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Une analyse ensembliste des opérateurs sur l'espace de Banach ℓ_∞/c_0

A set-theoretical analysis of operators on the Banach space ℓ_∞/c_0

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Résumés

Une analyse ensembliste des opérateurs sur l'espace de Banach ℓ_∞/c_0

Résumé

On étudie des opérateurs linéaires et continus sur l'espace ℓ_∞/c_0 et des espaces liés. On commence par examiner des opérateurs $T : \ell_\infty/c_0 \to \ell_\infty/c_0$. En particulier, on s'intéresse à la possibilité de représenter ses fragments de la forme

$$T_{B,A} : \ell_\infty(A)/c_0(A) \to \ell_\infty(B)/c_0(B)$$

pour $A, B \subseteq \mathbb{N}$ infinis, par des applications de $\ell_\infty(A)$ dans $\ell_\infty(B)$, des matrices $A \times B$, des fonctions continues de $B^* = \beta B \setminus B$ dans A^*, ou des bijections de B dans A. On montre plusieurs exemples. On définit et on étudie de nouvelles classes d'opérateurs. Pour certaines d'entre elles on obtient des représentations satisfactoires. Pour d'autres classes, on montre que c'est impossible. On montre des automorphismes de ℓ_∞/c_0 qui ne se relèvent pas à des opérateurs sur ℓ_∞ et on montre que sous OCA + MA tout automorphisme sans *fontaines* ou sans *entonnoirs* est induit par une bijection localement (au sens qu'on vient de spécifier). Cet axiome supplémentaire est nécessaire, comme témoignent des contre-exemples de plusieurs types construits avec HC.

Ensuite, on regarde les plongements isomorphiques de $\ell_\infty(\ell_\infty/c_0)$ dans ℓ_∞/c_0. On démontre que sous PFA la structure locale des opérateurs de chaque coordonnée a une grande influence sur l'opérateur original, ce qui entraîne l'impossibilité de plonger $\ell_\infty(\ell_\infty/c_0)$ dans ℓ_∞/c_0 sous PFA par des opérateurs dans plusieurs classes bien connues. Cela contraste avec la construction de Drewnowski et Roberts d'un tel plongement sous HC comme opérateur de composition.

Finalement, on présente une démonstration moderne d'une version légèrement améliorée d'un résultat de Kadec et Pełczyński sur des séquences de mesures de Radon. On montre des applications de ce résultat aux opérateurs sur ℓ_∞/c_0.

Mots-clefs

Espace de Banach, compactification de Čech-Stone des entiers, automorphisme, plongement, Hypotèse du Continu, *Open Coloring Axiom*, *Proper Forcing Axiom*, ℓ_∞-somme, mesure.

A set-theoretical analysis of operators on the Banach space ℓ_∞/c_0

Abstract

We investigate linear bounded operators on ℓ_∞/c_0 and related spaces. We begin by studying operators $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ focusing on the possibility of representing their fragments of the form

$$T_{B,A} : \ell_\infty(A)/c_0(A) \to \ell_\infty(B)/c_0(B)$$

for $A, B \subseteq \mathbb{N}$ infinite by means of operators from $\ell_\infty(A)$ into $\ell_\infty(B)$, infinite $A \times B$-matrices, continuous maps from $B^* = \beta B \setminus B$ into A^*, or bijections from B to A. We present many examples, introduce and investigate several classes of operators, for some we obtain satisfactory representations and for others we show that it is impossible. We show that there are automorphisms of ℓ_∞/c_0 which cannot be lifted to operators on ℓ_∞ and assuming OCA + MA we show that every automorphism of ℓ_∞/c_0 with no *fountains* or with no *funnels* is locally, i.e., for some infinite $A, B \subseteq \mathbb{N}$ as above, induced by a bijection from B to A. This additional set-theoretic assumption is necessary as we show that the Continuum Hypothesis implies the existence of counterexamples of diverse flavours.

Later, we look into isomorphic embeddings of the ℓ_∞-sum of ℓ_∞/c_0 into ℓ_∞/c_0. We show that under PFA the local structure of the induced coordinate operators has great influence over the original operator. As an application, we show the impossibility under PFA of embedding $\ell_\infty(\ell_\infty/c_0)$ into ℓ_∞/c_0 by means of some well-known classes of operators, contrasting with Drewnowski and Roberts' construction under CH of such an embedding as a composition operator.

Finally, we present a modern proof of a slightly improved version of a result by Kadec and Pełczyński on sequences of Radon measures. We show an application of this result to operators from ℓ_∞/c_0 into itself.

Keywords

Banach space, Čech-Stone compactification of the integers, automorphism, embedding, Continuum Hypothesis, Open Coloring Axiom, Proper Forcing Axiom, ℓ_∞-sum, measure.

Contents

Introduction

L'algèbre de Boole $\wp(\mathbb{N})/\mathit{Fin}$, c'est-à-dire le quotient de l'algèbre des parties de $\mathbb{N}$ par l'idéal des ensembles finis, et son espace de Stone dual $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, ont été largement étudiés sous différentes extensions de la théorie des ensembles classique de Zermelo et Frænkel plus l'axiome du choix (ZFC). Une facette de cette recherche qui a eu un succès remarquable concerne les automorphismes de cette algèbre. W. Rudin a démontré que sous l'Hypothèse du Continu (HC) ces automorphismes peuvent être assez compliqués (Cf. Théorème 4.7 de [50]), pendant que S. Shelah a montré dans [54]-§4 que c'est consistent avec ZFC que tous les automorphismes de cette algèbre sont triviaux, ce qui veut dire qu'ils sont tous induits par des bijections entre des sous-ensembles cofinis de $\mathbb{N}$ (ce qu'on appelle des presque permutations de $\mathbb{N}$). La preuve originale de Shelah fait appel à un usage sophistiqué de la condition de chaîne d'oracle, mais après il a réussi avec J. Stēprans ([52]) à en déduire le même résultat sous un axiome de forcing puissant, le *Proper Forcing Axiom* (PFA, introduit dans [54]).

Un moment important dans le développement de la théorie des automorphismes de $\wp(\mathbb{N})/\mathit{Fin}$ a été l'entrée en jeu du *Open Coloring Axiom* (OCA, introduit par S. Todorčević dans [56]) [1], ce qui est arrivé avec le travail de B. Veličković [64] où il a démontré que OCA avec l'axiome de Martin (MA) entraînent la trivialité de tous les automorphismes de cette algèbre. Une porte fut ouverte pour une multitude d'applications d'OCA dans la théorie de $\wp(\mathbb{N})/\mathit{Fin}$ ([64, 58, 25, 16, 17]). L'importance d'OCA dans cette théorie peut être apprécié de plusieurs points de vue. À l'égard de la théorie des ensembles c'est intéressant parce que même si OCA est une conséquence de PFA, il ne faut pas des grands cardinaux pour démontrer la consistance relative d'OCA, comme c'est le cas pour PFA (voir [63]). Du point de vue de l'applicabilité d'OCA, en étant un énoncé de type Ramsey son usage ne requiert pas de connaissances de la méthode de forcing. Mais surtout, OCA a été prouvé comme un principe assez puissant pour décider beaucoup de questions importantes concernant l'algèbre $\wp(\mathbb{N})/\mathit{Fin}$. En effet, en contraste avec la vue qu'on a de cet objet sous HC principalement grâce aux travaux de Parovičenko ([40]), OCA a servi comme prisme à travers duquel on peut voir l'ordre et l'élégance de sa structure ([64, 58, 25, 16, 17]).

Cette recherche a eu un grand effet sur des structures mathématiques plus complexes. Par exemple, il entraîne directement l'indécidabilité de la question de savoir si tous les automorphismes de l'algèbre de Banach ℓ_∞/c_0 sont induits par des presque permutations de $\mathbb{N}$. Indirectement, cette recherche a été source d'inspiration pour l'application

1. OCA s'énonce de la façon suivante : soit X un espace séparable et métrisable et soit $[X]^2 = K_0 \cup K_1$ une partition tel que K_0 est ouvert dans la topologie produit. Alors, soit il existe un sous-ensemble non-dénombrable $Y \subseteq X$ tel que $[Y]^2 \subseteq K_0$, soit $X = \bigcup_{n\in\mathbb{N}} X_n$ où $[X_n]^2 \subseteq K_1$ pour tout $n \in \mathbb{N}$.

d'OCA dans le contexte des algèbres C^*, où il y a des quotients analogues. Le résultat principal dans ce domaine est l'indécidabilité de la structure des automorphismes de l'algèbre de Calkin d'opérateurs sur l'espace de Hilbert modulo les opérateurs compacts ([44, 26]).

Ce travail résulte du souci de comprendre l'effet que la combinatoire de $\wp(\mathbb{N})/$*Fin* a sur une autre structure intimement liée, à savoir, l'espace de Banach ℓ_∞/c_0. Cela est une question naturelle étant donné la relation canonique existante entre ces deux objets : on rappelle que l'espace de Banach ℓ_∞/c_0 est une copie isométrique de l'espace $C(\mathbb{N}^*)$ des fonctions continues sur l'espace de Stone de $\wp(\mathbb{N})/$*Fin*. Néanmoins, ce n'est pas une question triviale, parce qu'il y a beaucoup d'applications linéaires continues sur ℓ_∞/c_0 qui ne sont pas induites par des homomorphismes de l'algèbre $\wp(\mathbb{N})/$*Fin*.

Il existe des travaux qui ont déjà donné quelques pas dans la direction générale d'étudier ℓ_∞/c_0 d'un point de vue ensembliste. Par exemple, on sait que de façon analogue au cas de $\wp(\mathbb{N})/$*Fin*, ZFC ne décide pas beaucoup de questions importantes concernant cet espace (e.g. [10, 9, 59, 34]), tandis que HC fournit quelques réponses grâce, principalement, à la possibilité de réaliser des constructions inductives de longueur du continu (e.g. [22, 11]). D'ailleurs, étant donné la grande influence que PFA et certains de ses fragments ont sur $\wp(\mathbb{N})/$*Fin*, c'est assez naturel d'espérer qu'ils auront une forte influence sur ℓ_∞/c_0 aussi, fournissant peut-être une théorie aussi élégante. En fait, cet espoir vient aussi de la constatation de la grande puissance que PFA a sur d'autres structures non-liées ([57, 38]) et d'un très récent développement dans le domaine même de ℓ_∞/c_0 (Cf. [21]). Dans ce travail on cherche à savoir si PFA ou OCA peuvent être en effet utilisés dans ce contexte.

Cette monographie est composée de trois chapitres et une annexe. Le gros de l'effort de cette investigation, contenu dans le chapitre 1, a été dédié à comprendre des automorphismes de ℓ_∞/c_0. Pour décrire le travail plus précisément on doit d'abord donner plus de contexte et introduire de la terminologie.

Dans le cas de l'algèbre de Boole $\wp(\mathbb{N})/$*Fin* et l'un de ses automorphismes h, les conditions suivantes sont équivalentes pour tous $A, B \subseteq \mathbb{N}$ cofinis :

- Il y a un isomorphisme $H : \wp(A) \to \wp(B)$ tel que $[H(C)]_{Fin} = h([C]_{Fin})$ pour tout $C \subseteq A$ (h peut être relevé à $\wp(\mathbb{N})$) ;
- Il y a un isomorphisme G : *FinCofin*$(A) \to$ *FinCofin*(B) de l'algèbre de Boole des sous-ensembles finis et cofinis de A dans l'algèbre de Boole correspondante à B, tel que $[\bigcup\{G(n) : n \in C\}]_{Fin} = h([C]_{Fin})$ pour tout $C \subseteq A$ (h est induit par un presque automorphisme de *FinCofin*$(\mathbb{N})$) ;
- Il y a une bijection $\sigma : B \to A$ tel que $[\{n \in B : \sigma(n) \in C\}]_{Fin} = h([C]_{Fin})$ pour tout $C \subseteq A$ (h est trivial).

Une autre caractéristique des relèvements des automorphismes de $\wp(\mathbb{N})/$*Fin*, c'est-à-dire des homomorphismes de $\wp(\mathbb{N})$ satisfaisant les propriétés précédents, est que

- Tout isomorphisme de $\wp(A)$ dans $\wp(B)$ pour $A, B \subseteq \mathbb{N}$ infinis est continu par rapport à la topologie produit dans $\{0,1\}^A$ et dans $\{0,1\}^B$.

De plus, si on identifie les points de $\mathbb{N}^*$ avec des ultrafiltres de $\wp(\mathbb{N})/$*Fin*, la dualité de Stone donne :

- pour tout endomorphisme h de $\wp(\mathbb{N})/\mathit{Fin}$ il existe une fonction continue ψ : $\mathbb{N}^* \to \mathbb{N}^*$ telle que
$$\chi_{h([A]_{\mathit{Fin}})^*} = \chi_{A^*} \circ \psi,$$
pour tout $A \subseteq \mathbb{N}$.

Les notions correspondantes dans le cas des applications sur ℓ_∞/c_0 sont présentées dans la définition suivante :

Définition 0.0.1. Si $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ est une application linéaire et continue, et $A, B \subseteq \mathbb{N}$ sont cofinis, alors on dit que

1. T est relevable (ou peut être relevé) si, et seulement si, il y a une application linéaire et continue $S : \ell_\infty(A) \to \ell_\infty(B)$ telle que pour tout $f \in \ell_\infty$ on a
$$T([f]_{c_0}) = [S(f)]_{c_0}$$
2. T est une application matricielle si, et seulement si, il existe une application $S : c_0(A) \to c_0(B)$ donnée par une matrice réelle $(b_{ij})_{i\in B, j\in A}$ telle que pour tout $f \in \ell_\infty(A)$ on a
$$T([f]_{c_0}) = [(\sum_{j\in A} b_{ij} f(j))_{i\in B}]_{c_0}$$
3. T est une application triviale si, et seulement si, il existe un réel $r \in \mathbb{R}$ **différent de zéro** et une bijection $\sigma : B \to A$ tels que pour tout $f \in \ell_\infty(A)$ on a
$$T([f]_{c_0}) = [rf \circ \sigma]_{c_0}$$
4. T est canonisable [2] au long de $\psi : \mathbb{N}^* \to \mathbb{N}^*$ si, et seulement si, ψ est une fonction continue et **surjective** et il existe un réel r **différent de zéro** tels que pour tout $f^* \in C(\mathbb{N}^*)$ on a
$$\hat{T}(f^*) = rf^* \circ \psi.$$

Pour des applications relevables et matricielles on utilisera des phrases plus complexes comme “automorphisme relevable” ou “plongement matriciel”, ce qui signifie que l'application est relevable ou matricielle, respectivement, et qu'en plus elle a la propriété additionnelle.

En contraste avec le cas de $\wp(\mathbb{N})/\mathit{Fin}$, nos résultats montrent que les relations entre ces notions sont loin d'être des équivalences :

[Automorphisme trivial] $\Rightarrow$ / $\not\Leftarrow$ [Automorphisme matriciel] $\Leftrightarrow$ [Automorphisme relevable avec un relèvement continu dans B_{ℓ_∞}] $\Rightarrow$ / $\not\Leftarrow$ [Automorphisme relevable] $\Rightarrow$ / $\not\Leftarrow$ [Automorphisme]

2. Il pourrait être raisonnable de considérer ici la possibilité d'avoir aussi $T(f^*) = gf^* \circ \psi$ pour tout $f^* \in C(\mathbb{N}^*)$ et pour une certaine fonction continue différente de zéro $g \in C(\mathbb{N}^*)$. Pourtant, dans le contexte de $\mathbb{N}^*$ toutes les fonctions continues sont “localement constantes” (A.1.2), donc c'est inutile d'inclure cette possibilité dans notre analyse.

Aucune des implications ou des contre-exemples a besoin d'axiomes supplémentaires de la théorie des ensembles. Les parties non-triviales du tableau sont les suivantes :

- Il y a des automorphismes de ℓ_∞/c_0 qui ne se relèvent pas à des applications linéaires sur ℓ_∞ (1.4.16) ;
- Il y a des automorphismes de ℓ_∞/c_0 qui se relèvent mais qui ne sont pas des applications matricielles et aucun de ses relèvements est continu dans B_{ℓ_∞} avec la topologie produit (1.4.13) ;
- Les automorphismes de ℓ_∞/c_0 qui ont des relèvements continus avec la topologie produit sont exactement les automorphismes matriciels (1.2.15).

On remarque que la question de savoir si tout automorphisme est canonisable est exclue au départ à cause du fait qu'il existe beaucoup de matrices d'isomorphismes de c_0 qui ne sont pas de matrices de presque permutations modulo c_0. (1.2.6).

Compte tenu des résultats absolus mentionnés auparavant et de l'exclusion de la possibilité de canoniser tout opérateur, on choisit de regarder des versions "locales" de ces propriétés. C'est-à-dire, on regarde si elles sont valides dans un certain sens pour des copies de ℓ_∞/c_0 de la forme $\ell_\infty(A)/c_0(A)$ pour $A \subseteq \mathbb{N}$ infini. Comme ces propriétés dépendent du lien entre ℓ_∞/c_0 et $\mathbb{N}^*$ ou entre ℓ_∞/c_0 et $\mathbb{N}$, on suit l'approche de Drewnowski et Roberts de [22], qui trouve des motivations dans l'analyse fonctionnelle :

Définition 0.0.2. Soit $A \subseteq \mathbb{N}$ infini. On définit $P_A : \ell_\infty/c_0 \to \ell_\infty(A)/c_0(A)$ et $I_A : \ell_\infty(A)/c_0(A) \to \ell_\infty/c_0$ par

$$P_A([f]_{c_0}) = [f|A]_{c_0(A)}, \quad I_A([g]_{c_0(A)}) = [g \cup 0_{\mathbb{N}\setminus A}]_{c_0}$$

pour tout $f \in \ell_\infty$ et tout $g \in \ell_\infty(A)$. Si $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ est un opérateur borné et $A, B \subseteq \mathbb{N}$ sont deux sous-ensembles infinis, alors la localisation de T à (A, B) est l'opérateur $T_{B,A} : \ell_\infty(A)/c_0(A) \to \ell_\infty(B)/c_0(B)$ défini par

$$T_{B,A} = P_B \circ T \circ I_A.$$

En outre, en vue d'itérer les résultats de localisation (comme dans [22]), il est utile de chercher des résultats qui sont valides *localement à gauche* ou *localement à droite*, et non seulement *quelque part* :

Définition 0.0.3. Soit $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ un opérateur borné et soit $\mathbb{P}$ une des propriétés suivantes : "relevable", "application matricielle", "triviale", "canonisable".

1. On dit que T est $\mathbb{P}$ quelque part si, et seulement si, il existe des sous-ensembles infinis $A \subseteq \mathbb{N}$ et $B \subseteq \mathbb{N}$ tels que $T_{B,A}$ est $\mathbb{P}$.
2. On dit que T est $\mathbb{P}$ localement à droite si, et seulement si, pour tout $A \subseteq \mathbb{N}$ infini il existe des sous-ensembles infinis $A_1 \subseteq A$ et $B \subseteq \mathbb{N}$ tels que T_{B,A_1} est $\mathbb{P}$.
3. On dit que T est $\mathbb{P}$ localement à gauche si, et seulement si, pour tout $B \subseteq \mathbb{N}$ infini il existe des sous-ensembles infinis $B_1 \subseteq B$ et $A \subseteq \mathbb{N}$ tels que $T_{B_1,A}$ est $\mathbb{P}$.

Drewnowski et Roberts ont démontré dans [22] que tout opérateur $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ a une sorte de propriété locale, à savoir, pour tout $A \subseteq \mathbb{N}$ infini il existe un sous-ensemble infini $A_1 \subseteq A$ tel que pour tout $[f]_{c_0} \in \ell_\infty(A_1)/c_0(A_1)$ on a $T_{A_1,A_1}([f]_{c_0}) = [rf]_{c_0}$ pour certain réel $r \in \mathbb{R}$. Pourtant, la possibilité d'avoir $T_{A_1,A_1} = 0$ n'est pas exclue, et en fait elle est assez commune. Donc, on cherche à obtenir des localisations qui sont des plongements [3] ou des isomorphismes et non seulement des automorphismes. Dans cette direction et justifiant dans quelque mesure notre approche, on obtient le théorème suivant qui affirme que, contrairement aux versions globales, les versions locales des notions de la définition 0.0.1 se comportent comme les notions correspondantes booléennes.

Théorème 0.0.1. *Soit $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ un automorphisme. Alors, les affirmations suivantes sont équivalentes :*

1. *T est quelque part relevable à un isomorphisme*
2. *T est quelque part un isomorphisme matriciel*
3. *T est quelque part relevable a un isomorphisme qui est continu avec la topologie produit*
4. *T est quelque part trivial.*

Démonstration. (1) implique (2) par 1.4.10 ; l'équivalence de (2) et (3) est 1.2.15 ; (2) implique (4) par 1.4.8 ; clairement, (4) implique (1). □

En fait, la même démonstration donne les équivalences précédents pour un T qui est un plongement et pour des localisations à droite qui sont des plongements. En générale, une condition nécessaire pour avoir des propriétés locales a droite non-triviales, c'est que T ait un noyau petit. Ceci arrive, par exemple, lorsque T est injectif. Similairement, pour obtenir des propriétés locales à gauche on doit supposer que l'image de T est grande, par exemple que T est surjectif. Mais contrairement à la remarque d'au-dessus, un opérateur surjectif peut être relevable globalement sans être nulle part une application matricielle (1.4.12), ou globalement une application matricielle sans être nulle part triviale (1.4.6).

Malgré les parallélismes, ces notions ont un caractère assez différent des notions booléennes correspondantes : l'image d'un sous-espace de la forme $\{[f] \in \ell_\infty/c_0 : f|(\mathbb{N} \setminus A) = 0\}$ pour $A \subseteq \mathbb{N}$ infini n'est habituellement pas de la forme $\{[f] \in \ell_\infty/c_0 : f|(\mathbb{N} \setminus B) = 0\}$ pour $B \subseteq N$ infini, même si $T_{B,A}$ est trivial. D'ailleurs, le fait que $T_{B,A}$ soit trivial ou canonisable ne fournit aucune information sur $T^{-1}_{A,B}$, comme c'est le cas pour les automorphismes de $\wp(\mathbb{N})/$*Fin*.

La proposition suivante donne davantage des raisons pour considérer les notions locales d'au-dessus :

Proposition 0.0.4. *Soient $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ un opérateur borné et $A, B \subseteq \mathbb{N}$ deux sous-ensembles infinis. Supposons que $T_{B,A}$ est canonisable au long d'un homéomorphisme. Alors, T fixe une copie complémentée de ℓ_∞/c_0 dont l'image sous T est complémentée dans ℓ_∞/c_0.*

Démonstration. Voir la démonstration du corollaire 2.4 de [22]. □

3. Un plongement au sens d'espace de Banach. Ils sont aussi appelés des opérateurs bornés au-dessous.

En fait, cette proposition est aussi vraie avec la même démonstration si on affaiblit l'hypothèse sur B d'être un ouvert-fermé de $\mathbb{N}^*$ à être un fermé de $\mathbb{N}^*$ homéomorphe à $\mathbb{N}^*$. Mais pour être sûrs d'induire un sous-espace et non seulement un quotient qui doit être fixé, on insiste que A^* soit ouvert-fermé. Dans le contexte d'autres espaces $C(K)$, cet approche est assez féconde pour obtenir des copies complémentées du $C(K)$ entier dans n'importe quelle copie isomorphe de $C(K)$ (par exemple, pour un $C(K)$ avec K métrisable voir [43] ; pour ℓ_∞ voir [28] ; et pour $C([0, \omega_1])$ voir [31]).

