Topology and its Applications 49 (1993) 1–13
North-Holland

OCA and automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$

Boban Velickovic

Mathematics Department, York University, North York, Ont., Canada M3J 1P3

Received 25 June 1990
Revised 20 May 1991 and 6 June 1991

Abstract

Velickovic, B., OCA and automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$, Topology and its Applications 49 (1993) 1–13.

Let OCA denote the Open Coloring Axiom. We show that OCA + $\mathrm{MA}_{\aleph_1}$ implies that $\mathcal{P}(\omega)/\mathrm{fin}$ and $P(\omega_1)/\mathrm{fin}$ have only trivial automorphisms. Under PFA for any infinite cardinal κ all automorphisms of $\mathcal{P}(\kappa)/\mathrm{fin}$ are trivial.

Keywords: Boolean algebras, Open Coloring Axiom, Proper Forcing Axiom, Martin's axiom, Borel liftings, trivial automorphisms.

AMS (MOS) Subj. Class: 03G05, 03E35, 54A35.

Introduction

The Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ and its dual space $\beta\omega\setminus\omega$ are well studied both from the algebraic and the topological point of view (see [8] for a survey of known results). Under the Continuum Hypothesis $\mathcal{P}(\omega)/\mathrm{fin}$ has $2^{2^{\aleph_0}}$ automorphisms. On the other hand Shelah [5, § 4] proved the consistency that every automorphism φ of $\mathcal{P}(\omega)/\mathrm{fin}$ is trivial, i.e., there exist finite sets a, $b \subseteq \omega$ and a bijection $e : \omega\setminus a \to \omega\setminus b$ such that for every $x \subseteq \omega$, $\varphi[x] = [e''(x)]$, where $[y]$ denotes the equivalence class of y modulo the ideal of finite subsets of ω. Clearly, there are only $2^{\aleph_0}$ such automorphisms. Shelah's construction uses the oracle chain condition and is quite

Correspondence to: Dr. B. Velickovic, Mathematics Department, York University, North York, Ont., Canada M3J 1P3.

0166-8641/93/\$06.00 © 1993—Elsevier Science Publishers B.V. All rights reserved

involved. Subsequently, Shelah and Steprans [6] have shown that the same conclusion follows from the Proper Forcing Axiom (PFA) by adapting Shelah's original proof to the PFA context. This paper takes a different approach by studying automorphisms of $\mathcal{P}(\omega)/\text{fin}$ via Ramsey forcing axioms, i.e., axioms that assert the existence of large homogeneous sets in certain kinds of partitions. More specifically we shall consider the following version of the Open Coloring Axiom (OCA):

If X is a set of reals and

$$[X]^2 = K_0 \cup K_1$$

is a partition with K_0 open in the product topology then either there exists an uncountable 0-homogeneous subset of X, or else X can be covered by countably many 1-homogeneous sets.

This version of OCA was introduced and proved relatively consistent with ZFC + $\text{MA}_{\aleph_1}$ by Todorcevic [7] who extended and refined the previous work of Abraham, Rubin and Shelah [1]. For other applications of OCA and related axioms the reader should consult [7, § 8] and [10]. The main result of this paper is that under OCA + $\text{MA}_{\aleph_1}$ all automorphisms of $\mathcal{P}(\omega)/\text{fin}$ are trivial. The argument involves a construction of several open colorings of sets of reals. The key application of OCA is obtained by associating to an automorphism φ an open coloring of a set of reals which cannot have uncountable 0-homogeneous sets. The existence of countably many 1-homogeneous sets which cover the domain of the partition is used to produce countably many Borel functions and an infinite set a such that for every $x \subseteq a$, one of the functions guesses the value of $\varphi[x]$. This is shown to imply the triviality of φ on a. Some ideas from this paper have recently been used by Just [2] to derive consequences of OCA to Boolean algebras of the form $\mathcal{P}(\omega)/I$ for various simply defined ideals I on ω.

The paper is organized as follows. In Section 1 we prove that the existence of countably many Borel liftings of an automorphism φ implies that φ is trivial, thus extending the previous result from [9]. This is then used in Section 2 to prove the main theorem. In Section 3 it is shown that $\text{MA}_{\aleph_1}$ does not imply that all automorphisms are trivial, thus answering a question from [6]. Starting with a model of a fragment of PFA, a generic nontrivial automorphism is introduced without adding new subsets of ω_1. In Section 4 the results from Section 2 are extended to show that, assuming PFA, for any cardinal κ, all automorphisms of $\mathcal{P}(\kappa)/\text{fin}$ are trivial.

Our notation is mostly standard and can be found, for example, in [3]. Given an automorphism φ of $\mathcal{P}(\omega)/\text{fin}$ we shall write $\varphi \restriction a$ for $\varphi \restriction \mathcal{P}(a)/\text{fin}$. We shall say that φ is trivial on a provided $\varphi \restriction a$ is induced by some function $e : a \to \omega$. We shall refer ambiguously to $\mathcal{P}(a)$ and 2^a by identifying a set with its characteristic function.

I would like to thank Stevo Todorcevic for listening to the arguments and for suggesting many improvements in the presentation and the referee for a very critical and useful report.

1. Borel liftings

In this section we extend the following ZFC result from [9].

Theorem 1.1. *Let φ be an automorphism of $\mathcal{P}(\omega)/\text{fin}$. Suppose that there exists a dense G_δ subset X of $\mathcal{P}(\omega)$ and a continuous function $F: X \to \mathcal{P}(\omega)$ such that $\varphi[a] = [F(a)]$ for every $a \in X$. Then φ is trivial.*

Let us fix an automorphism φ of $\mathcal{P}(\omega)/\text{fin}$ and a map $F: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $\varphi[x] = [F(x)]$, for every subset x of ω. The following result will be needed in the proof of Theorem 2.1. Its proof does not use any additional set theoretic assumptions.

Theorem 1.2. *Suppose there exist Borel functions $F_n: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, for $n < \omega$, such that for every $a \subseteq \omega$ there exists $n < \omega$ such that $F(a) =_* F_n(a)$. Then φ is trivial.*

Proof. Let $\mathcal{I}$ be the set of all $a \subseteq \omega$ such that φ is trivial on a. Then $\mathcal{I}$ is an ideal and our goal is to show that it is equal to $\mathcal{P}(\omega)$. We shall need the following fact.

