Forcing axioms and cardinal arithmetic

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Outline

1. Inner models of forcing axioms
2. Definable well orderings of the reals
   - The coding
   - The well ordering
3. Härtig quantifier
4. $\omega$-sequences
5. Open problems
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2. Definable well orderings of the reals
   - The coding
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4. $\omega$-sequences
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Inner models of forcing axioms

Problem

Is there an inner model theory for forcing axioms? To what extent is a model of a forcing axiom determined by its cardinal structure? By a *model* we mean a transitive model containing all the ordinals.

Question

If $V \subseteq W$ are models of some strong forcing axiom and $V$ and $W$ have the same cardinals. Is $\text{ORD}^{\omega_1} \cap V = \text{ORD}^{\omega_1} \cap W$?
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The motivation for this investigation comes from the following observation I made in 1986.

**Theorem (V.)**

If $V$ satisfies MM and $M$ is an inner model of $V$ such that $\aleph_2^M = \aleph_2^V$ then $\mathcal{P}(\omega_1)^V \subseteq M$.

In fact, this is a consequence of stationary set reflection and the following result of Gitik.

**Theorem (Gitik)**

Suppose $M \subseteq V$ are models of set theory and there is a real $x \in V \setminus M$. Then the set $([\omega_2]^{\aleph_0})^V \setminus M$ is stationary in $V$. 
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### Theorem (V.)

If $V$ satisfies MM and $M$ is an inner model of $V$ such that $\kappa_2^M = \kappa_2^V$ then $\mathcal{P}(\omega_1)^V \subseteq M$.

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### Theorem (Gitik)

Suppose $M \subseteq V$ are models of set theory and there is a real $x \in V \setminus M$. Then the set $([\omega_2]^{\aleph_0})^V \setminus M$ is stationary in $V$. 
Proof

For each $\alpha \in [\omega_1, \omega_2)$ choose in $M$ a closed unbounded set $C_\alpha \subseteq [\alpha]^{\aleph_0}$. Let $S = \bigcup_\alpha C_\alpha$. By Gitik’s result if $\mathbb{R}^V \not\subseteq M$ then $E = [\omega_2]^{\aleph_0} \setminus S$ is stationary. By SSR* there is $\alpha < \omega_2$ such that $E \cap [\alpha]^{\aleph_0}$ is stationary, which is a contradiction. Thus, $\mathbb{R}^M = \mathbb{R}^V$. But then by almost disjoint coding $\mathcal{P}(\omega_1)^V \subseteq M$. \hfill \qed
Why require $V$ and $W$ to have the same cardinals? We want to avoid situations like the following.

**Example**

- Suppose $V$ satisfies say MM and has a supercompact cardinal. We can first collapse cardinals and then force MM all over again to obtain a generic extension $W$ of $V$ such that $V$ and $W$ both satisfy MM but otherwise have little in common.
Why require $V$ and $W$ to have the same cardinals? We want to avoid situations like the following.

**Example**

- If $V$ has a proper class of completely Jónsson cardinals and $G$ is generic for the class stationary tower forcing $\mathbb{P}_\infty$ and $W = V[G]$ then there is an elementary embedding $j : V \to W$

  and we can arrange $cp(j)$ to be arbitrary high and of cofinality $\omega$ in $W$. 
We saw that forcing axioms imply that the continuum is $\aleph_2$ and more generally that $\kappa^{\aleph_1} = \kappa$, for all regular $\kappa \geq \aleph_2$. Now we ask how closely do forcing axioms tie the reals and more generally $\omega_1$-sequences of ordinals to ordinals. This is closely related to the question of definitability of well orderings of the reals one can obtain from forcing axioms.

One convenient way to express this is by using the Härting quantifier.
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One convenient way to express this is by using the Härtig quantifier.
**Definition The Härting quantifier**

The logic \( \mathcal{L}(I) \) is obtained by augmenting first-order logic with the binary quantifier \( I \). If \( \mathcal{M} = (M, \ldots) \) is a structure,

\[
\mathcal{M} = (M, \ldots) \models lxy (\phi(x), \psi(y))
\]

iff

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|\{ b \in M : \mathcal{M} \models \phi(b) \}| = |\{ c \in M : \mathcal{M} \models \psi(c) \}|.
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One is then interested in the expressive power of the logic \( \mathcal{L}(I) \) under the assumption of strong forcing axioms.

