and we can use the standard almost disjoint coding to code a subset of ω_1 by the sequence $\vec{r}$ and a real. Now, for an ordinal $\theta < \omega_2$, let M_θ be the unique transitive model M of T containing $\vec{C}$ such that $\mathrm{ORD} \cap \mathrm{M} = \theta$, if it exists; otherwise let $M_\theta = \emptyset$. Notice that the function $\theta \mapsto M_\theta$ is Δ_1 in the parameter $\vec{C}$: $M = M_\theta$ if and only if for every transitive model N of enough set theory that contains $\vec{C}$, θ and M, $N \models M = M_\theta$, if and only if there is one such model N.

Let $<_*$ be the antilexicographic ordering on the class $[\mathrm{ORD}]^3$ of increasing triples of ordinals. For a real r let θ_r be the least θ such that $r \in M_\theta$ and let $(\beta_r, \gamma_r, \delta_r)$ be the $<_*$-least triple of ordinals smaller than θ_r such that $M_{\theta_r} \models (\beta_r, \gamma_r, \delta_r)$ codes r. Finally, let $r \lhd s$ iff either $\theta_r < \theta_s$ or $\theta_r = \theta_s$ and $(\beta_r, \gamma_r, \delta_r) <_* (\beta_s, \gamma_s, \delta_s)$.

We can now define a well-ordering $\prec$ of $\mathcal{P}(\omega_1)$ as follows. For a and b subsets of ω_1, we define $a \prec b$ iff the $\lhd$-least real coding a (from the sequence $\vec{r}$ of almost disjoint reals defined from $\vec{C}$) is $\lhd$-smaller than the $\lhd$-least real coding b. By an argument similar to the above, $\prec$ is also Δ_1 in the parameter $\vec{C}$. □

Remark It is open whether even MM implies that there is a definable well ordering of the reals without parameters. On the other hand, Asperó [4] showed that it is consistent with PFA that there is a well-ordering of the reals definable over $H_{\aleph_2}$ without parameters and Larson [24] showed the same result with MM in place of PFA.

6 Inner models of forcing axioms

The results of the previous section suggest that there should be a structure theory for models of strong forcing axioms such as PFA or MM. In order to make this question precise the following conjecture was formulated by Caicedo and the author.

Conjecture 6.1 *Assume $V \subseteq W$ are two models of* PFA *or* MM *which have the same cardinals. Then* $\mathrm{ORD}^{\omega_1} \cap \mathrm{V} = \mathrm{ORD}^{\omega_1} \cap \mathrm{W}$.

We fix the cardinals to avoid situations like:

- If V satisfies MM and there is a supercompact cardinal then we can collapse $\aleph_2$ and force MM again. If W is the resulting generic extension then V and W have little in common.
- If V has a measurable cardinal κ and W is the result of iterating ω-many times a normal measure on κ then $V \subset W$ are elementarily equivalent, but V is not closed under ω-sequences.
- If V has a proper class of completely Jónsson cardinals, $\mathcal{P}_\infty$ is the class stationary tower forcing (see [25]), G is $\mathcal{P}_\infty$-generic over V, and $W = V[G]$, then there is an elementary embedding $j : V \to W$ and we can arrange that $\mathrm{cp}(j)$ is arbitrarily high and $\mathrm{cof}^W(\mathrm{cp}(j)) = \omega$.

While Conjecture 6.1 is still open, in this section we present some partial results. We start with the following observation. The proof is essentially the same as the proof that MM implies $\kappa^{\aleph_1} = \kappa$, for all regular $\kappa \geq \aleph_2$, see [16].

Proposition 6.2 *Assume W is a set generic extension of V and W satisfies* SRP*. Assume moreover that V and W have the same ordinals of cofinality ω and ω_1 and $\mathcal{P}(\omega_1)^V = \mathcal{P}(\omega_1)^W$. Then* $\mathrm{ORD}^{\omega_1} \cap \mathrm{V} = \mathrm{ORD}^{\omega_1} \cap \mathrm{W}$.

