Maharam Algebras

Boban Veličković

Equipe de Logique, Université de Paris 7, 2 Place Jussieu, 75251 Paris, France

Abstract

Maharam algebras are complete Boolean algebras carrying a positive continuous submeasure. They were introduced and studied by Maharam in [24] in relation to Von Neumann's problem on the characterization of measure algebras. The question whether every Maharam algebra is a measure algebra has been the main open problem in this area for around 60 years. It was finally resolved by Talagrand [31] who provided the first example of a Maharam algebra which is not a measure algebra. In this paper we survey some recent work on Maharam algebras in relation to the two conditions proposed by Von Neumann: weak distributivity and the countable chain condition. It turns out that by strengthening either one of these conditions one obtains a ZFC characterization of Maharam algebras. We also present some results on Maharam algebras as forcing notions showing that they share some of the well known properties of measure algebras.

Key words: measure algebra, Maharam algebra, ccc, weakly distributive, P-ideal dichotomy
1991 MSC: Primary 03Exx, Secondary 28Axx

1 Introduction

We say that a complete Boolean $\mathcal{B}$ algebra is a *measure algebra* if it admits a strictly positive σ-additive probability measure. It is easy to see that every measure algebra $\mathcal{B}$ satisfies the following.

1. $\mathcal{B}$ has the *countable chain condition* (ccc), i.e. if $\mathcal{A} \subseteq \mathcal{B}$ is such that $a \wedge b = \mathbf{0}$, for every $a, b \in \mathcal{A}$ such that $a \neq b$, then $\mathcal{A}$ is at most countable.
2. $\mathcal{B}$ is *weakly distributive*, i.e. if $\{b_{n,k}\}_{n,k}$ is a double sequence of elements of $\mathcal{B}$ then the following weak distributivity law holds:

Email address: `boban@logique.jussieu.fr` (Boban Veličković).

$$\bigwedge_n \bigvee_k b_{n,k} = \bigvee_{f:\mathbb{N}\to\mathbb{N}} \bigwedge_n \bigvee_{i<f(n)} b_{n,i}$$

In 1937 Von Neumann asked if these two conditions are sufficient to characterize measure algebras (see [25]). A more general version of this question is to find an algebraic characterization of measure algebras. In her work on Von Neumann's problem Maharam [24] formulated the notion of a continuous submeasure and found an algebraic characterization for a complete Boolean algebra to carry one. Recall that a *submeasure* on Boolean algebra $\mathcal{B}$ is a function $\nu : \mathcal{B} \to [0, 1]$ such that

(1) $\nu(\mathbf{0}) = 0$
(2) If $x \leq y$ then $\nu(x) \leq \nu(y)$
(3) $\nu(x \vee y) \leq \nu(x) + \nu(y)$

We say that ν is *positive* if $\nu(a) > 0$, for every $a \in \mathcal{B} \setminus \{\mathbf{0}\}$. If $\mathcal{B}$ is complete the role of σ-additivity is played by the following continuity condition.

(4) $\nu(x_n) \to \nu(\inf_n x_n)$, whenever $\{x_n\}_n$ is a decreasing sequence.

A submeasure ν satisfying (4) is called *continuous*. If a complete Boolean algebra $\mathcal{B}$ carries a positive continuous submeasure then we call it a *Maharam algebra*. Maharam [24] showed that every Maharam algebra is weakly distributive and satisfies the ccc. Therefore Von Neumann's original question decomposes into two questions.

Question 1 *Is every Maharam algebra a measure algebra?*

Question 2 *Is every ccc weakly distributive complete Boolean algebra a Maharam algebra?*

Over the years a significant amount of work has been done on Question 1, which was known to be equivalent to the famous Control Measure Problem, i.e. the question whether every countably additive vector valued measure μ defined on a σ-algebra of sets and taking values in an F-space, i.e. a completely metrizable topological vector space, admits a *control measure*, i.e. a countable additive scalar measure λ having the same null sets as μ. For instance, Kalton and Roberts [21] showed that a submeasure μ defined on a (not necessarily complete) Boolean algebra $\mathcal{B}$ is equivalent to a measure if and only if it is uniformly exhaustive. Recall that a submeasure μ on a Boolean algebra $\mathcal{B}$ is called *exhaustive* if for every sequence $\{a_n\}_n$ of disjoint elements of $\mathcal{B}$ we have $\lim_n \mu(a_n) = 0$. μ is *uniformly exhaustive* if for every $\epsilon > 0$ there is an integer n such that there is no sequence of n disjoint elements of $\mathcal{B}$ of μ-submeasure $\geq \epsilon$. Clearly, every continuous submeasure on a complete Boolean algebra is exhaustive. If μ is a positive submeasure on a Boolean algebra $\mathcal{B}$ one can define a metric d on $\mathcal{B}$ by setting $d(a,b) = \mu(a\Delta b)$. If μ is exhaustive then the metric completion $\bar{\mathcal{B}}$ of $\mathcal{B}$ equipped with the natural boolean algebraic structure is a complete Boolean algebra and μ has a unique extension $\bar{\mu}$ to a continuous submeasure on $\bar{\mathcal{B}}$. Thus $\bar{\mathcal{B}}$ is a Maharam algebra. It follows that Question 1 is equivalent to the question whether every exhaustive submeasure on a Boolean algebra $\mathcal{B}$ is uniformly exhaustive. Significant advances towards the solution of Question 1 were made by Roberts [28] and Farah [7]. Finally, in the Fall of 2005 by using some of their ideas Talagrand [31] produced a remarkable example of an exhaustive submeasure which is not uniformly exhaustive. As a consequence he obtained the following.

Theorem 1.1 [31] *There is a Maharam algebra which is not a measure algebra.*

Concerning Question 2 already Maharam [24] observed that a Souslin tree provides a counterexample. It is well known that Souslin trees may or may not exist depending on additional axioms of set theory (see for instance [22]). Various other examples relatively consistent MA + ¬CH were provided by Glówczyński [15], the author [34] and others.

In the positive direction, in late 2003 it was shown by Balcar, Jech and Pazák [3] and the author [36] that a positive answer to Question 2 is relatively consistent with ZFC assuming the consistency of a supercompact cardinal. This was achieved by deriving it from the P-ideal dichotomy, a combinatorial statement which was introduced by Abraham and Todorcevic [1] and later generalized by Todorcevic [32]. These results build heavily on the work of Quickert [27] and [26] who defined a natural P-ideal associated to a ccc weakly distributive complete Boolean algebra and noticed that the P-ideal dichotomy may be useful in this context. In particular, Quickert showed that under the P-ideal dichotomy there is no nonatomic ccc forcing notion with the Sacks property and that every ccc weakly distributive complete Boolean algebra satisfies the σ-finite chain condition. In fact, by slightly strengthening the countable chain condition one can obtain a ZFC characterization of Maharam algebras. Say that a complete Boolean algebra $\mathcal{B}$ is *indestructibly ccc* if it remains ccc in any generic extension of the universe of set theory. By using the fact that any instance of the P-ideal dichotomy can by forced by a σ-distributive forcing and such forcing preserves weak distributivity it follows that every indestructibly ccc weakly distributive complete Boolean algebra is a Maharam algebra. A particular instance of this result for σ-finite chain condition Boolean algebras was derived by Todorcevic [33]. It should also be mentioned that a version of this result for Souslin posets was obtained by Farah and Zapletal [10] from Solecki's result [30] about analytic P-ideals. Finally, we mention that it was shown by Farah and the author [8] that some large cardinal assumptions are necessary for the consistency of the positive answer to Question 2.

Concerning weak distributivity there is a natural two player infinite game $G(\mathcal{B})$ on a complete Boolean algebra $\mathcal{B}$ such that $\mathcal{B}$ is weakly distributive if and only if Player I does not have a winning strategy in $G(\mathcal{B})$. If in addition Player II has a winning strategy in this game we call $\mathcal{B}$ *strategically weakly distributive*. Games of this type were introduced by Gray [16] and studied by Jech [19], Dobrinen [6] and others. We will show in this paper that a complete Boolean algebra $\mathcal{B}$ is a Maharam algebra if and only if it is ccc and strategically weakly distributive. A related but apparently different result was obtained by Fremlin [12, Theorem 8F] who considered a slightly different game which is more difficult for Player II.

In light of Talagrand's solution of Maharam's problem it is natural to ask if some known properties of measure algebras are shared by all Maharam algebras. In this direction, it was shown by the author in [36] that every nonatomic Maharam algebra adds a splitting real and by Farah and the author in [9] that the product of two nonatomic Maharam algebras adds a Cohen real. Both of these properties are well known for measure algebras. The problem remains to find a combinatorial property of measure algebras which distinguishes them among all Maharam algebras. At the moment it is not even known if the Maharam algebra constructed by Talagrand contains a nonatomic complete subalgebra which is a measure algebra. If the answer to this is negative this would give a counterexample to a well known problem of Prikry who asked if it is relatively consistent that every ccc forcing adds a Cohen or a random real. For more information about this problem the reader is referred to [35].

The purpose of this paper is to survey some of the main results related to the characterization of Maharam algebras and their combinatorial properties. It reflects the author's personal taste and preferences and is not meant to be exhaustive. A more detailed account of Maharam algebras can be found in Fremlin survey paper [12]. The paper is organized as follows. In §2 we survey some counterexamples to Question 2 under various additional set theoretic assumptions. We present the examples of Glówczyński [15], the author [34] and Farah and the author [8]. In §3 we develop some facts about the sequential topology and present several characterizations of Maharam algebras. We prove a result of Balcar, Jech and Pazák [3] and the author [36] stating that under the P-ideal dichotomy every ccc weakly distributive complete Boolean algebra is a Maharam algebra and then discuss some corollaries of the proof. We present a result of Balcar, Glówczyński and Jech [2] saying that a ccc complete Boolean algebra $\mathcal{B}$ is a Maharam algebra iff the sequential topology τ_s on $\mathcal{B}$ is Hausdorff. We then use this to give another characterization of Maharam algebras: a complete Boolean algebra is a Maharam algebra iff it is ccc and strategically weakly distributive. In §4 we discuss properties of Maharam algebras as forcing notions. We present a result of the author from [36] saying that every nonatomic Maharam algebra adds a splitting real and a result of Farah and the author [9] saying that the product of any two nonatomic Maharam algebras adds a Cohen real. Finally, in §5 we discuss some open problems and directions for future research.