**Question**

What is the complexity of the set of validities \( V_I \) of the logic \( \mathcal{L}(I) \)?
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4 \( \omega \)-sequences

5 Open problems
Typically one proves that a forcing axiom implies $2^\aleph_1 = \aleph_2$ by showing that there is a definable (with parameters) injection of $\mathcal{P}(\omega_1)/\text{NS}_{\omega_1}$ into $\omega_2$.

For instance, assuming MRP we saw that to each $\delta < \omega_2$ of cofinality $\omega_1$ we can assign the equivalence class modulo $\text{NS}_{\omega_1}$ $[A_\delta]$ of a subset $A_\delta$ of $\omega_1$ such that for every $A \subseteq \omega_1$ there is $\delta$ such that $[A] = [A_\delta]$. 
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To get a well ordering of $\mathcal{P}(\omega)$ fix a partition $\omega_1 = \bigcup_n S_n$ into disjoint stationary sets. Given a real $r \in \mathcal{P}(\omega)$ let $\delta_r$ be the least $\delta$ such that $[A_\delta] = [\bigcup_{n \in r} S_n]$. So, let

$$r < s \text{ iff } \delta_r < \delta_s.$$ 

$<$ is a well ordering of $\mathcal{P}(\omega)$ and is $\Delta_2$-definable over $(H_{\aleph_2}, \in)$.

Why $\Delta_2$?

Because saying that a subset of $\omega_1$ is stationary costs us a quantifier.
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We will discuss the following two theorems.

**Theorem (Caicedo, V.)**

Assume $V \subseteq W$ are two models of set theory, $\aleph_2^V = \aleph_2^W$ and BPFA holds in both $V$ and $W$. Then $\mathcal{P}(\omega_1)^W \subseteq V$.

**Theorem (Caicedo, V.)**

Assume BPFA. Then there is a $\Delta_1$-definable well-ordering of $\mathcal{P}(\omega_1)$ with parameter a subset of $\omega_1$. The length of the well-ordering is $\omega_2$. 


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We start by defining the oscillation of sets of integers.

Given \( x, y, z \subseteq \omega \) we first define an equivalence relation \( \sim_x \) on \( \omega \setminus x \), by setting \( n \sim_x m \iff [n, m] \cap x = \emptyset \). The \( \sim_x \)-equivalence classes are the intervals between the consecutive members of \( x \). Let \( (I_k)_k \) be the increasing enumeration of those classes which intersect both \( y \) and \( z \).
In the case we are interested in \( t \) is finite. We define the oscillation \( o(x, y, z) : t \rightarrow \{0, 1\} \) by

\[
osc(x, y, z)(k) = 0 \text{ iff } \min(y \cap I_k) \leq \min(z \cap I_k)
\]
The coding

Fix a $C$-sequence $\vec{C} = (C_\xi : \xi < \omega_1 \& \lim(\xi))$, i.e., $C_\xi$ is cofinal in $\xi$ of order type $\omega$, for all limit $\xi < \omega_1$.

We code reals by triples of ordinals $< \omega_2$ and then use almost disjoint coding to code subsets of $\omega_1$ by reals.

Suppose $\omega_1 < \beta < \gamma < \delta$ are limit ordinals and $N \subseteq M \subseteq \delta$ countable sets of ordinals.

Assume $\{\omega_1, \beta, \gamma\} \subset N$, that $\sup(\xi \cap N) < \sup(\xi \cap M)$ and $\sup(\xi \cap M)$ is a limit ordinal, for every $\xi \in \{\omega_1, \beta, \gamma, \delta\}$. We define a finite $\{0, 1\}$-sequence

$$s_{\beta \gamma \delta}(N, M).$$

using the oscillation of $N$ and $M$ relative to $\beta, \gamma$ and $\delta$ and the fixed parameter $\vec{C}$. 
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Let $\pi_M : M \to \tilde{M}$ be the transitive collapse. Let $\alpha_M = \pi_M(\omega_1)$, $\beta_M = \pi_M(\beta)$, $\gamma_M = \pi_M(\gamma)$ and $\delta_M = \tilde{M} = ot(M)$. Each of these is a countable limit ordinal.