Les détails de la discussion précédente sont inclus dans le chapitre 1, ainsi qu'une étude des propriétés locales des automorphismes de ℓ_∞/c_0 et les relations entre elles. Une fois qu'on a fait cette exploration préliminaire et qu'on a prouvé le théorème 0.0.1, il ne reste qu'à déterminer si les automorphismes de ℓ_∞/c_0 sont canonisables quelque part au long d'un homéomorphisme. Une réponse positive signifie qu'ils ont une structure locale semblable à celle des homéomorphismes de $\mathbb{N}^*$: ils seraient tous triviaux en supposant OCA + MA tandis que sous HC, par exemple, ils ne le seraient pas. Si bien on n'a pas réussi complètement à attendre ce but, on a fait des progrès importants dans la deuxième moitié du chapitre 1.

La canonisation des automorphismes $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ (ou les $\hat{T} : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ correspondants) rencontre des problèmes au moins d'autant difficulté que celle de comprendre les fonctions continues d'un fermé de $\mathbb{N}^*$ vers $\mathbb{N}^*$, et non seulement des autohoméomorphismes de $\mathbb{N}^*$. Pour bien comprendre cela, on rappelle qu'une application linéaire continue sur $C(\mathbb{N}^*)$ est représentable comme une fonction $\tau : \mathbb{N}^* \to M(\mathbb{N}^*)$ continue dans la topologie faible-étoile (voir Théorème 1 dans VI.7 de [23]), où $M(\mathbb{N}^*)$ désigne l'espace de Banach des mesures de Radon sur $\mathbb{N}^*$ muni de la norme de variation totale, et identifié avec le dual de $C(\mathbb{N}^*)$ grâce au théorème de représentation de Riesz (Cf. [51]). Souvent, les points de $\mathbb{N}^*$ (identifiés avec les mesures de Dirac) sont envoyés par cette fonction vers des mesures sans atomes, ou vers des mesures avec plusieurs atomes, en donnant lieu à des fonctions partielles multivaluées vers $\mathbb{N}^*$. On obtient $\tau(x)$ comme $T^\star(\delta_x)$ pour chaque $x \in \mathbb{N}^*$, et la représentation est donnée par

$$\hat{T}(f^*)(x) = \int f^* \,\mathrm{d}\tau(x)$$

pour chaque $f^* \in C(\mathbb{N}^*)$. Les multifonctions, avec des valeurs possiblement vides, sont données par

$$\varphi_\varepsilon^T(y) = \{x \in \mathbb{N}^* : |T^\star(\delta_y)(\{x\})| \geq \varepsilon\}$$

pour tout $\varepsilon > 0$ ou par $\varphi^T(y) = \bigcup_{\varepsilon>0} \varphi_\varepsilon^T$. Une condition sur T équivalente à être quelque part canonisable au long d'un homéomorphisme est l'existence de sous-ensembles infinis $A, B \subseteq \mathbb{N}$ et d'un homéomorphisme $\psi : B^* \to A^*$ tels que

$$T^\star(\delta_y)|A^* = r\delta_{\psi(y)}$$

pour certain $r \in \mathbb{R}$ différent de zéro. Cela veux dire en particulier que $\varphi^T(y) \cap A^* = \{\psi(y)\}$, c'est-à-dire ψ est une sélection homéomorphe de φ^T. En principe, on pourrait rencontrer deux obstacles pour l'existence d'une telle sélection, à savoir, l'intérieur de $\bigcup_{y\in B^*} \varphi^T(y)$ pourrait être vide ou l'intérieur de $\{y \in B^* : \varphi^T(y) \neq \emptyset\}$ pourrait être vide pour un sous-ensemble infini $B \subseteq \mathbb{N}^*$. On appelle fontaines et entonnoirs, respectivement, à des versions plus fortes de ces obstacles incluant des mesures non-atomiques. On introduit aussi des classes d'opérateurs correspondantes pour lesquelles ces obstacles ne peuvent pas arriver par définition (opérateurs sans fontaines 1.3.13 et sans entonnoirs 1.3.18, respectivement). On obtient des conditions suffisantes pour la canonisation des opérateurs dans chaque une de ces classe :

- Tout automorphisme sur ℓ_∞/c_0 sans fontaines est canonisable au long d'une fonction quasi-ouverte localement à gauche ;
- Tout automorphisme sur ℓ_∞/c_0 sans entonnoirs est canonisable au long d'une fonction quasi-ouverte localement à droite.

Ici, on appelle quasi-ouverte à une fonction continue dont l'image de tout ensemble ouvert a l'intérieur non-vide (1.3.20).

Le deuxième résultat est en fait une conséquence d'un travail de G. Plebanek [45], tandis que la démonstration du premier prend une partie considérable du premier chapitre. La possibilité d'obtenir ces résultats dépend des propriétés spéciales des plongements et des opérateurs surjectifs. L'une de ces propriétés est une amélioration d'un théorème de Cengiz ("P" dans [12]) obtenu par Plebanek (théorème 3.3 dans [45]) qui garantie que l'image de $\varphi^T_{\|T\|\|T^{-1}\|}$ couvre $\mathbb{N}^*$ si T est un plongement. Pourtant, l'ensemble de y pour lesquels $\varphi^T(y)$ est non-vide pourrait être rare, donc on exclut cette possibilité en supposant que T n'a pas de entonnoirs. Par ailleurs, on montre que si T est surjectif, alors soit $\varphi^T(y)$ est non-vide pour chaque y, soit il existe un sous-ensemble infini $A \subseteq \mathbb{N}^*$ tel que $\bigcup\{\varphi^T(y) : y \in A^*\}$ est rare. La deuxième possibilité est exclue pour des opérateurs sans fontaines.

À ce point il nous reste encore le problème de réduire une fonction quasi-ouverte à un homéomorphisme entre des ouverts-fermés de $\mathbb{N}^*$. Les résultats de I. Farah [25] nous permettent de conclure que OCA + MA implique que toute fonction quasi-ouverte définie d'un ouvert-fermé de $\mathbb{N}^*$ vers un ouvert-fermé de $\mathbb{N}^*$ est quelque part un homéomorphisme. Donc, d'après les résultats de Veličković [64], elle est quelque part induite par une bijection entre des sous-ensembles infinis de $\mathbb{N}$. On obtient ainsi

- (OCA + MA) Tout automorphisme de ℓ_∞/c_0 sans fontaines est trivial localement à gauche (1.6.4) ;
- (OCA + MA) Tout automorphisme de ℓ_∞/c_0 sans entonnoirs est trivial localement à droite (1.6.4).

Vers la fin du chapitre on montre que les résultats précédents sont optimales dans plusieurs sens, grâce à des constructions sous HC. D'abord on rencontre un obstacle pour améliorer les résultats absolus de sélection mentionnés au-dessus (1.5.6, 1.5.9) en remplaçant "fonction quasi-ouverte" par "homéomorphisme" :

> (HC) Il existe un plongement sans fontaines et sans entonnoirs qui est présent partout, globalement canonisable au long d'une fonction quasi-ouverte mais qui n'est null part canonisable au long d'un homéomorphisme (1.6.11).

Ici, "présent partout" est une version faible de surjectivité ($P_A \circ T \neq 0$ pour tout sous-ensemble infini $A \subseteq \mathbb{N}$; voir 1.3.16). Les automorphismes T de ℓ_∞/c_0 satisfassent que $P_A \circ T$ est présent partout et $T \circ I_A$ est un plongement pour tout sous-ensemble infini $A \subseteq \mathbb{N}$. En outre, on obtient :

(HC) Il existe un automorphisme de ℓ_∞/c_0 qui est nulle part canonisable au long d'une fonction quasi-ouverte, et en particulier au long d'un homéomorphisme (1.6.6).

Le dernier exemple n'est pas une construction directe, mais on a d'autres exemples qui sont plus concrets, même si moins forts (1.6.9), qui dépendent de l'existence dans $\mathbb{N}^*$ d'un P-ensemble rare qui est une rétraction de $\mathbb{N}^*$ (construction due à van Douwen et van Mill [60]). Nos résultats n'excluent pas la possibilité qu'il soit consistent que tout plongement de ℓ_∞/c_0 dans lui-même n'ait pas des entonnoirs (voir section 1.7). Par contre, il existe des opérateurs dans ZFC qui sont surjectifs et qui ont des fontaines (1.3.3). Bien sûr, sous HC il existe des autohoméomorphismes de $\mathbb{N}^*$ qui ne sont triviaux nulle part et qui donnent lieu à des exemples d'opérateurs globalement canonisables qui ne sont relevables nulle part (1.6.10).

Si on continue avec le parallélisme entre la théorie de $\wp(\mathbb{N})/$***Fin*** et la théorie naissante de ℓ_∞/c_0, alors le deuxième chapitre correspond à une incursion dans le pas naturel suivant : examiner les sous-espaces de ℓ_∞/c_0. En particulier, on se demande sur la possibilité de trouver une copie isomorphe de $\ell_\infty(\ell_\infty/c_0)$ dans ℓ_∞/c_0. Une source d'intérêt dans cette question est liée à des propriétés d'universalité de ℓ_∞/c_0. Il est connu que HC implique que tout espace de Banach de densité [4] au plus la cardinalité du continu $\mathfrak{c}$ se plonge isométriquement dans cet espace, alors qu'il est consistent avec ZFC qu'il n'existe pas de tels espaces universels (see [53, 9, 10]). La question de plongeabilité de $\ell_\infty(\ell_\infty/c_0)$ dans ℓ_∞/c_0 est aussi liée à une autre question concernant l'espace ℓ_∞/c_0, celle de savoir si ce dernier espace est primaire [5] (Cf. [36]). Il se trouve que l'existence d'un plongement $T : \ell_\infty(\ell_\infty/c_0) \to \ell_\infty/c_0$ avec image complémenté est le pas crucial dans la preuve de Drewnowski et Roberts du fait que sous HC l'espace ℓ_∞/c_0 est primaire ([22]). Donc cette question est importante dans la recherche d'un modèle où ℓ_∞/c_0 pourrait ne pas être primaire.

Notre analyse des plongements de $\ell_\infty(\ell_\infty/c_0)$ dans ℓ_∞/c_0 suit l'approche du premier chapitre, i.e., on regarde la structure locale de ces opérateurs. On profite de la contribution récente et importante faite par A. Dow ([21]) pour démontrer le résultat principale du chapitre, lequel affirme que sous PFA il n'existe pas de plongement $T : \ell_\infty(\ell_\infty/c_0) \to \ell_\infty/c_0$ tel que les opérateurs coordonnés induits sont tous quelque part canonisables au long d'une fonction quasi-ouverte (2.3.2). Finalement, on discute ce résultat dans le contexte de celui de Dow et en regardant des classes d'opérateurs bien connues aussi que celles introduites dans le premier chapitre.

Le dernier chapitre traite un sujet apparemment séparé, car ici on s'occupe des séquences de mesures de Radon sur un espace compact. Cependant, un lien avec les autres chapitres existe grâce au fait que tout opérateur relevable $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ est déterminé par une séquence de mesures de Radon sur $\beta\mathbb{N}$. Dans ce chapitre on fournit une nouvelle preuve d'un vieux résultat de Kadec and Pełczyński qui affirme que toute telle séquence a une sous-séquence qui est nette au sens qu'elle est la somme d'une séquence

4. La densité d'un espace de Banach X est le plus petit cardinal d'un sous-ensemble dense de X.

5. Un espace de Banach X est dit primaire si pour tout décomposition de X comme somme directe $X = A \oplus B$, au moins un de deux termes est isomorphe à X.

faiblement convergente et une séquence de mesures qui sont portées par des ensembles disjoints et dont la variation totale est constante. Pour illustrer la force de ce résultat, on montre une application dans le contexte des opérateurs sur ℓ_∞/c_0, en déduisant un autre type de canonisation : au lieu de trouver une copie de ℓ_∞/c_0 où l'opérateur est canonique, on trouve une copie de ℓ_∞ dans ℓ_∞/c_0 où il est canonique. De fait, pour un opérateur non-faiblement compact de ℓ_∞/c_0 dans lui-même on trouve une copie isométrique de ℓ_∞ dans ℓ_∞/c_0 tel que la restriction à ce sous-espace est l'identité plus un opérateur faiblement compact, modulo un erreur aussi petit qu'on veut.

Finalement, on regroupe dans l'annexe quelques résultats déjà connus qui sont relevants pour notre étude, ainsi que quelques-unes des démonstrations les plus ennuyeuses, qui sont toutes des énoncés liés à des opérateurs matriciels.

Introduction

The Boolean Algebra $\wp(\mathbb{N})/\mathit{Fin}$, that is, the quotient of the algebra of subsets of $\mathbb{N}$ by the ideal of finite sets, together with its dual Stone space $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, have been thoroughly studied under different extensions of the classical axioms of Zermelo and Frænkel with the Axiom of Choice (ZFC). One aspect where this research has been particularly successful is concerned with the automorphisms of the algebra. W. Rudin first proved that under the Continuum Hypothesis (CH) these automorphisms can be quite complicated (see Theorem 4.7 of [50]), but S. Shelah later showed in [54]-§4 that it is consistent that they are all trivial, that is, they may all be induced by an almost permutation of $\mathbb{N}$ (i.e., by a bijection between cofinite subsets of $\mathbb{N}$). Shelah's original proof involved an elaborate use of the oracle chain condition, but together with J. Steprāns he was later able to obtain in [52] the same result using a powerful forcing axiom, namely, the Proper Forcing Axiom (introduced in [54]).

An important moment in the development of the theory of automorphisms of $\wp(\mathbb{N})/\mathit{Fin}$ was the coming into play of the Open Coloring Axiom (as introduced by S. Todorčević in [56])[6] with B. Veličković's proof that together with Martin's Axiom (MA) they imply that all automorphisms are trivial (see [64]). This opened the door to a host of applications of the Open Coloring Axiom (OCA) in the theory $\wp(\mathbb{N})/\mathit{Fin}$ (see [64, 58, 25, 16, 17]). The importance of OCA in this theory can be seen from several standpoints. From a set-theoretic point of view, it is interesting because even if OCA is a consequence of the Proper Forcing Axiom (PFA), large cardinals are not required to prove the relative consistency of OCA, as is the case for PFA (see [63]). From the point of view of the applicability of OCA, being a Ramsey type statement its use requires no specialized knowledge of forcing. But most importantly, OCA has proven to be a powerful principle capable of deciding many important questions relating to the Boolean algebra $\wp(\mathbb{N})/\mathit{Fin}$. Indeed, in contrast to the view we have of this structure under CH mainly thanks to Parovičenko's result ([40]), OCA has served as a prism through which we are able to see its elegant and ordered structure. (e.g. [64, 58, 25, 16, 17]).

This research has also had a profound impact on more complex mathematical structures. For example, it directly implies the undecidability of the question whether the only automorphisms of the Banach algebra ℓ_∞/c_0 are those induced by almost permutations of $\mathbb{N}$. Indirectly, this research has served as inspiration for the successful application of OCA in the context C^*–algebras, where analogous quotient structures

6. The Open Coloring Axiom states the following: Let X be a separable metric space and $[X]^2 = K_0 \cup K_1$ a partition such that K_0 is open in the product topology. Then either there is an uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq K_0$, or $X = \bigcup_{n\in\mathbb{N}} X_n$, where $[X_n]^2 \subseteq K_1$, for all $n \in \mathbb{N}$.

occur. The main result in this area is the undecidability of the the structure of the automorphisms of the Calkin algebra of operators on the Hilbert space modulo the compact operators ([26, 44]).

The present work is the result of an effort at understanding the effect that the combinatorics of $\wp(\mathbb{N})/\mathit{Fin}$ has in another related structure, namely, the Banach space ℓ_∞/c_0. This is a natural question to consider because of the canonical relationship between these two objects: recall that the Banach space ℓ_∞/c_0 is an isometric copy of the Banach space $C(\mathbb{N}^*)$ of all continuous functions on the Stone space of $\wp(\mathbb{N})/\mathit{Fin}$. Notice, however, that there are many more linear operators on ℓ_∞/c_0 than those induced by homomorphisms of the Boolean algebra $\wp(\mathbb{N})/\mathit{Fin}$, so it is not a trivial question.

Some work has already been advanced in the general direction of studying ℓ_∞/c_0 from a set-theoretic point of view. For example, we know that, as in the case of $\wp(\mathbb{N})/\mathit{Fin}$, ZFC alone cannot decide many basic questions concerning this space (see [10, 9, 59, 34]), while CH provides some answers mainly thanks to the possibility of carrying out inductive constructions of length continuum (e.g. [22, 11]). On the other hand, given the great impact that PFA and some of its fragments has on $\wp(\mathbb{N})/\mathit{Fin}$, it is quite natural to expect that it will have a strong influence over ℓ_∞/c_0 too, providing perhaps a similarly elegant theory. Actually, this hope comes also from the great power that PFA has shown to have over other unrelated structures (see [57, 38]) and from a very recent development in the realm of ℓ_∞/c_0 itself (see [21]). Our focus in the present work was to investigate whether OCA or PFA could indeed be successfully used in this context.

This monograph is divided into three chapters and one appendix. The central part of our investigation, contained in Chapter 1, is aimed at understanding automorphisms of ℓ_∞/c_0. In order to describe our work in more detail we need to introduce some background and terminology, which we proceed to do in what follows.

In the case of the Boolean algebra $\wp(\mathbb{N})/\mathit{Fin}$ and one of its automorphisms h the following conditions are equivalent for every two cofinite sets $A, B \subseteq \mathbb{N}$:

- There is an isomorphism $H : \wp(A) \to \wp(B)$ such that $[H(C)]_{\mathit{Fin}} = h([C]_{\mathit{Fin}})$ for all $C \subseteq A$ (h lifts to $\wp(\mathbb{N})$);
- There is an isomorphism $G : \mathit{FinCofin}(A) \to \mathit{FinCofin}(B)$ from the Boolean algebra of finite and cofinite subsets of A onto the corresponding Boolean algebra for B, such that $[\bigcup\{G(n) : n \in C\}]_{\mathit{Fin}} = h([C]_{\mathit{Fin}})$ for all $C \subseteq A$ (h is induced by an almost automorphism of $\mathit{FinCofin}(\mathbb{N})$);
- There is a bijection $\sigma : B \to A$ such that $[\{n \in B : \sigma(n) \in C\}]_{\mathit{Fin}} = h([C]_{\mathit{Fin}})$ for all $C \subseteq A$ (h is trivial).

Another special feature of liftings of automorphisms on $\wp(\mathbb{N})/\mathit{Fin}$, i.e., homomorphisms of $\wp(\mathbb{N})$ satisfying the properties above, is that

- Every isomorphism from $\wp(A)$ into $\wp(B)$ for $A, B \subseteq \mathbb{N}$ infinite is continuous with respect to the product topologies on $\{0,1\}^A$ and $\{0,1\}^B$.

Moreover, if we identify points of $\mathbb{N}^*$ with ultrafilters in $\wp(\mathbb{N})/\mathit{Fin}$, the Stone duality gives that:

- for every endomorphism h of $\wp(\mathbb{N})/\mathit{Fin}$ there is a continuous map $\psi : \mathbb{N}^* \to \mathbb{N}^*$ such that
$$\chi_{h([A]_{\mathit{Fin}})^*} = \chi_{A^*} \circ \psi$$
for every $A \subseteq \mathbb{N}$.

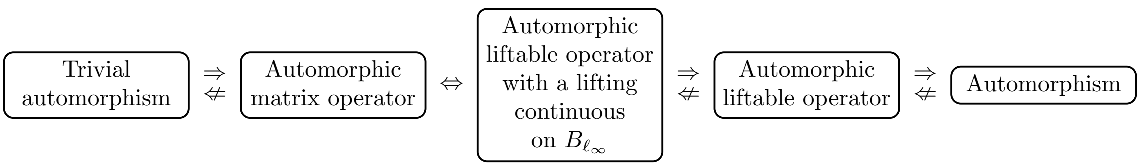
The corresponding notions for operators on ℓ_∞/c_0 are summarized in the following:

Definition 0.0.5. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear bounded operator, $A, B \subseteq \mathbb{N}$ cofinite, then we say that

1. T is liftable (can be lifted) if, and only if, there is a linear bounded $S : \ell_\infty(A) \to \ell_\infty(B)$ such that for all $f \in \ell_\infty$ we have
$$T([f]_{c_0}) = [S(f)]_{c_0}$$
2. T is a matrix operator if, and only if, there is an operator $S : c_0(A) \to c_0(B)$ given by a real matrix $(b_{ij})_{i\in B, j\in A}$ such that for all $f \in \ell_\infty(A)$ we have
$$T([f]_{c_0}) = [(\sum_{j\in A} b_{ij} f(j))_{i\in B}]_{c_0}.$$
3. T is a trivial operator if, and only if, there is a **nonzero** real $r \in \mathbb{R}$, and a bijection $\sigma : B \to A$ such that for all $f \in \ell_\infty(A)$ we have
$$T([f]_{c_0}) = [rf \circ \sigma]_{c_0}.$$
4. T is canonizable[7] along $\psi : \mathbb{N}^* \to \mathbb{N}^*$ if, and only if, ψ is a **surjective** continuous mapping and there is a **nonzero** real r such that for all $f^* \in C(\mathbb{N}^*)$ we have
$$\hat{T}(f^*) = rf^* \circ \psi.$$

In the case of liftable and matrix operators we will be using more complex phrases like automorphic liftable operator, embedding matrix operator etc., meaning that the operator is liftable or matrix respectively and it has the additional property.

In contrast with the case of $\wp(\mathbb{N})/\mathit{Fin}$ our results show that the relationships among these notions are far from equivalences:



None of the implications or counterexamples to the reverse implications require additional set-theoretic axioms. The nontrivial parts of the above chart are the following facts:

7. It would be reasonable to consider here also the possibility of having for all $f^* \in C(\mathbb{N}^*)$ the condition $T(f^*) = gf^* \circ \psi$, for some continuous nonzero $g \in C(\mathbb{N}^*)$. However, in the context of $\mathbb{N}^*$ all continuous functions are "locally constant" (A.1.2) so, as we shall see in a moment, in the context of our analysis there is no sense of introducing such a property.

- There are automorphisms of ℓ_∞/c_0 which are not liftable to a linear operator on ℓ_∞ (1.4.16);
- There are automorphisms of ℓ_∞/c_0 which are liftable but they are not matrix operators and none of their liftings are continuous on B_{ℓ_∞} in the product topology (1.4.13);
- Automorphisms of ℓ_∞/c_0 which have liftings to ℓ_∞ continuous in the product topology are exactly the automorphic matrix operators (1.2.15).

Note that the question of canonizing globally all automorphisms other than trivial is outright excluded by the clear fact that there are many matrices of isomorphisms on c_0 which are not matrices of almost permutations modulo c_0 (1.2.6).

In the light of the above absolute results and the exclusion of the possibility of a global canonization, we choose to look at "local" versions of the above properties of operators. By local we mean that they hold in some sense for copies of ℓ_∞/c_0 of the form $\ell_\infty(A)/c_0(A)$ for an infinite $A \subseteq \mathbb{N}$. Since these properties depend on the link between ℓ_∞/c_0 and $\mathbb{N}^*$ or ℓ_∞/c_0 and $\mathbb{N}$, we adopt the approach of Drewnowski and Roberts from [22] which has functional analytic motivations and applications:

Definition 0.0.6. Suppose $A \subseteq \mathbb{N}$ is infinite. We define $P_A : \ell_\infty/c_0 \to \ell_\infty(A)/c_0(A)$ and $I_A : \ell_\infty(A)/c_0(A) \to \ell_\infty/c_0$ by

$$P_A([f]_{c_0}) = [f|A]_{c_0(A)}, \quad I_A([g]_{c_0(A)}) = [g \cup 0_{\mathbb{N}\setminus A}]_{c_0}$$

for all $f \in \ell_\infty$ and all $g \in \ell_\infty(A)$. Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear bounded operator and $A, B \subseteq \mathbb{N}$ two infinite sets. The localization of T to (A, B) is the operator $T_{B,A} : \ell_\infty(A)/c_0(A) \to \ell_\infty(B)/c_0(B)$ given by

$$T_{B,A} = P_B \circ T \circ I_A.$$

Furthermore, in order to be able to iterate the use of several localization results (like in the case of [22]) it is useful to have right-local or left-local results and not just somewhere local results:

Definition 0.0.7. Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear bounded operator. Let $\mathbb{P}$ be one of the properties "liftable", "matrix operator", "trivial", "canonizable".