Lemma 1.3. *$\mathcal{I}$ is not a maximal nonprincipal ideal.*

Proof. Suppose that the lemma fails and fix, by [4, p. 112], a dense G_δ subset X of $\mathcal{P}(\omega)$ such that $F_n \restriction X$ is continuous, for all n.

Claim 1. *There is a decomposition $\omega = a_0 \cup a_1$ into disjoint sets and $t_i \subseteq a_i$, for $i \in \{0, 1\}$, such that if $x \subseteq a_i$ then $x \cup t_{1-i} \in X$.*

Proof. Fix a decreasing sequence $\langle U_n: n < \omega \rangle$ of dense open sets such that $X = \bigcap \{U_n: n < \omega\}$. Build inductively an increasing sequence $\langle n_i: i < \omega \rangle$ of natural numbers with $n_0 = 0$, and a sequence $\langle s_i: i < \omega \rangle$ of sets such that $s_i \subseteq [n_i, n_{i+1})$ and for every $x \subseteq \omega$ if $x \cap [n_i, n_{i+1}) = s_i$ then $x \in U_i$. Then let for $\varepsilon = 0, 1$:

$$a_\varepsilon = \bigcup \{[n_i, n_{i+1}): i \equiv \varepsilon \bmod 2\}$$

and

$$t_\varepsilon = \bigcup \{s_i: i \equiv \varepsilon \bmod 2\}.$$

Then this choice of a_ε and t_ε, for $\varepsilon \in \{0, 1\}$ works.

Fix a decomposition as in Claim 1. Suppose for definiteness that φ is nontrivial on a_0 and define, for each n, a function G_n by:

$$G_n(x) = F_n(x \cup t_1) \cap F(a_0).$$

Then G_n is a continuous function on $\mathcal{P}(a_0)$ and for every $x \subseteq a_0$ there exists n such that $G_n(x) =_* F(x)$. Let $\mathcal{T}$ be the restriction of $\mathcal{I}$ to $\mathcal{P}(a_0)$. For every $a \in \mathcal{T}$ pick a 1-1 function $e_a: a \to \omega$ which induces φ on a and define the function E_a on $\mathcal{P}(a)$ by letting $E_a(x) = e_a''(x)$. Then E_a is continuous and $E_a(x) =_* F(x)$, for all $x \subseteq a$. For $n, m < \omega$, consider the set

$$D^a_{n,m} = \{x: E_a(x) \setminus m = G_n(x) \setminus m\}.$$

Then $\mathcal{P}(a)=\bigcup_{n,m<\omega} D^a_{n,m}$ and hence there are some n, $m<\omega$ and a clopen subset U of $\mathcal{P}(a)$ such that $D^a_{n,m}$ is dense in U. Since $D^a_{n,m}$ is closed it follows that $U\subseteq D^a_{n,m}$. Let $\{\langle i_n, l_n, m_n, s_n, t_n\rangle : n<\omega\}$ be an enumeration of all quintuples $\langle i, l, m, s, t\rangle$ such that $i, l, m<\omega$, $t\subseteq i$, and s is a function with $\operatorname{dom} s\in\omega$ and $s:\operatorname{dom} s\to\omega$. Define a function H_n on $\mathcal{P}(a_0)$ by:

$$H_n(x)=(G_{l_n}((x\setminus i_n)\cup t_n)\setminus m_n)\cup s_n''(x\cap\operatorname{dom} s_n).$$

Thus, one obtains a countable family $\{H_n: n<\omega\}$ of continuous functions on $\mathcal{P}(a_0)$ such that for every $a\in\mathcal{T}$ there exists n such that $H_n(x)=e_a''(x)$, for all $x\subseteq a$. Let

$$\mathcal{T}_n=\{a\in\mathcal{T}: e_a''(x)=H_n(x), \text{ for all } x\subseteq a\}.$$

Assume for some $n<\omega$, $\mathcal{T}_n$ is cofinal in $\mathcal{T}, \subseteq_*$. Since $e_a\restriction a\cap b=e_b\restriction a\cap b$, for every $a, b\in\mathcal{T}_n$, it follows that letting

$$e=\bigcup\{e_a: a\in\mathcal{T}_n\},$$

one obtains a function which induces φ on every $a\in\mathcal{T}$. Hence e induces $\varphi\restriction a_0$, since φ is an automorphism. Contradiction.

If none of the $\mathcal{T}_n$ is cofinal we can find a decomposition $a_0=\bigcup_{n<\omega} b_n$ into disjoint sets such that $b_n\in\mathcal{T}$, for all n, and there is no $b\in\mathcal{T}$ such that $b_n\subseteq_* b$, for all n. Let $\mathcal{U}$ be the set of all $b\subseteq a_0$ which are almost disjoint from the b_n. Then $\mathcal{U}$ is a subideal of $\mathcal{T}$ which is σ-directed under $\subseteq_*$. Let $\mathcal{U}_n=\mathcal{T}_n\cap\mathcal{U}$. Then there exists n such that $\mathcal{U}_n$ is cofinal in $\mathcal{U}, \subseteq_*$. Let e be the union of the e_a, for $a\in\mathcal{U}_n$. Then e is a function which induces φ on a for every $a\in\mathcal{U}$. The following claim implies that φ is trivial on a_0.

Claim 2. *There exists $k<\omega$ such that e induces φ on $a_0\setminus\bigcup_{i<k} b_i$.*

Proof. It suffices to show that the set

$$T=\{n<\omega: e\restriction b_m \text{ does not induce } \varphi\restriction b_m\}$$

is finite. For then e induces $\varphi\restriction a$ for every a in the ideal generated by $\mathcal{U}$ and $\{b_m: m\notin T\}$. Since this ideal is dense in $\mathcal{P}(u)$, where $u=a\setminus\{b_m: m\in T\}$, and φ is an automorphism it follows that e induces φ on u.