Our notation is mostly standard and can be found in [22] for terms relating to set theory and in [11] for notions from measure theory.

2 Examples

We start by presenting some examples of ccc weakly distributive complete Boolean algebras obtained under additional set-theoretic assumptions which are not Maharam algebras.

Example 1 A Souslin tree is an ω_1 tree with no uncountable chains or antichains. It is well known that Souslin trees may or may not exist. For instance, the combinatorial principle $\diamondsuit$ which holds in the constructible universe L implies that there is a Souslin tree. On the other hand, MA + ¬ CH implies that there are no Souslin trees (see [22]). The regular open algebra of a Souslin tree is a ccc weakly distributive complete Boolean algebra which is not a Maharam algebra (see [24]).

Example 2 Sacks forcing $\mathcal{S}$ consists of all perfect subtrees of $2^{<\omega}$ ordered under inclusion. A key property of this forcing is called the *Sacks property.*

Definition 2.1 *A forcing notion $\mathcal{P}$ has the* Sacks property *iff for every $\mathcal{P}$-name τ for an element of ω^ω and for every condition $p \in \mathcal{P}$ there is a condition $q \leq p$ and a sequence $\langle I_n : n < \omega \rangle$ such that $I_n \in [\omega]^{2^n}$, for all n, and $q \Vdash \tau \in \Pi_n I_n$.*

Clearly the Sacks property implies weak distributivity. The forcing notion $\mathcal{S}$ itself is not ccc, but it is possible, under suitable assumptions, to extract a ccc suborder of $\mathcal{S}$ which still has the Sacks property. This was first done by Jensen [20] who constructed a ccc suborder of $\mathcal{S}$ in order to 'construct' a nonconstructible Π^1_2-singleton. He used $\diamondsuit$ and a fusion argument. This construction was later extended by the author [34]. For a partial ordering $\mathcal{P}$ let CCC($\mathcal{P}$) denote the following statement.

> For every family $\mathcal{D}$ of $2^{\aleph_0}$ dense open subsets of $\mathcal{P}$ there is a ccc perfect suborder $\mathcal{Q}$ of $\mathcal{P}$ such that $D \cap \mathcal{Q}$ is dense in $\mathcal{Q}$, for every $D \in \mathcal{D}$.

Here *perfect suborder* means that the incompatibility relation of $\mathcal{Q}$ is the restriction of the incompatibility relation of $\mathcal{P}$. In [34] the following was shown.

Theorem 2.2 [34] *Suppose $\kappa > \aleph_1$ is regular and $\kappa^{<\kappa} = \kappa$. Then there is a generic extension in which Martin's Axiom holds, $2^{\aleph_0} = \kappa$, and* CCC($\mathcal{S}$) *holds.*

Proposition 2.3 *Assume* CCC($\mathcal{S}$). *Then there is a ccc perfect suborder of* $\mathcal{S}$ *which has the Sacks property.*

PROOF: In order to do this we have to define a family of $2^{\aleph_0}$ dense open subsets of $\mathcal{S}$. Suppose $\bar{A} = \langle A_n : n < \omega\rangle$ is a sequence of countable antichains in $\mathcal{S}$. Let $D_{\bar{A}}$ be the set of all $T \in \mathcal{S}$ such that either there is n such that T is incompatible with all members of A_n or for all n there is $X_n \subseteq A_n$ with $|X_n| \leq 2^n$ such that $T \leq \bigvee X_n$. By a standard fusion argument one shows that each $D_{\bar{A}}$ is dense and open in $\mathcal{S}$. Let

$$\mathcal{D} = \{D_{\bar{A}} : \bar{A} \text{ a sequence of countable antichains }\}.$$

By applying CCC($\mathcal{S}$) to $\mathcal{D}$ we obtain a ccc perfect suborder $\mathcal{Q}$ of $\mathcal{S}$ in which the sets $D_{\bar{A}}$ are all dense. It follows that $\mathcal{Q}$ has the Sacks property. $\Box$

Example 3 We now present a construction of Glówczyński from [15]. In order to obtain the assumptions of the theorem one needs a measurable cardinal, but the advantage is that the construction is very simple.

Theorem 2.4 [15] *Assume Martin's Axiom holds and there is an uncountable cardinal* $\kappa < 2^{\aleph_0}$ *which carries an* $\aleph_1$*-saturated* σ*-ideal* $\mathcal{I}$. *Then* $\mathcal{P}(\kappa)/\mathcal{I}$ *is a ccc weakly distributive complete Boolean algebra which is not a Maharam algebra.*

Remark. This situation is easy to obtain: start with a measurable cardinal κ and force, using the standard ccc poset, MA + $\kappa < 2^{\aleph_0}$. The dual of the measure on κ in the ground model generates an $\aleph_1$-saturated σ-complete ideal $\mathcal{I}$.

PROOF: Let $\mathcal{B} = \mathcal{P}(\kappa)/\mathcal{I}$. Clearly, $\mathcal{B}$ is ccc.

Claim 2.5 $\mathcal{B}$ *is weakly distributive.*

PROOF: Fix a double sequence $\{a_{n,k}\}_{n,k}$ of elements of $\mathcal{B}$. We may assume without loss of generality that $\bigvee_k a_{n,k} = \mathbf{1}$, for each n. We can find subsets $A_{n,k}$ of κ, for each n and k, such that $a_{n,k} = [A_{n,k}]_{\mathcal{I}}$ and such that the family $\{A_{n,k}\}_k$ is a partition of κ, for each n. For each $\alpha < \kappa$, define a function $g_\alpha \in \omega^\omega$ by:

$$g_\alpha(n) = \min\{k : \alpha \in A_{n,k}\}.$$

Since we have MA_κ there is a function $g \in \omega^\omega$ such that $g_\alpha \leq_* f$, for each α. Here $\leq_*$ denotes domination modulo finite sets. Define functions $f_l \in \omega^\omega$ by $f_l(k) = \max\{f(k), l\}$ and let

$$B_l = \{\alpha : g_\alpha(n) \leq f_l(n), \text{ for all } n\}.$$

It follows that $\kappa = \bigcup_l B_l$. Thus, letting $b_l = [B_l]_{\mathcal{I}}$, for each l, we have that $\bigvee_l b_l = \mathbf{1}$, and on the other hand $b_l \leq \bigwedge_n \bigvee_{k<f_l(n)} a_{n,k}$. $\Box$

Claim 2.6 *$\mathcal{B}$ is not a Maharam algebra.*

PROOF: Suppose towards contradiction that $\mathcal{B}$ is a Maharam algebra and fix a continuous submeasure μ on $\mathcal{B}$. By lifting μ to κ we obtain a continuous submeasure $\bar{\mu}$ on $\mathcal{P}(\kappa)$ which gives value 0 to singletons and value 1 to the whole space. Notice that we can replace κ by any set of reals D of size κ. Since we have MA_κ, any subset of D is a G_δ-set. Therefore if Y is a subset of D of $\bar{\mu}$-submeasure 0 then, for every $\epsilon > 0$, there is an open set $G \supseteq Y$ such that $\bar{\mu}(G) < \epsilon$. But now we can repeat the proof of the classical result that under MA_κ any set of reals of size κ is Lebesgue null (see [22]). $\Box$

Example 4 We now present an example from [8] showing that if there is a singular strong limit cardinal κ such that $2^\kappa = \kappa^+$ and $\Box_\kappa$ holds then there is a complete Boolean algebra $\mathcal{B}$ of size κ^+ which is weakly distributive and ccc but is not a Maharam algebra. In fact, it suffices to build $\mathcal{B}$ which is not a measure algebra but such that every complete subalgebra of $\mathcal{B}$ of size at most κ is a measure algebra. The point of this example is that if there is no such cardinal κ then there is an inner model with a measurable cardinal λ of Mitchell order $o(\lambda) = \lambda^{++}$ (see [13]). Therefore the consistency of the positive answer to Question 2 requires at least these large cardinal assumptions. The construction relies heavily on the one of Gitik and Shelah from [14].

Theorem 2.7 [8] *Assume κ is a singular strong limit cardinal of uncountable cofinality such that $2^\kappa = \kappa^+$ and $\Box_\kappa$ holds. Then there is a complete Boolean algebra $\mathcal{B}$ of size κ^+ which is ccc and weakly distributive but is not a Maharam algebra.*

PROOF: We build a complete Boolean algebra $\mathcal{B}$ of size κ^+ which is not a measure algebra, but any complete subalgebra of $\mathcal{B}$ of strictly smaller size is a measure algebra. The fact that small subalgebras are measure algebras clearly implies that $\mathcal{B}$ is ccc and weakly distributive. On the other hand, it is well known (see ([11, p. 584]) that any Maharam algebra which is not a measure algebra contains a countably generated subalgebra which is not a measure algebra. Thus $\mathcal{B}$ cannot be a Maharam algebra.

To begin the construction, note that using the assumptions of the theorem we can obtain a nonreflecting stationary set $S \subseteq \kappa^+$ consisting of limit ordinals of cofinality ω such that $\diamondsuit(S)$ holds and such that there is a $\Box_\kappa$-sequence $(C_\alpha : \alpha < \kappa^+ \& \lim(\alpha))$ such that $C_\alpha \cap S = \emptyset$, for all limit α (see [8] for details).

For a set I let $\mathcal{A}_I$ be the Boolean algebra of clopen subsets of $\{0,1\}^I$. If $I \subseteq J$ there is a natural projection $\pi^J_I : \{0,1\}^J \to \{0,1\}^I$ and we can consider $\mathcal{A}_I$ as a subalgebra of $\mathcal{A}_J$ by identifying a set $A \in \mathcal{A}_I$ with $(\pi^J_I)^{-1}(A)$. We shall consider finitely additive probability measures on various $\mathcal{A}_I$. Note that if μ is such a measure then it naturally extends to a σ-additive measure $\bar{\mu}$ on the σ-

algebra $\mathcal{B}_I$ of Baire subsets of $\{0,1\}^I$ (see [11]). We let $\mathcal{I}_{\bar{\mu}}$ denote the σ-ideal of $\bar{\mu}$-null sets. Given two finitely additive measures μ and ν on a Boolean algebra $\mathcal{A}$ we say μ and ν are *equivalent* and write $\mu \sim \nu$ if for every sequence $(a_n)_n$ of elements of $\mathcal{A}$ we have $\lim \mu(a_n) = 0$ iff $\lim \nu(a_n) = 0$. In case $\mathcal{A} = \mathcal{A}_I$ this means that the measures $\bar{\mu}$ and $\bar{\nu}$ have the same null sets. Finally, if $I \subseteq J$ and μ is a finitely additive measure on $\mathcal{A}_J$ let $\mu \restriction \mathcal{A}_I$ be the induced measure on $\mathcal{A}_I$.