1. We say that T is somewhere $\mathbb{P}$ if, and only if, there are infinite $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ such that $T_{B,A}$ has $\mathbb{P}$.
2. We say that T is right-locally $\mathbb{P}$ if, and only if, for every infinite $A \subseteq \mathbb{N}$ there are infinite $A_1 \subseteq A$ and $B \subseteq \mathbb{N}$ such that T_{B,A_1} has $\mathbb{P}$.
3. We say that T is left-locally $\mathbb{P}$ if, and only if, for every infinite $B \subseteq \mathbb{N}$ there are infinite $B_1 \subseteq B$ and $A \subseteq \mathbb{N}$ such that $T_{B_1,A}$ has $\mathbb{P}$.

Drewnowski and Roberts proved in [22] that every operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ has some sort of local property, namely, for every infinite $A \subseteq \mathbb{N}$ there is an infinite $A_1 \subseteq A$ such that for all $[f]_{c_0} \in \ell_\infty(A_1)/c_0(A_1)$ we have $T_{A_1,A_1}([f]_{c_0}) = [rf]_{c_0}$ for some real $r \in \mathbb{R}$. However, this does not exclude the possibility of $T_{A_1,A_1} = 0$, which actually is quite common. Therefore, we focus on obtaining localizations which are isomorphic embeddings [8] or isomorphisms (instead of only automorphisms). In this direction, we obtain the following theorem which states that, in contrast to the global versions, the local versions of the notions from Definition 0.0.5 behave like the Boolean counterparts, thus providing some justification for this approach.

Theorem 0.0.8. *Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is an automorphism. Then, the following are equivalent*

1. *T is somewhere a liftable isomorphism,*
2. *T is somewhere an isomorphic matrix operator,*
3. *T is somewhere a liftable isomorphism with a lifting which is continuous in the product topology,*
4. *T is somewhere trivial.*

Proof. The implication from (1) to (2) follows from 1.4.10; the equivalence of (2) and (3) is 1.2.15; the implication from (2) to (4) follows from 1.4.8; the fact that (4) implies (1) is clear. □

Actually, the above equivalences hold (with the same proof) in the case of T being an isomorphic embedding and for right-localizations which are isomorphic embeddings. In general, to hope for nontrivial right-local properties we need to assume that the kernel is small, for example that T is injective. Similarly, for isomorphic left-local properties one needs to assume that the image of T is big, for example that T is surjective. But in contrast to the just mentioned remark, a surjective operator can be globally liftable but nowhere a matrix operator (1.4.12) or can be globally a matrix operator but nowhere trivial (1.4.6).

One should note, however, that the notion of e. g., somewhere trivial automorphism on ℓ_∞/c_0, has quite a different character than being somewhere trivial automorphism of $\wp(\mathbb{N})/\textit{Fin}$. This is because the images of subspaces of the form $\{[f] \in \ell_\infty/c_0 : f|(\mathbb{N} \setminus A) = 0\}$ for $A \subseteq \mathbb{N}$ are usually not of the form $\{[f] \in \ell_\infty/c_0 : f|(\mathbb{N} \setminus B) = 0\}$ for $B \subseteq N$, even if $T_{B,A}$ is trivial. Also, trivialization or canonization of $T_{B,A}$ does not yield any information about $T^{-1}_{A,B}$ as in the case of automorphisms of $\wp(\mathbb{N})/\textit{Fin}$.

Further justification for the above local notions comes from the following:

Proposition 0.0.9. *Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear bounded operator and $A, B \subseteq \mathbb{N}$ are two infinite sets. Suppose that $T_{B,A}$ is canonical along a homeomorphism. Then, T fixes a complemented copy of ℓ_∞/c_0 whose image under T is complemented in ℓ_∞/c_0.*

Proof. See the proof of Corollary 2.4 of [22]. □

In fact, the above proposition would also be true with the same proof if we weakened the hypothesis on B from clopen to closed subset of $\mathbb{N}^*$ homeomorphic to $\mathbb{N}^*$. But to make sure that A induces a subspace and not just a quotient which is to be fixed we must insist on A^* to be clopen. In the context of other $C(K)$ spaces, this approach is quite

8. By isomorphic embedding we mean an operator which is an isomorphism onto its closed range. Sometimes these operators are called bounded below.

fruitful for obtaining complemented copies of the entire $C(K)$ inside any isomorphic copy of the $C(K)$ (for example, for $C(K)$ with K metrizable see [43], for ℓ_∞ see [28], and for $C([0,\omega_1])$ see [31]).

Details of the above discussion are found in Chapter 1, as well as a study of these local properties of automorphisms of ℓ_∞/c_0 and the relationships between them. Having carried out this preliminary study and having proven Theorem 0.0.8 one is left with deciding whether automorphisms of ℓ_∞/c_0 are somewhere canonizable along homeomorphisms. If they are, their local structure is similar to that of homeomorphisms of $\mathbb{N}^*$ i.e., assuming OCA + MA they would be trivial and, for example, under CH not. Although we were not completely successful in attaining this goal, some significant progress was made in the second half of Chapter 1.

Canonizing automorphisms $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ (or corresponding $\hat{T} : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$) encounters, however, problems at least as difficult as understanding continuous maps defined on closed subsets of $\mathbb{N}^*$ with ranges in $\mathbb{N}^*$ (not only autohomeomorphisms of $\mathbb{N}^*$). To better understand why this is so, let us recall that linear bounded operators on $C(\mathbb{N}^*)$ can be represented as weakly* continuous mappings $\tau : \mathbb{N}^* \to M(\mathbb{N}^*)$ (see Theorem 1 in VI.7 of [23]), where $M(\mathbb{N}^*)$ denotes the Banach space of all Radon measures on $\mathbb{N}^*$ with the total variation norm identified by the Riesz representation theorem with the dual to $C(\mathbb{N}^*)$ (see [51]). Often the points of $\mathbb{N}^*$ (identified with the Dirac measures) are sent by this map to measures that do not have atoms, and if they have atoms they may have many of them giving rise to partial multivalued functions into $\mathbb{N}^*$. One obtains $\tau(x)$ as $T^\star(\delta_x)$ for each $x \in \mathbb{N}^*$ and the representation is given by

$$\hat{T}(f^*)(x) = \int f^* \, \mathrm{d}\tau(x)$$

for every $f^* \in C(\mathbb{N}^*)$. The multifunctions, possibly of empty values, are given by

$$\varphi^T_\varepsilon(y) = \{x \in \mathbb{N}^* : |T^\star(\delta_y)(\{x\})| \geq \varepsilon\}$$

for any $\varepsilon > 0$ or by $\varphi^T(y) = \bigcup_{\varepsilon>0} \varphi^T_\varepsilon$. An equivalent condition for T being somewhere canonizable along a homeomorphism is the existence of infinite $A, B \subseteq \mathbb{N}$ and a homeomorphism $\psi : B^* \to A^*$ such that

$$T^\star(\delta_y)|A^* = r\delta_{\psi(y)}$$

for some nonzero $r \in \mathbb{R}$, which in particular means that $\varphi^T(y) \cap A^* = \{\psi(y)\}$, or in other words that ψ is a homeomorphic selection from φ^T. Right up front there could be two basic obstacles for the existence of such a selection, namely $\bigcup_{y \in B^*} \varphi^T(y)$ could have empty interior or $\{y \in B^* : \varphi^T(y) \neq \emptyset\}$ could have empty interior for an infinite $B \subseteq \mathbb{N}^*$. We call these obstacles (in stronger versions including nonatomic measures) fountains and funnels, respectively. We also introduce two classes of operators (fountainless operators, Definition 1.3.13, and funnelless operators, Definition 1.3.18) for which by definition the above obstacles cannot arise, respectively, and we obtain some reasonable sufficient conditions for the canonization:

- Every automorphism on ℓ_∞/c_0 which is fountainless is left-locally canonizable along a quasi-open mapping (1.5.6);

- Every automorphism on ℓ_∞/c_0 which is funnelless is right-locally canonizable along a quasi-open mapping (1.5.9);

where quasi-open means that the image of every open set has nonempty interior (1.3.20).

The second result is in fact a consequence of a study by G. Plebanek [45], the proof of the first, however, takes a considerable part of Chapter 1. The possibility of obtaining these results is based on special properties of isomorphic embeddings and surjections. One ingredient is an improvement of a theorem of Cengiz ("P" in [12]) obtained by Plebanek (Theorem 3.3. in [45]) which guarantees that the range of $\varphi^T_{\|T\|\|T^{-1}\|}$ covers $\mathbb{N}^*$ if T is an isomorphic embedding. However, in this result the set of y's where $\varphi^T(y)$ is nonempty could be nowhere dense, so we exclude this possibility by assuming that T has no funnels. On the other hand we prove that if T is surjective, then either for each y the set $\varphi^T(y)$ is nonempty or else there is an infinite $A \subseteq \mathbb{N}^*$ such that $\bigcup\{\varphi^T(y) : y \in A^*\}$ is nowhere dense, the second possibility being excluded if T has no fountains.

Then one is still left with the problem of reducing a quasi-open map to a homeomorphism between two clopen sets. The results of I. Farah [25] allow us to conclude that OCA + MA implies that a quasi-open mapping defined on a clopen subset of $\mathbb{N}^*$ and being onto a clopen subset of $\mathbb{N}^*$ is somewhere a homeomorphism and so by results of Veličković [64] it is somewhere induced by a bijection between two infinite subsets of $\mathbb{N}$. Hence we obtain:

- (OCA + MA) Every fountainless automorphism of ℓ_∞/c_0 is left-locally trivial (1.6.4)
- (OCA + MA) Every funnelless automorphism of ℓ_∞/c_0 is right-locally trivial (1.6.4)

Toward the end of the chapter we show that the above results are optimal in many directions by means of several constructions under CH. First, an obstacle to improving our above-mentioned ZFC selection results (1.5.6, 1.5.9) by replacing quasi-open by a homeomorphism between clopen sets is the following example:

> (CH) There is a fountainless and funnelless everywhere present isomorphic embedding globally canonizable along quasi-open map which is nowhere canonizable along a homeomorphism (1.6.11).

Here everywhere present is a weak version of a surjective operator ($P_B \circ T \neq 0$ for any infinite $B \subseteq \mathbb{N}$; see 1.3.16). Automorphisms T have the property that $P_A \circ T$ is everywhere present and $T \circ I_A$ is an isomorphic embedding for any infinite $A \subseteq \mathbb{N}$. Moreover we have the following:

> (CH) There is an automorphism of ℓ_∞/c_0 which is nowhere canonizable along a quasi-open map, in particular along a homeomorphism (1.6.6).

The above example is not a direct construction, but we have more concrete and slightly weaker examples (1.6.9) based on the existence in $\mathbb{N}^*$ of nowhere dense P-sets which are retracts of $\mathbb{N}^*$, due to van Douwen and van Mill ([60]). It is not excluded by our results (see Section 1.7) that consistently all isomorphic embeddings on ℓ_∞/c_0 are funnelless, however there are ZFC surjective operators which are not fountainless (1.3.3). And, of course, assuming CH there are well familiar nowhere trivial homeomorphisms of $\mathbb{N}^*$ which provide examples of a globally canonizable operator which is nowhere liftable (1.6.10).

If we continue with the parallelism between the theory of $\wp(\mathbb{N})/\mathit{Fin}$ and the incipient theory of ℓ_∞/c_0, then Chapter 2 corresponds to an incursion into the next step, that of examining the subspaces of ℓ_∞/c_0. In particular, we study the possibility of finding an isomorphic copy of the ℓ_∞-sum of ℓ_∞/c_0 inside ℓ_∞/c_0. One of the sources of interest in this question is related to the universality properties of ℓ_∞/c_0. It is known that CH implies that every Banach space of density [9] at most continuum can be isometrically embedded into this space, while it is consistent with ZFC that there is no such universal space (see [53, 9, 10]). The question of the embeddability of the ℓ_∞-sum of ℓ_∞/c_0 into ℓ_∞/c_0 is also related to another question concerning the space ℓ_∞/c_0, namely, whether this latter space is primary [10] (see [36]). It turns out that finding an embedding $T : \ell_\infty(\ell_\infty/c_0) \to \ell_\infty/c_0$ with complemented range was the crucial step in the proof in [22] of the fact that under the Continuum Hypothesis the space ℓ_∞/c_0 is indeed primary. So this question is important in the search for models where ℓ_∞/c_0 may fail to be primary.

Our analysis of embeddings of $\ell_\infty(\ell_\infty/c_0)$ into ℓ_∞/c_0 follows the approach set in Chapter 1, i.e., looking at the local structure of these operators. We then take advantage of A. Dow's important recent contribution ([21]) to prove the main result of the chapter, which says that under PFA there is no embedding $T : \ell_\infty(\ell_\infty/c_0) \to \ell_\infty/c_0$ such that the coordinate operators it defines are all somewhere canonical along a quasi-open mapping (2.3.2). Finally, we discuss this result by relating it to Dow's and by looking at some well-known classes of operators as well as some of those introduced in Chapter 1.

Chapter 3 deals with a seemingly separate subject, as it deals with sequences of Radon measures over a compact space. Nevertheless, the link to the other chapters exists thanks to the realization that every liftable operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is determined by a sequence of Radon measures over $\beta\mathbb{N}$. In this chapter we give a new proof of an old result by Kadec and Pełczyński which says that every such a sequence has a subsequence which is neat in the sense that it is the sum of a weakly converging sequence and a sequence of measures with constant total variation and with disjoint supports. To exemplify the strength of this result we show an application in the context of operators on ℓ_∞/c_0, proving a different kind of canonization result: instead of finding a copy of ℓ_∞/c_0 where the operator is canonical, we obtain a copy of ℓ_∞ inside ℓ_∞/c_0 where it is canonical. In fact, for a non-weakly compact operator we find an isometrically isomorphic copy of ℓ_∞ inside ℓ_∞/c_0 such that the restriction to this subspace is the identity plus a weakly compact operator, modulo an ε-error.

Finally, in the appendix we gather some previously known results that are relevant to our study as well as some of the more tedious proofs which are mostly of statements

9. The density character of a Banach space X is the least cardinality of a dense subset of X.
10. A Banach space X is said to be primary if for every direct sum decomposition $X = A \oplus B$ at least one of the summands is isomorphic to X.

relating to operators given by matrices.

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*   *   *

Notation

$\mathbb{N}$	The set of non-negative integers
$\wp(A)$	The set of parts of A
Fin	The ideal of finite subsets of $\mathbb{N}$
FinCofin	The Boolean algebra of finite and cofinite subsets of $\mathbb{N}$
$[A] = [A]_{\mathit{Fin}}$	The equivalence class of A with respect to ***Fin***
$A =_* B$	$A \triangle B \in$***Fin***
$A \subseteq_* B$	$A \setminus B \in$***Fin***
$\beta\mathbb{N}$	The Čech-Stone compactification of the integers and the Stone space of $\wp(\mathbb{N})$, regarded as the set of ultrafilters over $\mathbb{N}$
$\mathbb{N}^*$	The Čech-Stone remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ and the Stone space of $\wp(\mathbb{N})/$***Fin***, regarded as the set of nonprincipal ultrafilters over $\mathbb{N}$
βA	The clopen set in $\beta\mathbb{N}$ defined by $\{x \in \beta\mathbb{N} : A \in x\}$
A^*	The clopen set in $\mathbb{N}^*$ defined by $\beta A \setminus A$
βf	The element of $C(\beta\mathbb{N})$ which extends $f \in \ell_\infty$
f^*	The element of $C(\mathbb{N}^*)$ obtained by restricting βf to $\mathbb{N}^*$
$[f] = [f]_{c_0}$	The equivalence class of $f \in \ell_\infty$ with respect to c_0

Since any element of $C(\mathbb{N}^*)$ or $C(\beta\mathbb{N})$ is of the form f^* or βf, for some $f \in \ell_\infty$ respectively, we may use this convention when talking about general elements of these spaces. However, not all continuous functions on $\mathbb{N}^*$ or linear operators on $C(\mathbb{N}^*)$ are induced by corresponding objects in $\mathbb{N}$ or ℓ_∞. So for the passage from an endomorphism h of $\wp(\mathbb{N})/$***Fin*** to a continuous self-mapping on $\mathbb{N}^*$ or from a linear operator T on ℓ_∞/c_0 to a linear operator on $C(\mathbb{N}^*)$ we will use $\hat{h}$ and $\hat{T}$, respectively.

$[T]$	The operator on ℓ_∞/c_0 induced by an operator $T : \ell_\infty \to \ell_\infty$ which preserves c_0 (i.e., $T[c_0] \subseteq c_0$), that is, $[T]([f]_{c_0}) = [T(f)]_{c_0}$ for any $f \in \ell_\infty$
βT	The operator on $C(\beta\mathbb{N})$ induced by an operator $T : \ell_\infty \to \ell_\infty$, that is, $\beta T(\beta f) = \beta(T(f))$ for any $f \in \ell_\infty$
T^*	The operator on $C(\mathbb{N}^*)$ induced by an operator $T : \ell_\infty \to \ell_\infty$ which preserves c_0 (i.e., $T[c_0] \subseteq c_0$) that is $T^*(f^*) = (T(f))^*$ for any $f \in \ell_\infty$
$\hat{T}$	The operator from $C(\mathbb{N}^*)$ into itself which corresponds to $T : \ell_\infty/c_0 \to \ell_\infty/c_0$, i.e., $\hat{T}(f^*) = g^*$ where $[g] = T([f])$

$\hat{h}$	The continuous selfmap of $\mathbb{N}^*$ which corresponds via the Stone duality to an endomorphism h of $\wp(\mathbb{N})/\mathit{Fin}$, i.e., $\hat{h}(x) = h^{-1}[x]$ when we identify points of $\mathbb{N}^*$ with the ultrafilters of $\wp(\mathbb{N})/\mathit{Fin}$
T_ψ	The composition operator $T_\psi : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ which maps f to $f \circ \psi$, for some continuous $\psi : \mathbb{N}^* \to \mathbb{N}^*$
P_B	The operator $P_A : \ell_\infty/c_0 \to \ell_\infty(A)/c_0(A)$ given by $P_A([f]_{c_0}) = [f\|A]_{c_0(A)}$
I_A	The operator $I_A : \ell_\infty(A)/c_0(A) \to \ell_\infty/c_0$ given by $I_A([g]_{c_0(A)}) = [g \cup 0_{\mathbb{N}\setminus A}]_{c_0}$
$T_{B,A}$	The operator $T_{B,A} : \ell_\infty(A)/c_0(A) \to \ell_\infty(B)/c_0(B)$ given by $T_{B,A} = P_B \circ T \circ I_A$

The remaining often used symbols are:

$\mathbb{R}$	The field of real numbers
χ_A	The characteristic function of the set A
$\overline{A}$	The closure of the set A in a topological space
$\\|f\\|$	The supremum norm of the function $f \in C(K)$, for some compact K
$\|\mu\|$	The total variation of the signed measure μ
μ^+	The positive part of the signed measure μ relative to a Jordan decomposition
μ^-	The negative part of the signed measure μ relative to a Jordan decomposition
$B(K)$	The σ-algebra of Borel subsets of a topological space K
$M(K)$	The Banach space of Radon measures on a compact Hausdorff K with the total variation norm, identified with the dual space to $C(K)$ via the Riesz representation theorem
$T^\star$	The dual or adjoint operator of T, i.e., $T^\star(\mu)(f) = \mu(T(f))$. $T^\star$ acts on the spaces of Radon measures if T acts on a space of continuous functions (we stress the difference between T^* and $T^\star$)
δ_x	The Dirac measure concentrated on x
$\mu\|F$	the restriction of a measure $\mu \in M(\mathbb{N}^*)$ to a Borel subset $F \subseteq \mathbb{N}^*$, i.e., $\mu\|F$ is an element of $M(\mathbb{N}^*)$ such that $(\mu\|F)(G) = \mu(G \cap F)$ for any Borel $G \subseteq \mathbb{N}^*$
B_X	The unit ball of the Banach space X
$\varphi^T_\varepsilon(y)$	The set $\{x \in \mathbb{N}^* : \|T^\star(\delta_y)(\{x\})\| > \varepsilon\}$, where T is an operator on $C(\mathbb{N}^*)$ and $\varepsilon > 0$
$\varphi^T(y)$	The set $\bigcup_{\varepsilon>0} \varphi^T_\varepsilon(y)$, where T is an operator on $C(\mathbb{N}^*)$

Chapter 1

Automorphisms of the Banach space ℓ_∞/c_0 [1]

1.1 Introduction

This chapter is dedicated to the analysis of operators from ℓ_∞/c_0 into itself. We begin in Section 2 by studying liftings of such operators, i.e. operators $R : \ell_\infty \to \ell_\infty$ for which $R[c_0] \subseteq c_0$. We characterize those operators which are determined by their action on c_0 as those which are given by a certain type of infinite matrix and as those which are continuous in the product topology when restricted to a bounded set. We also obtain that liftings of operators on ℓ_∞/c_0 can be decomposed as the sum of an operator which is determined by its action on c_0 plus an operator which is small in a certain sense.

Section 3 starts by introducing the ideal of locally null operators on $C(\mathbb{N}^*)$, i.e., those operators T such that for every infinite $A \subseteq \mathbb{N}$ there is an infinite $A_1 \subseteq A$ such that every A_1^*-supported $f^* \in C(\mathbb{N}^*)$ is in the kernel of T. Then, we go into a detailed study of the weakly* continuous mapping from $\mathbb{N}^*$ into $M(\mathbb{N}^*)$ associated with every linear bounded operator from $C(\mathbb{N}^*)$ into itself. In particular, we look at the mapping which associates to every $y \in \mathbb{N}^*$ the set of atoms of the measure $T^\star(\delta_y)$, and we show that it behaves well locally. Later, we introduce what we call fountainless and funnelless operators.

In Section 4 we return to liftable operators and the relation between an operator and its lifting. We then explore the local properties of liftable operators, obtaining sufficient conditions for an operator to be locally trivial. We construct several ZFC examples that clarify the relationships between the concepts of liftable, matrix and trivial operator, and the corresponding local versions.

In Section 5 we look into the possibility of canonizing operators along continuous maps. Building upon our understanding of the maps associated with the adjoint operator (developed in Section 3), we are able to find sufficient conditions for an operator to be left-locally canonizable along a quasi-open map. On the other hand, by taking advantage of previously known results we obtain corresponding sufficient conditions for an operator to be right-locally canonizable along a quasi-open map.

Finally, in the last section we contrast how the local structure of operators on ℓ_∞/c_0 is afected by different extensions of ZFC. In particular, we find that under OCA + MA fountainless and funnelless automorphisms are locally trivial, while using

1. The material of this chapter corresponds to the article [32].

CH we are able to construct several examples that draw a very different picture: there are automorphisms which are nowhere canonizable along a quasi-open map, there are automorphisms which are fountainless and funnelless and are nowhere trivial, among others.