Now, suppose T were infinite. For each $m\in T$ we pick an infinite subset c_m of b_m such that $e''(c_m)\cap F(c_m)=_*\emptyset$. By shrinking the c_m we can arrange that, furthermore, for every $m, k\in T$, $e''(c_m)\cap F(c_k)=_*\emptyset$. We then find d such that for every $m\in T$, $F(c_m)\subseteq_* d$ and $e''(c_m)\cap d=_*\emptyset$ and let c be such that $F(c)=_* d$. It follows that $c_m\subseteq_* c$, for each $m\in T$ and hence we can pick $i_m\in c_m\cap c$ such that $e(i_m)\notin F(c)$. Let $b=\{i_m: m\in T\}$. Then $b\in\mathcal{U}$ and hence $F(b)=_* e''(b)$. On the other hand $b\subseteq c$ and hence $F(b)\subseteq_* F(c)$. But $e''(b)\cap F(c)=\emptyset$. Contradiction. $\square$

Proof of Theorem 1.2 (continued). Now to prove the theorem assume φ is nontrivial and build inductively disjoint sets a_n, $n<\omega$ and x_n, $n<\omega$ such that for every $n<\omega$:

(a) $x_n\subseteq a_n$,

(b) φ is nontrivial on $\omega\setminus\bigcup_{i\le n} a_i$,

(c) for every $x\subseteq\omega\setminus\bigcup_{i\le n} a_i$

$$F_n\left(\bigcup_{i\le n} x_i\cup x\right)\cap F(a_n)\ne_* F(x_n).$$

Suppose $\langle a_i\colon i<n\rangle$ and $\langle x_i\colon i<n\rangle$ have been constructed. Let $c_n=\omega\setminus\bigcup_{i<n} a_i$ and $z_n=\bigcup_{i<n} x_i$. Then find, using Lemma 1.3 above, a decomposition $c_n=d_n\cup e_n$ such that φ is nontrivial on both d_n and e_n. For $y\subseteq d_n$ define:

$$H_n(y)=\{x\subseteq e_n\colon F_n(z_n\cup y\cup x)\cap F(d_n)=_* F(y)\}.$$

Then, clearly, $H_n(y)$ is a Borel subset of $\mathcal{P}(e_n)$.

Claim. *There exists $y\subseteq d_n$ such that $H_n(y)$ is not comeager.*

Proof. Suppose that $H_n(y)$ is comeager for every $y\subseteq d_n$. Then $\mathrm{Gr}(\varphi\restriction d_n)$ is equal to the set of all pairs $\langle y, u\rangle$ in $\mathcal{P}(d_n)\times\mathcal{P}(\omega)$ such that

$$\{x\subseteq e_n\colon F_n(z_n\cup y\cup x)\cap F(d_n)=_* u\}$$

is comeager. Therefore $\mathrm{Gr}(\varphi\restriction d_n)$ is analytic and can be uniformized on a comeager set by a continuous function. Thus, by Theorem 1.1, φ is trivial on d_n. Contradiction.

Now fix $y\subseteq d_n$ and a basic clopen set $[s]$ in $\mathcal{P}(e_n)$ such that $H_n(y)$ is meager in $[s]$. Let $u_0=s^{-1}\{0\}$, $u_1=s^{-1}\{1\}$, and $u=u_0\cup u_1$. As in the proof of Lemma 1.3 find a decomposition $e_n\setminus\mathrm{dom}\, s=e_n^0\cup e_n^1$ and subsets $t_0\subseteq e_n^0$ and $t_1\subseteq e_n^1$ such that for every $i\in\{0,1\}$ and $x\subseteq e_n^i$, $u\cup x\cup t_{1-i}\notin H_n(y)$. Then, there is $i\in\{0,1\}$ such that φ is nontrivial on e_n^i. Let us assume, for concreteness, that this is true for $i=0$. Then set

$$a_n=d_n\cup u_0\cup e_n^1,$$

and

$$x_n=y\cup u_1\cup t_1.$$

This completes the inductive construction. In the end let $x=\bigcup_{n<\omega} x_n$. It follows that for every $n<\omega$, $F_n(x)\cap F(a_n)\ne_* F(x_n)$, contradicting the fact that for some $n<\omega$, $F_n(x)=_* F(x)$. $\square$

2. Trivial automorphisms

The objective of this section is to prove the following theorem which is the main result of this paper.

Theorem 2.1. $(\mathrm{OCA}+\mathrm{MA}_{\aleph_1})$. *Every automorphism of $\mathcal{P}(\omega)/\mathrm{fin}$ is trivial.*

Proof. The proof will consist of a sequence of lemmas which put together imply the theorem. Let us fix for the rest of this section an automorphism φ and a function

$F:\mathcal{P}(\omega)\to\mathcal{P}(\omega)$ such that $\varphi[x]=[F(x)]$, for every subset x of ω. Call a family $\mathcal{A}$ of almost disjoint infinite subsets of ω *neat* if there is a 1-1 map $e:\omega\to 2^{<\omega}$ such that if $a\in\mathcal{A}$ and $n, m\in a$ then $e(n)\subseteq e(m)$ or $e(m)\subseteq e(n)$. Thus, $\bigcup e''(a)$ is an infinite branch through $2^{<\omega}$, for every $a\in\mathcal{A}$. The following lemma is the key application of OCA in the proof.

Lemma 2.2. *Let $\mathcal{A}$ be a neat almost disjoint family. Then φ is trivial on all but countably many $c\in\mathcal{A}$.*

Proof. Let $e:\omega\to 2^{<\omega}$ be a function witnessing that $\mathcal{A}$ is neat. Let X be the set of all pairs $\langle a, b\rangle$ of subsets of ω such that there exists $c\in\mathcal{A}$ such that $b\subseteq a\subseteq c$, and define the partition:

$$[X]^2=K_0\cup K_1$$

by $\{\langle a, b\rangle, \langle\bar a, \bar b\rangle\}\in K_0$ iff

(a) $\bigcup e''a\neq\bigcup e''\bar a$

(b) $a\cap\bar b=\bar a\cap b$,

(c) $F(a)\cap F(\bar b)\neq F(\bar a)\cap F(b)$.

Then K_0 is open in the product of the separable metric topology τ on X obtained by identifying $\langle a, b\rangle$ with $\langle a, b, F(a), F(b)\rangle$.

Claim. *There are no uncountable 0-homogeneous subsets of X.*

Proof. Suppose Y is an uncountable 0-homogeneous set. Let d be the union of all b such that for some a the pair $\langle a, b\rangle$ belongs to Y. Let $\langle a, b\rangle$ be such a pair. By (b) in the definition of K_0 it follows that $d\cap a=b$ and hence $F(d)\cap F(a)=_* F(b)$. We can find an uncountable $Z\subseteq Y$ and $n<\omega$ such that for every $\langle a, b\rangle\in Z$, $(F(d)\cap F(a))\Delta F(b)\subseteq n$ and $F(b)\setminus n\subseteq F(a)$. Then there are distinct $\langle a, b\rangle$ and $\langle\bar a, \bar b\rangle$ in Z such that $F(a)\cap n=F(\bar a)\cap n$ and $F(b)\cap n=F(\bar b)\cap n$. It then follows that $F(a)\cap F(\bar b)=F(\bar a)\cap F(b)$ which contradicts the fact that $\{\langle a, b\rangle, \langle\bar a, \bar b\rangle\}\in K_0$.