PROOF: Let $\mathcal{P} \in V$ be a forcing notion such that $W = V[G]$, for some V-generic filter over $\mathcal{P}$. Let $\kappa = |\mathcal{P}|^+$. Then obviously $\mathcal{P}$ satisfies the κ-chain condition. Let $S = \{\alpha < \kappa : \mathrm{cof}(\alpha) = \omega\}$ and fix in V a partition $S = \bigcup\{S_\xi : \xi < \kappa\}$ into disjoint stationary sets. By the κ-cc the S_ξ are still stationary in W. Fix, also in V, a partition $\omega_1 = \bigcup\{E_\xi : \xi < \omega_1\}$ into disjoint stationary sets. Then, by our assumption, the E_ξ are also stationary in W. Now, work in W and suppose $A \in W$ is a subset of κ of cardinality ω_1. We want to show that $A \in V$. To this end let $f : \omega_1 \to \kappa$ be a $1-1$ enumeration of A and consider the following set

$$T = \{X \in [\kappa]^{\aleph_0} : \sup(X \cap \omega_1) \in E_\xi \text{ iff } \sup(X) \in S_{f(\xi)}, \text{ for all } \xi\}.$$

Let θ be a sufficiently large regular cardinal and let T^* be the set of all countable elementary submodels M of H_θ such that $M \cap \kappa \in T$.

Claim 6.3 *T^* is projective stationary.*

PROOF: Let E be a stationary subset of ω_1. We need to show that $T^*_E = \{M \in T^* : \sup(M \cap \omega_1) \in E\}$ is stationary in $[H_\theta]^\omega$. Let $F : [H_\theta]^{<\omega} \to H_\theta$ be an algebra on H_θ. We need to find $M \in T^*$ which is closed under F. First we find $\xi < \omega_1$ such that $E \cap E_\xi$ is stationary. Since $S_{f(\xi)}$ is a stationary subset of κ we can find an elementary submodel N of $(H_\theta, \in, F)$ such that $\delta = N \cap \kappa \in S_{f(\xi)}$. Build a continuous increasing chain $\langle M_\eta : \eta < \omega_1\rangle$ of countable elementary submodels of N such that $M_0 \cap \delta$ is cofinal in δ (we can do this since $\mathrm{cof}(\delta) = \omega$) and such that if we let $\alpha_\eta = \sup(M_\eta \cap \omega_1)$ the sequence $\langle \alpha_\eta : \eta < \omega_1\rangle$ is strictly increasing. Then $D = \{\alpha_\eta : \eta < \omega_1\}$ is a club in ω_1 and since $E \cap E_\xi$ is stationary there is η such that $\alpha_\eta \in D \cap E \cap E_\xi$. Then $M_\eta \in T^*_E$, as desired. □

Now, by SRP we can find a continuous $\in$-chain $\langle M_\eta : \eta < \omega_1\rangle$ of countable elementary submodels of H_θ such that $M_\eta \in T^*$, for all η. Let $\alpha_\eta = \sup(M_\eta \cap \omega_1)$ and $\delta_\eta = \sup(M_\eta \cap \kappa)$. Then $\{\alpha_\eta : \eta < \omega_1\}$ is a club in ω_1 and $\{\delta_\eta : \eta < \omega_1\}$ is a club in some $\delta < \kappa$ with $\mathrm{cof}(\delta) = \omega_1$. Since $M_\eta \in T^*$ we have that $\alpha_\eta \in E_\xi$ iff $\delta_\eta \in S_{f(\xi)}$, for all ξ. By our assumption $\mathrm{cof}^V(\delta) = \omega_1$ and the notion of stationary subset of δ is the same in V and W. Since A is equal to the set of $\zeta < \kappa$ such that S_ζ is stationary in δ it follows that $A \in V$. □

In [41] Viale defined a family of covering principles which he used to address problems related to Conjecture 6.1. We will need a generalization of the notion of a covering matrix introduced in §3.

Definition 6.4 *Let θ and κ be two regular cardinals with $\theta < \kappa$. an infinite cardinal. A θ-*covering matrix *for κ is a sequence $\mathcal{K} = \{K(\xi,\beta) : \xi < \theta, \beta < \kappa\}$ such that*

(1) $K(\xi,\beta)$ is a closed subset of β in the order topology, for all ξ and β,
(2) $K(\xi,\beta) \subseteq K(\eta,\beta)$, for all $\xi < \eta$ and all β,
(3) $\beta = \bigcup\{K(\xi,\beta) : \xi < \theta\}$, for all β,
(4) for all $\alpha < \beta$ and ξ there is η such that $K(\xi,\alpha) \subseteq K(\eta,\beta)$.