Fix a nonreflecting stationary set $S \subseteq \kappa^+$ consisting of ordinals of cofinality ω, a $\diamondsuit(S)$-sequence $(D_\alpha : \alpha \in S)$ and a $\square_\kappa$-sequence $(C_\alpha : \alpha < \kappa^+ \& \lim(\alpha))$ such that $C_\alpha \cap S = \emptyset$, for all limit α.

We construct by induction a sequence μ_α, for $\alpha < \kappa^+$, such that μ_α is a finite probability measure on $\mathcal{A}_\alpha$ and the following conditions hold:

(1) If $\alpha < \beta$ then $\mu_\alpha \sim \mu_\beta \restriction \mathcal{A}_\alpha$.
(2) If $\alpha = \beta + 1$ for some β, then μ_α is the product of μ_β and the uniform probability measure on $\{0,1\}$ at the β-th coordinate.
(3) If $\alpha \notin S$ is a limit ordinal or $\alpha \in S$ and D_α is bounded in α, then μ_α is the product measure of μ_{α_0} and the $\mu_{\alpha_{\xi+1}} \restriction \mathcal{A}_{[\alpha_\xi, \alpha_{\xi+1})}$, where $\{\alpha_\xi : \xi < \delta\}$ is the increasing enumeration of C_α.
(4) If $\alpha \in S$ and D_α is unbounded in α fix a sequence $\{\alpha_n\}_n$ cofinal in α consisting of elements of D_α. Now let ν_n be the measure on $\mathcal{A}_{[\alpha_n, \alpha_n+1)}$ which is the product of $\mu_{\alpha_{n+1}} \restriction \mathcal{A}_{(\alpha_n, \alpha_{n+1})}$ and, at coordinate α_n, the probability measure ρ_n on $\{0,1\}$ which gives measure $1/2^{n+1}$ to $\{0\}$ and $1 - 1/2^{n+1}$ to $\{1\}$. Finally, let μ_α be the product measure of μ_{α_0} and the ν_n, for $n < \omega$.

It is easy to show by induction that condition (1) holds. The main point is showing that once the construction is completed we have the following.

Claim 2.8 *There is no finitely additive probability measure μ on $\mathcal{A}_{\kappa^+}$ such that for all $\alpha < \kappa^+$, $\mu \restriction \mathcal{A}_\alpha \sim \mu_\alpha$.*

PROOF: Assume otherwise and fix such a μ. For every α let E_α be the set of all $f \in \{0,1\}^{\kappa^+}$ such that $f(\alpha) = 1$. Then $E_\alpha \in \mathcal{A}_{\kappa^+}$, but we also consider it as an element of $\mathcal{A}_I$, for all I such that $\alpha \in I$. Formally, we are identifying E_α and $\pi_I^{\kappa^+}(E_\alpha)$.

Claim 2.9 *For every $\epsilon > 0$ there is $\delta > 0$ such that for every $A \in \mathcal{A}_\alpha$ if $\mu(A) > \epsilon$ then $\mu(A \setminus E_\alpha) > \delta$.*

PROOF: Recall that $\mu_{\alpha+1}$ is the product of μ_α and the uniform probability measure on $\{0,1\}$ at the α coordinate. Therefore if $A \in \mathcal{A}_\alpha$ then

$$\mu_{\alpha+1}(A \setminus E_\alpha) = \frac{1}{2}\mu_\alpha(A).$$

Since $\mu \restriction \mathcal{A}_{\alpha+1} \sim \mu_{\alpha+1}$ the claim follows. $\square$

Now for a given $\alpha < \kappa^+$ define the function $f_\alpha \in \omega^\omega$ by letting $f_\alpha(n)$ be the least m such that for every $A \in \mathcal{A}_\alpha$ if $\mu(A) \geq 1/n$ then $\mu(A \setminus E_\alpha) \geq 1/m$. By Claim 2.9 this function is well defined. Since $\kappa^+ > 2^{\aleph_0}$ there is an unbounded subset X of κ^+ and a function $f \in \omega^\omega$ such that $f_\alpha = f$, for all $\alpha \in X$.

Claim 2.10 *Suppose $\{\alpha_n\}_n$ is a strictly increasing sequence of elements of X. Then*

$$\lim_{n\to\infty} \mu(\bigcap_{i<n} E_{\alpha_i}) = 0.$$

PROOF: Assume otherwise and fix an integer k such that for $\mu(\bigcap_{i<n} E_{\alpha_i}) > 1/k$, for all n. Let $l = f(k)$. Since $f_{\alpha_n} = f$, for every n, and the sequence $\{\alpha_n\}_n$ is strictly increasing, it follows that for every n

$$\mu(\bigcap_{i<n+1} E_{\alpha_i}) < \mu(\bigcap_{i<n} E_{\alpha_i}) - 1/l,$$

which is a contradiction. $\square$

Now since $(D_\alpha : \alpha \in S)$ is a $\diamondsuit(S)$-sequence there is $\alpha \in S$ such that $D_\alpha = X \cap \alpha$ and this set is unbounded in α. It follow that at stage α we were using case (4) of our construction. Let $\{\alpha_n\}_n$ be the chosen sequence of elements of D_α. By the definition of μ_α the sets E_α are stochastically independant and $\mu_\alpha(E_{\alpha_n}) = 1 - 1/2^{n+1}$. But then $\lim_n \mu_\alpha(\bigcap_{i<n} E_{\alpha_i}) \geq \prod_n(1 - 1/2^{n+1}) > 0$. Therefore $\mu \restriction \mathcal{A}_\alpha \nsim \mu_\alpha$, a contradiction. $\square$

Now, extend each μ_α to a σ-additive measure $\bar{\mu}_\alpha$ on the σ-algebra $\mathcal{B}_\alpha$ of Baire subsets of $\{0,1\}^\alpha$. Let $\mathcal{I}_{\bar{\mu}_\alpha}$ be the σ-ideal of $\bar{\mu}_\alpha$-null sets. It follows by our construction that if $\alpha < \beta$ then $\mathcal{I}_{\bar{\mu}_\alpha} = \mathcal{I}_{\bar{\mu}_\beta} \restriction \mathcal{B}_\alpha$ in the obvious sense. Now let $\mathcal{I} = \bigcup_{\alpha<\kappa^+} \mathcal{I}_{\bar{\mu}_\alpha}$ and $\mathcal{C} = \mathcal{B}_{\kappa^+}/\mathcal{I}$. It follows that $\mathcal{C}$ is a complete Boolean algebra of size κ^+ which is not a measure algebra but any complete subalgebra of $\mathcal{C}$ of size at most κ is a measure algebra. Finally to see that $\mathcal{B}$ is not a Maharam algebra note that any Maharam algebra which is not a measure algebra contains a countably generated subalgebra which is not a measure algebra (see [11, p. 584]). $\square$

3 Characterizations of Maharam algebras

The goal of this section is to present several characterizations of Maharam algebras both in ZFC and under additional set theoretic assumptions. We will show that it is relatively consistent with ZFC that the two conditions isolated by Von Neumann, weak distributivity and the countable chain condition, are sufficient to characterize Maharam algebras. We then show that by strengthening either one of these two conditions one can obtain a ZFC characterization. We start by reviewing some facts about the sequential topology defined by Maharam [24].

Definition 3.1 *Let $\mathcal{B}$ be a σ-complete Boolean algebra. We say that an infinite sequence $\{x_n\}_n$ of elements of a $\mathcal{B}$* converges strongly *to x (in symbols $x_n \to x$) iff*

$$\limsup_n x_n = \liminf_n x_n = x$$

It is fairly easy to verify.

(1) If $x_n = x$, for all n, then $x_n \to x$
(2) If $x_n \to x$, then any subsequence of $\{x_n\}_n$ also converges strongly to x.
(3) A monotone sequence is strongly convergent
(4) $x_n \to \mathbf{0}$ iff $\limsup_n x_n = \mathbf{0}$
(5) If $x_n \to x$ then $-x_n \to -x$
(6) $\inf_n x_n \vee \inf_n y_n = \inf_{n,m} x_n \vee y_m$

Given a subset A of $\mathcal{B}$ let us define $\bar{A}$ as the set of all x in $\mathcal{B}$ such that there is a sequence $\{x_n\}_n$ of elements of A such that $x_n \to x$. We now have:

(1) $\overline{\{\mathbf{0}\}} = \{\mathbf{0}\}$
(2) $A \subseteq \overline{A}$
(3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Note that in general $\bar{\bar{A}}$ may be different from $\bar{A}$. In order to get a topology we would have to iterate this operation ω_1 times. However, Maharam ([24, Theorem 2]) showed that this is not necessary if $\mathcal{B}$ satisfies the following condition.

(I) *Given any double sequence $\{x_{n,k}\}_{n,k}$ which, for each fixed n decreases monotonically to $\mathbf{0}$ as $k \to \infty$, there exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that:*

$$\bigwedge_i \bigvee_n x_{n,f(i,n)} = \mathbf{0}$$

Moreover, it is easy to see that if $\mathcal{B}$ is weakly distributive and has the countable chain condition then it satisfies (I). In this case let τ_s be the induced topology. Note that the boolean operations are not in general continuous as binary functions, but they are separately continuous, i.e. if we fix x then the function $y \mapsto x \vee y$ is continuous in y. Similarly for $\wedge$ and Δ.