Now, by OCA we can find a decomposition $X=\bigcup_{n<\omega}X_n$ where X_n is 1-homogeneous, for all n. Fix for each n a countable subset D_n of X_n which is dense in X_n in the sense of τ. For each $\langle a, b\rangle\in X$ pick $\sigma(a)\in\mathcal{A}$ such that $b\subseteq a\subseteq\sigma(a)$. Let

$$\mathcal{B}=\{\sigma(a):\langle a, b\rangle\in D_n \text{ for some } n<\omega\}.$$

We shall show that φ is trivial on every $c\in\mathcal{A}\setminus\mathcal{B}$. Thus, fix any such c and decompose it into two disjoint sets $c=c_0\cup c_1$ such that for every $i\in\{0, 1\}$, $n<\omega$, and $\langle a, b\rangle\in X_n$ if $a\subseteq c_i$ then for every $m<\omega$ there exists $\langle\bar a, \bar b\rangle\in D_n$ such that:

(a) $a\cap\bar b=\bar a\cap b$,

(b) $a\cap m=\bar a\cap m$ and $b\cap m=\bar b\cap m$,

(c) $F(a)\cap m=F(\bar a)\cap m$ and $F(b)\cap m=F(\bar b)\cap m$.

This is done as follows. An increasing sequence $\langle n_i : i < \omega \rangle$ is constructed by induction. Let $n_0 = 0$. Suppose $\langle n_i : i \leq k \rangle$ has been defined. Then n_{k+1} is chosen sufficiently large such that for every $x, y, u, v \subseteq n_k$ and every $i \leq k$ if there exist $\langle a, b \rangle \in X_i$ such that $a \cap n_k = x$, $b \cap n_k = y$, $F(a) \cap n_k = u$ and $F(b) \cap n_k = v$ then there exist $\langle a, b \rangle \in D_i$ with the same property such that in addition $a \cap c \subseteq n_{k+1}$. This is possible since a is almost disjoint from c whenever there is b such that $\langle a, b \rangle \in D_n$. Finally, let

$$c_0 = \bigcup \{ c \cap [n_k, n_{k+1}) : k \text{ is even} \}$$

and let $c_1 = c \setminus c_0$.

Now define the function $F_n : \mathcal{P}(c_0) \to \mathcal{P}(\omega)$, for $n < \omega$, by:

$$F_n(b) = \bigcup \{ F(c_0) \cap F(\bar{b}) : \langle \bar{a}, \bar{b} \rangle \in D_n \text{ and } \bar{a} \cap b = c_0 \cap \bar{b} \}.$$

Clearly, F_n is a Borel function for all n. We claim that if $\langle c_0, b \rangle \in X_n$ then $F_n(b) =_* F(b)$. This follows easily from the properties of the decomposition $c = c_0 \cup c_1$. Thus, by Theorem 1.2, φ is trivial on c_0. A similar argument shows that φ is trivial on c_1, and hence it is also trivial on c. □

The rest of the proof consists of showing that if φ is nontrivial then there exists an uncountable neat almost disjoint family $\mathcal{A}$ such that φ is nontrivial on every $a \in \mathcal{A}$. We first show, by a standard application of Martin's axiom, that *any* uncountable almost disjoint family suffices.

Lemma 2.3. *Let $\mathcal{A}$ be an uncountable almost disjoint family of infinite subsets of ω. Then there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ and for every $a \in \mathcal{B}$ a decomposition $a = a_0 \cup a_1$ such that the family $\mathcal{B}_i = \{a_i : a \in \mathcal{B}\}$ is neat for $i \in \{0, 1\}$.*

Proof. By shrinking $\mathcal{A}$, if necessary, we may assume that every clopen subset of 2^ω which intersects $\mathcal{A}$ in fact has uncountable intersection with $\mathcal{A}$. Define the poset $\mathcal{P}$ as follows. Elements of $\mathcal{P}$ are tuples of the form $\langle e^0, e^1, n, A, D \rangle$ where:

(a) $n < \omega$ and $e^i : n \to 2^{<\omega}$ is a 1-1 map for $i \in \{0, 1\}$,

(b) A is a finite subset of $\mathcal{A}$ such that for every distinct $a, b \in A$, $a \cap b \subseteq n$,

(c) $D = \{ f^a : a \in \mathcal{A} \}$, where for every $a \in A$, $f^a : a \cap n \to \{0, 1\}$,

(d) for every $k, l < n$ if there exists $a \in A$ such that $k, l \in a$ and $f^a(k) = f^a(l) = i$ for some $i \in \{0, 1\}$ then

$$e^i(k) \subseteq e^i(l) \quad \text{or} \quad e^i(l) \subseteq e^i(k).$$

For $p \in \mathcal{P}$ write $p = \langle e_p^0, e_p^1, n_p, A_p, D_p \rangle$. Say that $p \leq q$ iff $n_q \leq n_p$, $e_q^i \subseteq e_p^i$ for $i \in \{0, 1\}$, $A_q \subseteq A_p$, and for every $a \in A_q$, $f_q^a \subseteq f_p^a$. It is clear that a sufficiently generic filter in $\mathcal{P}$ produces an uncountable $\mathcal{B} \subseteq \mathcal{A}$, a decomposition for each $a \in \mathcal{B}$, and functions e_i witnessing that $\mathcal{B}_i$ is neat, for $i \in \{0, 1\}$. Thus, in order to finish the proof of the lemma it suffices to establish the following.