$\tau_{\mathcal{K}}$ is the least ordinal τ such that $\mathrm{ot}(K(\xi,\beta)) < \tau$, for all ξ and β. $\mathcal{K}$ is trivial *if $\tau_{\mathcal{K}} = \kappa$.*

Thus, what we called a covering matrix for κ^+ in Definition 3.10 is an ω-covering matrix $\mathcal{K}$ for κ^+ such that $\tau_{\mathcal{K}} = \kappa$.

We say that a θ-covering matrix for κ $\mathcal{K}$ is *downward coherent* if in addition it satisfies the following coherence property.

(5) For every $\alpha < \beta < \kappa$ and $\xi < \theta$ there is η such that $K(\xi,\beta) \cap \alpha \subseteq K(\eta,\alpha)$.

Given a family $\mathcal{A}$ of subsets of κ we say that $\mathcal{A}$ is *covered* by $\mathcal{K}$ if for every $X \in \mathcal{A}$ there is $K \in \mathcal{K}$ such that $X \subseteq K$.

Definition 6.5 (Covering Property (CP)) *Given two regular cardinal $\theta < \kappa$, the* covering property $\mathrm{CP}(\theta, \kappa)$ *is the following statement.*

> *For every θ-covering matrix $\mathcal{K} = \{K(\xi, \beta) : \xi < \theta, \beta < \kappa\}$ for κ there is an unbounded subset A of κ such that $[A]^\theta$ is covered by $\mathcal{K}$.*

$\mathrm{CP}(\kappa)$ *abbreviates* $\mathrm{CP}(\omega, \kappa)$ *and* CP *states that* $\mathrm{CP}(\kappa)$ *holds, for all regular* $\kappa > \mathfrak{c}$.

It should be pointed out that the proofs of both Theorem 3.9 and Theorem 4.16 can be factored through CP. Namely, in [41] Viale proved the following.

Theorem 6.6 *(1) Both* MRP *and* PID *imply* CP.
(2) CP *implies $\kappa^{\aleph_0} = \kappa$, for all regular $\kappa \geq \mathfrak{c}$. In particular,* CP *implies* SCH.
□

Proposition 6.7 *Let κ be a regular cardinal. Then there is a downward coherent κ-covering matrix $\mathcal{K}$ for κ^+ with $\tau_{\mathcal{K}} = \kappa$.*

PROOF: For each limit ordinal $\alpha < \kappa^+$ fix a club C_α in α of minimal order type. If $\alpha = \beta + 1$ is a successor ordinal let $C_\alpha = \{\beta\}$. Recall the definition of Todorčević's ρ function, see for instance [37, pp. 271]. The function $\rho : [\kappa^+]^2 \to \kappa$ is defined recursively as follows:

$$\rho(\alpha, \beta) = \sup\{\mathrm{ot}(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha), \rho(\xi, \alpha) : \xi \in C_\beta \cap \alpha\},$$

where we let $\rho(\alpha, \alpha) = 0$, by convention. The function ρ has the following properties, see [37, Lemmas 9.1.1 and 9.1.2].

(1) For every $\nu < \kappa$ and $\alpha < \kappa^+$ the set $P_\nu(\alpha) = \{\xi < \alpha : \rho(\xi, \alpha) < \nu\}$ has cardinality at most $|\nu| + \aleph_0$.
(2) For every $\alpha < \beta < \gamma < \kappa^+$,
  (a) $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$,
  (b) $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\alpha, \gamma)\}$.

Let now $K(\nu, \alpha) = \overline{P_\nu(\alpha)}$, for all $\nu < \kappa$ and $\alpha < \kappa^+$. Then the sequence $\mathcal{K} = \{K(\nu, \alpha) : \nu < \kappa, \alpha < \kappa^+\}$ is a downward coherent κ-covering matrix for κ^+ and $\tau_{\mathcal{K}} = \kappa$. □

In [41] Viale also showed the following.