Now, if $\mathcal{B}$ is a measure algebra with a measure μ this topology is in fact metrizable and the metric d is given by $d(x, y) = \mu(x\Delta y)$, where Δ is the symmetric difference. In this case $(\mathcal{B}, \Delta)$ is a topological group with the sequential topology. Maharam showed that $\mathcal{B}$ carries a continuous positive submeasure, i.e. is a Maharam algebra, iff $(\mathcal{B}, \tau_s)$ be metrizable. Moreover, she found an algebraic condition equivalent to the metrizability of $\mathcal{B}$. Maharam's characterization was improved by Balcar, Glówczyński and Jech [2] who showed that $\mathcal{B}$ is a Maharam algebra assuming only that the sequential topology τ_s is Hausdorff.

We will start by restating Maharam's result in modern terminology. Recall that an ideal $\mathcal{I}$ of subsets of a set X is called a *P-ideal* if for every sequence $\{A_n : n < \omega\}$ of elements of $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subseteq_* A$, for all n. Here $\subseteq_*$ denotes inclusion modulo finite sets. We say that a set Y is *orthogonal* to a family $\mathcal{A}$ provided the intersection of Y with any element of $\mathcal{A}$ is finite. In this case we write $Y \perp \mathcal{A}$ and let $\mathcal{A}^\perp$ denote the collection of all subsets Y of X which are orthogonal to $\mathcal{A}$. Note that $\mathcal{A}^\perp$ is always an ideal. Finally we say that an ideal $\mathcal{I}$ on a set X is *countably generated over* FIN if there is a countable family $\{A_n\}_n$ of elements of $\mathcal{I}$ such that for any $A \in \mathcal{I}$ there is n such that $A \subseteq_* A_n$. Let $\mathcal{B}$ be a ccc weakly distributive complete Boolean algebra. Following Quickert [27] let us define the following ideal $\mathcal{I}$ of countable subsets of $\mathcal{B}$:

> *$X \in \mathcal{I}$ if there is a maximal antichain A of elements of $\mathcal{B}$ such that every member of A is compatible with at most finitely many elements of X.*

Note that $X \in \mathcal{I}$ iff for every enumeration $\{x_n\}_n$ of X $\lim_n x_n = \mathbf{0}$ in the sequential topology. Now we can restate Maharam's result.

Theorem 3.2 [24] *A ccc weakly distributive complete Boolean algebra $\mathcal{B}$ carries a continuous submeasure if and only if the dual $\mathcal{I}^\perp$ of the Quickert ideal of $\mathcal{B}$ is countably generated over* FIN. □

Fix a Boolean algebra and let $\mathcal{I}$ be its Quickert ideal. Note that a set A is in $\mathcal{I}^\perp$ iff the closure $\bar{A}$ of A in the sequential topology does not contain $\mathbf{0}$. The following two facts are from [26].

Lemma 3.3 [26] *If $\mathcal{B}$ is weakly distributive and ccc then $\mathcal{I}$ is a P-ideal.*

Proof: Suppose $X_n \in \mathcal{I}$, for all n. Fix for each n a maximal antichain A_n such that each element of A_n is compatible with at most finitely many members of X_n. By the ccc of $\mathcal{B}$ A_n is countable for all n. Now, using the weak distributivity of $\mathcal{B}$ we can find a maximal antichain A which *weakly refines* each A_n, i.e. such that for each $a \in A$ and each n there are at most finitely many members of A_n compatible with a. Note that each element of A is compatible with at most finitely many members of X_n, for each n. Now, fix an enumeration $\{a_n\}_n$ of A and let X be the union of the sets $X_n \setminus \{x \in X_n : x \wedge \bigvee_{i=0}^n a_i \neq \mathbf{0}\}$. Then $X \in \mathcal{I}$ as witnessed by the antichain A and $X_n \subseteq_* X$, for all n. □

Lemma 3.4 [26] *If $\mathcal{B}$ is ccc then there is no uncountable X such that $[X]^{\leq\omega} \subseteq \mathcal{I}$.*

Proof: Suppose X is an uncountable subset of $\mathcal{B}$. Since $\mathcal{B}$ satisfies the ccc there is $b \in \mathcal{B}$ such that every nonzero $a \leq b$ is compatible with uncountably many members of X. Then again, using the ccc of $\mathcal{B}$ there is a countable $A \subseteq X$ such that every nonzero $a \leq b$ is compatible with infinitely many elements of A. $\square$

We are now in the position to present the following strengthening of Maharam's theorem which was proved by Balcar, Jech and Pazák [3] and the author [36] .

Theorem 3.5 [3][36] *Let $\mathcal{B}$ be a weakly distributive ccc complete Boolean algebra and let $\mathcal{I}$ be the Quickert ideal of $\mathcal{B}$. Assume $\mathcal{B} \setminus \{\mathbf{0}\}$ can be covered by countably many sets orthogonal to $\mathcal{I}$. Then $\mathcal{B}$ is a Maharam algebra.*

Proof: Now, by our assumption we can write $\mathcal{B} \setminus \{\mathbf{0}\} = \bigcup_n X_n$, where X_n is orthogonal to $\mathcal{I}$, for all n. By replacing X_n by its upward closure we may assume that if $a \in X_n$ and $a \leq b$ then $b \in X_n$. Second, we can replace X_n by its topological closure. Note that the topological closure of X_n is still upward closed. Finally, by replacing X_n by $\bigcup_{i \leq n} X_i$, we may assume that $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$. Let $U_n = \mathcal{B} \setminus X_n$. Let $\mathcal{U}$ be the collection of all open sets which are downward closed and contain $\mathbf{0}$. Then, each U_n belongs to $\mathcal{U}$, $U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots$ and $\bigcap\{U_n \colon n < \omega\} = \{\mathbf{0}\}$.

Our first goal is to improve this sequence in order to have the additional property that $U_{n+1} \vee U_{n+1} \subseteq U_n$, for every n, where $U \vee V = \{u \vee v \colon u \in U \text{ and } v \in V\}$. For this, it is clearly sufficient to show that for every $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V \vee V \subseteq U$.

For a subset V of $\mathcal{B}$ let $V(x) = \{a \in V \colon x \vee a \in V\}$. Using the separate continuity of $\vee$ we see that if V is open then so is $V(x)$, for every x. If V is also downward closed then so is $V(x)$ and finally if $x \in V$ then $V(x)$ is nonempty since it contains $\mathbf{0}$. So, if $V \in \mathcal{U}$ and $x \in V$ then $V(x) \in \mathcal{U}$.

Lemma 3.6 *For every $V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $W \vee W \subseteq V$.*

Proof: Assume otherwise and fix a $V \in \mathcal{U}$ for which this fails. Let $V_0 = V \cap U_0$. Then, by our assumption, there are x_0 and y_0 in V_0 such that $x_0 \vee y_0 \notin V$. Let $V_1 = V_0(x_0) \cap V_0(y_0) \cap U_1$. Then $V_1 \in \mathcal{U}$ and therefore there are x_1 and y_1 in V_1 such that $x_1 \vee y_1 \notin V$. Let $V_2 = V_1(x_1) \cap V_1(y_1) \cap U_2$. Then again, $V_2 \in \mathcal{U}$. We pick x_2 and y_2 in V_2 such that $x_2 \vee y_2 \notin V$. We proceed by recursion. Given $V_n \in \mathcal{U}$ we can pick x_n and y_n in V_n such that $x_n \vee y_n \notin V$. We then let $V_{n+1} = V_n(x_n) \cap V_n(y_n) \cap U_{n+1}$. Notice that for every n and k we have $x_n \vee x_{n+1} \vee x_{n+2} \ldots \vee x_{n+k} \in U_n$.

Claim 3.7 *$\{x_n\colon n<\omega\}$ and $\{y_n\colon n<\omega\}$ belong to $\mathcal{I}$.*

PROOF: Since the statement is symmetric let us assume, towards contradiction, that $\{x_n\colon n<\omega\}$ is not in $\mathcal{I}$. Fix a nonzero b in $\mathcal{B}$ such that every nonzero $a\leq b$ is compatible with infinitely many x_n. Since $\mathcal{B}$ is weakly distributive we can pick a strictly increasing function f in ω^ω such that

$$c=\bigwedge_n \bigvee_{i=f(n)+1}^{f(n+1)} x_i \neq \mathbf{0}.$$

Let $z_n=\bigvee_{i=f(n)+1}^{f(n)} x_i$. Then, by our construction, we have that $z_n\in U_{f(n)}$ and on the other hand $c\leq z_n$, for all n. Since the U_n are downward closed it follows that $c\in\bigcap\{U_l\colon l<\omega\}=\{\mathbf{0}\}$, a contradiction. $\square$

Now, since both $\{x_n\colon n<\omega\}$ and $\{y_n\colon n<\omega\}$ are in $\mathcal{I}$ it follows that $\{x_n\vee y_n\colon n<\omega\}$ is in $\mathcal{I}$, as well. This means that the sequence $\{x_n\vee y_n\}_n$ strongly converges to $\mathbf{0}$ and V was supposed to be a neighborhood of $\mathbf{0}$. Therefore, for almost all n, $x_n\vee y_n\in V$, a contradiction. $\square$

Now, using Lemma 3.6 we can improve the original decreasing sequence $U_0\supseteq U_1\supseteq\ldots$ to have in addition that $U_{n+1}\vee U_{n+1}\subseteq U_n$, for all n. At this point we could use Maharam's theorem, but it is equally easy to define a continuous submeasure directly. First, let us define a function $\varphi\colon\mathcal{B}\to[0,1]$ by:

$$\varphi(a)=\inf\{2^{-n}\colon a\in U_n\}$$

Now we define a submeasure $\mu\colon\mathcal{B}\to[0,1]$ as follow

$$\mu(b)=\inf(\{\sum_{i=1}^{l}\varphi(a_i)\colon b\leq\bigvee_{i=1}^{l}a_i\}\cup\{1\})$$

Lemma 3.8 *μ is a positive continuous submeasure on $\mathcal{B}$.*

PROOF: It is clear that if $a\leq b$ then $\mu(a)\leq\mu(b)$ and that $\mu(a\vee b)\leq\mu(a)+\mu(b)$, for every $a,b\in\mathcal{B}$. We need to show that μ is positive on every nonzero element of $\mathcal{B}$ and that it is continuous. The following fact is immediate.