1.2 Operators on ℓ_∞ preserving c_0

1.2.1 Operators given by c_0-matrices

A linear operator R on ℓ_∞ which preserves c_0 (i.e., $R[c_0] \subseteq c_0$) defines, of course, an operator on c_0. In the case of the Boolean algebra $\wp(\mathbb{N})$, any Boolean automorphism preserves *FinCofin*$(\mathbb{N})$ and its restriction to *FinCofin*$(\mathbb{N})$ completely determines the automorphism. The analogous fact does not hold for linear automorphisms on ℓ_∞, for example there are many distinct automorphisms of ℓ_∞ which do not move c_0 (see 1.2.16). However, the restrictions to c_0 of operators on ℓ_∞ which preserve c_0 will play an important role, and in some cases will determine a given operator. So let us establish a transparent representation of operators on c_0:

Proposition 1.2.1. *$R : c_0 \to c_0$ is a linear bounded operator if, and only if, there exists an $\mathbb{N} \times \mathbb{N}$ matrix $(b_{ij})_{i,j\in\mathbb{N}}$ such that*

1. *every row is in ℓ_1,*
2. *if we write $b_i = (b_{ij})_j$, then $\{\|b_i\|_{\ell_1} : i \in \mathbb{N}\}$ is a bounded set,*
3. *every column is in c_0,*

and such that for every $f \in c_0$ we have

$$R(f) = \begin{pmatrix} b_{00} & b_{01} & \cdots \\ b_{10} & b_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ \vdots \end{pmatrix}.$$

Proof. Use the fact that $c_0^\star = \ell_1$ and put $b_i = R^\star(\delta_i)$, where δ_i is the functional corresponding to the i-th coordinate for each $i \in \mathbb{N}$. □

This representation corresponds to representing endomorphisms of *FinCofin*$(\mathbb{N})$ by finite-to-one functions from $\mathbb{N}$ into itself. Such endomorphisms induce operators on c_0 whose matrix satisfies the above characterization and where every row has one entry equal to 1 and the remaining entries equal to 0. Matrices define some operators on ℓ_∞ as well, of course:

Proposition 1.2.2. *Let $(b_{ij})_{i,j\in\mathbb{N}}$ be a matrix and let $b_i = (b_{ij})_j$ be the i-th row, for every $i \in \mathbb{N}$. Then,*

$$R(f) = \begin{pmatrix} b_{00} & b_{01} & \cdots \\ b_{10} & b_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ \vdots \end{pmatrix}$$

defines an linear bounded operator $R : \ell_\infty \to \ell_\infty$ if, and only if, $b_i \in \ell_1$, for all $i \in \mathbb{N}$, and $\{\|b_i\|_{\ell_1} : i \in \mathbb{N}\}$ is a bounded set.

Proof. Use the fact that $\ell_1 \subseteq \ell_\infty^\star$ and put $b_i = R^\star(\delta_i)$, where δ_i is the functional corresponding to the i-th coordinate for each $i \in \mathbb{N}$. □

Definition 1.2.3. (i) We say that a matrix is a c_0-matrix if it satisfies conditions (1)–(3) of Proposition 1.2.1.

(ii) We say that a linear bounded operator $R : \ell_\infty \to \ell_\infty$ is given by a c_0-matrix if there exists a c_0-matrix $(b_{ij})_{i,j\in\mathbb{N}}$ such that

$$R(f) = \begin{pmatrix} b_{00} & b_{01} & \dots \\ b_{10} & b_{11} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ \vdots \end{pmatrix},$$

for every $f \in \ell_\infty$.

Corollary 1.2.4. *Suppose that $R : \ell_\infty \to \ell_\infty$ is a linear bounded operator which preserves c_0 and is given by*

$$R(f) = \begin{pmatrix} b_{00} & b_{01} & \dots \\ b_{10} & b_{11} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ \vdots \end{pmatrix},$$

where $(b_{ij})_{i,j\in\mathbb{N}}$ is a real matrix. Then $(b_{ij})_{i,j\in\mathbb{N}}$ is a c_0-matrix.

Proof. If such an operator on ℓ_∞ was not given by a c_0-matrix, then some of the columns of the corresponding matrix would not be in c_0 by 1.2.2 and by 1.2.1. Then the operator would not preserve c_0. □

Proposition 1.2.5. *If a linear bounded operator $R : \ell_\infty \to \ell_\infty$ is given by a c_0-matrix, then $R = (R|c_0)^{\star\star}$.*

Proof. Appendix A.2.5. □

The next easy example distinguishes between the classes of trivial operators and those given by c_0-matrices.

Example 1.2.6. Let $a_0, a_1 \in \mathbb{R}$ be such that $a_0 a_1 \neq 0$ and $a_0 \neq a_1$. Let $M = (b_{ij})_{i,j\in\mathbb{N}}$ be a real matrix defined by

$$b_{(2i)(2i)} = a_0 \qquad b_{(2i+1)(2i+1)} = a_1$$

and $b_{ij} = 0$ for all other $i, j \in \mathbb{N}$.

It is clear that M is a c_0-matrix and that the operator R it defines is an automorphism. We claim that it is not, however, of the form $rf \circ \sigma$, for any real r and any almost permutation σ of $\mathbb{N}$. Indeed, if for every $f \in \ell_\infty$ we have that $R(f)$ is eventually equal to $rf \circ \sigma$, for some $r \in \mathbb{R}$ and some σ almost permutation of $\mathbb{N}$, then in particular $R(\chi_{\mathbb{N}})$ is eventually constant and equal to r, which contradicts our choice of a_0 and a_1.

1.2.2 Falling and weakly compact operators

Let us recall the following characterization of weakly compact operators on c_0:

Theorem 1.2.7. *Let $R : c_0 \to c_0$ be a linear bounded operator and let $(b_{ij})_{i,j\in\mathbb{N}}$ be the corresponding matrix. The following are equivalent:*

1. *R is weakly compact.*
2. *$R^{\star\star}[\ell_\infty] \subseteq c_0$.*
3. *$\|b_i\|_{\ell_1} \to 0$.*

Proof. Appendix A.2.6 □

Proposition 1.2.8. *Let $R : \ell_\infty \to \ell_\infty$ be an operator given by a c_0-matrix. Then, R is weakly compact if, and only if, $R[\ell_\infty] \subseteq c_0$.*

Proof. If R is weakly compact, then $R|c_0$ must be as well, and so by 1.2.7 we have $(R|c_0)^{\star\star}[\ell_\infty] \subseteq c_0$ but $(R|c_0)^{\star\star} = R$ by 1.2.5. In the other direction, use the fact that every operator defined on a Grothendieck Banach space into a separable Banach space is weakly compact (Theorem 1 (v) of [14]). □

Definition 1.2.9. A c_0-matrix operator $R : \ell_\infty \to \ell_\infty$ is called falling if, and only if, for every $\varepsilon > 0$ there is a partition $A_0, ..., A_{k-1}$ of $\mathbb{N}$ such that

$$\sum_{j\in A_m} |b_{ij}| < \varepsilon$$

for all $m < k$ and $i \in \mathbb{N}$ sufficiently large.

Proposition 1.2.10. *Every operator on ℓ_∞ which is given by a c_0-matrix and is weakly compact is falling.*

Proof. Use Theorem 1.2.7. □

Proposition 1.2.11. *There is a falling, non-weakly compact operator on ℓ_∞ given by a c_0-matrix.*

Proof. Let $R : \ell_\infty \to \ell_\infty$ be given by the matrix

$$b_{ij} = \begin{cases} 1/(i+1), & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$$

for all $i \in \mathbb{N}$. By 1.2.7 this is not a weakly compact operator. Given $k \in \mathbb{N}$ if we consider $A_m = \{lk + m : l \in \mathbb{N}\}$, for $m < k$ then,

$$\sum_{j\in A_m} |b_{ij}| \leq \left(\frac{i+1}{k}\right)\left(\frac{1}{i+1}\right) = 1/k,$$

so the operator is falling. □

1.2.3 Antimatrix operators

The behaviour opposite to operators given by a c_0-matrix is the subject of the following:

Definition 1.2.12. A linear bounded operator $R : \ell_\infty \to \ell_\infty$ will be called an *antimatrix* operator if, and only if, $R[c_0] = \{0\}$.

Using the isometry between ℓ_∞ and $C(\beta\mathbb{N})$, an operator R on ℓ_∞ can be associated with an operator βR on $C(\beta\mathbb{N})$ and these operators can be associated with weak* continuous functions from $\beta\mathbb{N}$ into the Radon measures $M(\beta\mathbb{N})$ on $\beta\mathbb{N}$ (see Theorem 1 in VI 7. of [23]). Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, such functions are determined by their values on $\mathbb{N}$. The following characterizations will be useful later on:

Lemma 1.2.13. *Suppose $R : \ell_\infty \to \ell_\infty$ is a bounded linear operator such that $R[c_0] \subseteq c_0$. Then,*

(a) R is given by a c_0-matrix if, and only if, $R^\star(\delta_n)$ is concentrated on $\mathbb{N}$ for all $n \in \mathbb{N}$, that is, $R^\star(\delta_n) \in \ell_1$, $\forall n \in \mathbb{N}$.

(b) R is an antimatrix operator if, and only if, $R^\star(\delta_n)$ is concentrated on $\mathbb{N}^*$ for all $n \in \mathbb{N}$.

Proof. **(a)** Assume R is given by a c_0-matrix $(b_{ij})_{i,j\in\mathbb{N}}$. Then, for every $f \in \ell_\infty$ we have $R^\star(\delta_n)(f) = R(f)(n) = b_n(f)$, where b_n is the nth row of $(b_{ij})_{i,j\in\mathbb{N}}$. So $R^\star(\delta_n) = b_n$ and by definition of c_0-matrix, we have that $b_n \in \ell_1$.

Conversely, assume $R^\star(\delta_n) \in \ell_1$. Let M be the matrix formed by putting $R^\star(\delta_n)$ as the nth row. Then, R is induced by M. Moreover, since $R[c_0] \subseteq c_0$, we know that M is a c_0-matrix.

(b) Suppose $R^\star(\delta_n)$ is not concentrated on $\mathbb{N}^*$ for some $n \in \mathbb{N}$. Then, there exists an $m \in \mathbb{N}$ such that $R^\star(\delta_n)(\{m\}) \neq 0$. Then, $R(\chi_{\{m\}})(n) = R^\star(\delta_n)(\chi_{\{m\}}) \neq 0$. Therefore, $\chi_{\{m\}} \in c_0$ is a witness to the fact that $R[c_0] \neq \{0\}$, so R is not an antimatrix operator.

Conversely, assume $R^\star(\delta_n)$ is concentrated on $\mathbb{N}^*$, for every $n \in \mathbb{N}$. Fix $f \in c_0$. Then, for every $n \in \mathbb{N}$ we have $R(f)(n) = R^\star(\delta_n)(\beta f) = \int \beta f \, \mathrm{d}R^\star(\delta_n) = \int_{\mathbb{N}^*} \beta f \, \mathrm{d}R^\star(\delta_n) = 0$, because $\beta f|\mathbb{N}^* = 0$. □

Thus a typical example of an antimatrix operator is one defined by $R(f) = ((\beta f(x_i))_{i\in\mathbb{N}})$, where $(x_i)_{i\in\mathbb{N}}$ is any sequence of nonprincipal ultrafilters.

Proposition 1.2.14. *If $R : \ell_\infty \to \ell_\infty$ is such that $R[c_0] \subseteq c_0$, then $R = S_0 + S_1$, where S_0 is given by a c_0-matrix and S_1 is an antimatrix operator.*

Proof. As $R[c_0] \subseteq c_0$ there is a matrix $(b_{ij})_{i,j\in\mathbb{N}}$ which satisfies 1.2.1. Define S_0 as multiplication by this matrix, i.e. $S_0 = (R|c_0)^{\star\star}$ by 1.2.5. Now $S_1 = R - S_0$ is antimatrix, so we obtain the desired decomposition. □

1.2.4 Product topology continuity of operators

The importance of operators on ℓ_∞ given by c_0-matrices is expressed in the following theorem which exploits the fact that ℓ_∞ is the bidual space of c_0. In the theorem below, the weak* topology on ℓ_∞ is given by the duality $\ell_1^\star = \ell_\infty$ and τ_p denotes the product topology in $\mathbb{R}^\mathbb{N}$.

Theorem 1.2.15. *Let $R : \ell_\infty \to \ell_\infty$ be a linear bounded operator. The following are equivalent:*

1. $R = (R|c_0)^{\star\star}$.
2. R *is given by a* c_0*-matrix.*
3. R *is* w^**-*w^**-continuous and* $R[c_0] \subseteq c_0$.
4. $R|B_{\ell_\infty} : (B_{\ell_\infty}, \tau_p) \to (\ell_\infty, \tau_p)$ *is continuous and* $R[c_0] \subseteq c_0$.

Proof. Appendix A.2.10. □

Thus, the nonzero antimatrix operators are discontinuous in the product topology. Such discontinuities are not, however, incompatible with being an automorphism or having a nice behaviour on c_0.

Theorem 1.2.16. *There are discontinuous automorphisms of ℓ_∞ preserving c_0. There are different automorphisms on ℓ_∞ which agree on c_0. They can be the identity on c_0.*

Proof. Let $(A_i)_{i\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets. Let x_i be any nonprincipal ultrafilter such that $A_i \in x_i$ for all $i \in \mathbb{N}$. For a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ define

$$R_\sigma(f)(n) = f(n) - \beta f(x_i) + \beta f(x_{\sigma(i)}),$$

where $i \in \mathbb{N}$ is such that $n \in A_i$. First note that $R_{\sigma^{-1}} \circ R_\sigma = R_\sigma \circ R_{\sigma^{-1}} = Id$ and so R_σ is an automorphism. One verifies that $R_\sigma|c_0$ is the identity for any permutation σ, in particular $R_\sigma - Id \neq 0$ is antimatrix for any permutation σ different than the identity and hence R_σ is discontinuous by 1.2.15.

□

In this proof we really decompose ℓ_∞ as a direct sum $X \oplus Y$, both factors necessarily isomorphic to ℓ_∞: the first of the functions constant on each set A_i and the second of the functions equal to zero in each point x_i. Since the second factor contains c_0, the automorphisms of the first factor induce automorphisms of ℓ_∞ which do not move c_0. This lack of continuity is also present in homomorphisms of $\wp(\mathbb{N})$ (3.2.3. of [26]) but not its automorphisms.

1.3 Operators on ℓ_∞/c_0

1.3.1 Ideals of operators on ℓ_∞/c_0

As usual by an (left, right) ideal we will mean a collection $\mathcal{I}$ of operators such that $T + S \in \mathcal{I}$ whenever $T, S \in \mathcal{I}$ and $S \circ R, R \circ S \in \mathcal{I}$ ($R \circ S \in \mathcal{I}$, $S \circ R \in \mathcal{I}$) whenever $S \in \mathcal{I}$ and R is any operator on ℓ_∞/c_0. We say that an operator T on ℓ_∞/c_0 factors through ℓ_∞ if, and only if, there are operators $R_1 : \ell_\infty/c_0 \to \ell_\infty$ and $R_2 : \ell_\infty \to \ell_\infty/c_0$ such that $T = R_2 \circ R_1$. It is clear that operators which factor through ℓ_∞ form a two-sided ideal. Also it is well known that weakly compact operators form a two-sided proper ideal (VI 4.5. of [23]). We introduce another class of operators:

Definition 1.3.1. An operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is locally null if, and only if, for every infinite $A \subseteq \mathbb{N}$ there is an infinite $A_1 \subseteq A$ such that

$$T \circ I_{A_1} = 0.$$

Locally null should really be right-locally null, but left-locally null is just null, so there is no need of using the word "right".

Proposition 1.3.2. *Locally null operators form a proper left ideal which contains all weakly compact operators and all operators which factor through ℓ_∞.*

Proof. It is clear that locally null operators form a proper left ideal.

Let us prove that every weakly compact operator on ℓ_∞/c_0 is locally null. We will use the fact that an operator T on a $C(K)$-space is weakly compact if, and only if, $\|T(f_n)\| \to 0$ whenever $(f_n)_{n\in\mathbb{N}} \subseteq C(K)$ is a bounded pairwise disjoint sequence (i.e., $f_n \cdot f_m = 0$ for $n \neq m$) (see Corollary VI–17 of [13]).

Let $A \subseteq \mathbb{N}$ be infinite. Consider $\{A_\xi : \xi < \omega_1\}$, a family of almost disjoint infinite subsets of A. Notice that by the weak compactness of T we have that the set of $\alpha \in \omega_1$ such that $T \circ I_{A_\alpha} \neq 0$ must be at most countable, so take α outside this set.

Now let $T = R_2 \circ R_1$ where $R_1 : \ell_\infty/c_0 \to \ell_\infty$ and $R_2 : \ell_\infty \to \ell_\infty/c_0$. Let $\mu_n = R_1^\star(\delta_n)$. Let $A \subseteq \mathbb{N}$ be infinite. As the supports of μ_n's are c.c.c. and there are continuum many pairwise disjoint clopen subsets of A^*, there is an infinite $A_1 \subseteq A$ such that $|\mu_n|(A_1^*) = 0$ for every $n \in \mathbb{N}$. It follows that $R_1 \circ I_{A_1} = 0$, which completes the proof.

□

Proposition 1.3.3. *There is a locally null operator on ℓ_∞/c_0 which factors through ℓ_∞ and is surjective. There is no surjective weakly compact operator.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a discrete subset of $\mathbb{N}^*$. Define $R : \ell_\infty/c_0 \to \ell_\infty$ by $R([f]_{c_0}) = (f^*(x_n))_{n\in\mathbb{N}}$. It is well-known that the closure of $\{x_n : n \in \mathbb{N}\}$ in $\mathbb{N}^*$ is homeomorphic to $\beta\mathbb{N}$. So by the Tietze extension theorem R is onto ℓ_∞. Furthermore, $Q \circ R$ is surjective, where $Q : \ell_\infty \to \ell_\infty/c_0$ is the quotient map. As no clopen subset A^* of $\mathbb{N}^*$ is separable, below every infinite A there is an infinite $A_1 \subseteq A$ such that no x_n belongs to A_1^*. Then, $R \circ I_{A_1} = 0$ which proves that R is locally null, and so $Q \circ R$ as well.

Weakly compact operators on an infinite dimensional $C(K)$ cannot be surjective because weakly compact subsets of an infinite dimensional Banach space have empty interior if the space is not reflexive. So countable unions of them are of the first Baire category, and in particular, the images of the balls cannot cover an infinite dimensional Banach space $C(K)$. □

See 1.6.8 for more information on the ideal of locally null operators under CH.

1.3.2 Local behaviour of functions associated with the adjoint operator

In general, for a linear bounded operator T acting on the Banach space $C(K)$ for a compact K, the function which sends $x \in K$ to $\|T^\star(\delta_x)\|$ is lower semicontinuous (e.g., Lemma 2.1 of [45]) and may be quite discontinuous.

Proposition 1.3.4. *Suppose $F \subseteq \mathbb{N}^*$ is a nowhere dense retract of $\mathbb{N}^*$. There is a linear bounded operator T on $C(\mathbb{N}^*)$ such that the function $\alpha : \mathbb{N}^* \to \mathbb{R}$ defined by*

$$\alpha(y) = \|T^\star(\delta_y)\|$$

for every $y \in \mathbb{N}^*$, is discontinuous in every point of F.

Proof. Define T by putting

$$T(f) = f - f \circ r,$$

where $r : \mathbb{N}^* \to F$ is the retraction onto F. Then $T(f)(y) = f(y) - f(r(y))$ and so $T^\star(\delta_y) = \delta_y - \delta_{r(y)}$. Hence $\alpha = 2\chi_{\mathbb{N}^*\setminus F}$. Since F is nowhere dense, the set of discontinuities of α is F. □

By Lemma 4.1. of [45], for every lower semicontinuous function, every $\varepsilon > 0$ and every open $U \subseteq \mathbb{N}^*$ there is an open $V \subseteq U$ such that the function's oscillation on V is smaller than ε. Hence, by A.1.1 there is a dense open subset of $\mathbb{N}^*$ where the function which sends $y \in \mathbb{N}^*$ to $\|T^\star(\delta_y)\|$ is locally constant. In the case of $\mathbb{N}^*$ we have not only the local stabilization of the values of $\|T^\star(\delta_y)\|$ but the local stabilization of the Hahn decompositions of the measures $T^\star(\delta_y)$:

Lemma 1.3.5. *Suppose $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is bounded linear and $B \subseteq \mathbb{N}$ is infinite. Then, there are an infinite $B_1 \subseteq_* B$, a real number s, and partitions $\mathbb{N} = C_n \cup D_n$ into infinite sets, such that for every $y \in B_1^*$ we have:*

(i) $s = \|T^\star(\delta_y)\|$

(ii) *if $T^\star(\delta_y) = \mu^+ - \mu^-$ is the Jordan decomposition of the measure, then $\mu^-(C_n^*) < 1/4(n+1)$ and $\mu^+(D_n^*) < 1/4(n+1)$.*

Proof. We construct by induction a $\subseteq_*$-decreasing sequence of infinite sets $(A_n)_{n\in\mathbb{N}}$, $y_n \in A_n^*$, and partitions $\mathbb{N} = C_n \cup D_n$ into infinite sets such that for every $n \in \mathbb{N}$ we have:

1. $\sup\{\|T^\star(\delta_y)\| : y \in A_n^*\} - \|T^\star(\delta_{y_n})\| < 1/6(n+1)$
2. $\|T^\star(\delta_{y_n})\| - T^\star(\delta_{y_n})(C_n^*) + T^\star(\delta_{y_n})(D_n^*) < 2/6(n+1)$
3. For all $y \in A_{n+1}^*$ we have $|T^\star(\delta_y)(C_n^*) - T^\star(\delta_{y_n})(C_n^*)| < 1/6(n+1)$ and $|T^\star(\delta_y)(D_n^*) - T^\star(\delta_{y_n})(D_n^*)| < 1/6(n+1)$.

This is arranged as follows. Put $A_0 = B$ and assume we have constructed A_n. Take $y_n \in A_n^*$ such that $\|T^\star(\delta_{y_n})\| > \sup\{\|T^\star(\delta_y)\| : y \in A_n^*\} - 1/6(n+1)$. Take a Hahn decomposition $\mathbb{N}^* = H_n^+ \cup H_n^-$ for the measure $T^\star(\delta_{y_n})$. By the regularity, we may choose an infinite $C_n \subseteq \mathbb{N}$ such that $|T^\star(\delta_{y_n})(H_n^+) - T^\star(\delta_{y_n})(C_n^*)| < 1/6(n+1)$. If we put $D_n = \mathbb{N} \setminus C_n$, we obtain $|T^\star(\delta_{y_n})(H_n^-) - T^\star(\delta_{y_n})(D_n^*)| < 1/6(n+1)$. Therefore,

$$\|T^\star(\delta_{y_n})\| = T^\star(\delta_{y_n})(H_n^+) - T^\star(\delta_{y_n})(H_n^-) < T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_{y_n})(D_n^*) + 2/6(n+1),$$

and so (2) holds.

By the weak* continuity of $T^\star$, the set of points which satisfy the condition in (3) is an open neighbourhood of y_n, so we may take $A_{n+1} \subseteq_* A_n$ satisfying (3). This ends the induction.

Notice that $|T^\star(\delta_{y_n})(C_n^*)| \leq \|T\|$ for every $n \in \mathbb{N}$, and so there exists a convergent subsequence of $(T^\star(\delta_{y_n})(C_n^*))_{n\in\mathbb{N}}$. The same is true for the D_n's and so we may assume that both of these sequences converge. Let us define

$$s^+ = \lim_{n\to\infty} T^\star(\delta_{y_n})(C_n^*) \qquad s^- = \lim_{n\to\infty} T^\star(\delta_{y_n})(D_n^*).$$

Now let $B_1 \subseteq \mathbb{N}$ be infinite such that $B_1 \subseteq_* A_n$, for all $n \in \mathbb{N}$. We will show that for every $y \in B_1^*$ we have $\|T^\star(\delta_y)\| = s$, where $s = s^+ - s^-$.

So let us fix $y \in B_1^*$. Notice that from (3) we obtain that

$$s = \lim_{n\to\infty} (T^\star(\delta_y)(C_n^*) - T^\star(\delta_y)(D_n^*)) = \lim_{n\to\infty} T^\star(\delta_y)(\chi_{C_n^*} - \chi_{D_n^*}).$$

Therefore, $s \leq \|T^\star(\delta_y)\|$.

Now, by (1) and (2) the following holds for every $n \in \mathbb{N}$:

$$\begin{aligned} T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_{y_n})(D_n^*) &> \|T^\star(\delta_{y_n})\| - 2/6(n+1) \\ &> \sup\{\|T^\star(\delta_z)\| : z \in A_n^*\} - 1/2(n+1) \\ &\geq \|T^\star(\delta_y)\| - 1/2(n+1). \end{aligned}$$

Therefore, $s = \lim_{n\to\infty} T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_y)(D_n^*) \geq \|T^\star(\delta_y)\|$. This proves the first statement of the lemma.