Claim. *$\mathcal{P}$ is a ccc poset.*

Proof. Suppose X is an uncountable subset of $\mathcal{P}$. By thinning down, if necessary, we can assume that there exist $n<\omega$, and $e^0, e^1: n \to 2^{<\omega}$ such that $n_p = n$ and $e^i_p = e^i$ for $i \in \{0, 1\}$ and all $p \in X$. In addition, we may assume that the A_p, $p \in X$ form a Δ-system with root A and that for each $a \in A$ there is $f^a: a \cap n \to \{0, 1\}$ such that $f^a_p = f^a$ for all $p \in X$. Fix now any distinct p and q in X. We shall construct a condition r below both p and q. Set $A_r = A_p \cup A_q$, and let $n_r \geqslant n$ be sufficiently large such that $a \cap b \subseteq n_r$ for every distinct $a, b \in A_r$. Suppose now $a \in A_r$. We define $f^a_r: a \cap n_r \to \{0, 1\}$ as follows. If $a \in A_p$ let:

$$f^a_r(k) = \begin{cases} f^a_p(k), & \text{if } k < n, \\ 0, & \text{if } n \leqslant k \leqslant n_r. \end{cases}$$

If $a \in A_q \setminus A_p$ let:

$$f^a_r(k) = \begin{cases} f^a_q(k), & \text{if } k < n, \\ 1, & \text{if } n \leqslant k < n_r. \end{cases}$$

Then set $D_r = \{f^a_r: a \in A_r\}$. Finally, we construct 1-1 functions e^0_r and e^1_r as follows. Set $e^0_r \restriction n = e^0_p$. Since for each $a \in A_p$, $(e^0_p)''(a \cap n)$ is a chain in $2^{<\omega}$ and $\{a \setminus n: a \in A_p\}$ is a family of disjoint sets we can define $e^0_r \restriction [n, n_r)$ in such a way that $(e^0_r)''(a \cap n_r)$ is a chain for every $a \in A_p$. We define e^1_r taking care of $A_q \setminus A_p$ in a similar fashion. Then, clearly, $r = \langle e^0_r, e^1_r, n_r, A_r, D_r \rangle$ is a lower bound for p and q. $\square$

As in the proof of Theorem 1.2 we shall consider the following ideal:

$$\mathcal{J} = \{a \subseteq \omega: \varphi \text{ is trivial on } a\}.$$

Fix, for the rest of the proof, for each a in $\mathcal{J}$ a function $e_a: a \to \omega$ inducing $\varphi \restriction a$. Recall that an ideal on ω containing all finite sets is called *dense* provided every infinite subset of ω contains an infinite member of the ideal, and it is called a *P-ideal* iff it is countably directed under $\subseteq_*$.

Lemma 2.4. *If $\mathcal{J}$ is a dense P-ideal then φ is trivial.*

Proof. Let us define the partition

$$[\mathcal{J}]^2 = K_0 \cup K_1$$

by $\{a, b\} \in K_0$ iff there exists $n \in a \cap b$ such that $e_a(n) \neq e_b(n)$. Note that K_0 is open in the topology on $\mathcal{J}$ obtained by identifying a with e_a.

Claim. *There are no uncountable* 0*-homogeneous subsets of $\mathcal{J}$.*

Proof. Suppose that H is an uncountable 0-homogeneous subset of $\mathcal{J}$. Since $\mathcal{J}$ is countably directed under $\subseteq_*$, by enlarging the members of H, we may assume that they form an $\subseteq_*$-increasing ω_1-sequence. Define the poset $\mathcal{P}$ as follows: $p \in \mathcal{P}$ iff $p = \langle s_p, A_p \rangle$ where $s_p \in 2^{<\omega}$, $A_p \in [H]^{<\omega}$ and for every distinct $a, b \in A_p$ there is $k \in a \cap b$ such that $s(e_a(k)) \neq s(e_b(k))$. Say that $p \leqslant q$ iff $s_q \subseteq s_p$ and $A_q \subseteq A_p$.

Subclaim. *$\mathcal{P}$ is a ccc poset.*

Proof. Suppose X is an uncountable subset of $\mathcal{P}$. We may assume without loss of generality that all the s_p are equal to some s and that the A_p are disjoint and of the same size for $p \in X$. For each $p \in X$ let a_p be the minimal element of A_p under $\subseteq_*$ and pick m_p such that for every a in A_p

$$e_{a_p} \restriction (a_p \setminus m_p) \subseteq e_a \quad \text{and} \quad e_{a_p}^{-1}(\mathrm{dom}(s)) \subseteq m_p.$$

Now, find distinct p and q in X such that for some $m < \omega$, $m_p = m_q = m$ and $e_{a_p} \restriction m = e_{a_q} \restriction m$. Since $\{a_p, a_q\} \in K_0$ there is $k \in a_p \cap a_q$ such that $e_{a_p}(k) \neq e_{a_q}(k)$. It follows that $k \geqslant m$ and that $e_{a_p}(k)$ and $e_{a_q}(k)$ are not in dom s. Hence we can extend s to some t such that $t(e_{a_p}(k)) \neq t(e_{a_q}(k))$. Then $\langle t, A_p \cup A_q \rangle$ is a lower bound for p and q.

Now, applying $\mathrm{MA}_{\aleph_1}$ to a suitable family of dense subsets of $\mathcal{P}$, one obtains a function $\sigma : \omega \to \{0, 1\}$ and an uncountable $H^* \subseteq H$ such that for every distinct $a, b \in H^*$ there is $n \in a \cap b$ such that $\sigma(e_a(n)) \neq \sigma(e_b(n))$. Let now $y = \sigma^{-1}\{1\}$ and let x be such that $F(x) =_* y$. Fix $b \in H^*$. Then $e_b''(b \cap x) =_* \mathrm{ran}(e_b) \cap y$ and therefore for some $n_b < \omega$ for every $k \in b \setminus n_b$, $\sigma(e_b(k)) = 1$ iff $e_b(k) \in y$ iff $k \in x$. Now find distinct $a, b \in H^*$ such that $n_a = n_b = n$ and $e_a \restriction n = e_b \restriction n$. It follows for every $k \in a \cap b$, $\sigma(e_a(k)) = \sigma(e_b(k))$, contradicting the fact that $\{a, b\} \in K_0$.