Theorem 6.8 *Let $V \subseteq W$ be two models of set theory with the same reals and cardinals. Assume W satisfies* CP. *Then V and W have the same ordinals of cofinality ω.*

PROOF: Assume otherwise and let κ be the least uncountable cardinal which is regular in V and has cofinality ω in W. Then by our assumption $\kappa > \mathfrak{c}$. In V fix a downward coherent κ-covering matrix $\mathcal{K} = \{K(\xi, \alpha) : \xi < \kappa, \alpha < \kappa^+\}$ for κ^+ such that $\tau_{\mathcal{K}} = \kappa$. Now, since $\mathrm{cof}^W(\kappa) = \omega$ we can fix an increasing ω-sequence $\{\xi_n : n < \omega\}$ of ordinals converging to κ. Let $L(n, \alpha) = K(\xi_n, \alpha)$, for all n and α. Since $(\kappa^+)^V = (\kappa^+)^W$ it follows that $\mathcal{L} = \{L(n, \alpha) : n < \omega, \alpha < \kappa^+\}$ is a downward coherent ω-covering matrix for κ^+ in W. Now, in W, apply $\mathrm{CP}(\omega, \kappa^+)$ to $\mathcal{L}$ to get an unbounded subset A of κ^+ such that $[A]^{\aleph_0}$ is covered by $\mathcal{L}$. Let α be such that $\mathrm{ot}(A \cap \alpha) = \kappa$. Since $\mathrm{ot}(L(n, \alpha)) < \kappa$ we can pick $\xi_n \in A \setminus L(n, \alpha)$, for all n. Let $X = \{\xi_n : n < \omega\}$. By our construction X is not covered by $L(n, \alpha)$, for any n. Since $\mathcal{L}$ is a downward coherent ω-covering matrix it follows that X is not covered by any member of $\mathcal{L}$, a contradiction. $\square$

Concerning agreement on ordinals of cofinality ω_1 Viale [41] also proved the following.

Theorem 6.9 *Let $V \subseteq W$ be two models of set theory with the same cardinals. Assume W satisfies* MM *and that all limit cardinals are strong limits. Then V and W have the same ordinals of cofinality ω_1.* $\square$

7 Open problems

In §4 we showed that PID implies that the bounding number $\mathfrak{b}$ is at most $\aleph_2$. Assuming the existence of a supercompact cardinal one can produce models of PID in which the continuum is either $\aleph_1$ or $\aleph_2$. However, the following is still open.

Question 1 *Does* PID *imply that $\mathfrak{c} \leq \aleph_2$?*

In §5 we showed that BPFA implies the existence of a well ordering of the reals which is Δ_1-definable in parameter a subset of ω_1. Concerning definable (without parameters) well orderings of the reals, as we already mentioned, Asperó [4] and Larson [24] showed that the existence of such well orderings is compatible with PFA and MM. However, it is possible that the existence of such well orderings as an outright consequence of MM.

Question 2 *Does* MM *imply the existence of a well ordering of the reals which is definable without parameters?*

Finally, we restate Conjecture 6.1 we discussed in §6.

Question 3 *Assume $V \subseteq W$ are two models of set theory with the same cardinals and W satisfies* MM. *Is $\mathrm{ORD}^{\omega_1} \cap V = \mathrm{ORD}^{\omega_1} \cap W$?*

References

[1] U. Abraham. Proper forcing. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer-Verlag, Berlin, New York, 2008.

[2] U. Abraham and M. Magidor. Cardinal arithmetic. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer-Verlag, Berlin, New York, 2008.

[3] U. Abraham and S. Todorčević. Partition Properties of ω_1 Compatible with CH. *Fundamenta Mathematicae*, 152:165–180, 1997.

[4] D. Asperó. Guessing and non-guessing of canonical functions. *Ann. Pure Appl. Logic*, 146(2-3):150–179, 2007.

[5] J. Bagaria. Bounded forcing axioms as principles of generic absoluteness. *Arch. Math. Logic*, 39(6):393–401, 2000.

[6] B. Balcar, T. Jech, and T. Pazák. Complete ccc boolean algebras, the order sequential topology, and a problem of von neumann. *Bulletin of the London Mathematical Society*, 37:885–898, 2005.

[7] J. E. Baumgartner. Applications of the proper forcing axiom. In *Handbook of set-theoretic topology*, pages 913–959. North-Holland, Amsterdam, 1984.

[8] J. E. Baumgartner and A D. Taylor. Saturation properties of ideals in generic extensions.i. *Transactions of the American Mathematical Society*, 270(2):557–574, 1982.

[9] A. Caicedo and B. Veličković. Bounded proper forcing axiom and well orderings of the reals. *Mathematical Research Letters*, 13(2-3):393–408, 2006.

[10] G. Cantor. Ein beitrag zur mannifaltigkeitslehre. *J.f. Math.*, 84:242–258, 1878.