To check that (ii) holds, let us fix $y \in B_1^*$ and the Jordan decomposition for the measure $T^\star(\delta_y) = \mu^+ - \mu^-$. By going to a subsequence if necessary, we may assume that $|s - (T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_{y_n})(D_n^*))| < 1/6(n+1)$, for every $n \in \mathbb{N}$.

Observe that since $C_n^* \cup D_n^* = \mathbb{N}^*$, we have that

$$\begin{aligned} T^\star(\delta_y)(C_n^*) - T^\star(\delta_y)(D_n^*) &= |T^\star(\delta_y)|(C_n^*) + |T^\star(\delta_y)|(D_n^*) - 2\mu^-(C_n^*) - 2\mu^+(D_n^*) \\ &= \|T^\star(\delta_y)\| - 2\mu^-(C_n^*) - 2\mu^+(D_n^*), \end{aligned}$$

for every $n \in \mathbb{N}$. Then, by (3) we obtain

$$\begin{aligned} 2(\mu^-(C_n^*) + \mu^+(D_n^*)) &= s - (T^\star(\delta_y)(C_n^*) - T^\star(\delta_y)(D_n^*)) \\ &\leq s - (T^\star(\delta_{y_n})(C_n^*) - T^\star(\delta_{y_n})(D_n^*) - 2/6(n+1)) \\ &< 1/6(n+1) + 2/6(n+1) = 1/2(n+1). \end{aligned}$$

Since both $\mu^-(C_n^*)$ and $\mu^+(D_n^*)$ are non negative, they are both strictly less than $1/4(n+1)$ and (ii) is proved. □

Corollary 1.3.6. *Suppose $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is bounded linear and $B \subseteq \mathbb{N}$ is infinite. Then, there are an infinite $B_1 \subseteq_* B$ and a Borel partition $\mathbb{N}^* = X \cup Y$ such that X and Y form a Hahn decomposition of $T^\star(\delta_y)$, for every $y \in B_1^*$.*

Proof. Let $B_1 \subseteq_* B$ and $C_n, D_n \subseteq \mathbb{N}$ be as in 1.3.5. Let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive integers such that $\frac{1}{4(n_k+1)} < 1/2^k$, $\forall k \in \mathbb{N}$.

Let $F_i = \bigcap_{k\geq i} C^*_{n_k}$, $X = \bigcup_{i\in\mathbb{N}} F_i$ and $Y = \mathbb{N}^* \setminus X$. Fix $y \in B_1^*$ and let $T^\star(\delta_y) = \mu^+ - \mu^-$ be a Jordan decomposition of the measure. Since $F_i \subseteq F_{i+1}$ and $F_i \subseteq C^*_{n_i}$, for every $i \in \mathbb{N}$, by 1.3.5 we have that

$$\mu^-(F_{i_0}) \leq \mu^-(F_i) \leq \mu^-(C_{n_i}) < \frac{1}{4(n_i+1)},$$

for every $i_0 \in \mathbb{N}$ and every $i \geq i_0$. Therefore, $\mu^-(F_{i_0}) = 0$, for every $i_0 \in \mathbb{N}$, and so $\mu^-(X) = 0$.

On the other hand, we have $Y \subseteq \mathbb{N}^* \setminus F_i = \bigcup_{k\geq i} D^*_{n_k}$, for every $i \in \mathbb{N}$. Therefore, $\mu^+(Y) \leq \sum_{k\geq i} \mu^+(D^*_{n_k}) < \sum_{k\geq i} \frac{1}{4(n_k+1)} < \sum_{k\geq i} 1/2^k$, for every $i \in \mathbb{N}$. It follows that $\mu^+(Y) = 0$. □

Corollary 1.3.7. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is a linear bounded operator and $B \subseteq \mathbb{N}$ is infinite. Then, there is an infinite $B_1 \subseteq B$ such that the functions which map $y \in B_1^*$ to the positive part, to the negative part, and to the total variation measure of the measure $T^\star(\delta_y)$, respectively, are all weak* continuous. In particular, T is left-locally a regular operator.*

Proof. Let $B_1 \subseteq B$ and s be as in 1.3.5. If $s = 0$, then the first part of the corollary is trivially true and $P_{B_1} \circ T = 0$ is positive. Otherwise, consider the operator $\frac{1}{s}T$. We have that $\|(\frac{1}{s}T)^\star(\delta_y)\| = 1$ for all $y \in B_1^*$. Now apply the second part of Lemma 2.2 of [46] which says that on the dual sphere sending the measure μ to its total variation $|\mu|$ is weakly* continuous. Since the positive and the negative parts of μ can be obtained from μ and $|\mu|$, and $|s\mu| = s|\mu|$ for nonnegative s, using the weak* continuity of $T^\star$ we conclude the first part of the corollary.

For the second part we will define two positive operators T^+ and T^- such that $T^+ - T^- = P_{B_1} \circ T$. For every $y \in B_1^*$ we have that $(P_{B_1} \circ T)^\star(\delta_y) = T^\star(P^\star_{B_1}(\delta_y)) = T^\star(\delta_y)$. So for every $y \in B_1^*$ define

$$T^+(f)(y) = \int f \,\mathsf{d}(T^\star(\delta_y))^+, \qquad T^-(f)(x) = \int f \,\mathsf{d}(T^\star(\delta_y))^-.$$

It is clear that $P_{B_1} \circ T = T^+ - T^-$. The linearity of T^+ and T^- follows from general properties of the integral. To see that they are bounded, notice that for every $f \in C(\mathbb{N}^*)$ with $\|f\| \leq 1$ we have that $\|T^+(f)\| \leq \sup_{y\in B_1^*}(T^\star(\delta_y))^+(\mathbb{N}^*)$. If $\{(T^\star(\delta_y))^+(\mathbb{N}^*) : y \in B_1^*\}$ were unbounded, the set $F_n = \{y \in B_1^* : (T^\star(\delta_y))^+(\mathbb{N}^*) \geq n\}$ would be nonempty for every $n \in \mathbb{N}$. But by the w^*-continuity of the map $y \mapsto (T^\star(\delta_y))^+$, each F_n is closed. Since the F_n's form a decreasing chain of nonempty closed sets, there exists $y \in \bigcap_{n\in\mathbb{N}} F_n$. But this is impossible.

The same argument shows that T^- is bounded.

□

Definition 1.3.8. Let X, Y be topological spaces. A function $\varphi : X \to \wp(Y)$ is called *upper semicontinuous* if for every open set $V \subseteq Y$ the following set is open in X

$$\{x \in X : \varphi(x) \subseteq V\}.$$

Our main interest in multifunctions is related to the following

Definition 1.3.9. Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear bounded operator and $\varepsilon > 0$. We define

$$\varphi_\varepsilon^T(y) = \{x \in \mathbb{N}^* : |T^\star(\delta_y)(\{x\})| \geq \varepsilon\},$$
$$\varphi^T(y) = \bigcup_{\varepsilon>0} \varphi_\varepsilon^T(y).$$

Proposition 1.3.10. *There is a linear bounded $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ such that $\varphi_{1/2}^T$ is not upper semicontinuous.*

Proof. Consider the operator from the proof of 1.3.4. We have $\varphi_{1/2}(y) = \emptyset$ if $y \in F$ and $\varphi_{1/2}(y) = \{r(y), y\}$ whenever $y \in \mathbb{N}^* \setminus F$. So for example taking $V = \mathbb{N}^* \setminus F$ we obtain that $\{y : \varphi_{1/2}(y) \subseteq V\} = F$ which is not open, but closed nowhere dense. □

However we obtain the left-local upper semicontinuity:

Lemma 1.3.11. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator and let $B \subseteq \mathbb{N}$ be infinite. Then, there exists $B_1 \subseteq_* B$ such that $\varphi_\varepsilon^T | B_1^*$ is upper semicontinuous for every $\varepsilon > 0$.*

Proof. Let $B_1 \subseteq_* B$ and C_n, D_n be as in 1.3.5. Fix $y \in B_1^*$ and an open $V \subseteq \mathbb{N}^*$ such that $\varphi_\varepsilon^T(y) \subseteq V$. Then, for every $x \in \mathbb{N}^* \setminus V$ we find a clopen neighbourhood U_x of x as follows. First notice that $|T^\star(\delta_y)(\{x\})| < \varepsilon$. Let $N_x \in \mathbb{N}$ be such that $|T^\star(\delta_y)(\{x\})| < \varepsilon - 1/N_x$. Now, using the regularity of the measure $T^\star(\delta_y)$, find U_x such that $|T^\star(\delta_y)|(U_x) < \varepsilon - 1/N_x$. We may assume that U_x is included in either $C_{N_x}^*$ or $D_{N_x}^*$.

Since $\mathbb{N}^* \setminus V$ is compact, we may find $x_0, \ldots, x_{k-1} \in \mathbb{N}^* \setminus V$ such that $\mathbb{N}^* \setminus V \subseteq \bigcup_{i<k} U_{x_i}$. Using the weak*-continuity of $T^\star$, we now find a clopen neighbourhood of y, say E^*, which we may assume to be included in B_1^*, such that for every $z \in E^*$ we have

$$|T^\star(\delta_z)(U_{x_i})| < \varepsilon - 1/N_{x_i}, \text{ for each } i < k.$$

This is possible because $|T^\star(\delta_y)(U_x)| \leq |T^\star(\delta_y)|(U_x) < \varepsilon - 1/N_x$.

We claim that for every $z \in E^*$ and every $x \in \bigcup_{i<k} U_{x_i}$ we have $|T^\star(\delta_z)(\{x\})| < \varepsilon$, that is $\varphi_\varepsilon^T(z) \subseteq V$. So fix $z \in E^*$ and $x \in U_{x_i}$. Let $T^\star(\delta_z) = \mu_z^+ - \mu_z^-$ be a Jordan decomposition of the measure. Notice that $|T^\star(\delta_z)|(U_{x_j}) \leq |T^\star(\delta_z)(U_{x_j})| + 2\mu_z^-(U_{x_j})$ and $|T^\star(\delta_z)|(U_{x_j}) \leq |T^\star(\delta_z)(U_{x_j})| + 2\mu_z^+(U_{x_j})$. So if $U_{x_i} \subseteq C_{N_{x_i}}^*$, since $\mu_z^-(U_{x_i}) \leq \mu_z^-(C_{N_{x_i}}^*) < 1/4(N_{x_i} + 1)$, we have that

$$\begin{aligned} |T^\star(\delta_z)(\{x\})| &\leq |T^\star(\delta_z)(U_{x_j})| + 2\mu_z^-(U_{x_j}) \\ &< \varepsilon - 1/N_{x_j} + 1/2(N_{x_j} + 1) \\ &< \varepsilon \end{aligned}$$

If $U_{x_i} \subseteq D_{N_{x_i}}^*$, we use the fact that $\mu_z^+(U_{x_i}) \leq \mu_z^+(D_{N_{x_i}}^*) < 1/4(N_{x_j} + 1)$ to obtain the same result.

□

1.3.3 Fountains and funnels

The property of being locally null can be expressed using a topological property of $T^\star$.

Proposition 1.3.12. *A bounded linear operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is locally null if, and only if, there is a nowhere dense set $F \subseteq \mathbb{N}^*$ such that $T^\star(\delta_y)$ is concentrated on F for every $y \in \mathbb{N}^*$.*

Proof. Suppose T is locally null. If we set $D = \bigcup\{A^* : T \circ I_A = 0\}$, then D is an open dense set. Suppose $|T^\star(\delta_y)|(D) > \varepsilon$ for some $y \in \mathbb{N}^*$ and some $\varepsilon > 0$. By the regularity of the measure we may find a compact $G \subseteq D$ such that $|T^\star(\delta_y)|(G) > \varepsilon$. We may further find finitely many $A_0, \ldots, A_{n-1} \subseteq \mathbb{N}$ such that $T \circ I_{A_i} = 0$ for all $i < n$ and $\sum_{i<n} |T^\star(\delta_y)|(A_i^*) > \varepsilon$. Choose $i < n$ such that $|T^\star(\delta_y)|(A_i^*) > \varepsilon/n$ and a function f with support included in A_i^* such that $T^\star(\delta_y)(f) > \varepsilon/n$. Then, $T(f)(y) \neq 0$, which contradicts the hypothesis. Therefore, $T^\star(\delta_y)$ is concentrated on $F = \mathbb{N}^* \setminus D$, for every $y \in \mathbb{N}^*$.

Conversely, suppose F is a nowhere dense set such that for every $y \in \mathbb{N}^*$ the measure $T^\star(\delta_y)$ is concentrated on F. Given an infinite $A \subseteq \mathbb{N}$, take $A_1 \subseteq_* A$ infinite such that $A_1^* \cap F = \emptyset$. Then, $|T^\star(\delta_y)|(A_1^*) = 0$ and it follows that $T \circ I_{A_1} = 0$. □

As in the previous proposition, many results in the following parts of the chapter will show the important role played by nowhere dense sets of $\mathbb{N}^*$ in the context of operators on $C(\mathbb{N}^*)$. It is this fact that leads to the definitions of fountains, funnels and fountainless and funnelless operators:

Definition 1.3.13. An operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is called fountainless or without fountains if, and only if, for every nowhere dense set $F \subseteq \mathbb{N}^*$ the set

$$G = \{y \in \mathbb{N}^* : T^\star(\delta_y) \text{ is nonzero and concentrated on } F\}$$

is nowhere dense. A fountain for T is a pair (F, U) with $F \subseteq \mathbb{N}^*$ nowhere dense and $U \subseteq \mathbb{N}^*$ open such that all the measures $T^\star(\delta_y)$ for $y \in U$ are concentrated on F.

Lemma 1.3.14. *Let T be fountainless and let $B \subseteq \mathbb{N}$ be infinite. If $P_B \circ T$ is locally null, then $P_B \circ T = 0$.*

Proof. By 1.3.12 there is a nowhere dense $F \subseteq \mathbb{N}^*$ such that for every $y \in B^*$ we have that $(P_B \circ T)^\star(\delta_y)$, which is equal to $T^\star(\delta_y)$, is concentrated on F. By 1.3.13 the set $G = \{y \in B^* : T^\star(\delta_y) \neq 0\}$ is nowhere dense. But this means that for every $f \in C(\mathbb{N}^*)$ we have $T(f)(x) = 0$ if $x \in B^* \setminus G$. Since $B^* \setminus G$ is dense in B^* we conclude that $P_B \circ T = 0$.

□

Corollary 1.3.15. *Suppose that T is locally null and has no fountains, then $T = 0$.*

Proof. Put $B = \mathbb{N}$ in 1.3.14. □

Definition 1.3.16. We say that an operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is everywhere present if, and only if, for every infinite $B \subseteq \mathbb{N}$ we have that $P_B \circ T \neq 0$.

In the following lemma we obtain a kind of left dual to an improvement of a theorem of Cengiz ("P" in [12]) obtained by Plebanek (Theorem 3.3. in [45]) which implies that if T is an isomorphic embedding then every $x \in \mathbb{N}^*$ is in $\varphi^T(y)$ for some $y \in \mathbb{N}^*$.

Lemma 1.3.17. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is an everywhere present fountainless operator. Then, for every infinite $B \subseteq \mathbb{N}$ there exists an infinite $B_1 \subseteq_* B$ such that $\varphi^T(y) \neq \emptyset$, for every $y \in B_1^*$.*

Proof. Given an infinite $B \subseteq \mathbb{N}$, let $B_1 \subseteq_* B$ and C_n, $D_n \subseteq \mathbb{N}$ be as in 1.3.5. Suppose that $y_0 \in B_1^*$ is such that $\varphi^T(y_0) = \emptyset$. For every $n \in \mathbb{N}$ we find an open covering of $\mathbb{N}^*$ as follows. Given $x \in \mathbb{N}^*$, find by the regularity of the measure $T^\star(\delta_{y_0})$ a clopen neighbourhood of x, say U_x, such that $|T^\star(\delta_{y_0})|(U_x) < 1/2(n+1)$ and U_x is included in either C_n^* or D_n^*.

By the compactness of $\mathbb{N}^*$ obtain for each $n \in \mathbb{N}$ an open covering $\{U_{n,i} : i < j_n\}$ of $\mathbb{N}^*$ such that for each $i < j_n$ we have that

1. $|T^\star(\delta_{y_0})(U_{n,i})| \leq |T^\star(\delta_{y_0})|(U_{n,i}) < 1/2(n+1)$, and
2. either $U_{n,i} \subseteq C_n^*$ or $U_{n,i} \subseteq D_n^*$.

By the weak* continuity of $T^\star$ there are open neighbourhoods V_n of y_0 such that $|T^\star(\delta_y)(U_{n,i})| < 1/2(n+1)$ holds for all $y \in V_n$ and all $i < j_n$. Let V^* be a clopen subset of $\bigcap_{n\in\mathbb{N}} V_n \cap B_1^*$ and consider the family $\mathcal{A} \subseteq \wp(\mathbb{N})$ of those sets A such that for each $n \in \mathbb{N}$ we have $A^* \subseteq U_{n,i_n}$ for some $i_n < j_n$. We claim that $|T^\star(\delta_y)|(A^*) = 0$ for every $y \in V^*$ and every $A \in \mathcal{A}$.

So fix $y \in V^*$, $A \in \mathcal{A}$ and $n \in \mathbb{N}$. We will show that $|T^\star(\delta_y)|(A^*) < 1/(n+1)$. Let $T^\star(\delta_y) = \mu^+ - \mu^-$ be a Jordan decomposition of the measure. By 1.3.5 we have that $\mu^-(C_n^*) < 1/4(n+1)$ and $\mu^+(D_n^*) < 1/4(n+1)$. Assume without loss of generality that $U_{n,i_n} \subseteq C_n^*$. Then,

$$\begin{array}{rcl} |T^\star(\delta_y)|(A^*) & \leq & |T^\star(\delta_y)|(U_{n,i_n}) \\ & = & T^\star(\delta_y)(U_{n,i_n}) + 2\mu^-(U_{n,i_n}) \\ & \leq & |T^\star(\delta_y)(U_{n,i_n})| + 2\mu^-(C_n^*) \\ & < & 1/2(n+1) + 2/4(n+1) \\ & = & 1/(n+1) \end{array}$$

So the claim is proved.

Notice that this implies that $(P_V \circ T)(f) = 0$ for every $f \in C(\mathbb{N}^*)$ whose support is included in A^*, for some $A \in \mathcal{A}$. Therefore, if $\mathcal{A}$ is a dense family, by 1.3.14 we would have that $(P_V \circ T)(g) = 0$ for all $g \in C(\mathbb{N}^*)$, but this would contradict the hypothesis that T is everywhere present.

We prove that $\mathcal{A}$ is a dense family. For a fixed infinite $E_0 \subseteq \mathbb{N}$, we may define by induction a $\subseteq_*$- decreasing sequence (E_n) of infinite sets by choosing $\emptyset \neq E_{n+1}^* \subseteq E_n^* \cap U_{n,i_n}$, for some $i_n < j_n$ (this is possible because $\{U_{n,i} : i < j_n\}$ is an open covering of $\mathbb{N}^*$ for each $n \in \mathbb{N}$). Take A such that $A \subseteq_* E_n$, for all $n \in \mathbb{N}$. It is clear that $A \subseteq_* E_0$ and $A \in \mathcal{A}$.

□

Let us introduce a dual notion to a fountain:

Definition 1.3.18. An operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is called funnelless or without funnels if, and only if, for every nowhere dense set $F \subseteq \mathbb{N}^*$ there is a nowhere dense $G \subseteq \mathbb{N}^*$ such that for all $y \in F$ the measure $T^\star(\delta_y)$ is concentrated on G. A funnel for T is a pair (U, F) with $F \subseteq \mathbb{N}^*$ nowhere dense and $U \subseteq \mathbb{N}^*$ open such that there is no proper closed subset of U where all the measures $T^\star(\delta_y)|U$ for $y \in F$ are concentrated.

1.3.4 Operators induced by continuous maps and nonatomic operators

Definition 1.3.19. Suppose that $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a continuous map. Then $T_\psi : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is given for every $f \in C(\mathbb{N}^*)$ by

$$T_\psi(f) = f \circ \psi.$$

Definition 1.3.20. A continuous map $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is called quasi-open if, and only if, the image of every nonempty open set under ψ has nonempty interior.

Proposition 1.3.21. *Suppose that $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a continuous map. Then T_ψ is fountainless if, and only if, ψ is quasi-open.*

Proof. Notice that for every $y \in \mathbb{N}^*$ we have $T_\psi^\star(\delta_y) = \delta_{\psi(y)}$. Notice also that for every subset $X \subseteq \mathbb{N}^*$ the following holds

$$|\delta_{\psi(y)}|(X) \neq 0 \quad \text{iff} \quad \psi(y) \in X \quad \text{iff} \quad y \in \psi^{-1}[X].$$

Therefore, if ψ is quasi-open and $F \subseteq \mathbb{N}^*$ is nowhere dense, we have that $\{y \in \mathbb{N}^* : |T_\psi^\star(\delta_y)|(\mathbb{N}^* \setminus F) = 0\} = \psi^{-1}[F]$ is nowhere dense, and so T_ψ is fountainless. On the other hand if T_ψ is fountainless, consider $\psi[U]$ where U is open. If $\psi[U]$ were nowhere dense, then $\{y \in \mathbb{N}^* : |T_\psi^\star(\delta_y)|(\mathbb{N}^* \setminus \psi[U]) = 0\} = \psi^{-1}[\psi[U]]$ would be nowhere dense, which contradicts the fact that $U \subseteq \psi^{-1}[\psi[U]]$. □

Proposition 1.3.22. *Let $\psi : \mathbb{N}^* \to \mathbb{N}^*$ be a continuous map. Then T_ψ is funnelless if, and only if, ψ sends nowhere dense sets into nowhere dense sets.*

Proof. Suppose T_ψ is funnelless and let $F \subseteq \mathbb{N}^*$ be nowhere dense. Let $G \subseteq \mathbb{N}^*$ be nowhere dense such that $T_\psi^\star(\delta_y)$ is concentrated on G for every $y \in F$. Then, as in the proof of 1.3.21, we have that $F \subseteq \{y \in \mathbb{N}^* : |T_\psi^\star(\delta_y)|(\mathbb{N}^* \setminus G) = 0\} = \psi^{-1}[G]$.

Now suppose ψ sends nowhere dense sets into nowhere dense sets and let $F \subseteq \mathbb{N}^*$ be nowhere dense. Then, $F \subseteq \psi^{-1}[\psi[F]] = \{y \in \mathbb{N}^* : |T_\psi^\star(\delta_y)|(\mathbb{N}^* \setminus \psi[F]) = 0\}$, which means that $T_\psi^\star(\delta_y)$ is concentrated on $\psi[F]$, for every $y \in F$. □

Definition 1.3.23. An operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is nonatomic if, and only if, for every $y \in \mathbb{N}^*$ the measure $T^\star(\delta_y)$ is nonatomic.

Proposition 1.3.24. *Every positive nonatomic operator on ℓ_∞/c_0 is locally null.*

Proof. Since for every $y \in \mathbb{N}^*$ the measure $T^\star(\delta_y)$ has no atoms, by the regularity of $T^\star(\delta_y)$ and the compactness of $\mathbb{N}^*$ we may find for each $n \in \mathbb{N}$ a finite open covering

$(U_i(y,n))_{i<j(y,n)}$ of $\mathbb{N}^*$ by clopen sets such that $|T^\star(\delta_y)|(U_i(y,n)) < 1/2(n+1)$ holds for all $i < j(y,n)$.

Now by the weak* continuity of $T^\star$, we may choose for each $n \in \mathbb{N}$ an open neighbourhood $V_n(y)$ of y such that for all $z \in V_n(y)$ we have

$$|T^\star(\delta_z)|(U_i(y,n)) = |T^\star(\delta_z)(U_i(y,n))| < 1/2(n+1)$$

for all $i < j(y,n)$. The first equality follows from the hypothesis that T is positive.