Fact 1 *Suppose $n_1<n_2<\ldots<n_k$ and $a_i\in U_{n_i+1}$, for $i=1,\ldots k$. Then $\bigvee_{i=1}^k a_i\in U_{n_1}$.* $\square$

From this it follows that if $a\notin U_n$ then $\mu(a)\geq 2^{-n}$, therefore μ is positive. Finally, to see that μ is continuous notice that by Lemma 1 of [24] it suffices to prove that if $\{a_n\colon n<\omega\}\in\mathcal{I}$ then $\lim_n\mu(a_n)=0$. To show this fix an integer k. Since $\{a_n\colon n<\omega\}\in\mathcal{I}$ and U_k is large, there is n such that $a_l\in U_k$ for all $l\geq n$. This means that

$$\mu(a_l)\leq\varphi(a_l)\leq 2^{-k}$$

for all $l \geq n$. Since k was abritrary it follows that $\lim_n(a_n) = 0$, as desired. This finishes the proof of Lemma 3.8 and Theorem 3.5. $\square$

For a given uncountable cardinal κ we consider the following statement.

$(*)_\kappa$: *Let X be a set of size at most κ and let $\mathcal{I}$ be a P-ideal of countable subsets of X. Then one of the following two alternatives holds:*
(a) *there is an uncountable subset Y of X such that $[Y]^{\leq\omega} \subseteq \mathcal{I}$, or*
(b) *we can write $X = \bigcup_n X_n$, where X_n is orthogonal to $\mathcal{I}$ for each n.*

The *P-ideal dichotomy* is the statement that $(*)_\kappa$ holds for all cardinals κ. This principle follows from the Proper Forcing Axiom and was first studied by Abraham and Todorcevic [1] who proved that $(*)_{2^{\aleph_0}}$ is relatively consistent with CH. Later, Todorcevic [32] extended this result by showing that the full version of the P-ideal dichotomy is relatively consistent with GCH assuming the existence of a supercompact cardinal. In particular, what is shown in [32] is the following.

Lemma 3.9 *Let $\mathcal{I}$ be a P-ideal of countable subsets of a set X. Assume that X is not covered by countably many sets orthogonal to $\mathcal{I}$. Then there is proper σ-distributive forcing notion $\mathcal{P}$ which adds an uncountable subset Y of X such that $[Y]^{\leq\omega} \subseteq \mathcal{I}$.*

Say that a partial ordering $\mathcal{P}$ is *indestructibly ccc* if $\mathcal{P}$ remains ccc in any generic extension of the universe of set theory. For example, forcing notions which are σ-linked, satisfy the σ-finite chain condition [1] and Souslin forcings [2] are all indestructibly ccc. On the other hand, a Souslin tree T is not indestructibly ccc since forcing with T itself destroys the ccc-ness of T. Now, using Theorem 3.5, Lemma 3.4 and Lemma 3.9 we immediately have the following.

Corollary 3.10 *Every indestructibly ccc weakly distributive complete Boolean algebra is a Maharam algebra.* $\square$

The particular instance of this corollary for Boolean algebras satisfying the σ-finite chain condition was derived directly by Todorcevic [33]. Finally, by assuming the full P-ideal dichotomy we have the following.

Corollary 3.11 *Assume the P-ideal dichotomy. Then every weakly distributive ccc complete Boolean algebra is a Maharam algebra.* $\square$

In [29] Shelah showed that every ccc Souslin forcing which is nowhere weakly distributive adds a Cohen real. On the other hand, it is well known (see [18])

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[1] A partial ordering $\mathcal{P}$ satisfies the *σ-finite chain condition* if it can be written as $\mathcal{P} = \bigcup_n A_n$ such that every antichain in A_n is finite, for all n.
[2] A forcing notion $(\mathcal{P}, \leq)$ is called *Souslin* if $\mathcal{P}$ is an analytic set of reals and both the ordering $\leq$ and the incompatibility relation $\perp$ are analytic subsets of $\mathcal{P}^2$.

that the ccc of Souslin forcing notions is indestructible. Thus we have the following result which was also obtained by Farah and Zapletal [10] by using Solecki's result [30] about analytic P-ideals.

Corollary 3.12 *Let $\mathcal{S}$ be a nonatomic Souslin ccc forcing notion. Then either there is $p \in \mathcal{S}$ such that forcing with $\mathcal{S}$ below p adds a Cohen real or else the regular open algebra $RO(\mathcal{S})$ is a Maharam algebra.* $\square$

We now turn back to the sequential topology τ_s on a complete Boolean algebra $\mathcal{B}$ and present a result of B. Balcar, W. Glówczyński, and T. Jech [2] saying that $\mathcal{B}$ is a Maharam algebra iff τ_s is Hausdorff. In order to prove this we will need a lemma which is similar to Lemma 3.6. Let as before $\mathcal{U}$ denote the family of open sets which are nonempty and downward closed. The difference between Lemma 3.13 and Lemma 3.6 is that now we do not assume that there is a decreasing family $\{U_n\}_n$ of sets which are in $\mathcal{U}$ and such that $\bigcap_n U_n = \{\mathbf{0}\}$. In fact, Lemma 3.6 is not true without this additional assumption.

Lemma 3.13 [2] *Assume $\mathcal{B}$ is ccc and weakly distributive. Then for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V \vee V \vee V \subseteq U \vee U$.*

Proof: Assume otherwise and fix a $U \in \mathcal{U}$ for which this fails. For any $V \subseteq \mathcal{B}$ and $x \in \mathcal{B}$ let as before $V(x)$ denote the set of all $y \in \mathcal{B}$ such that $x \vee y \in V$. So we have that if $V \in \mathcal{U}$ and $x \in V$ then $V(x) \in \mathcal{U}$.

We define recursively a sequence $\{V_n\}_n$ of sets in $\mathcal{U}$ and three sequences $\{x_n\}_n$, $\{y_n\}_n$ and $\{z_n\}_n$ as follows. Let $V_0 = U$. Then, by our assumption, there are x_0, y_0 and z_0 in V_0 such that $x_0 \vee y_0 \vee z_0 \notin U \vee U$. Let $V_1 = V_0(x_0) \cap V_0(y_0) \cap V_0(z_0)$. Then $V_1 \in \mathcal{U}$ and therefore there are x_1, y_1 and z_1 in V_1 such that $x_1 \vee y_1 \vee z_1 \notin U \vee U$. Let $V_2 = V_1(x_1) \cap V_1(y_1) \cap V_1(z_1)$. We proceed by recursion. Given $V_n \in \mathcal{U}$ we can pick x_n, y_n and z_n in V_n such that $x_n \vee y_n \vee z_n \notin U \vee U$. We then let $V_{n+1} = V_n(x_n) \cap V_n(y_n) \cap V_n(z_n)$. Once we complete the construction let $x = \limsup_n x_n$, $y = \limsup_n y_n$ and $z = \limsup_n z_n$. Notice that $x \vee y \vee z \geq \limsup_n (x_n \vee y_n \vee z_n)$ and since $x_n \vee y_n \vee z_n \notin U \vee U$, for all n, and $U \vee U$ is downward closed and topologically open it follows that $x \vee y \vee z \notin U \vee U$.

On the other hand, notice that by our construction for every $l < m < n$ we have

$$\bigvee_{i=0}^{l-1} x_i \vee \bigvee_{i=l}^{m-1} y_i \vee \bigvee_{i=m}^{n-1} z_i \in U.$$

Now, first find l such that $a = x - \bigvee_{i=1}^{l-1} x_i \in U$, then let $U_1 = U(a)$ and find $m > l$ such that $b = y - \bigvee_{i=l}^{m-1} y_i \in U_1$, and finally let $U_2 = U_1(b)$ and find $n > m$ such that $c = z - \bigvee_{m}^{n-1} z_i \in U_2$. So we have that $a \vee b \vee c \in U$. Therefore we have that

$$x \vee y \vee z \leq (\bigvee_{i=0}^{l-1} x_i \vee \bigvee_{i=l}^{m-1} y_i \vee \bigvee_{i=m}^{n-1} z_i) \vee (a \vee b \vee c) \in U \vee U.$$

Since $U \vee U$ is downward closed it follows that $x \vee y \vee z \in U \vee U$, a contradiction. $\square$

Theorem 3.14 [2] *Suppose $\mathcal{B}$ is a ccc complete Boolean algebra for which the sequential topology is Hausdorff. Then $\mathcal{B}$ is a Maharam algebra.*

PROOF: We first prove that $\mathcal{B}$ is weakly distributive. So, fix a sequence $\{A_n\}_n$ of maximal antichains in $\mathcal{B}$. Enumerate each A_n as $\{a_{n,k}\}_k$. Now, given a nonzero $x \in \mathcal{B}$ we need to find $y \leq x$ which is compatible with at most finitely many elements of A_n, for each n. Fix disjoint τ_s-open sets U and V such that $\mathbf{0} \in U$ and $c \in V$. Now, define a decreasing sequence $\{x_n\}_n$ of elements of V as follows. Let $x_0 = x$. Given x_n consider the sequence $\{x_n \wedge \bigvee_{i=0}^{k} a_{n,i}\}_k$. Since A_n is a maximal antichain this sequence converges to x_n and since V is open and $x_n \in V$ there is k_n such that $x_n \wedge \bigvee_{i=0}^{k_n} a_{n,i} \in V$. Let $x_{n+1} = x_n \wedge \bigvee_{i=0}^{k_n} a_{n,i}$. Note that the sequence $\{x_n\}_n$ is decreasing, so it converges to $y = \bigwedge_n x_n$. Since the x_n do not belong to U and U is open it follows that $y \notin U$. So in particular y is not zero and it is clearly compatible with at most finitely many members of A_n, for each n.