Now, by OCA, there is a decomposition $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ where for every $n < \omega$, $\mathcal{I}_n$ is 1-homogeneous. Since $\mathcal{I}$ is a P-ideal, there is n such that $\mathcal{I}_n$ is cofinal in $\mathcal{I}, \subseteq_*$. Let e be the union of the e_a, for $a \in \mathcal{I}_n$. it follows that for every $a \in \mathcal{I}$, $e \restriction a =_* e_a$, and, since $\mathcal{I}$ is dense, that e induces φ. $\square$

Remark. Recall that the ideal $\mathcal{T}$ containing all finite sets is called a $P_{\aleph_1}$-ideal if for every subset A of $\mathcal{T}$ of size $\aleph_1$ there exists $b \in \mathcal{T}$ such that $a \subseteq_* b$ for all $a \in A$. Note that OCA itself implies the following general fact. If $\mathcal{T}$ is a $P_{\aleph_1}$-ideal, and $\{f_a : a \in \mathcal{T}\}$ is a family of functions such that $f_a : a \to \omega$ for every $a \in \mathcal{T}$ and $f_b \restriction a =_* f_a$ whenever $a, b \in \mathcal{T}$ and $a \subseteq b$ then there exists $f : \omega \to \omega$ such that $f \restriction a =_* f_a$ for every $a \in \mathcal{T}$.

The following lemma finishes the proof of Theorem 2.1.

Lemma 2.5. *If φ is nontrivial then there exists an uncountable almost disjoint family $\mathcal{A}$ such that φ is nontrivial on every $a \in \mathcal{A}$.*

Proof. By Lemma 2.4 we may assume that there exists a decomposition $\omega = \bigcup_{n<\omega} a_n$ into disjoint infinite sets from $\mathcal{I}$ such that there does not exist a in $\mathcal{I}$ almost containing a_n for all n. Given $f \in \omega^\omega$ let $b_f = \bigcup \{a_n \cap f(n) : n < \omega\}$.

Claim. *There exists $f \in \omega^\omega$ such that φ is nontrivial on b_f.*

Proof. Assume otherwise and let $\mathcal{T}$ be the collection of all $b \subseteq \omega$ which are almost disjoint from the a_n. Then it follows from either $\mathrm{MA}_{\aleph_1}$ or OCA [7] that $\mathcal{T}$ is a $P_{\aleph_1}$-subideal of $\mathcal{I}$. By the remark following the proof of Lemma 2.4 there exists $e: \omega \to \omega$ such that $e \restriction b =_* e_b$ for every $b \in \mathcal{T}$. We claim that there exists $k < \omega$ such that e induces φ on $\omega \setminus \bigcup_{i<k} a_i$, which contradicts the nontriviality of φ. This is done in exactly the same way as in the proof of Lemma 1.3.

Note that Claim actually shows that for every $f \in \omega^\omega$ there exists $g \in \omega^\omega$ such that $b_g \setminus b_f$ is nontrivial. We can then easily construct an $<_*$-increasing sequence f_α; $\alpha < \omega_1$ in ω^ω such that φ is nontrivial on $b_{f_{\alpha+1}} \setminus b_{f_\alpha}$ for every $\alpha < \omega_1$. □

3. Martin's axiom and automorphisms

The purpose of this section is to show that Martin's axiom does not suffice to prove that all automorphisms are trivial. This answers a question raised in [6, 10]. We start with a model of PFA, generically add a nontrivial automorphism of $\mathcal{P}(\omega)/\mathrm{fin}$ and show that the forcing is sufficiently mild to preserve $\mathrm{MA}_{\aleph_1}$. In fact, we use only a fragment of PFA whose consistency does not require any large cardinal assumptions.

Theorem 3.1. *Assume PFA. Then there exists a poset $\mathcal{P}$ such that $V^{\mathcal{P}}$ satisfies* MA+ $2^{\aleph_0} = \aleph_2$ *and there exists a nontrivial automorphism of $\mathcal{P}(\omega)/\mathrm{fin}$.*

Proof. Fix a decomposition $\omega = \bigcup_{n<\omega} I_n$ into disjoint intervals with $\mathrm{card}(I_n) = n$. Define the poset $\mathcal{P}$ as follows: $p \in \mathcal{P}$ iff

(a) p is a 1-1 function with $\mathrm{dom}(p) \subseteq \omega$,

(b) $p''(I_n \cap \mathrm{dom}(p)) \subseteq I_n$, for all n,

(c) $\sup_n \mathrm{card}(I_n \setminus \mathrm{dom}(p)) = \infty$.

Say that $p \leqslant q$ iff $q \subseteq_* p$, i.e., p contains q except for finitely many points. let G be a $\mathcal{P}$-generic filter. In $V[G]$ define $\varphi: \mathcal{P}(\omega)/\mathrm{fin} \to \mathcal{P}(\omega)/\mathrm{fin}$ by setting

$$\varphi[a] = \begin{cases} [p''(a)], & \text{if } a \subseteq \mathrm{dom}(p) \text{ and } p \in G, \\ [\omega \setminus p''(\omega \setminus a)], & \text{if } \omega \setminus a \subseteq \mathrm{dom}(p) \text{ and } p \in G. \end{cases}$$

It is easily checked that φ is well defined and that it is an automorphism of $\mathcal{P}(\omega)/\mathrm{fin}$. A genericity argument shows that φ is nontrivial. Notice that $\mathcal{P}$ is σ-closed. We shall show that, assuming PFA, it does not add ω_1-sequences, and hence $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$ holds in the extension. □

Lemma 3.2. *$\mathcal{P}$ is $\aleph_1$-Baire.*

Proof. Let $\mathcal{D}$ be a collection of $\aleph_1$ dense open subsets of $\mathcal{P}$. We shall show that $\bigcap \mathcal{D}$ is nonempty. A similar argument show that it is actually dense. Define the poset $\mathcal{A}(\mathcal{P})$ as follows. Elements of $\mathcal{A}(\mathcal{P})$ are pairs (p, n) where $p \in \mathcal{P}$ and $n < \omega$. Say that $(p, n) \leq (q, m)$ iff $q \subseteq p$, $m \leq n$, and $\mathrm{dom}(p) \cap m = \mathrm{dom}(q) \cap m$.