We have thus constructed for each $n \in \mathbb{N}$ an open covering $\{V_n(y) : y \in \mathbb{N}^*\}$ of $\mathbb{N}^*$. By the compactness of $\mathbb{N}^*$, for each $n \in \mathbb{N}$ take $y_0(n), ..., y_{m(n)-1}(n) \in \mathbb{N}^*$ such that

$$\mathbb{N}^* \subseteq \bigcup_{l<m(n)} V_n(y_l).$$

Now consider the family $\mathcal{A}$ of those sets $A \subseteq \mathbb{N}$ such that given $n \in \mathbb{N}$, for each $l < m(n)$ there is $i < j(y_l, n)$ such that A^* is included in $U_i(y_l, n)$. As in the proof of 1.3.17, it is easy to see that $\mathcal{A}$ is dense and that for every $z \in \mathbb{N}^*$ and every $A \in \mathcal{A}$ we have that $|T^\star(\delta_z)|(A^*) = 0$. Therefore if $f \in C(\mathbb{N}^*)$ is A^*- supported we have $T(f) = 0$, as required.

□

1.4 Operators on ℓ_∞/c_0 and operators on ℓ_∞

1.4.1 Operators induced by operators on ℓ_∞

Definition 1.4.1. Suppose that $R : \ell_\infty \to \ell_\infty$ is a linear operator which preserves c_0, then $[R] : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear operator defined by

$$[R]([f]_{c_0}) = [R(f)]_{c_0},$$

for every $f \in \ell_\infty$. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear operator, then a lifting $R : \ell_\infty \to \ell_\infty$ is any linear operator such that $[R] = T$.

Note that our terminology is slightly different than the one used in the literature concerning the trivialization of endomorphisms of $\wp(\mathbb{N})/$*Fin*. This is due to the fact that we do not use nonlinear liftings of linear operators.

Lemma 1.4.2. *Let $R_0, R_1 : \ell_\infty \to \ell_\infty$ be linear operators which preserve c_0. Then,*

1. $[R_0 + \alpha R_1] = [R_0] + \alpha[R_1]$*, for every real* α.
2. $[R_1 \circ R_0] = [R_1] \circ [R_0]$.

Proof. Fix $f \in \ell_\infty$. Then,

$$[R_0+\alpha R_1)]([f]_{c_0}) = [(R_0+\alpha R_1)(f)]_{c_0} = [R_0(f)]_{c_0}+\alpha[R_1(f)]_{c_0} = ([R_0]+\alpha[R_1])([f]_{c_0})$$

$$\text{and } [R_1] \circ [R_0]([f]_{c_0}) = [R_1]([R_0(f)]_{c_0}) = [R_1(R_0(f))]_{c_0} = [R_1 \circ R_0]([f]_{c_0}).$$

□

Proposition 1.4.3. *Let $R_0, R_1 : \ell_\infty \to \ell_\infty$ be linear operators which preserve c_0. Then,*

1. *If $[R_0] = 0$, then R is weakly compact.*
2. *If $[R_0] = [R_1]$, then $R_0 - R_1$ is weakly compact.*

Proof. $[R_0] = 0$ means that the image of R_0 is included in c_0. However, ℓ_∞ is a Grothendieck space and all operators from such spaces into separable spaces are weakly compact (Theorem 1 of [14]). For part (2) apply 1.4.2 and part (1) to $R_0 - R_1$. □

So there could be many liftings of the same operator but they all differ by a weakly compact perturbation. When we look at ℓ_∞/c_0 as $C(\mathbb{N}^*)$, then liftings correspond to extensions.

Lemma 1.4.4. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be liftable to $R : C(\beta\mathbb{N}) \to C(\beta\mathbb{N})$. Then, for every $y \in \mathbb{N}^*$ we have*

$$R^\star(\delta_y)|\mathbb{N}^* = T^\star(\delta_y).$$

Proof. If $Q : C(\beta\mathbb{N}) \to C(\mathbb{N}^*)$ is the restriction map, then the dual of the lifting relation $T \circ Q = Q \circ R$ is $Q^\star \circ T^\star = R^\star \circ Q^\star$. Notice that $Q^\star$ acts on measures on $\mathbb{N}^*$ by extending them to $\beta\mathbb{N}$ with $\mathbb{N}$ having measure zero. So for every $y \in \mathbb{N}^*$ we have $T^\star(\delta_y) = (Q^\star \circ T^\star)(\delta_y)|\mathbb{N}^* = (R^\star \circ Q^\star)(\delta_y)|\mathbb{N}^* = R^\star(\delta_y)|\mathbb{N}^*$. □

1.4.2 Local properties of liftable operators on ℓ_∞/c_0

Proposition 1.4.5. *If $R : C(\beta\mathbb{N}) \to C(\beta\mathbb{N})$ is a positive falling operator, then the operator $[R] : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is nonatomic and locally null.*

Proof. By the definition (1.2.9), given $\varepsilon > 0$ we have a cofinite set $B \subseteq \mathbb{N}$ and a partition $\{A_1, ..., A_k\}$ of $\mathbb{N}$ such that

$$R^\star(\delta_i)(\beta A_m) = |R^\star(\delta_i)|(\beta A_m) < \varepsilon$$

for every $m \leq k$ and every $i \in B$. As any δ_y, for $y \in \mathbb{N}^*$, is in the weak* closure of $\{\delta_n : n \in B\}$, it follows by the weak* continuity of $R^\star$ that $R^\star(\delta_y)(\beta A_m) < \varepsilon$, for every $y \in \mathbb{N}^*$ and every $m \leq k$. But by 1.4.4 and the positivity of R we have $[R]^\star(\delta_y)(A_m^*) = R^\star(\delta_y)(A_m^*) \leq R^\star(\delta_y)(\beta A_m) < \varepsilon$, for every every $y \in \mathbb{N}^*$ and every $m \leq k$. As $\{A_1^*, ..., A_k^*\}$ is a partition of $\mathbb{N}^*$, we conclude that $[R]^\star(\delta_y)$ is nonatomic for every $y \in \mathbb{N}^*$. By 1.3.24, $[R]$ is locally null.

□

Corollary 1.4.6. *There is a matrix operator T which has fountains and such that whenever $T \circ I_A \neq 0$, we have that $T \circ I_A$ is not canonizable along any continuous map. In particular T is nowhere trivial.*

Proof. The operator R from 1.2.11 is a non-weakly compact, positive, falling operator on ℓ_∞. Its range is not included in c_0 by 1.2.8 (actually, the characteristic function of a subset of $\mathbb{N}$ of positive density is sent to an element not in c_0). So $T = [R] \neq 0$. On the other hand, by 1.4.5 we know that T is locally null, so by 1.3.15 it follows that T has fountains.

Now note that by 1.4.5 we have that $(T \circ I_A)^\star(\delta_y) = T^\star(\delta_y)|A^*$ is nonatomic or zero for every $y \in \mathbb{N}^*$, so the second part of the corollary follows. □

Proposition 1.4.7. *If $R : \ell_\infty \to \ell_\infty$ is an antimatrix operator, then the operator $[R] : \ell_\infty/c_0 \to \ell_\infty/c_0$ factors through ℓ_∞ and so is locally null.*

Proof. Let $\mu_n = R^\star(\delta_n)$. Since R is antimatrix (Definition 1.2.12) we may consider μ_n as a measure on $\mathbb{N}^*$. Consider $S : \ell_\infty/c_0 \to \ell_\infty$ given by $S([f]_{c_0}) = (\mu_n(\beta f))_{n\in\mathbb{N}^*}$ for every $f \in \ell_\infty$, and $Q : \ell_\infty \to \ell_\infty/c_0$ the quotient map. S is well-defined since the measures μ_n are null on $\mathbb{N}$. We have for every $f \in \ell_\infty$ that

$$(Q \circ S)([f]_{c_0}) = [(\mu_n(\beta f))_{n\in\mathbb{N}}]_{c_0} = [R(f)]_{c_0} = [R]([f]_{c_0}),$$

so $(Q \circ S)$ is $[R]$. To conclude that $[R]$ is locally null use 1.3.2. □

Theorem 1.4.8. *If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a matrix operator which is an isomorphic embedding, then it is right-locally trivial.*

Proof. Let $R : \ell_\infty \to \ell_\infty$ be given by a c_0-matrix such that $[R] = T$. Let $(b_{ij})_{i,j\in\mathbb{N}}$ be the matrix corresponding to R. Let $M > 0$ be such that $\|T([f]_{c_0})\| \geq M\|[f]_{c_0}\|$, for every $f \in \ell_\infty \setminus c_0$. Notice that this condition is equivalent to the statement that $\limsup_{n\to\infty} |R(f)(n)| \geq M$, for every $f \in \ell_\infty$ such that $\limsup_{n\to\infty} |f(n)| = 1$.

Fix an infinite $\tilde{A} \subseteq \mathbb{N}$.

CLAIM: $\lim_{i\to\infty} \max\{|b_{ij}| : j \in \tilde{A}\} \neq 0$.

Assume otherwise. We will construct an $f \in \ell_\infty$ such that $\limsup_{n\to\infty} |f(n)| = 1$ and $\limsup_{n\to\infty} |R(f)(n)| < M$.

Let $m_i = \min\{k \in \mathbb{N} : \sum_{j\geq k} |b_{ij}| < 1/(i+1)\}$, for every $i \in \mathbb{N}$. We shall construct by induction two strictly increasing sequences of integers $(i_n)_{n\in\mathbb{N}}$ and $(j_n)_{n\in\mathbb{N}}$, with $j_n \in \tilde{A}$ for every $n \in \mathbb{N}$. Let $i_0 = 0$ and $j_0 = \min \tilde{A}$. If we have constructed i_l, j_l, for $l \leq n$, take $i_{n+1} > i_n$ such that $\max\{|b_{ij}| : j \in \tilde{A}\} < \frac{1}{(n+2)^2}$, for every $i \geq i_{n+1}$; take $j_{n+1} \in \tilde{A}$ such that $j_{n+1} > \max\{m_l : l < i_{n+1}\}$ and $j_{n+1} > j_n$.

Now let f be the characteristic function of $\{j_n : n \in \mathbb{N}\}$ and let $N \in \mathbb{N}$ be such that $\frac{N}{(N+1)^2} < M/4$ and $1/N < M/4$. Fix $k \geq i_N$. Then, $k \geq N$ and also, $i_n \leq k < i_{n+1}$, for some $n \geq N$. Consider the following:

$$\begin{aligned}
|R(f)(k)| = |\textstyle\sum_{j\in\mathbb{N}} b_{kj} f(j)| &\leq \textstyle\sum_{j<m_k} |b_{kj} f(j)| + \sum_{j\geq m_k} |b_{kj}| \\
&\leq \textstyle\sum_{j<m_k} |b_{kj} f(j)| + 1/k \\
&\leq \textstyle\sum_{l<n} |b_{kj_l}| + 1/N \qquad \text{(because } j_{n+1} > m_k) \\
&\leq n \cdot \max\{|b_{kj}| : j \in \tilde{A}\} + 1/N \\
&\leq \tfrac{n}{(n+1)^2} + 1/N \qquad \text{(because } k \geq i_n) \\
&< M/2
\end{aligned}$$

This contradicts the definition of M, and so the claim is proved.

Let $\delta > 0$ and let $B_0 \subseteq \mathbb{N}$ be infinite such that $\max\{|b_{ij}| : j \in \tilde{A}\} > \delta$, for every $i \in B_0$.

We shall construct by induction three strictly increasing sequences of integers, $(i_n)_{n\in\mathbb{N}}$, $(j_n)_{n\in\mathbb{N}}$, $(k_n)_{n\in\mathbb{N}}$, satisfying the following for every $n \in \mathbb{N}$:

1. $|b_{i_n j_n}| > \delta$
2. $j_n \in \tilde{A}$

3. $k_n \leq j_n < k_{n+1}$
4. $\sum_{j<k_n} |b_{i_n j}| < \frac{1}{2(n+1)}$
5. $\sum_{j\geq k_{n+1}} |b_{i_n j}| < \frac{1}{2(n+1)}$

Let $k_0 = 0$ and $i_0 = \min(B_0)$. Let $j_0 \in \tilde{A}$ be such that $|b_{i_0 j_0}| > \delta$. Let $k_1 > j_0$ be such that $\sum_{j\geq k_1} |b_{i_0 j}| < 1$.

Assume we have constructed i_l, j_l and k_{l+1}, satisfying 1–5 for every $l \leq n$. Let N be such that $\sum_{j<k_{n+1}} |b_{ij}| < \min\{\delta, \frac{1}{2(n+2)}\}$, for every $i \geq N$ (it exists because it is a c_0-matrix). Let $i_{n+1} \in B_0 \setminus N$. Let $j_{n+1} \in \tilde{A}$ be such that $|b_{i_{n+1}j_{n+1}}| > \delta$ (it exists because $i_{n+1} \in B_0$). Notice that $j_{n+1} \geq k_{n+1}$ because $|b_{i_{n+1}j}| < \delta$, for every $j < k_{n+1}$. Let $k_{n+2} > j_{n+1}$ be such that $\sum_{j\geq k_{n+2}} |b_{i_{n+1}j}| < \frac{1}{2(n+2)}$. This ends the inductive construction.

Now, $\delta < |b_{i_n j_n}| \leq \sup\{|b_{ij}| : i, j \in \mathbb{N}\}$, for every $n \in \mathbb{N}$. Therefore, by going to a subsequence we may assume that $b_{i_n j_n}$ converges to some r with $|r| \geq \delta$. Let $A = \{j_n : n \in \mathbb{N}\}$ and $B = \{i_n : n \in \mathbb{N}\}$. Let $\sigma : B \to A$ be given by $\sigma(i_n) = j_n$, for each $n \in \mathbb{N}$.

CLAIM: $(P_B \circ T \circ I_A)([f]_{c_0(A)}) = [rf \circ \sigma]_{c_0(B)}$, for every $f \in \ell_\infty(A)$.

Note that what we need to show is that $\lim_{n\to\infty} |R(f)(i_n) - rf(\sigma(i_n))| = 0$, for every $f \in \ell_\infty(A)$. So fix $f \in \ell_\infty(A)$ and fix an arbitrary $\varepsilon > 0$. Let M' be such that $\|T^\star(\delta_n)\| \leq M'$, for every $n \in \mathbb{N}$ (it exists by definition of c_0-matrix). Let N_0 be such that $|b_{i_n j_n} - r| < \frac{\varepsilon}{3\|f\|}$, for all $n \geq N_0$, and let N_1 be such that $1/(N_1+1) < \frac{\varepsilon}{3\|f\|}$. Then, for every $n \geq N_0 + N_1$ we have

$$\begin{aligned}
|R(f)(i_n) - rf(\sigma(i_n))| &= |\textstyle\sum_{j\in\mathbb{N}} b_{i_n j} f(j) - rf(j_n)| \\
&\leq \textstyle\sum_{j<k_n} |b_{i_n j} f(j)| + \sum_{\substack{k_n \leq j < k_{n+1} \\ j \neq j_n}} |b_{i_n j} f(j)| \\
&\qquad + \textstyle\sum_{j\geq k_{n+1}} |b_{i_n j} f(j)| + |b_{i_n j_n} f(j_n) - rf(j_n)| \\
&< \frac{\|f\|}{2(n+1)} + 0 + \frac{\|f\|}{2(n+1)} + \|f\| |b_{i_n j_n} - r| \\
&< \frac{\|f\|}{N_1+1} + \|f\| \frac{\varepsilon}{3\|f\|} \\
&< \varepsilon/3 + \varepsilon/3 < \varepsilon.
\end{aligned}$$

This concludes the proof.

□

Corollary 1.4.9. *Every liftable isomorphic embedding $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is right-locally trivial.*

Proof. Since T is liftable, there exist $R_0, R_1 : \ell_\infty \to \ell_\infty$ an antimatrix operator and one given by a c_0-matrix, respectively, such that $T = [R_0 + R_1] = [R_0] + [R_1]$. Fix an infinite $A \subseteq \mathbb{N}$. By 1.4.7, take an infinite $A_0 \subseteq A$ such that $T \circ I_{A_0} = [R_1] \circ I_{A_0}$. Then, $[R_1] \circ I_{A_0}$ is a matrix operator which is an isomorphic embedding, so by 1.4.8 there exist infinite $A_1 \subseteq A_0$ and $B \subseteq \mathbb{N}$ such that $P_B \circ T \circ I_{A_1} = P_B \circ [R_1] \circ I_{A_1}$ is trivial. □

Corollary 1.4.10. *Every liftable isomorphic embedding $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is right-locally an isomorphic matrix operator.*

Proof. By 1.4.9 it suffices to recall that a trivial operator is an isomorphic matrix operator. □

Corollary 1.4.11. *Let $\mathbb{P}$ be one of the following properties: isomorphically liftable, isomorphically matrix, trivial, canonizable along ψ. Suppose that $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ is locally null. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is right-locally $\mathbb{P}$ (left-locally $\mathbb{P}$, somewhere $\mathbb{P}$), then $S + T$ is right-locally $\mathbb{P}$ (left-locally $\mathbb{P}$, somewhere $\mathbb{P}$).*

Proof. First we will note that if the localization $T_{B,A}$ of T to (A, B) has $\mathbb{P}$, then for every infinite $A' \subseteq A$ there is an infinite $B' \subseteq B$ such that the localization $T_{B',A'}$ of T to (A', B') has $\mathbb{P}$.

In the case where $T_{B,A}$ is isomorphically liftable, by 1.4.9 it is enough to notice that a trivial operator is isomorphically liftable. Similarly, if $T_{B,A}$ is isomorphically matrix, by 1.4.8 it is enough to notice that a trivial operator is isomorphically matrix.

If $T_{B,A}$ is trivial, it is enough to take $B' = \sigma^{-1}[A']$, where $\sigma : B \to A$ is the bijection witnessing the triviality of $T_{B,A}$. Similarly, if $T_{B,A}$ is canonizable along ψ, we take $B' \subseteq B$ such that $(B')^* = \psi^{-1}[(A')^*]$.

Now, given a localization $T_{B,A}$ with property $\mathbb{P}$, take an infinite $A' \subseteq A$ such that $S \circ I_{A'} = 0$. By the above, there exists $B' \subseteq B$ such that $T_{B',A'} = (S + T)_{B',A'}$ has $\mathbb{P}$. □

If we do not assume that the operator is bounded below, then there is no hope of obtaining local trivialization anywhere:

Proposition 1.4.12. *There is a surjective operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ which is globally liftable but is nowhere a nonzero matrix operator.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a discrete sequence of nonprincipal ultrafilters and consider the typical antimatrix operator $R : \ell_\infty \to \ell_\infty$ given by $R(f) = ((\beta f)(x_n))_{n\in\mathbb{N}}$. Let $T = [R]$. By 1.3.3 we know that T is surjective. Suppose for some infinite $A, B \subseteq \mathbb{N}$ there is $S : \ell_\infty(A) \to \ell_\infty(B)$ given by a c_0-matrix and such that $[S] = T_{B,A}$. Let us denote by $R_{B,A}$ the operator which maps $f \in \ell_\infty(A)$ into $R(f \cup 0_{\mathbb{N}\setminus A})|B$. By 1.4.3 we have that $S - R_{B,A}$ is weakly compact, and since R is an antimatrix operator we have that $R|c_0 = 0$, so $S|c_0(A)$ is weakly compact. Therefore, by 1.2.7 and 1.2.5 we have that the image of S is included in $c_0(B)$ and so $T_{B,A} = [S] = 0$.

□

1.4.3 Lifting operators on ℓ_∞/c_0

In the case of the Boolean algebra $\wp(\mathbb{N})/$*Fin*, any endomorphism which can be lifted to a homomorphism of $\wp(\mathbb{N})$ is induced by a homomorphism of *FinCofin*$(\mathbb{N})$. However, in the case of ℓ_∞/c_0, like for ℓ_∞ (1.2.16), there exist automorphisms which are not determined by its values on c_0:

Proposition 1.4.13. *There are liftable operators such that all their liftings are discontinuous and are not induced by its action on c_0, i.e., are not matrix operators. Moreover, such operators can be automorphisms of ℓ_∞/c_0.*

Proof. Let $(A_i)_{i\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets. For each $i \in \mathbb{N}$, let x_i be any nonprincipal ultrafilter such that $A_i \in x_i$. For a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ consider the automorphism $R_\sigma : \ell_\infty \to \ell_\infty$ from the proof of 1.2.16 which is given by

$$R_\sigma(f)(n) = f(n) - \beta f(x_i) + \beta f(x_{\sigma(i)}),$$

where $i \in \mathbb{N}$ is such that $n \in A_i$. Recall that $R_\sigma \circ R_{\sigma^{-1}} = Id_{\ell_\infty}$, so by 1.4.2 we have that $[R_\sigma] \circ [R_{\sigma^{-1}}] = [Id_{\ell_\infty}] = Id_{\ell_\infty/c_0}$. It follows that the operators $[R_\sigma]$ are automorphisms of ℓ_∞/c_0.

Now suppose that $S : \ell_\infty \to \ell_\infty$ is a continuous lifting of $[R_\sigma]$. By 1.2.15 the operator S is given by a c_0-matrix, and by 1.4.3 we have that $S - R_\sigma$ is a weakly compact operator into c_0. Note that $R_\sigma|c_0 = Id_{c_0}$, therefore $S|c_0 = Id_{c_0} + W$, where $W : c_0 \to c_0$ is the restriction of $S - R_\sigma$ to c_0 and so is weakly compact. By 1.2.5 we have

$$S = (S|c_0)^{\star\star} = Id_{c_0}^{\star\star} + W^{\star\star} = Id_{\ell_\infty} + W^{\star\star},$$

and so $S^\star = Id_{M(\beta\mathbb{N}^*)} + U$, where U is weakly compact by the Gantmacher theorem. Therefore, $Id_{M(\beta\mathbb{N}^*)} + U - R_\sigma^\star$ is a weakly compact operator, and so is $Id_{M(\beta\mathbb{N}^*)} - R_\sigma^\star$. We will show that this is impossible by showing that the bounded sequence of measures $(\delta_{x_i})_{i\in\mathbb{N}}$ is not mapped onto a relatively weakly compact set.

A simple calculation gives that $R_\sigma^\star(\delta_x) = \delta_x - \delta_{x_i} + \delta_{x_{\sigma(i)}}$, if $x \in A_i^*$. It follows that $R_\sigma^\star(\delta_{x_i}) = \delta_{x_{\sigma(i)}}$, for every $i \in \mathbb{N}$. So $(Id_{M(\beta\mathbb{N}^*)} - R_\sigma^\star)(\delta_{x_i}) = \delta_{x_i} - \delta_{x_{\sigma(i)}}$, which by the Dieudonne-Grothendieck theorem implies that $Id_{M(\beta\mathbb{N}^*)} - R_\sigma^\star$ is not weakly compact unless σ moves only finitely many $i \in \mathbb{N}$, as the sequence $(x_i)_{i\in\mathbb{N}}$ is discrete.

□

Unlike in the case of the algebra $\wp(\mathbb{N})/$*Fin*, nonliftable automorphisms of ℓ_∞/c_0 exist in ZFC. Before proving this we need one:

Lemma 1.4.14. *Suppose $R : \ell_\infty \to \ell_\infty$ preserves c_0. If R is not weakly compact, then $[R]$ is not weakly compact either.*

Proof. If R is not weakly compact, then there is an infinite $A \subseteq \mathbb{N}$ such that R restricted to $\ell_\infty^0(A) = \{f \in \ell_\infty : f|(\mathbb{N} \setminus A) = 0\}$ is an isomorphism onto its range (see Prop. 1.2. from [49] and Corollary VI–17 of [13]). Consider $X = R^{-1}[c_0]$, a closed subspace of ℓ_∞ containing c_0. Note that $X \cap \ell_\infty^0(A)$ is separable as $R[X \cap \ell_\infty^0(A)] \subseteq c_0$ and R is an isomorphism on $\ell_\infty^0(A)$. By the standard argument using the Stone-Weierstrass theorem with respect to simple functions one can find a countable Boolean algebra $\mathfrak{B}$ of subsets of A such that $X \cap \ell_\infty^0(A)$ is included in the closure of the span of $\{\chi_B : B \in \mathfrak{B}\}$.