Let, as before, $\mathcal{U}$ denote the family of all topologically open downward closed nonempty subsets of $\mathcal{B}$. We claim that for every nonzero $a \in \mathcal{B}$ there is a sequence $\{U_n\}_n$ of members of $\mathcal{U}$ such that $a - \bigvee \bigcap_n U_n \neq \mathbf{0}$. To see this fix $a \neq \mathbf{0}$. Since τ_s is Hausdorff we can find $U \in \mathcal{U}$ such that $a \notin U \vee U$. Using Lemma 3.13 we can pick a decreasing sequence $\{U_n\}_n$ of members of $\mathcal{U}$ such that $U_0 = U$, and $U_{n+1} \vee U_{n+1} \vee U_{n+1} \subseteq U_n \vee U_n$, for every n. It follows that

$$\underbrace{U_{n+k} \vee U_{n+k} \vee \ldots \vee U_{n+k}}_{k+2-\text{times}} \subseteq U_n \vee U_n$$

We claim that the sequence $\{U_n\}_n$ works. Suppose towards contradiction that $a \leq \bigvee \bigcap_n U_n$. Since $\mathcal{B}$ is ccc we can find a sequence $\{b_n\}_n$ of elements of $\bigcap_n U_n$ such that $a \leq \bigvee_n b_n$. Since U_1 is a downward closed neighborhood of $\mathbf{0}$ there is an n such that $a - \bigvee_{i=0}^{n} b_i \in U_1$. Since $b_i \in U_n$, for each i, we have that $\bigvee_{i=0}^{n} b_i \in U_1 \vee U_1$. Therefore, $a \in U_1 \vee U_1 \vee U_1 \subseteq U \vee U$, a contradiction.

Now, we can choose a maximal antichain A and, for each $a \in A$, a decreasing sequence $\{U_n^a\}_n$ of members of $\mathcal{U}$ such that a is incompatible with every element of $\bigcap_n U_n^a$. Since $\mathcal{B}$ is ccc the antichain A is countable and it follows that $\{U_n^a : n \in \omega \text{ and } a \in A\}$ is a countable family of members of $\mathcal{U}$ whose intersection is $\{\mathbf{0}\}$. Therefore by Theorem 3.5 $\mathcal{B}$ is a Maharam algebra. $\square$

As an application of this result, we now show that by keeping the ccc and strengthening weak distributivity we can obtain another characterization of Maharam algebras. For this we will need the concept of a weak distributivity game $G(\mathcal{B})$ on a complete Boolean algebra $\mathcal{B}$ which is played as follows.

I	a_0, A_0		A_1		A_2		...	
II		a_1		a_2		a_3		...

Player I starts by playing a nonzero $a_0 \in \mathcal{B}$ and a maximal antichain A_0 in $\mathcal{B}$. Then Player II chooses a nonzero element $a_1 \leq a_0$ which is contained in the union of finitely many elements of A_0. Player I then plays another maximal antichain A_1 and Player II plays a nonzero $a_2 \leq a_1$ contained in the union of finitely many elements of A_1. The game proceeds in this fashion ω moves. In the end we say that Player I wins iff $\bigwedge_n a_n = \mathbf{0}$; otherwise Player II wins. A *winning strategy* τ for one of the players is a rule which tells him what to play at any given position such that if he follows τ he wins the game. The following fact is well known and easy to prove (see e.g. [19]).

Proposition 3.15 *Suppose $\mathcal{B}$ is a complete ccc Boolean algebra. Then $\mathcal{B}$ is weakly distributive iff Player I does not have a winning strategy in $G(\mathcal{B})$.* □

Clearly, at most one player can have a winning strategy in this game. Note that if $\mathcal{B}$ is a Maharam algebra then Player II has an easy winning strategy. Namely, fix a strictly positive continuous submeasure μ on $\mathcal{B}$. Suppose I plays a_0, A_0 in the first move. Let $\alpha = \mu(a_0)$. II just makes sure to play elements a_n such that $\mu(a_n) > \alpha/2$. At stage n suppose we have a_n such that $\mu(a_n) > \alpha/2$ and I plays a maximal antichain A_n. Let $\{x_k\}_k$ be an enumeration of A_n. Since μ is continuous there is k such that $\mu(a_0 \wedge \bigvee_{i=0}^k x_i) > \alpha/2$. Player II then plays $a_{n+1} = a_n \wedge \bigvee_{i=0}^k x_i$. In the end, since μ is continuous we have that $\mu(\bigwedge_n a_n) \geq \alpha/2$, so Player II wins this run of the game. Now, let us say that a complete Boolean algebra $\mathcal{B}$ is *strategically weakly distributive* if Player II has a winning strategy in $G(\mathcal{B})$. We now show that by strengthening weak distributivity to strategic weak distributivity in Question 2 we obtain a positive result in ZFC.

Theorem 3.16 *Suppose $\mathcal{B}$ is a ccc strategically weakly distributive complete Boolean algebra. Then $\mathcal{B}$ is a Maharam algebra.*

PROOF: By Theorem 3.14 it suffices to show that the sequential topology τ_s on $\mathcal{B}$ is Hausdorff. We first show that there are two disjoint τ_s-open sets which separate $\mathbf{1}$ and $\mathbf{0}$. Let us fix a winning strategy σ for Player II in $G(\mathcal{B})$. We say that a position $p = (a_0, A_0, a_1, A_1, \ldots, A_n)$ is consistent with σ if $a_k = \sigma(a_0, A_0, a_1, A_1, \ldots, A_{k-1})$, for each $k \geq 1$.

Claim 3.17 *There is a position p in $G(\mathcal{B})$ consistent with σ such that for every two positions q and r of odd length consistent with σ and extending p we have $\sigma(q) \wedge \sigma(r) \neq \mathbf{0}$.*

Proof: Otherwise we could build a tree $\{p_s : s \in 2^{<\omega}\}$ of positions of odd length consistent with σ and such that $\sigma(p_{s^\frown 0}) \wedge \sigma(p_{s^\frown 1}) = \mathbf{0}$, for every $s \in \{0,1\}^{<\omega}$. If $\alpha \in \{0,1\}^\omega$ then $p_\alpha = \bigcup_n p_{\alpha\restriction n}$ is an infinite run of the game in which Player II follows σ and therefore wins. Let $\{a_{\alpha,n}\}_n$ be the sequence of moves played by Player II in p_α. It follows that $a_\alpha = \bigwedge_n a_{\alpha,n} \neq \mathbf{0}$. Since $\sigma(p_{s^\frown 0}) \wedge \sigma(p_{s^\frown 1}) = \mathbf{0}$, for each s, it follows that $\{a_\alpha : \alpha \in \{0,1\}^\omega\}$ is a antichain of size $2^{\aleph_0}$ in $\mathcal{B}$ which contradicts the ccc of $\mathcal{B}$. □

Now fix a position as in Claim 3.17. We may assume it is Player I's turn to play. Let U be the set of all $a \in \mathcal{B}$ such that there exists a maximal antichain A in $\mathcal{B}$ such that $\sigma(p^\frown A) \leq a$ and let V be the set $\{-a : a \in U\}$. Clearly, $\mathbf{1} \in V$ and $\mathbf{0} \in U$. Note that by our assumption on p U and V are disjoint. We claim that they are both open. To see this for U suppose $\{x_n\}_n$ is a sequence converging to $\mathbf{0}$. We can fix a maximal antichain A such that every element of A is compatible with at most finitely many of the x_n. Let $a = \sigma(p^\frown A)$. Then $a \in V$ and $-a \in U$. Since U is closed downward it follows that all but finitely many of the x_n belong to U. Thus, a tail of every sequence converging to $\mathbf{0}$ is contained in U and so $\mathbf{0}$ belongs to the interior of U. A similar argument shows that V contains a tail of any sequence converging to $\mathbf{1}$, i.e. $\mathbf{1}$ is in the interior of V. Given any nonzero $a \in \mathcal{B}$ Player I can start the game by playing $a_0 = a$, so a similar argument gives two disjoint open sets separating a and $\mathbf{0}$. This implies that τ_s is Hausdorff and thus by Theorem 3.14 $\mathcal{B}$ is a Maharam algebra. □

Remark In [12] Fremlin considered the following related game $\mathcal{G}^*(\mathcal{B})$ played on a complete Boolean algebra $\mathcal{B}$.

I	A_0		A_1		A_2		...	
II		a_1		a_2		a_3		...

In the n-th round of the game Player I plays a maximal antichain A_n in $\mathcal{B}$ and Player II responds by playing $a_n \in \mathcal{B}$ which is contained in the union of finitely many members of A_n. The game proceeds in this fashion ω moves. In the end we say that Player II wins iff $(a_n)_n$ converges to $\mathbf{1}$ in the sequential topology. Fremlin shows that a complete Boolean algebra $\mathcal{B}$ is a Maharam algebra iff Player II has a winning strategy in $\mathcal{G}^*(\mathcal{B})$. Clearly, if Player II has a winning strategy in $\mathcal{G}^*(\mathcal{B})$ then $\mathcal{B}$ is ccc. Thus, Fremlin's result follows from Theorem 3.16. In addition Theorem 3.16 implies that Problems 11J and 11K from [12] have negative answers.

4 Maharam algebras as forcing notions

In this section we considered Maharam algebras as forcing notions and ask if they have some of the common properties of measure algebras. We first present a result from [36] saying that every nonatomic Maharam algebra adds a splitting real and then a result from [9] saying that the product of any two nonatomic Maharam algebras adds a Cohen real.

Let $\mathcal{B}$ be a complete Boolean algebra. Recall that $\mathcal{B}$ adds a *splitting real* if there is a $\mathcal{B}$ name τ for a set of integers such that the maximal condition $\mathbf{1}_{\mathcal{B}}$ forces that τ is neither contained nor disjoint from an infinite set of integers from the ground model. It is well known that any nonatomic measure algebra adds a splitting real. The following result shows that Maharam algebras have the same property.

Theorem 4.1 [36] *Let $\mathcal{B}$ be a nonatomic Maharam algebra. Then $\mathcal{B}$ adds a splitting real.*

We will need the following lemma.