[11] J. P. Cohen. The Independence of the Continuum Hypothesis. volume 50, pages 1143–1148, 1963.

[12] J. Cummings. Notes on singular cardinal combinatorics. *Notre Dame J. of Formal Logic*, 46(3):251–282, 2005.

[13] K. Devlin and S. Shelah. A weak form of diamond which follows from $2^{\aleph_0} < 2^{\aleph_1}$. *Israel Journal of Mathematics*, 29:239–247, 1978.

[14] W. B. Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1:139–178, 1970.

[15] T. Eisworth and P. Nyikos. Antidiamond principles and topological applications. *Trans. American Math. Society*, page to appear.

[16] M. Foreman, M. Magidor, and S. Shelah. Martin's Maximum, saturated ideals and nonregular ultrafilters. *Ann.of Math. (2)*, 127(1)(1):1–47, 1988.

[17] M. Foreman and S. Todorčević. A new löwenheim-skolem theorem. *Transactions of the American Mathematical Society*, 357(5):1693–1715, 2005.

[18] D. Gale and F. M. Stewart. Infinite games with perfect information. *Annals of Mathematics Studies*, 28:245–266, 1953.

[19] K. Gödel. *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory.* Princeton University Press, 1940.

[20] K. Gödel. What is Cantor's continuum problem? *American Mathematical Monthly*, 54:515–525, 1947.

[21] K. Gödel. What is Cantor's continuum problem? In P. Benacerraf and H. Putnam, editors, *Philosophy of mathematics selected readings*, pages 470–485. Cambridge University press, 1983.

[22] M. Goldstern and S. Shelah. The bounded proper forcing axiom. *J. Symbolic Logic*, 60(1):58–73, 1995.

[23] T. Jech. *Set theory, The Third Millennium Edition, Revised and Expanded.* Springer, New York, Berlin, 2002.

[24] P. B. Larson. Martin's Maximum and definability in $H(\aleph_2)$. *Annals of Pure and Applied Logic*, to appear.

[25] P. B. Larson. *The Stationary Tower: Notes on a Course by W. Hugh Woodin.* AMS, 2004.

[26] D. A. Martin and R.M. Solovay. Internal Cohen extensions. *Annals of Mathematical Logic*, 2:143–178, 1970.

[27] J. T. Moore. A five element basis for the uncountable linear orders. *Annals of Mathematics*, 163(2):669–688, 2006.

[28] Justin Tatch Moore. Set mapping reflection. *J. Math. Log.*, 5(1):87–97, 2005.

[29] A. Sharon. MRP and square principles. unpublished, 23 pages, 2006.

[30] S. Shelah. Semiproper Forcing Axiom implies Martin's Maximum but not PFA^+. *Journal of Symbolic Logic*, 52(2):360–367, 1987.

[31] S. Shelah. *Cardinal Arithmetic.* Oxford University Press, 1994.

[32] S. Shelah. Reflection implies SCH. *Fundamenta Mathematicae*, 198:95–111, 2008.

[33] J. H. Silver. On the singular cardinal problem. *Proceedings of the International Congress of Mathematicians, Vancouver, B.C., 1974*, 1:265–268, 1975.

[34] R. M. Solovay and S. Tennenbaum. Iterated Cohen extensions and Souslin's problem. *Ann. of Math. (2)*, 94:201–245, 1971.

[35] S. Todorčević. A dichotomy for P-ideals of countable sets. *Fundamenta Mathematicae*, 166(3):251–267, 2000.

[36] S. Todorčević. Generic absoluteness and the continuum. *Math. Res. Lett.*, 9(4):465–471, 2002.

[37] S. Todorčević. *Walks on Ordinals and Their Characteristics*, volume 263 of *Progress in Mathematics*. Birkhauser Verlag AG, 2007.

[38] B. Veličković. Forcing axioms and stationary sets. *Advances in Mathematics*, 94(2)(2):256–284, 1992.

[39] B. Veličković. CCC Forcing and Splitting Reals. *Israel Journal of Mathematics*, 147:209–220, 2005.

[40] M. Viale. The Proper Forcing Axiom and the Singular Cardinal Hypothesis. *Journal of Symbolic Logic*, 71(2):473–479, 2006.

[41] M. Viale. A family of covering properties. *Mathematical Research Letters*, 15(2):221–238, 2008.

[42] W. H. Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*. Walter de Gruyter and Co., 1999.