Let $(D_\xi)_{\xi<\omega_1}$ be a family of pairwise almost disjoint infinite subsets of A. For each $\xi < \omega_1$ take $x \in D_\xi^*$ and let E_ξ be infinite such that $E_\xi^* \subseteq \bigcap\{B^* : B \in \mathfrak{B} \cap x\} \cap \bigcap\{\mathbb{N}^* \setminus B^* : B \in \mathfrak{B} \setminus x\}$ (it exists by A.1.1 and because $\mathfrak{B}$ is countable). Now take $u_\xi, v_\xi \in E_\xi^*$ distinct. It follows that no element of $\mathfrak{B}$ separates any of the pairs (u_ξ, v_ξ). Therefore, $\beta f(u_n) = \beta f(v_n)$ for every $f \in X \cap \ell_\infty^0(A)$.

For every $\xi < \omega_1$ choose $g_\xi \in \ell_\infty^0(A)$ with support in D_ξ such that $\|g_\xi\| = 1$, $g_\xi(u_\xi) = 1$ and $g_\xi(v_\xi) = -1$. Notice that $R(g_\xi) \notin c_0$, for every $\xi < \omega$. This implies that $\|[R(g_\xi)]_{c_0}\| > 0$ for every $\xi < \omega_1$, so there exists $n \in \mathbb{N}$ such that for infinitely many $\xi < \omega_1$ we have $\|[R(g_\xi)]_{c_0}\| > 1/n$. Since the g_ξ^* are pairwise disjoint, $[R]$ is not weakly compact by Corollary VI–17 of [13]. □

Proposition 1.4.15. *Every weakly compact operator on ℓ_∞/c_0 with nonseparable range is nonliftable. Such operators exist.*

Proof. Suppose that $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ is weakly compact with nonseparable range and $R : \ell_\infty \to \ell_\infty$ is such that $[R] = S$ and R preserves c_0. R must be weakly compact itself by 1.4.14. In particular, the image of the unit ball under R is weakly compact. Since weakly compact subsets of ℓ_∞ are norm separable (Corollary. 4.6 of [48]), we have that the image of R is separable. But this implies that the image of $[R] = S$ is separable as well, contradicting the hypothesis.

Now we construct a weakly compact operator $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ with nonseparable range which is weakly compact. The construction is based on the fact that ℓ_∞/c_0 contains an isometric copy of $\ell_2(2^\omega)$. This follows from the result of Avilés in [6] which states that the unit ball in $\ell_2(2^\omega)$ with the weak topology (equivalently weak* topology) is a continuous image of $A(2^\omega)^{\mathbb{N}}$, where $A(2^\omega)$ is the one point compactification of the discrete space of size 2^ω. On the other hand, by Theorem 2.5 and Example 5.3. in [8] we have that $A(2^\omega)^{\mathbb{N}}$ is a continuous image of $\mathbb{N}^*$. Hence $C(B_{\ell_2(2^\omega)})$ embeds isometrically into $C(\mathbb{N}^*)$ and so does $\ell_2(2^\omega)$. So let $S_1 : \ell_2(2^\omega) \to \ell_\infty/c_0$ be an isomorphism onto its range.

To complete the construction, it is enough to take a surjective operator $S_2 : \ell_\infty/c_0 \to \ell_2(2^\omega)$ and consider $S = S_1 \circ S_2$. This is because any operator into a reflexive Banach space is weakly compact (Corollary VI.4.3 of [23]) and weakly compact operators form a two sided ideal (Theorem VI.4.5 of [23]).

The existence such of a surjective operator follows from the complementation of ℓ_∞ in ℓ_∞/c_0 and the existence of a surjective operator $T : \ell_\infty \to \ell_2(2^\omega)$ which was proved in [47] Proposition 3.4. and remark 2 below it. It is based on a construction of an isomorphic copy of $\ell_2(2^\omega)$ inside $\ell_\infty^\star$ (proposition 3.4 of [47]). Once we have an isomorphic embedding $T : \ell_2(2^\omega) \to \ell_\infty^\star$ we consider

$$T^\star \circ J : \ell_\infty \to \ell_\infty^{\star\star} \to \ell_2(2^\omega)^\star,$$

where $J : \ell_\infty \to \ell_\infty^{\star\star}$ is the canonical embedding. We have that $(T^\star \circ J)^\star = J^\star \circ T^{\star\star} = T$ using the reflexivity of $\ell_2(2^\omega)$ to identify it with $\ell_2(2^\omega)^{\star\star}$. But T is one-to-one with closed range, so $T^\star \circ J$ must be onto as required.

□

Theorem 1.4.16. *There is an automorphism $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ which cannot be lifted to a linear operator on ℓ_∞.*

Proof. Consider $T_1 = Id + S$ where S is any weakly compact operator on ℓ_∞/c_0 from the previous proposition. Since S is strictly singular, T_1 is a Fredholm operator of Fredholm index 0 (see Proposition 2.c.10 of [37]), i.e., its kernel is finite dimensional of dimension n and its range is of the same finite codimension n. Since finite dimensional subspaces of Banach spaces are complemented we can write

$$T_1 : Ker(T_1) \oplus X \to Range(T_1) \oplus Y$$

where Y is of finite dimension n and X has the same finite codimension n. Let $U : Ker(T_1) \to Y$ be an isomorphism and define $T : Ker(T_1) \oplus X \to Range(T_1) \oplus Y$

by $T(z,x) = (T_1(x), U(z))$. It follows that T satisfies

$$T = T_1 + U = Id + S + U$$

Having null kernel and being surjective it is an automorphism of ℓ_∞/c_0. Now let us show that T cannot be lifted to an operator on ℓ_∞/c_0. $S+U$ is weakly compact with nonseparable range as a sum of an operator with this property and a finite rank operator, so it cannot be lifted by the previous proposition. Since the sum of two liftable operators is liftable and *Id* is liftable, it follows that T is an automorphism which cannot be lifted. □

Proposition 1.4.17. *If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is canonizable along a homeomorphism $\psi : \mathbb{N}^* \to \mathbb{N}^*$ and ψ is a nontrivial homeomorphism (i.e., it is not induced by a bijection of two coinfinite subsets of $\mathbb{N}$), then T is not liftable.*

Proof. We may assume that $\hat{T} = T_\psi$, that is, the constant r of Definition 0.0.5 is 1. Suppose $R : \ell_\infty \to \ell_\infty$ is a lifting of T. Let $R = R_1 + R_2$ (see 1.2.14), where R_1 is an operator given by a c_0-matrix and R_2 is an antimatrix operator.

CLAIM 1: There cannot exist disjoint functions $f,\, g \in \ell_\infty$ (i.e., such that $f{\cdot}g = 0$) and $\varepsilon > 0$ such that $\int \beta f \,\mathrm{d}R_1^\star(\delta_i),\ \int \beta g \,\mathrm{d}R_1^\star(\delta_i) > \varepsilon$ for infinitely many $i \in \mathbb{N}$.

Indeed, in such a case, we would find an infinite $\tilde{B} \subseteq \mathbb{N}$ such that for every $y \in \tilde{B}^*$, we would have $\int \beta f \,\mathrm{d}(R_1^\star(\delta_y)),\ \int \beta g \,\mathrm{d}R_1^\star(\delta_y) \geq \varepsilon$. Since $T = T_\psi$, for all $y \in \mathbb{N}^*$ we have that $T^\star(\delta_y) = \delta_{\psi(y)}$, which implies that either $\int f^* \,\mathrm{d}T^\star(\delta_y) = 0$ or $\int g^* \,\mathrm{d}T_1^\star(\delta_y) = 0$. Therefore, we will obtain a contradiction if we can free $T^\star$ from the influence of $R_2^\star$ somewhere. By 1.2.7 and using an argument as in the proof of A.2.1, we find an infinite $B \subseteq \tilde{B}$ and pairwise disjoint finite $F_i \subseteq \mathbb{N}$ such that $|R_1^\star(\delta_i)|(\beta\mathbb{N} \setminus F_i) < \varepsilon/3\max\{\|f\|, \|g\|\}$ for $i \in B$. This implies that $\int_{F_i} \beta f \,\mathrm{d}R_1^\star(\delta_i)$, $\int_{F_i} \beta g \,\mathrm{d}R_1^\star(\delta_i) \geq 2\varepsilon/3$. Consider an uncountable almost disjoint family $\{B_\xi : \xi < \omega_1\}$ of subsets of B and sets $A_\xi = \bigcup\{F_i : i \in B_\xi\}$, for $\xi < \omega_1$. We have that $\int_{\beta A_\xi} \beta f \,\mathrm{d}R_1^\star(\delta_i)$, $\int_{\beta A_\xi} \beta g \,\mathrm{d}R_1^\star(\delta_i) \geq \varepsilon/3$ for each $i \in B_\xi$, since the measures are concentrated on $\mathbb{N}$ (see 1.2.13). Now, the sets A_ξ^* are pairwise disjoint and the measures $R_2^\star(\delta_i)$ are concentrated on $\mathbb{N}^*$, so there is a $\xi_0 < \omega_1$ such that $|R_2^\star(\delta_i)|(\beta A_{\xi_0}) = 0$ for all $i \in \mathbb{N}$. So by 1.4.4 and by the weak* continuity of $R_1^\star$, for every $y \in B_{\xi_0}^*$ we have

$$\int (f^*|A_{\xi_0}^*) \,\mathrm{d}T^\star(\delta_y) = \int_{\mathbb{N}^*} (\beta f|\beta A_{\xi_0}) \,\mathrm{d}(R_1^\star(\delta_y) + R_2^\star(\delta_y)) \geq \varepsilon/3 + 0 = \varepsilon/3.$$

A similar calculation works for $\beta g|\beta A_{\xi_0}$ which gives the desired contradiction since the restrictions of disjoint functions are disjoint. So the claim is proved.

Let $(b_{ij})_{i,j\in\mathbb{N}}$ be the matrix of R_1, i.e., $R_1^\star(\delta_i) = \sum_{j\in\mathbb{N}} b_{ij}\delta_j$ for all $i \in \mathbb{N}$. Let $j_i \in \mathbb{N}$ be such that b_{ij_i} has the biggest absolute value among the numbers $\{b_{ij} : j \in \mathbb{N}\}$ for all $i \in \mathbb{N}$.

CLAIM 2: The b_{ij_i}'s are separated from 0.

Assume otherwise. Then, we can find an infinite $B \subseteq \mathbb{N}$ such that for $i \in B$ the numbers b_{ij_i}'s converge to 0. If the sequence $(\|b_i\|_{\ell_1})_{i\in B}$ is not separated from zero, then by 1.2.7 there would be an infinite $B' \subseteq B$ such that the map $f \mapsto R_1(f)|B'$ is weakly compact and so $P_{B'} \circ [R_1] = 0$ by 1.2.5 and 1.2.7. This would then imply by 1.4.7 that $P_{B'} \circ T$ is locally null, which is impossible since $P_{B'} \circ T_\psi$ is an automorphism on $\psi[(B')^*]$-supported functions. Therefore, the sequence $(\|b_i\|_{\ell_1})_{i\in B}$ is separated from zero.

By A.2.1, there exist $\delta > 0$, an infinite $B_0 \subseteq B$ and finite $F_i \subseteq \mathbb{N}$ for $i \in B$ which are pairwise disjoint and such that $\sum_{j\in F_i} |b_{ij}| > \delta$ for all $i \in B_0$. Since b_{ij_i}'s converge to 0, one can partition each F_i into H_i and G_i such that $\sum_{j\in G_i} |b_{ij}| > \delta/4$ and $\sum_{j\in H_i} |b_{ij}| > \delta/4$ for sufficiently large $i \in B$ (construct G_i considering initial fragments $G_i(k)$ of F_i, for $k \leq |F_i|$ starting with $G_i(0) = \emptyset$ and increasing the previous fragment by one element. Since the jumps between $\sum_{j\in G_i(k)} |b_{ij}|$ and $\sum_{j\in G_i(k+1)} |b_{ij}|$ can be at most $|b_{ij_i}|$, which is eventually less than $\delta/4$, we can obtain the required G_i and $H_i = F_i \setminus G_i$ at some stage $k \leq |F_i|$). But then we can define two disjoint functions, f with support $\bigcup_{i\in B} G_i$ and g with support $\bigcup_{i\in B} H_i$, which contradict claim 1. Therefore, the claim is proved.

Now consider the matrix $(c_{ij})_{i,j\in\mathbb{N}}$ such that $c_{ij_i} = b_{ij_i}$, for $i \in \mathbb{N}$, and all other entries are zero. Write $R_1 = R_3 + R_4$ where R_3 is induced by $(c_{ij})_{i,j\in\mathbb{N}}$ and $R_4 = R_1 - R_3$. If R_4 were not weakly compact, we would have that the norms of its rows do not converge to zero (1.2.7). Then, using A.2.1 and an argument analogous to that of claim 2 we can construct disjoint functions which contradict claim 1. Thus $[R_4]$ must be zero by 1.2.5 and 1.2.7 and so $[R_1] = [R_3]$. Therefore, we may assume that R_1 is given by a matrix of a function from $\mathbb{N}$ into $\mathbb{N}$, that is, all entries of the matrix are equal to zero except for the b_{ij_i}'s, which are separated from zero by some $\delta > 0$.

CLAIM 3: There are cofinite sets $A, B \subseteq \mathbb{N}$ such that $J : B \to A$ given by $J(i) = j_i$ is a bijection.

If $\{j_i : i \in \mathbb{N}\}$ is coinfinite, say disjoint from an infinite $A \subseteq \mathbb{N}$, then $[R_1] \circ I_A = 0$ which together with the fact that $[R_2]$ is locally null (see 1.4.7) leads to a contradiction with the fact that T is an automorphism. Of course J cannot send infinite sets into one value, because it would give rise to a column of the matrix of R_1 which would not be in c_0, as the entries of the matrix are separated from 0, contradicting 1.2.1. If there are infinitely many values of J which are assumed on distinct integers, then there are two disjoint infinite sets $B_1 = \{i^1_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ and $B_2 = \{i^2_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ such that $j_{i^1_n} = j_{i^2_n}$. Define $J'(n) = j_{i^1_n} = j_{i^2_n}$ and put $f(i^2_n) = b_{i^1_n J'(n)}/b_{i^2_n J'(n)}$ and otherwise put the value of f to be 0. Note that whenever $A' \subseteq \{J'(n) : n \in \mathbb{N}\} = A$, we have that

$$\delta_{i^1_n}(R_1(\chi_{A'})) - f(i^2_n)\delta_{i^2_n}(R_1(\chi_{A'})) = 0,$$

for all $n \in \mathbb{N}$. Let $\eta : B_1^* \to B_2^*$ be the extension of the bijection from B_1 to B_2 sending i^1_n to i^2_n for all $n \in \mathbb{N}$. It follows that for every A-supported g and every $x \in B_1^*$ we have

$$(\delta_x - (\beta f)(\eta(x))\delta_{\eta(x)})([R_1]([g])) = 0.$$

Find $B_1' \subseteq B_1$ such that $\psi[(B_1')^* \cup \eta[(B_1')^*]] = (A')^*$ for some infinite $A' \subseteq A$ such that $[R_2] \circ I_{A'} = 0$ (by 1.4.7). This can be achieved since ψ is a homeomorphism. Considering the values $T_\psi(g^*)$ for A'-supported g's we obtain all functions supported by $B_1' \cup \eta[B_1']$. However, $[R_2]$ on such g's is zero, so we obtain a contradiction since the values of $[R_1]$ on such functions have the above restrictions. The claim is proved.

Thus R_1^* is T_ϕ where $\phi : \mathbb{N}^* \to \mathbb{N}^*$ is a trivial homeomorphism of $\mathbb{N}^*$. Therefore, there is $x \in \mathbb{N}^*$ such that $\psi^{-1}(x) \neq \phi^{-1}(x)$. It follows that there are infinite $B_1, B_2, A \subseteq \mathbb{N}$ such that $A^* = \phi[B_1^*]$, $A^* = \psi[B_2^*]$ and $B_1 \cap B_2 = \emptyset$. Using 1.4.7 take an infinite $A' \subseteq A$ such that $[R_2] \circ I_{A'} = 0$. Then, $T_\phi([\chi_{A'}])$ and $T_\psi([\chi_{A'}])$ have disjoint supports, so we cannot have $[R_1 + R_2] = T$, which completes the proof.

□

1.5 Canonizing operators acting along a quasi-open mapping

In [22] it was proved that for a linear bounded operator T on ℓ_∞/c_0 and an infinite $A \subseteq \mathbb{N}$ there is a real $r \in \mathbb{R}$ and an infinite $B \subseteq A$ such that

$$T(f)|B^* = rf$$

for every B-supported f. This gives, for example, that if P_1 and P_2 are complementary projections on ℓ_∞/c_0, then at least one of them canonizes as above for a nonzero r, in other words we obtain a local canonization along the identity on B^*. However, a big disadvantage of this result is that in general we cannot guarantee that the constant r is nonzero. If one works with an automorphism, this kind of result is of no use. For example, consider an infinite and coinfinite set $D \subseteq \mathbb{N}$ and the bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma[D] = \mathbb{N} \setminus D$, $\sigma[\mathbb{N} \setminus D] = D$ and σ^2 is the identity. Define an automorphism T of ℓ_∞/c_0 by $T([f]_{c_0}) = [f \circ \sigma]_{c_0}$. The above result gives an infinite $B \subseteq D$ such that $T([f]_{c_0})|B = 0[f]_{c_0}$ for every B-supported $f \in \ell_\infty$, which looses much of the information. So in this section we embark on finding a surjective $\psi : B^* \to A^*$ along which T may canonize with r nonzero as required in Definition 0.0.5. Note that a potential obstacle for finding such a canonization would be if $\bigcup \varphi^T[B^*]$ were nowhere dense. Actually, we have examples such that $\bigcup \varphi^T[\mathbb{N}^*]$ is nowhere dense and T is surjective (1.3.3, 1.3.12). So it is natural to assume that the surjections we consider are fountainless and that embeddings are funnelless. Under these assumptions we obtain a quasi-open ψ such that $T^\star(\delta_x)(\{\psi(x)\}) \neq 0$ holds locally, which is sufficient for the canonization by the following:

Theorem 1.5.1. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator and let $\tilde{A} \subseteq \mathbb{N}$ be infinite. If $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a quasi-open continuous function such that $\tilde{A}^* \subseteq \psi[\mathbb{N}^*]$, then there exist $r \in \mathbb{R}$ and clopen sets $A^* \subseteq \tilde{A}^*$ and $B^* = \psi^{-1}[A^*]$ such that $T_{B,A}(f^*) = r(f^* \circ \psi)|B^*$, for every $f \in \ell_\infty(A)$.*

Proof. Fix $\tilde{A}$ and ψ as above.

CLAIM: There exists an infinite $A \subseteq \tilde{A}$ and a clopen $E^* \subseteq \psi^{-1}[A^*]$ such that for every $y \in E^*$ there exists $r_y \in \mathbb{R}$ satisfying

$$T^\star(\delta_y)|A^* = r_y \delta_{\psi(y)}.$$

Suppose this does not hold. We will construct recursively sequences $(A_\xi)_{\xi<\omega_1}$, $(D_\xi)_{\xi<\omega_1}$ and $(E_\xi)_{\xi<\omega_1}$ of infinite subsets of $\mathbb{N}$, and a sequence $(a_\xi)_{\xi<\omega_1}$ of nonzero reals such that

1. $A_\eta \subseteq_* A_\xi \subseteq_* \tilde{A}$ and $D_\xi \subseteq_* A_\xi \setminus A_{\xi+1}$, for every $\xi < \eta < \omega_1$;

2. $E_\eta \subseteq_* E_\xi$, for every $\xi < \eta < \omega_1$;
3. $E^*_\xi \subseteq \psi^{-1}[A^*_\xi]$, for all $\xi < \omega_1$;
4. either $T^\star(\delta_y)(D^*_\xi) > a_\xi > 0$ for all $y \in E^*_{\xi+1}$, or $T^\star(\delta_y)(D^*_\xi) < a_\xi < 0$ for all $y \in E^*_{\xi+1}$.

Let $A_0 = \tilde{A}$ and $E^*_0 = \psi^{-1}[A^*_0]$. Let $\eta < \omega_1$ and suppose we have constructed A_ξ, D_ξ, E_ξ and a_ξ satisfying (1)–(4) for every $\xi < \eta$. If η is a limit ordinal, take an infinite E such that $E \subseteq_* E_\xi$ for every $\xi < \eta$. By hypothesis there exists a clopen $A^*_\eta \subseteq \psi[E^*]$. Put $E^*_\eta = \psi^{-1}[A^*_\eta] \cap E^*$. Now we may suppose we have A_ξ and E_ξ for every $\xi \leq \eta$, and D_ξ and a_ξ for every $\xi < \eta$.

Take an infinite A'_η such that $(A'_\eta)^* \subseteq \psi[E^*_\eta]$. By our assumption, there exist $y \in \psi^{-1}[(A'_\eta)^*] \cap E^*_\eta$ and $X \subseteq (A'_\eta)^* \setminus \{\psi(y)\}$ such that $T^\star(\delta_y)(X) \neq 0$. By the regularity of $T^\star(\delta_y)$, there exists an infinite $D_\eta \subseteq_* A'_\eta$ such that $\psi(y) \notin D^*_\eta$ and $T^\star(\delta_y)(D^*_\eta) \neq 0$. Let a_η be such that either $0 < a_\eta < T^\star(\delta_y)(D^*_\eta)$ or $0 > a_\eta > T^\star(\delta_y)(D^*_\eta)$.

By the weak* continuity of $T^\star$, there exists V a clopen neighbourhood of y such that either $T^\star(\delta_z)(D^*_\eta) > a_\eta$ for all $z \in V$, or $T^\star(\delta_z)(D^*_\eta) < a_\eta$ for all $z \in V$. Finally, choose $A_{\eta+1} = A'_\eta \setminus D_\eta$ and $E^*_{\eta+1} = \psi^{-1}[A^*_{\eta+1}] \cap V \cap E^*_\eta$ (notice that $y \in E^*_{\eta+1}$). This ends the construction.

Since $|a_\xi| > 0$ for every $\xi < \omega_1$, there must exist $n \in \mathbb{N}$ and an infinite $I \subseteq \omega_1$ such that $|a_\xi| > 1/n$, for every $\xi \in I$. Hence, we may choose $\xi_0, \dots, \xi_{k-1} \in I$, for some $k \in \mathbb{N}$, such that $a_{\xi_0}, \dots, a_{\xi_{k-1}}$ are all of the same sign and $|\sum_{i<k} a_{\xi_i}| > \|T^\star\|$. Assume $\xi_0 \geq \xi_i$ for $i < k$. Take $y \in E^*_{\xi_0+1}$. Then, since the $D^*_{\xi_i}$ are pairwise disjoint and since $y \in E^*_{\xi_i+1}$, for every $i < k$, we have

$$|T^\star(\delta_y)(\bigcup_{i<k} D^*_{\xi_i})| = |\sum_{i<k} T^\star(\delta_y)(D^*_{\xi_i})| > |\sum_{i<k} a_{\xi_i}| > \|T^\star\|.$$

This contradiction proves the claim.