Lemma 4.2 *Let $\mathcal{B}$ be a ccc complete Boolean algebra and let $\mathcal{I}$ be the Quickert ideal of $\mathcal{B}$. Suppose $\mathcal{B}$ does not add a splitting real below any condition. Then every infinite subset X of $\mathcal{B}$ has an infinite subset Y such that either $\bigwedge Y \neq \mathbf{0}$ or $Y \in \mathcal{I}$.*

PROOF: Fix an enumeration $X = \{b_n : n < \omega\}$ and let τ be the name for an element of 2^ω defined by $||\tau(n) = 1|| = b_n$. Since τ is forced not to be a splitting real there is an infinite $I_0 \subseteq \mathbb{N}$ and a nonzero c_0 such that $c_0 \Vdash$ "$\tau \restriction I_0$ is constant." We recursively build an antichain $\{c_\xi : \xi < \delta\}$ and a decreasing mod finite sequence $I_0 \supseteq_* \ldots \supseteq_* I_\xi \supseteq_* \ldots$ such that $c_\xi \Vdash_{\mathcal{B}}$ "$\tau \restriction I_\xi$ is almost constant", for all ξ. At a countable limit stage λ we first diagonalize to find an infinite J such that $J \subseteq_* I_\xi$, for all $\xi < \lambda$. If $\{c_\xi : \xi < \lambda\}$ is not already a maximal antichain, by using the fact that $\tau \restriction J$ is forced not to be a splitting real, we find c_λ incompatible with all the c_ξ, for $\xi < \lambda$, and an infinite $I_\lambda \subset J$ such that $c_\lambda \Vdash_{\mathcal{B}}$ "$\tau \restriction J$ is constant". Since $\mathcal{B}$ is ccc the construction must stop after countably many steps. At this stage we get an infinite I such that $\Vdash$ "$\tau \restriction I$ is almost constant". Let $Y = \{b_n : n \in I\}$. If there is $c \in \mathcal{B} \setminus \{\mathbf{0}\}$ and an integer n such that $c \Vdash_{\mathcal{B}} \tau \restriction (I \setminus n) \equiv 1$, then it follows that $c \leq \bigwedge Y$. Otherwise $\Vdash_{\mathcal{B}}$ "$Y \cap \dot{G}$ is finite ". $\square$

PROOF of Theorem 4.1: Let μ be a continuous submeasure on $\mathcal{B}$. If μ is uniformly exhaustive, by a theorem of Kalton and Roberts [21] (for a precise statement see Theorem 4.5 below) $\mathcal{B}$ is a measure algebra and therefore it adjoins a random real. Assume now $\mathcal{B}$ is not uniformly exhaustive and fix an $\epsilon > 0$ which witnesses this. We can now fix, for each n, a family $A_n =$

$\{a_{n,1} \ldots, a_{n,n}\}$ of pairwise disjoint sets of μ-submeasure $\geq \epsilon$. Note that by Lemma 4.2 and the continuity of μ if X is an infinite set of members of $\mathcal{B}$ each of μ-submeasure $\geq \epsilon$ then there is an infinite subset Y of X such that $\bigwedge Y \neq \mathbf{0}$. Fix a family $\{f_\xi : \xi < \omega_1\}$ of functions in $\prod_n \{1, \ldots, n\}$ such that for $\xi \neq \eta$ there is l such that $f_\xi(k) \neq f_\eta(k)$, for all $k \geq l$. We build a tower of infinite subsets of $\mathbb{N}$, $I_0 \supseteq_* I_1 \supseteq_* \ldots \supseteq_* I_\xi \supseteq_* \ldots$, for $\xi < \omega_1$, such that

$$b_\xi = \bigwedge \{a_{n,f_\xi(n)} : n \in I_\xi\} \neq \mathbf{0},$$

for each ξ. At a stage α we do the following. If α is a limit ordinal we first find an infinite set J such that $J \subseteq_* I_\xi$, for all $\xi < \alpha$; if $\alpha = \beta + 1$ let $J = I_\beta$. Now, look at the family $\{a_{n,f_\alpha(n)} : n \in J\}$. By Lemma 4.2 we can find an infinite $I_\alpha \subseteq_* J$ such that $b_\alpha = \bigwedge \{a_{n,f_\alpha(n)} : n \in I_\alpha\} \neq \mathbf{0}$. Notice that if $\xi \neq \eta$ then b_ξ and b_η are incompatible. Therefore $\{b_\xi : \xi < \omega_1\}$ is an uncountable antichain in $\mathcal{B}$, a contradiction. $\square$

For the following corollary we only need a version of the P-ideal dichotomy for ideals on $2^{\aleph_0}$ whose relative consistency with CH was proved by Abraham and Todorcevic [1] without any large cardinal assumptions. This follows from the fact that every nonatomic complete ccc Boolean algebra contains a nonatomic complete subalgebra of size at most $2^{\aleph_0}$.

Corollary 4.3 [36]*Assume ZFC is consistent. Then so is ZFC + CH + "every nonatomic weakly distributive ccc forcing adds a splitting real".* $\square$

The next result we present is motivated by the well known fact that the product of any two nonatomic measure algebras adds a Cohen real (see for instance [4]). It was shown by Farah and the author [9] that the same is true for Maharam algebras instead of measure algebras. Before we begin let us recall that a submeasure ν on a Boolean algebra is *pathological* if it does not dominate a positive nonzero finitely additive measure. The following is well known (see, for instance, [10, Lemma 2.5]).

Proposition 4.4 *Let ν be a continuous submeasure on a complete Boolean algebra $\mathcal{B}$. Then there is $b \in \mathcal{B}$ such that $\nu \restriction \mathcal{B}_b$ is pathological and $\nu \restriction \mathcal{B}_{b^\complement}$ is equivalent to a measure, where $\mathcal{B}_a$ denotes the restriction of $\mathcal{B}$ to a.*

We will also need the following important result of Kalton and Roberts from [21] (see also [23] and [12]).

Theorem 4.5 [21] *Let ν be a pathological submeasure on a Boolean algebra $\mathcal{B}$. Assume that $\nu(\mathbf{1}_\mathcal{B}) = 1$ and let $\alpha < 1/3$. Then for every integer n there is a sequence $(a_i)_{i=1}^n$ of pairwise disjoint elements of $\mathcal{B}$ such that $\nu(a_i) \geq \alpha$, for all i.*

Let $CO(2^\mathbb{N})$ denote the algebra of all clopen subsets of $2^\mathbb{N}$. For $A \subseteq (2^\mathbb{N})^2$ let

$A_x = \{y : (x,y) \in A\}$ and $A^y = \{x : (x,y) \in A\}$. The following theorem from [9] shows that for the product of two submeasure at one of which is pathological Fubini's theorem fails in the worst possible way. Special cases of this result appear in [5] and [10, Lemma 2.5].

Theorem 4.6 [9] *Let φ and ψ be exhaustive nonatomic submeasures on $CO(2^{\mathbb{N}})$. Assume that ψ is pathological and let $\epsilon > 0$. Then there is a clopen subset W of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that*

(1) $\psi(W_x^{\complement}) \leq \epsilon$, for every $x \in 2^{\mathbb{N}}$,
(2) $\varphi(W^y) \leq \epsilon$, for every $y \in 2^{\mathbb{N}}$.

PROOF: First note that if ν is an exhaustive nonatomic submeasure on $CO(2^{\mathbb{N}})$ then for every $\delta > 0$ there is an integer n and a partition $(A_i)_{i=1}^n$ of $2^{\mathbb{N}}$ into disjoint clopen sets such that $\nu(A_i) \leq \epsilon$, for all i. We fix exhaustive submeasures φ and ψ as in the statement of the theorem.

Claim 4.7 *For every $\epsilon > 0$ there is a clopen set $U \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that*

(1) $\psi(U_x) \geq \psi(2^{\mathbb{N}})/4$, for every $x \in 2^{\mathbb{N}}$,
(2) $\varphi(U^y) \leq \epsilon$, for every $y \in 2^{\mathbb{N}}$.

PROOF: Fix a partition $(A_i)_{i=1}^n$ of $2^{\mathbb{N}}$ into disjoint clopen sets such that $\varphi(A_i) \leq \epsilon$, for all i. Now, since ψ is pathological we can apply Theorem 4.5 to find a partition $(B_i)_{i=1}^n$ of $2^{\mathbb{N}}$ into clopen sets such that $\psi(B_i) \geq \psi(2^{\mathbb{N}})/4$, for all i. Let $U = \bigcup_{i=1}^n A_i \times B_i$. Then U is as desired. □

Claim 4.8 *Let W be a clopen subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ and $\epsilon > 0$. Then there is a clopen U such that $U \cap W = \emptyset$ and*

(1) $\psi(U_x) \geq \psi((W_x)^{\complement})/4$, for every $x \in 2^{\mathbb{N}}$,
(2) $\varphi(U^y) \leq \epsilon$, for every $y \in 2^{\mathbb{N}}$.

PROOF: Since W is clopen we can fix an integer n and a subset S of $2^n \times 2^n$ such that $(x,y) \in W$ iff $(x \restriction n, y \restriction n) \in S$. For each $s \in 2^{<\mathbb{N}}$ let $[s]$ denote the set $\{x \in 2^{\mathbb{N}} : s \subseteq x\}$. For $s \in 2^n$ let W_s denote W_x, for any (equivalently all) $x \in [s]$. Let Z_s be the complement of W_s and apply Claim 1 to get a clopen subset of $[s] \times Z_s$ such that

(1) $\psi((U_s)_x) \geq \psi(Z_s)/4$, for every $x \in [s]$,
(2) $\varphi((U_s)^y) \leq \epsilon/2^n$, for every $y \in Z_s$.

Let $U = \bigcup\{U_s : s \in 2^n\}$. Then U is as desired. □

To finish the proof of the theorem, using Claims 1 and 2 we construct a sequence $(U_n)_n$ of clopen subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that, letting $W_n = \bigcup\{U_i : i < n\}$, for every n we have

(1) $\psi((U_n)_x) \geq \psi((W_n)_x)^{\complement})/4$, for every $x \in 2^{\mathbb{N}}$,
(2) $\varphi((U_n)^y) \leq \epsilon/2^{n+1}$, for every $y \in 2^{\mathbb{N}}$.

We claim that there is n such that $\psi((W_n)_x^{\complement}) \leq \epsilon$, for all $x \in 2^{\mathbb{N}}$. To see that assume otherwise and let C_n be the set of all x such that $\psi((W_n)_x^{\complement}) > \epsilon$. Note that C_n is clopen and by our assumption it is nonempty, for all n. Moreover $C_{n+1} \subseteq C_n$ for all n. Choose x in $\bigcap_n C_n$. It follows that the sequence $((U_n)_x)_n$ is pairwise disjoint and $\psi((U_n)_x) \geq \epsilon/4$. This contradicts the fact that ψ is exhaustive. □

We are now ready to prove the following.