Claim. *$\mathcal{A}(\mathcal{P})$ is a proper poset.*

Proof. Let θ be a sufficiently large regular cardinal and let M be a countable elementary submodel of H_θ. Fix $(p, n) \in \mathcal{A}(\mathcal{P}) \cap M$. We find $(q, n) \leq (p, n)$ which is $(M, \mathcal{A}(\mathcal{P}))$-generic. Let $\langle D_n : n < \omega \rangle$ be an enumeration of dense open subsets of $\mathcal{A}(\mathcal{P})$ which are in M. Build inductively a decreasing sequence $\langle q_i : i < \omega \rangle$ of conditions in $\mathcal{P}$ and an increasing sequence $\langle m_i : i < \omega \rangle$ of integers. Let $q_0 = p$ and $m_0 = n$. Having constructed q_i let m_i be the least such that for some $k > m_{i-1}$, $\sup I_k \leq m_i$ and $\mathrm{card}(I_k \setminus \mathrm{dom}(q_n)) \geq i$. Let $\langle \sigma_j : j < l \rangle$ be an enumeration of all partial 1-1 functions $\sigma : m_i \to m_i$ such that $\mathrm{ran}(\sigma \restriction I_k) \subseteq I_k$, for all k. Build inductively a decreasing sequence $\langle r_j : j \leq l \rangle$ of conditions such that $\mathrm{dom}(r_j) \cap m_i = \emptyset$, for all $j \leq l$. Let $r_0 = q_i \restriction (\mathrm{dom}(q_i) \setminus m_i)$. If r_j has been constructed let $r_{j+1} \leq r_j$ be such that for every $k < i$ either $(\sigma_j \cup r_{j+1}, m_i) \in D_k$ or there is no $r \in \mathcal{P}$ such that $(r, m_i) \in D_k$ and $(r, m_i) \leq (\sigma_j \cup r_{j+1}, m_i)$. Set $q_{i+1} = r_l$. Finally, let $q = \bigcup_{i<\omega} q_i \restriction m_i$. One then easily verifies that (q, n) is as desired.

Now, for each $D \in \mathcal{D}$ let $\mathcal{A}(D) = \{(p, n) : p \in D\}$. Then, clearly, $\mathcal{A}(D)$ is a dense subset of $\mathcal{A}(\mathcal{P})$. For each $k < \omega$ let E_k be the set of all $(p, n) \in \mathcal{A}(\mathcal{P})$ such that there exists m such that $I_m \subseteq n$ and $\mathrm{card}(I_m \setminus \mathrm{dom}(p)) \geq k$. Let $\mathcal{F}$ be the union $\{\mathcal{A}(D) : D \in \mathcal{D}\}$ and $\{E_k : k < \omega\}$. Now, if H is an $\mathcal{F}$-generic filter in $\mathcal{A}(\mathcal{P})$ then the union of the first coordinates of elements of H is a condition in $\mathcal{P}$ which belongs to $\bigcap \mathcal{D}$. $\square$

In the proof of the consistency of PFA large cardinal assumptions are used. This is not necessary for the above result. To see this one performs a standard countable support iteration of length ω_2 consisting of proper posets forcing $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$. At limit states of cofinality ω_1 one forces with the poset $\mathcal{A}(\mathcal{P})$ as defined in the current model. One then shows that Lemma 3.2 holds in the extension. Thus, the poset $\mathcal{P}$ defined in the final model has the desired properties.

4. Automorphisms of $\mathcal{P}(\kappa)/\mathrm{fin}$

In this section we extend the main result of Section 2 to higher cardinals.

Theorem 4.1 ($\mathrm{MA}_{\aleph_1} + \mathrm{OCA}$). *Every automorphism of $\mathcal{P}(\omega_1)/\mathrm{fin}$ is trivial.*

Proof. Let φ be an automorphism of $\mathcal{P}(\omega_1)/\mathrm{fin}$. By Theorem 2.1, for every $\alpha < \omega_1$, $\varphi \restriction \alpha$ is trivial. Let T_α be the set of all functions $e : \alpha \to \omega_1$ such that e induces $\varphi \restriction \alpha$. Note that if $e, f \in T_\alpha$ then $e =_* f \bmod (\mathrm{fin})$. Let $T = \bigcup_{\alpha<\omega_1} T_\alpha$. Then T, ordered by inclusion, is a tree of height ω_1 and cardinality $\aleph_1$. Assuming that φ is nontrivial T has no ω_1-branches. If $f \in 2^{\omega_1}$ is an arbitrary function let

$$T(f) = \{f \circ e : e \in T\}.$$

Let $\mathcal{P}$ be the standard poset for adding $\aleph_1$ Cohen reals. Suppose G is $\mathcal{P}$-generic and let $g = \bigcup G$.

Claim. *$T(g)$ has no ω_1-branches in $V[G]$.*

Proof. Assume τ is a $\mathcal{P}$-name for a cofinal branch. For each $\alpha < \omega_1$ pick $p_\alpha \in \mathcal{P}$ and $t_\alpha \in T_\alpha$ such that $p_\alpha \Vdash \tau \restriction \alpha = \dot{g} \circ t_\alpha$. Let $D_\alpha = t_\alpha^{-1}(\mathrm{dom}(p_\alpha))$. Find an uncountable $X \subseteq \omega_1$ such that p_α, $\alpha \in X$ form a Δ-system with root p, and if $\alpha, \beta \in X$ and $\alpha < \beta$ then $\mathrm{ran}(t_\alpha) \cap \mathrm{dom}(p_\beta \backslash p) = \emptyset$. We claim that there exist $\alpha, \beta \in X$ such that $t_\alpha \restriction (\alpha \backslash D_\alpha)$ and $t_\beta \restriction (\beta \backslash D_\beta)$ are incompatible. Otherwise, letting

$$t = \bigcup_{\alpha \in X} t_\alpha \restriction (\alpha \backslash D_\alpha)$$

one obtains a function which induces φ. So, fix some $\alpha, \beta \in X$ with $\alpha < \beta$ and $\xi < \alpha$ such that $\xi \notin D_\alpha \cup D_\beta$ and $t_\alpha(\xi) \neq t_\beta(\xi)$. It follows that $t_\alpha(\xi) \notin \mathrm{dom}(p_\alpha) \cup \mathrm{dom}(p_\beta)$. Therefore there exists $q \in \mathcal{P}$ such that $p_\alpha \cup p_\beta \subseteq q$ and $q(t_\alpha(\xi)) \neq q(t_\beta(\xi))$. But then, clearly, q forces that $\dot{g} \circ t_\alpha$ and $\dot{g} \circ t_\beta$ are incomparable in $T(\dot{g})$. Contradiction.