Let

- $ht_{\alpha_M}(\alpha_N) = |C_{\alpha_M} \cap \alpha_N| = n(N, M) = n$
- $x = \{\pi_M(\xi) \cap \beta_M : \xi \in N \cap \beta\}$
- $y = \{\pi_M(\xi) \cap \gamma_M : \xi \in N \cap \gamma\}$
- $x = \{\pi_M(\xi) \cap \delta_M : \xi \in N \cap \delta\}$

If $osc(x \setminus n, y \setminus n, z \setminus n)$ is a sequence of length $\geq n$ let

$$s_{\beta,\gamma,\delta}(N, M) = osc(x \setminus n, y \setminus n, z \setminus n) \upharpoonright n.$$ 

In all other cases let $s_{\beta,\gamma,\delta}(N, M) = \ast$, i.e. undefined.
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The coding

$x = \{1, 3, 4, 5, 8, 9, 10\}$
Definition

Suppose $\omega_1 < \beta < \gamma < \delta < \omega_2$ and $\beta, \gamma$ and $\delta$ are of cofinality $\omega_1$. We say that the triple $(\beta, \gamma, \delta)$ codes a real $r$ if there is a continuous increasing sequence of countable sets $(N_\nu : \nu < \omega_1)$ whose union is $\delta$ such that for every countable limit ordinal $\nu$ there is $\nu_0 < \nu$ such that

$$r = \bigcup_{\nu_0 < \xi < \nu} s_{\beta \gamma \delta}(N_\xi, N_\nu).$$
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The point is that any such triple of ordinals \((\beta, \gamma, \delta)\) can code at most one real.

Why?

Suppose \((N_\nu : \nu < \omega_1)\) and \((N'_\nu : \nu < \omega_1)\) are two continuous increasing sequences of countable sets with union \(\delta\) witnessing that \((\beta, \gamma, \delta)\) codes \(r\) and \(r'\) respectively. Then there is a club \(D\) in \(\omega_1\) such that \(N_\nu = N'_\nu\), for all \(\nu \in D\).

Let \(\nu\) be a limit point of \(D\). Then by the definition of the coding there is \(\nu_0 < \nu\) such that

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    r = \bigcup_{\xi \in D \cap (\nu_0, \nu)} s_{\beta\gamma\delta}(N_\xi, N_\nu) = \bigcup_{\xi \in D \cap (\nu_0, \nu)} s_{\beta\gamma\delta}(N'_\xi, N'_\nu) = r'.
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r &= \bigcup_{\xi \in D \cap (\nu_0, \nu)} s_{\beta \gamma \delta}(N_\xi, N_\nu) = \bigcup_{\xi \in D \cap (\nu_0, \nu)} s_{\beta \gamma \delta}(N'_\xi, N'_\nu) = r'.
\end{align*}
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The point is that any such triple of ordinals \((\beta, \gamma, \delta)\) can code at most one real.

Why?

Suppose \((N_{\nu} : \nu < \omega_1)\) and \((N'_{\nu} : \nu < \omega_1)\) are two continuous increasing sequences of countable sets with union \(\delta\) witnessing that \((\beta, \gamma, \delta)\) codes \(r\) and \(r'\) respectively. Then there is a club \(D\) in \(\omega_1\) such that \(N_{\nu} = N'_{\nu}\), for all \(\nu \in D\).

Let \(\nu\) be a limit point of \(D\). Then by the definition of the coding there is \(\nu_0 < \nu\) such that

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Let $r$ be a given real. Use MRP to show $r$ is coded by some triple of ordinals. Define an open stationary set mapping $\Sigma_r$ on the set of all countable elementary submodels of $H_{\aleph_4}$ containing $\vec{C}$. For $M \in \text{dom}(\Sigma_r)$ we let $\Sigma_r(M)$ be

$$\{ N \in [M \cap \omega_4]^\omega : s_{\omega_2 \omega_3 \omega_4}(N, M \cap \omega_4) \subseteq r \}.$$ 

$\Sigma_r(M)$ is open in the Ellentuck topology. The difficult part is to show that $\Sigma_r(M)$ is $M$-stationary. This is done using Namba types games inside the transitive collapse $\tilde{M}$ of $M$. Inside $\tilde{M}$ $\pi_M(\omega_i)$, for $i = 1, \ldots, 4$ are cardinals, but from the outside they are countable ordinals, as witnessed by the $C$-sequence $\vec{C}$. 
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Let $\langle N_\nu : \nu < \omega_1 \rangle$ be a reflecting sequence for $\Sigma_r$ and let $N = \bigcup_{\nu<\omega_1} N_\nu$. Then $N$ is an elementary submodel of $H_{\omega_4}$ of size $\aleph_1$.