Theorem 4.9 [9] *Let $\mathcal{B}$ and $\mathcal{C}$ be two nonatomic Maharam algebras. Then $\mathcal{B} \times \mathcal{C}$ adds a Cohen real.*

Proof: First, we may assume without loss of generality that $\mathcal{B}$ and $\mathcal{C}$ are both countably generated. By the Loomis-Sikorski theorem we may also assume that there are continuous submeasures φ and ψ on the σ-algebra Bor of Borel subsets of $2^{\mathbb{N}}$ such that $\mathcal{B} = Bor/Null(\varphi)$ and $\mathcal{C} = Bor/Null(\psi)$. Since $\mathcal{B}$ and $\mathcal{C}$ are nonatomic, we may also assume that φ and ψ are positive on any nonempty clopen subset of $2^{\mathbb{N}}$. If both $\mathcal{B}$ and $\mathcal{C}$ are measure algebras, then $\mathcal{B} \times \mathcal{C}$ adds a Cohen real (see e.g., [4, Theorem 3.2.11]). Therefore we may assume that one of $\mathcal{B}$ and $\mathcal{C}$ is not a measure algebra. By Proposition 4.4 for every continuous submeasure θ on Bor we can find a Borel set A such that the restriction of θ to the algebra of Borel subsets of A is equivalent to a measure while the restriction of θ to the algebra of Borel subsets of $2^{\mathbb{N}} \setminus A$ is pathological. This means that without loss of generality we may assume that one of φ or ψ is a pathological submeasure. For concreteness, let us say that ψ is pathological. Our plan is to use Theorem 4.6. For $\epsilon > 0$, let us say that a clopen subset U of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is ϵ-*good* if

(1) $\psi(U_x^{\complement}) \leq \epsilon$, for every $x \in 2^{\mathbb{N}}$,
(2) $\varphi(U^y) \leq \epsilon$, for every $y \in 2^{\mathbb{N}}$.

Thus, by Theorem 4.6 for every $\epsilon > 0$ there is a clopen set U which is ϵ-good. Suppose now U is ϵ-good, for some $\epsilon > 0$. Note that for every Borel sets B and C such that $\varphi(B) > \epsilon$ and $\psi(C) > \epsilon$ we have $B \times C$ intersects both U and $U^{\complement}$. Given ρ, ϵ and δ such that $0 < \delta < \rho$ let us say that U is (ρ, δ)-*good* if for every Borel sets B and C with $\varphi(B) > \rho$ and $\psi(C) > \rho$ and $i \in \{0, 1\}$ there are Borel sets $B^i \subseteq B$ and $C^i \subseteq C$ such that $\varphi(B^i) > \delta$, $\psi(C^i) > \delta$ and such that $B^i \times C^i \subseteq U^i$, where $U^0 = U$ and $U^1 = U^{\complement}$.

Claim 4.10 *Suppose U is clopen and ϵ-good. Then for every $\rho > \epsilon$ there is $\delta > 0$ such that U is (ρ, δ)-good.*

PROOF: Since U is clopen there is n such that U depends on the first n

coordinates, i.e., there is a subset S of $2^n \times 2^n$ such that $(x, y) \in U$ iff $(x \restriction n, y \restriction n) \in S$. Let $\delta = \frac{\rho-\epsilon}{2^n}$. We will show that U is (ρ, δ)-good. Suppose B and C are Borel sets such that $\varphi(B) > \rho$ and $\psi(C) > \rho$. Let T be the set of all $u \in 2^n$ such that $\varphi(B \cap [u]) > \frac{\rho-\epsilon}{2^n}$. Let $B' = \{x \in B : x \restriction n \in T\}$. Similarly let R be the set of all $v \in 2^n$ such that $\psi(C \cap [v]) > \frac{\rho-\epsilon}{2^n}$ and let $C' = \{y \in C : y \restriction n \in R\}$. Then $\varphi(B') > \epsilon$ and $\psi(C') > \epsilon$. Since U is ϵ-good we have $B' \times C' \cap U \neq \emptyset$. Fix $(x, y) \in B' \times C' \cap U$. Let $u = x \restriction n$ and $v = y \restriction n$. It follows that $(u, v) \in S$, i.e. $[u] \times [v] \subseteq U$. Set $B^0 = B \cap [u]$ and $C^0 = C \cap [v]$. Then B^0 and C^0 are as desired. The construction of B^1 and C^1 is symmetric and is done in the same way. □

We now construct a $\mathcal{B} \times \mathcal{C}$-name for a Cohen real. We are going to build a decreasing sequence of positive reals $(\epsilon_n)_n$ converging to 0 and a sequence $(U_n)_n$ of clopen subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that U_n is $(2\epsilon_n, 2\epsilon_{n+1})$-good. To start let $\epsilon_0 = 1/2$ and let U_0 be any clopen subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which is 1/2-good. Given ϵ_n and U_n which is ϵ_n-good we first use Claim 3 to find ϵ_{n+1} such that U_n is $(2\epsilon_n, 2\epsilon_{n+1})$-good and then use Theorem 4.6 a to find a clopen subset U_{n+1} of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which is ϵ_{n+1}-good. By decreasing ϵ_{n+1} if necessary we may assume that it is less than $1/2^{n+1}$. This completes the construction of the ϵ_n and U_n. Now, since each U_n is clopen we can find an integer m_n and a subset S_n of $2^{m_n} \times 2^{m_n}$ such that $x \in U_n$ iff $x \restriction m_n \in S_n$.

Suppose now $\dot{G} \times \dot{H}$ is the canonical name for the $\mathcal{B} \times \mathcal{C}$-generic filter. Let τ denote a name for the set of all n such that there is $(u, v) \in S_n$ such that $[u] \times [v] \in \dot{G} \times \dot{H}$.

Claim 4.11 *The maximal condition in $\mathcal{B} \times \mathcal{C}$ forces that τ is Cohen generic over the ground model.*

PROOF: Let D be a dense set in the Cohen forcing $2^{<\mathbb{N}}$ and let (B, C) be a condition in $\mathcal{B} \times \mathcal{C}$. Let $\rho = \min\{\varphi(B), \psi(C)\}$. Find n such that $\rho > 2\epsilon_n$. Then find an integer m and $t \in 2^m$ such that $s^\frown t \in D$, for every $s \in 2^n$. We build decreasing sequences $(B_i)_{i=0}^m$ and $(C_i)_{i=0}^m$ of conditions in $\mathcal{B}$ and $\mathcal{C}$ respectively. We will have $\varphi(B_i) > 2\epsilon_{n+i}$ and $\psi(C_i) > 2\epsilon_{n+i}$, for all i. To start let $B_0 = B$ and $C_0 = C$. Suppose B_i and C_i have been constructed. Since U_{n+i} is $(2\epsilon_{n+i}, 2\epsilon_{n+i+1})$-good if $t(i) = 0$ we can choose B_{i+1} and C_{i+1} such that $\varphi(B_{i+1}) > 2\epsilon_{n+i+1}$ and $\psi(C_{i+1}) > 2\epsilon_{n+i+1}$ and such that $B_{i+1} \times C_{i+1} \cap U_{n+i} = \emptyset$. If $t(i) = 1$ we can choose B_{i+1} and C_{i+1} such that $B_{i+1} \times C_{i+1} \subseteq U_{n+i}$. Finally let $B^* = B_m$ and $C^* = C_m$. It follows that (B^*, C^*) forces that there is $s \in 2^n$ such that $\tau \restriction (n + m) = s^\frown t$ and therefore $\tau \restriction (n + m) \in D$. Since B and C were arbitrary it follows that τ is forced to be a Cohen real over the ground model. □

This completes the proof of Claim 4 and Theorem 4.9. □

Using the theorem of Shelah [29] saying that every nonatomic ccc nowhere weakly distributive Souslin forcing adds a Cohen real, Corollary 3.12 and Theorem 4.9 we now have the following immediate corollary.

Corollary 4.12 *If $\mathcal{P}$ and $\mathcal{Q}$ are nonatomic Souslin ccc forcing notions then $\mathcal{P} \times \mathcal{Q}$ adds a Cohen real.* $\square$

5 Open Questions

At the moment the only example of a Maharam algebra which is not a measure algebra is the one constructed by Talagrand [31] and not much is known about its properties.

Question 3 *Does the Maharam algebra constructed by Talagrand* [31] *contain a complete nonatomic subalgebra which is a measure algebra?*

This is related to the following well known problem of Prikry.

Question 4 *Is it relatively consistent with ZFC that every nonatomic ccc forcing notion adds either a Cohen or a random real?*

The next question asks if it is possible to generalize Talagrand's construction to higher cardinals.

Question 5 *Given an infinite cardinal κ is there a Maharam algebra of density κ which does not contain a measure subalgebra of density κ?*

Question 6 *Can forcing with a Maharam algebra add a minimal real, i.e. a real r such that for every real $s \in V[r]$ either $s \in V$ or $V[s] = V[r]$?*

Note that by the main result of [29] referred to at the end of Section 4 this is really asking if there is a Souslin ccc forcing which adds a minimal real.

Question 7 *Suppose a ccc forcing $\mathcal{P}$ does not add splitting reals. Is $\mathcal{P}$ necessarily weakly distributive?*

Finally, we mention a well known problem of Horn and Tarski [17]. Recall that a Boolean algebra $\mathcal{B}$ has the *σ-finite chain condition* if it can be written as $\mathcal{B} \setminus \{\mathbf{0}\} = \bigcup_n \mathcal{X}_n$ such that $\mathcal{X}_n$ does not contain infinite antichains, for each n. We say that $\mathcal{B}$ has the *σ-bounded chain condition* if there is such a partition such that for each n there is an integer k_n such that $\mathcal{X}_n$ does not contain any antichain of size at least k_n. Clearly, every Maharam algebra has the σ-finite chain condition; we can simply take $\mathcal{X}_n = \{a \in \mathcal{B} : \mu(a) \geq 1/n\}$, where μ is a continuous strictly positive submeasure on $\mathcal{B}$.

Question 8 *Is there a Boolean algebra which has the σ-finite chain condition but not the σ-bounded chain condition? In particular does every Maharam algebra have the σ-bounded chain condition?*

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