Thus, $T(g)$ is an Aronszajn tree in $V[G]$. let $\mathcal{Q}$ be the standard ccc poset which specializes $T(g)$. Now, applying $\mathrm{MA}_{\aleph_1}$ to a suitable family of $\aleph_1$ dense subsets of $\mathcal{P} * \mathcal{Q}$, one finds in V a function $f: \omega_1 \to 2$ such that $T(f)$ is special. Then, letting $a = f^{-1}\{1\}$ it follows that there is no $b \subseteq \omega_1$ such that $\varphi[b] = [a]$. Contradiction. $\square$

We now extend this result to all cardinals assuming PFA. This will follow from a more general result which may have some interest of its own. Before we state it let us make some preliminary definitions. Let I be an uncountable set and suppose that for every $a \in [I]^{\aleph_0}$ a function $e_a : a \to I$ is given. The system $\langle e_a : a \in [I]^{\aleph_0} \rangle$ is called *coherent* if for every $a, b \in [I]^{\aleph_0}$, $e_a \restriction a \cap b =_* e_b \restriction a \cap b$. It is called *trivial* provided there exists $e : I \to I$ such that if a is a countable subset of I then $e \restriction a =_* e_a$.

Theorem 4.2 (PFA). *Let κ be an uncountable cardinal and suppose $\langle e_a : a \in [\kappa]^{\aleph_0} \rangle$ is a nontrivial coherent system of functions. Then there exists $I \subseteq \kappa$ of size $\aleph_1$ such that $\langle e_a : a \in [I]^{\aleph_0} \rangle$ is nontrivial.*

Proof. Let $\mathcal{C}$ be the usual σ-closed collapse of κ to $\aleph_1$. Suppose G is a $\mathcal{C}$-generic filter.

Claim. *$\langle e_a : a \in [I]^{\aleph_0} \rangle$ is nontrivial in $V[G]$.*

Proof. Assume otherwise and let τ be a $\mathcal{C}$-name for a trivializing function. Let θ be a sufficiently large regular cardinal and pick a countable elementary submodel $M \prec H_\theta$ containing all the relevant information. Let $a = M \cap \kappa$. Build inductively a decreasing sequence $\langle p_n : n < \omega \rangle$ of conditions in $\mathcal{C} \cap M$ and a sequence $\langle \alpha_n : n < \omega \rangle$ of distinct ordinals in a as follows. Suppose p_n has been constructed. Since the system $\langle e_a : a \in [I]^{\aleph_0} \rangle$ is nontrivial in V, there exists an ordinal $\alpha_n \notin \{\alpha_i : i < n\}$ and two extensions p_n^0 and p_n^1 of p_n deciding the value of $\tau(\alpha_n)$ in different ways. By elementarity and the fact that all the relevant information is in M, we can assume that $\alpha_n, p_n^0, p_n^1 \in M$. Now, there exists $i \in \{0, 1\}$ such that $p_n^i \Vdash \tau(\alpha_n) \neq e_a(\alpha_n)$. Let p_{n+1} be equal to p_n^i for such i. This completes the inductive construction. Finally, let $p = \bigcup_{n<\omega} p_n$. It follows that p forces that $\tau \restriction a \neq_* e_a \bmod (\text{fin})$. Contradiction.

Now, in $V[G]$ fix an increasing sequence $\langle a_\alpha : \alpha < \omega_1 \rangle$ of countable subsets of κ such that $\kappa = \bigcup_{\alpha<\omega_1} a_\alpha$. Let T_α be the set of all functions $f : a_\alpha \to \bigcup_{\xi<\omega_1} a_\xi$ such that $f =_* e_{a_\alpha}$. Let $T = \bigcup_{\alpha<\omega_1} T_\alpha$. Then T, ordered under extension is a tree of size $\aleph_1$ with no ω_1-branches. Let $\mathcal{Q}$ be the standard poset for specializing T. Now, applying PFA to a suitable family of $\aleph_1$ dense subsets of $\mathcal{P} * \mathcal{Q}$, one finds in V a chain $\langle b_\alpha : \alpha < \omega_1 \rangle$ of countable subsets of κ such that the analogously defined tree is special. This implies that $\langle e_a : a \in [I]^{\aleph_0} \rangle$ is nontrivial, where $I = \bigcup_{\alpha<\omega_1} b_\alpha$. $\square$

The following is now immediate using Theorems 4.1 and 4.2 and the fact that $\text{OCA}+\text{MA}_{\aleph_1}$ follows from PFA.

Theorem 4.3 (PFA). *If κ is an infinite cardinal then all automorphisms of $\mathcal{P}(\kappa)/\text{fin}$ are trivial.*

Let us note that $\text{OCA}+\text{MA}_{\aleph_1}$ does not suffice to prove Theorem 4.3. To see that start with a model of $V = L$ and force with the standard ccc poset of size $\aleph_2$ which makes $\text{OCA}+\text{MA}_{\aleph_1}$ true. The combinatorial principles $\square_{\aleph_1}$ and $\diamondsuit^*_{\aleph_2}$ persist in the generic extension and can be used together with $\text{MA}_{\aleph_1}$ to build a nontrivial automorphism of $\mathcal{P}(\omega_2)/\text{fin}$.

References

[1] U. Abraham, M. Rubin and S. Shelah, On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types, Ann. Pure Appl. Logic 29 (1985) 123–206.

[2] W. Just, A weak form of AT from OCA, in: H. Judah, W. Just and H. Woodin, eds., Set Theory of the Continuum (Springer, Berlin, 1992) 281–292.

[3] K. Kunen, Set Theory (North-Holland, Amsterdam, 1980).

[4] Y. Moschovakis, Descriptive Set Theory, Studies in Logic and the Foundations of Mathematics 100 (North-Holland, Amsterdam, 1980).

[5] S. Shelah, Proper Forcing, Lecture Notes in Mathematics 940 (Springer, Berlin, 1982).

[6] S. Shelah and J. Steprans, PFA implies all automorphisms are trivial, Preprint.

[7] S. Todorcevic, Partition Problems in Topology, Contemporary Mathematics 84 (Amer. Math. Soc., Providence, RI, 1989).

[8] J. van Mill, Introduction to $\beta\omega$, in: K. Kunen and J. E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 503–567.

[9] B. Velickovic, Definable automorphisms of $\mathcal{P}(\omega)/\text{fin}$, Proc. Amer. Math. Soc. 96 (1986) 130–135.

[10] B. Velickovic, Applications of the Open Coloring Axiom, in: H. Judah, W. Just and H. Woodin, eds., Set Theory of the Continuum (Springer, Berlin, 1992) 137–154.