Let $\pi_N : N \rightarrow \tilde{N}$ be the transitive collapse. Then one can show that $(\pi_N(\omega_2), \pi_N(\omega_3), \pi_N(\omega_4))$ codes $r$. 
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The coding

Not every triple of ordinals $< \omega_2$ codes a real, but given $\omega_1 < \beta < \gamma < \delta < \omega_2$ of cofinality $\omega_1$ we can define a family of clopen partitions

$$\alpha = \bigcup_{s \in 2^n \cup \{\ast\}} K_{s}$$

for $\alpha < \omega_1$ and apply MRP to find a continuous increasing chain of countable sets $\langle N_\nu : \nu < \omega_1 \rangle$ with union $\delta$ such that for each limit $\nu$ the "pattern of oscillation" between $N_\xi$ and $N_\nu$ stabilizes on a tail of $\xi$'s below $\nu$.

So, we show that $(\beta, \gamma, \delta)$ "codes something", although not necessarily a real.
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To get a $\Delta_1$ well ordering of the reals with parameter the $C$-sequence $\vec{C}$ we proceed as follows.

We define a theory $T$ which is sufficient to do the coding and decoding. Transitive models of this theory of size $\aleph_1$ are uniquely determined by their ordinals. Let $M_\theta$ be the unique transitive model of $T$ such that $\text{ORD}^M = \theta$ (if it exists). The function

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Outline

1. Inner models of forcing axioms
2. Definable well orderings of the reals
   - The coding
   - The well ordering
3. Härtig quantifier
4. $\omega$-sequences
5. Open problems
Applications

Recall that $V_I$ is the set of Gödel numbers of the validities for the logic $\mathcal{L}(I)$ with the Härtig quantifier.

**Theorem (Väänänen)**

Assume $V = L$. Then $V_I$ is not $\Sigma^m_n$, for any finite $n, m$.

So, in the context of $V = L$ the Härtig quantifier is extremely powerful. Knowing when two sets have the same cardinality is sufficient to define well orderings, cardinals and compute correctly the powerset operation, i.e. the logic $\mathcal{L}(I)$ is as powerful as 2nd order logic.
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In the context of forcing axioms we have the following results.

Theorem (Caicedo, V.)
Assume BPFA. Then $V_I$ is not projective.

Using the result of Steel that PFA implies $AD^{L(R)}$ and a result of Solovay saying that $AD^{L(R)}$ and the existence of $R^\#$ imply that there is a real which is ordinal definable in $L(R^\#)$ but not ordinal definable in $L(R)$ we obtain the following.

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**Question**

If $V \subseteq W$ are models of some strong forcing axiom and $V$ and $W$ have the same cardinals. Is $\text{ORD}^{\omega_1} \cap V = \text{ORD}^{\omega_1} \cap W$?

This question is still open, but there are some partial results. Using ideas from his proof that PID implies SCH Viale has shown the following.
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This question is still open, but there are some partial results. Using ideas from his proof that PID implies SCH Viale has shown the following.
Theorem (Viale)

Assume $V \subseteq W$, the P-ideal dichotomy holds in $W$ and $V$ and $W$ have the same reals and cardinals. Let $\kappa$ be the least such that $(\kappa^\omega)^W \notin V$. Then $\kappa$ is not a regular cardinal in $V$. 
Theorem (Viale)

Assume $V \subseteq W$ are models with the same cardinals and reals. Assume that $W$ satisfies PID and $\kappa$ is the least such that $(\kappa^\omega)^W \not\subseteq V$ (so $\text{cof}(\kappa)^V = \omega$).

1. For every $W$-regular $\lambda < \kappa$ there is a stationary set in $\kappa^+$ consisting of ordinals of $V$ cofinality $\lambda$ which is not stationary in $W$.

2. If $\kappa = \aleph_\omega^V$ then $\kappa$ is $V$-Jónsson, i.e. any algebra on $\kappa$ which is in $V$ has a proper subalgebra of the same size in $W$. So, if $V$ and $W$ have the same bounded subsets of $\kappa$ then $\kappa$ is Jónsson in $W$. 
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Open Problems

- Assume If $V \subseteq W$, both $V$ and $W$ are models of PFA and have the same cardinals. Is $\text{ORD}^{\omega_1} \cap V = \text{ORD}^{\omega_1} \cap W$?
- Does PFA or MM imply the existence of a well-ordering of the reals which is definable without parameters? Paul Larson has shown that the existence of such a well-ordering is consistent with MM.
- Does the P-ideal dichotomy imply $2^{\aleph_0} \leq \aleph_2$? It is known that most standard forcing for adding a real destroy the P-ideal dichotomy.
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