Therefore, for every A-supported $f \in \ell_\infty$ and every $y \in E^*$ we have $T(f)(y) = T^\star(\delta_y)(f) = r_y f(\psi(y))$. In particular, $T(\chi_{A^*})(y) = r_y$, for every $y \in E^*$. This means that the function $y \mapsto r_y$ with domain E^* is continuous. Then, by A.1.2 it must be constant on some clopen $B^* \subseteq E^*$. This means that for some $r \in \mathbb{R}$ we have $T_{B,A}(f) = r(f \circ \psi)|B$, for every A-supported $f \in \ell_\infty$. By going to a subset of A we may choose A and B so that $\psi^{-1}[A^*] = B^*$

□

Theorem 1.5.2. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator and let $\tilde{A} \subseteq \mathbb{N}$ be infinite and $F \subseteq \mathbb{N}^*$ be closed. If $\psi : F \to \mathbb{N}^*$ is an irreducible continuous function, then there exist $r \in \mathbb{R}$ and an infinite $A \subseteq \tilde{A}$ such that $T(f^*)|\psi^{-1}[A^*] = r(f^* \circ \psi)|\psi^{-1}[A^*]$, for every A-supported $f \in \ell_\infty$.*

Proof. The proof is similar to that of 1.5.1, so we will skip identical parts. The main difference is that nonempty G_δ's of F do not need to have nonempty interior. However the irreducibility of the map onto $\mathbb{N}^*$ gives through Lemma A.1.5 that appropriate G_δ's have nonempty interior. Fix $\tilde{A}$, F and ψ as above.

CLAIM: There exists an infinite $A \subseteq \tilde{A}$ such that for every $y \in \psi^{-1}[A^*]$ there exists $r_y \in \mathbb{R}$ satisfying

$$T^\star(\delta_y)|A^* = r_y \delta_{\psi(y)}.$$

Suppose this does not hold. We will construct recursively sequences $(A_\xi)_{\xi<\omega_1}$ and $(D_\xi)_{\xi<\omega_1}$ of infinite subsets of $\mathbb{N}$, and a sequence $(a_\xi)_{\xi<\omega_1}$ of nonzero reals such that

1. $A_\eta \subseteq_* A_\xi \subseteq_* \tilde{A}$ and $D_\xi \subseteq_* A_\xi \setminus A_{\xi+1}$, for every $\xi < \eta < \omega_1$;
2. either $T^\star(\delta_y)(D^*_\xi) > a_\xi > 0$ for all $y \in \psi^{-1}[A^*_{\xi+1}]$, or $T^\star(\delta_y)(D^*_\xi) < a_\xi < 0$ for all $y \in \psi^{-1}[A^*_{\xi+1}]$.

Let $A_0 = \tilde{A}$. Let $\eta < \omega_1$ and suppose we have constructed A_ξ, D_ξ and a_ξ satisfying (1)–(2) for every $\xi < \eta$. If η is a limit ordinal, take an infinite A_η such that $A_\eta \subseteq_* A_\xi$ for every $\xi < \eta$. Now we may suppose we have A_ξ for every $\xi \le \eta$, and D_ξ and a_ξ for every $\xi < \eta$.

By our assumption, there exist $y \in \psi^{-1}[A^*_\eta]$ and $X \subseteq A^*_\eta \setminus \{\psi(y)\}$ such that $T^\star(\delta_y)(X) \neq 0$. By the regularity of $T^\star(\delta_y)$, there exists an infinite $D_\eta \subseteq_* A_\eta$ such that $\psi(y) \notin D^*_\eta$ and $T^\star(\delta_y)(D^*_\eta) \neq 0$. Let a_η be such that either $0 < a_\eta < T^\star(\delta_y)(D^*_\eta)$ or $0 > a_\eta > T^\star(\delta_y)(D^*_\eta)$.

By the weak* continuity of $T^\star$, there exists V a clopen neighbourhood of y in F such that either $T^\star(\delta_z)(D^*_\eta) > a_\eta$ for all $z \in V$, or $T^\star(\delta_z)(D^*_\eta) < a_\eta$ for all $z \in V$. V may be assumed to be included in $\psi^{-1}[A^*_\eta \setminus D^*_\eta]$ as $y \in V \cap \psi^{-1}[A^*_\eta \setminus D^*_\eta]$. Using the irreducibility of ψ and Lemma A.1.5 there is an infinite $A_{\eta+1} \subseteq \mathbb{N}$ such that $\psi^{-1}[A^*_{\eta+1}] \subseteq V \subseteq \psi^{-1}[A^*_\eta \setminus D^*_\eta]$. In particular $A_{\eta+1} \subseteq_* A_\eta \setminus D_\eta$ (note that y may not belong to $\psi^{-1}[A^*_{\eta+1}]$). This ends the construction. We finish the proof of the claim as in Theorem 1.5.1.

Therefore, for every A-supported $f \in \ell_\infty$ and every $y \in \psi^{-1}[A^*]$ we have $T(f^*)(y) = T^\star(\delta_y)(f) = r_y f(\psi(y))$. In particular, $T(\chi_{A^*})(y) = r_y$, for every $y \in \psi^{-1}[A^*]$. This means that the function $y \mapsto r_y$ with domain $\psi^{-1}[A^*]$ is continuous. Then, by A.1.7 it must be constant on some clopen set of F of the the form $\psi^{-1}[B^*]$ for an infinite $B \subseteq A$. This means that for some $r \in \mathbb{R}$ we have $T(f) = r(f \circ \psi)|\psi^{-1}[B]$, for every B-supported $f \in \ell_\infty$.

□

1.5.1 Left-local canonization of fountainless operators

Lemma 1.5.3. *Suppose that $B \subseteq \mathbb{N}$ is infinite, $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is fountainless and everywhere present. Then,*

$$F = \bigcup \{\varphi^T(y) : y \in B^*\}$$

has nonempty interior.

Proof. Suppose F is nowhere dense. Take $B_1 \subseteq_* B$ and $C_n, D_n \subseteq \mathbb{N}$ as in 1.3.5. In view of applying 1.3.14, we will find a dense family $\mathcal{A}$ such that $T(f^*)|B^*_1 = 0$ whenever the support of f is included in an element of $\mathcal{A}$. This would contradict the fact that T is everywhere present.

Fix a nonempty clopen $U \subseteq \mathbb{N}^*$ disjoint from F. Notice that for every $y \in B^*_1$ the measure $T^\star(\delta_y)$ has no atoms in U. Therefore, by the regularity of $T^\star(\delta_y)$ and the compactness of U we may find for each $n \in \mathbb{N}$ an open covering $(U_i(y,n))_{i<j(y,n)}$ of U by clopen sets such that $|T^\star(\delta_y)|(U_i(y,n)) < 1/2(n+1)$ holds for all $i < j(y,n)$. We may further assume that either $U_i(y,n) \subseteq C^*_n$ or $U_i(y,n) \subseteq D^*_n$, for each $n \in \mathbb{N}$ and each $i < j(y,n)$.

Now by the weak* continuity of $T^\star$, we may choose for each $n \in \mathbb{N}$ an open neighbourhood $V_n(y)$ of y such that for all $z \in V_n(y)$ we have

$$|T^\star(\delta_z)(U_i(y,n))| < 1/2(n+1) \tag{1.1}$$

for all $i < j(y,n)$.

We have thus constructed for each $n \in \mathbb{N}$ an open covering $\{V_n(y) : y \in B_1^*\}$ of B_1^*. By the compactness of B_1^*, for each $n \in \mathbb{N}$ take $y_0(n), ..., y_{m(n)-1}(n) \in B_1^*$ such that

$$B_1^* \subseteq \bigcup_{l<m(n)} V_n(y_l).$$

Now consider the family $\mathcal{A}_U$ of those sets $E \subseteq \mathbb{N}$ such that given $n \in \mathbb{N}$, for each $l < m(n)$ there is $i < j(y_l, n)$ such that E^* is included in $U_i(y_l, n)$. We claim that if $E \in \mathcal{A}_U$, then for every E^*- supported $f^* \in C(\mathbb{N}^*)$ we have that $T(f^*)|B_1^* = 0$.

Fix $E \in \mathcal{A}_U$ and $y \in B_1^*$. We show that for every $n \in \mathbb{N}$ we have $|T^\star(\delta_y)|(E^*) < 1/(n+1)$. Let $T^\star(\delta_y) = \mu^+ - \mu^-$ be a Jordan decomposition of the measure. By 1.3.5 we have that $\mu^-(C_n^*) < 1/4(n+1)$ and $\mu^+(D_n^*) < 1/4(n+1)$. By construction there exists $l < m(n)$ such that $y \in V_n(y_l)$, and by the definition of $\mathcal{A}_U$, there exists $i < j(y_l, n)$ such that $E^* \subseteq U_i(y_l, n)$. We may assume without loss of generality that $U_i(y_l, n) \subseteq C_n^*$. From this and from (1.1) above we obtain:

$$\begin{array}{rcl} |T^\star(\delta_y)|(E^*) & \leq & |T^\star(\delta_y)|(U_i(y_l,n)) \\ & = & T^\star(\delta_y)(U_i(y_l,n)) + 2\mu^-(U_i(y_l,n)) \\ & \leq & |T^\star(\delta_y)(U_i(y_l,n))| + 2\mu^-(C_n^*) \\ & < & 1/2(n+1) + 2/4(n+1) \end{array}$$

Therefore, $|T^\star(\delta_y)|(E^*) = 0$ for every $y \in B_1^*$. So if the support of $f^* \in C(\mathbb{N}^*)$ is included in E^*, we have that $0 = \int f^* \, dT^\star(\delta_y) = T^\star(\delta_y)(f^*) = T(f^*)(y)$, for every $y \in B_1^*$, as claimed.

Now, notice that it is enough to show that $\mathcal{A}_U$ is dense under U, as in that case $\mathcal{A} = \bigcup\{\mathcal{A}_U : U \subseteq \mathbb{N}^* \setminus F,\, U \text{ clopen}\}$ is the dense family we are after.

To see that $\mathcal{A}_U$ is dense under U, fix an infinite $E_{0,0} \subseteq \mathbb{N}$ such that $E_{0,0}^* \subseteq U$. We define by induction a $\subseteq_*$- decreasing sequence of infinite sets $\{E_{n,l} : n \in \mathbb{N},\, l < m(n)\}$ ordered lexicographically. Suppose (n', l') is successor of (n, l). Since $(U_i(y_l, n))_{i<j(y_l,n)}$ is an open covering of U, we may choose $i < j(y_l, n)$ such that $E_{n,l}^* \cap U_i(y_l, n) \neq \emptyset$. Take $\emptyset \neq E_{n',l'}^* \subseteq E_{n,l}^* \cap U_i(y_l, n)$. Finally, if we take an infinite E such that $E \subseteq_* E_{n,l}$ for every $n \in \mathbb{N}$, $l < m(n)$, then it is clear that $E \subseteq_* E_{0,0}$ and $E \in \mathcal{A}_U$.

□

Lemma 1.5.4. *Let $B \subseteq \mathbb{N}$ be infinite. Suppose $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is such that $\varphi^T(y) \neq \emptyset$ for each $y \in B^*$ and $\bigcup\{\varphi^T(y) : y \in V\}$ has nonempty interior for every open $V \subseteq \mathbb{N}^*$. Then, there is an infinite $B_1 \subseteq_* B$ and $\varepsilon > 0$ such that*

1. *$\varphi_\varepsilon^T(y) \neq \emptyset$ for each $y \in B_1^*$, and*
2. *$\bigcup\{\varphi_\varepsilon^T(y) : y \in D^*\}$ has nonempty interior for every infinite $D \subseteq_* B_1$.*

Proof. Suppose that (1) fails for all infinite $B' \subseteq_* B$ and all $\varepsilon > 0$. Let $B_0 \subseteq_* B$, C_n, $D_n \subseteq \mathbb{N}$ be given by Lemma 1.3.5. We will construct by induction a $\subseteq_*$- decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ such that $\varphi^T_{1/(n+1)}(y) = \emptyset$ for every $y \in B^*_{n+1}$. If we then take any $y \in \bigcap_{n\in\mathbb{N}} B^*_n$, we will have that $\varphi^T(y) = \emptyset$, which contradicts our hypothesis.

Assume we have already constructed B_n. Since we are assuming that (1) fails, there exists $y \in B^*_n$ such that $\varphi^T_{1/2(n+1)}(y) = \emptyset$. This means that $|T^\star(\delta_y)|(\{x\}) = |T^\star(\delta_y)(\{x\})| < 1/2(n+1)$, for every $x \in \mathbb{N}^*$.

By the regularity of the measure $T^\star(\delta_y)$ and by the compactness of $\mathbb{N}^*$, we may cover $\mathbb{N}^*$ by finitely many clopen $(U_i)_{i<k}$ such that $|T^\star(\delta_y)|(U_i) < 1/2(n+1)$, for each $i < k$. We may further assume that each U_i is included in either C^*_n or D^*_n. Since $T^\star$ is weak* continuous, we may find an open neighbourhood of y, say V, such that for every $z \in V$ we have $|T^\star(\delta_z)(U_i)| < 1/2(n+1)$, for each $i < k$. Take B_{n+1} such that $y \in B^*_{n+1} \subseteq B^*_n \cap V$. We claim that $\varphi^T_{1/(n+1)}(z) = \emptyset$, for every $z \in B^*_{n+1}$.

Fix any $z \in B^*_{n+1}$ and take $T^\star(\delta_z) = \mu^+ - \mu^-$ a Jordan decomposition of the measure. Recall that by Lemma 1.3.5 we have that $\mu^-(C^*_n) < 1/4(n+1)$ and $\mu^+(D^*_n) < 1/4(n+1)$. Now take any $x \in \mathbb{N}^*$. Let $i < k$ be such that $x \in U_i$ and assume without loss of generality that $U_i \subseteq C^*_n$. Then,

$$|T^\star(\delta_z)(\{x\})| \leq |T^\star(\delta_z)|(U_i) \leq |T^\star(\delta_z)(U_i)| + 2\mu^-(C^*_n) < 1/(n+1),$$

and the claim is proved. This finishes the proof of the first part, so let us assume that $\varepsilon_0 > 0$ and $B_0 \subseteq_* B$ are such that (1) holds.

To prove the second part, let us assume that for every $B' \subseteq_* B_0$ and every $\varepsilon > 0$ there exists an infinite $D \subseteq_* B_0$ with $\bigcup\{\varphi^T_\varepsilon(y) : y \in D^*\}$ nowhere dense. We may then find a $\subseteq_*$- descending sequence of infinite sets $(D_n)_{n\in\mathbb{N}}$ such that $D_n \subseteq_* B$ and $\bigcup\{\varphi^T_{1/n}(y) : y \in D^*_n\}$ is nowhere dense, for every $n \in \mathbb{N}$. Let $V \subseteq \bigcap_{n\in\mathbb{N}} D^*_n$ be a nonempty open. Then, since $\varphi^T(y) = \bigcup_{n\in\mathbb{N}} \varphi^T_{1/n}(y)$, we have

$$\bigcup\{\varphi^T(y) : y \in V\} = \bigcup_{n\in\mathbb{N}} \bigcup\{\varphi^T_{1/n}(y) : y \in V\} \subseteq \bigcup_{n\in\mathbb{N}} \bigcup\{\varphi^T_{1/n}(y) : y \in D^*_n\},$$

which is nowhere dense by A.1.8. This contradicts the hypothesis of the lemma. Therefore, there exist an infinite $B_1 \subseteq_* B_0$ and $\varepsilon_1 > 0$ which satisfy (2).

The lemma holds for B_1 and $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$. □

Lemma 1.5.5. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is fountainless and everywhere present. Then, for every infinite $B \subseteq \mathbb{N}$ there is an infinite $B_1 \subseteq_* B$ and a continuous quasi-open $\psi : B^*_1 \to \mathbb{N}^*$ such that*

$$T^\star(\delta_y)(\{\psi(y)\}) \neq 0$$

for all $y \in B^*_1$.

Proof. Since T is fountainless and everywhere present, by 1.3.17 and 1.5.3 we know that the hypothesis of 1.5.4 are satisfied. So find $\varepsilon > 0$ and an infinite $B_0 \subseteq_* B$ such that $\varphi^T_\varepsilon(y) \neq \emptyset$ for every $y \in B_0$, and $\bigcup\{\varphi^T_\varepsilon(y) : y \in D^*\}$ has nonempty interior for every infinite $D \subseteq_* B_0$. We may also assume that there exist C_n, $D_n \subseteq \mathbb{N}$ for every $n \in \mathbb{N}$ such that the statement in 1.3.5 holds for B_0 and C_n, D_n.

CLAIM 1: There exists an infinite $B_0' \subseteq_* B_0$ and a finite collection $(V_i)_{i<k}$ of almost disjoint infinite subsets of $\mathbb{N}$ such that for every $z \in (B_0')^*$ we have that $\varphi_\varepsilon^T(z) \subseteq \bigcup_{i<k} V_i^*$ and $|\varphi_\varepsilon^T(z) \cap V_i^*| = 1$, for each $i < k$.

We will construct recursively a $\subseteq_*$-descending sequence (A_n) of subsets of $\mathbb{N}$, $y_n \in A_n^*$, finite collections $(V_{n,i})_{i<k_n}$ of almost disjoint infinite subsets of $\mathbb{N}$ and open intervals $I_i^n \subseteq \mathbb{R}$, $i < k_n$, such that for every n we have

1. $\varphi_\varepsilon^T(z) \subseteq \bigcup_{i<k_n} V_{n,i}^*$, for all $z \in A_{n+1}^*$
2. For every $i < k_{n+1}$ there exists $j < k_n$ such that $V_{n+1,i} \subseteq_* V_{n,j}$
3. $|T^\star(\delta_z)(V_{n,i})| \in I_i^n$, for all $z \in A_{n+1}^*$ and all $i < k_n$
4. *length*$(I_i^n) = \varepsilon(2^{n+1}k_n)^{-1}$, for all $i < k_n$

Begin by noticing that for every $y \in \mathbb{N}^*$ the number of elements of $\varphi_\varepsilon^T(y)$ is finite, as it must be bounded by $\|T\|/\varepsilon$. Let $A_0 = B_0$ and fix any $y_0 \in A_0^*$ and let $\{x_i^0 : i < k_0\}$ be an enumeration of $\varphi_\varepsilon^T(y_0)$ (note that $k_0 \geq 1$). Let $N_0 \in \mathbb{N}$ be such that $1/N_0 < \varepsilon/8k_0$. By the regularity of the measure $T^\star(\delta_{y_0})$, we find for each $i < k_0$ a clopen neighbourhood $V_{0,i}^*$ of x_i^0 such that

$$|T^\star(\delta_{y_0})|(V_{0,i}^*) < |T^\star(\delta_{y_0})(\{x_i^0\})| + \varepsilon/8k_0.$$

We may assume that the $V_{0,i}$'s are almost disjoint and that each of them is almost included in either C_{N_0} or D_{N_0}.

For each $i < k_0$ we define $I_i^0 \subseteq \mathbb{R}$ to be the open interval with centre $|T^\star(\delta_{y_0})(\{x_i^0\})|$ and radius $\varepsilon/4k_0$. Then, by the above, $|T^\star(\delta_{y_0})|(V_{0,i}^*)$ lies in I_i^0 and, as we shall see, $|T^\star(\delta_{y_0})(V_{0,i}^*)|$ does so as well. Indeed, take $i < k_0$ and assume without loss of generality that $V_{0,i}^* \subseteq C_{N_0}^*$. If $T^\star(\delta_{y_0}) = \mu^* - \mu^-$ is a Jordan decomposition of the measure, then

$$|T^\star(\delta_{y_0})|(V_{0,i}^*) \leq T^\star(\delta_{y_0})(V_{0,i}^*) + 2\mu^-(V_{0,i}^*) \leq |T^\star(\delta_{y_0})(V_{0,i}^*)| + 2\mu^-(C_{N_0}^*).$$

From this it follows that $|T^\star(\delta_{y_0})|(V_{0,i}^*) - |T^\star(\delta_{y_0})(V_{0,i}^*)| < 1/2(N_0+1) < \varepsilon/8k_0$, and so $|T^\star(\delta_{y_0})(V_{0,i}^*)| \in I_i^0$.

By the upper semicontinuity of φ_ε^T (Lemma 1.3.11) and the weak*-continuity of $T^\star$, we now find a clopen neighbourhood of y_0, say A_1^*, which we may assume to be included in A_0^*, such that for every $z \in A_1^*$ we have

$$\varphi_\varepsilon^T(z) \subseteq \bigcup_{i<k_0} V_{0,i}^* \quad \text{and}$$

$$|T^\star(\delta_z)(V_{0,i}^*)| \in I_i^0, \text{ for each } i < k_0.$$

If we have that $|\varphi_\varepsilon^T(z) \cap V_{0,i}^*| = 1$, for every $z \in A_1^*$ and for each $i < k_0$, then the recursion stops and the claim is proved. Otherwise, choose $y_1 \in A_1^*$ a witness to this fact, and repeat the procedure to obtain open intervals I_i^1 with centre $|T^\star(\delta_{y_1})(\{x_i^1\})|$ and radius $\varepsilon/2^3k_1$, and clopen sets $V_{1,i}^*$ such that both $|T^\star(\delta_{y_1})|(V_{1,i}^*)$ and $|T^\star(\delta_{y_1})(V_{1,i}^*)|$ lie inside I_i^1, for each $i < k_1$. Notice that we may take each set $V_{1,i}$ as a subset of one of the $V_{0,j}$. Then, by the same argument using the upper semicontinuity of φ_ε^T and the weak*-continuity of $T^\star$ we obtain an infinite $A_2 \subseteq_* A_1$ such that for every $z \in A_2^*$ we have $\varphi_\varepsilon^T(z) \subseteq \bigcup_{i<k_1} V_{1,i}^*$ and $|T^\star(\delta_z)(V_{1,i}^*)| \in I_i^1$, for each $i < k_1$.

We claim that this process stops after finitely many steps. First notice that the failure to stop at step n is due to one of two reasons:

(a) there exists $y_{n+1} \in A^*_{n+1}$ such that $|\varphi^T_\varepsilon(y_{n+1}) \cap V^*_{n,i}| \geq 2$ for some $i < k_n$

(b) there exists $y_{n+1} \in A^*_{n+1}$ such that $\varphi^T_\varepsilon(y_{n+1}) \cap V^*_{n,i} = \emptyset$ for some $i < k_n$

Notice also that once condition (a) fails, it continues to fail in subsequent steps. So we may assume that we first only check for condition (a), and only after it does not occur do we check for condition (b).

By condition (2) in the construction, we have that every time (a) occurs there exists $i < k_0$ such that $|\varphi^T_\varepsilon(y_{n+1}) \cap V^*_{0,i}| \geq 2$. So for each $i < k_0$ consider $m_i \in \mathbb{N}$ such that $m_i \cdot \varepsilon \geq T^\star(\delta_{y_0})(\{x^0_i\})$ and suppose (a) has occurred at $n = \sum_{i<k_0} m_i$ many steps. Suppose that still (a) happens once more. Then, there exists $i_0 < k_0$ and $n_0 < \cdots < n_{m_{i_0}} = n$ such that $|\varphi^T_\varepsilon(y_{n_j+1}) \cap V^*_{0,i_0}| \geq 2$, for every $j \leq m_{i_0}$. Hence, if $x^{n+1}_0,\ x^{n+1}_1 \in \varphi^T_\varepsilon(y_{n+1}) \cap V^*_{n,i_n}$, for certain $i_n < k_n$, then $|T^\star(\delta_{y_{n+1}})|(\{x^{n+1}_0\}) + \varepsilon \leq |T^\star(\delta_{y_{n+1}})|(\{x^{n+1}_0, x^{n+1}_1\}) \leq |T^\star(\delta_{y_{n+1}})|(V^*_{n,i_n})$ and we obtain

$$\begin{aligned}
|T^\star(\delta_{y_{n+1}})|(\{x^{n+1}_0\}) &< \sup I^n_{i_n} - \varepsilon \\
&= |T^\star(\delta_{y_n})|(\{x^n_{i_n}\}) + \varepsilon(2^{n+2}k_n)^{-1} - \varepsilon \\
&< \ldots \\
&< |T^\star(\delta_{y_0})|(\{x^0_{i_0}\}) + \textstyle\sum_{j<n+1} \varepsilon(2^{j+2}k_j)^{-1} - (m_{i_0}+1)\cdot\varepsilon \\
&\leq (|T^\star(\delta_{y_0})|(\{x^0_{i_0}\}) - m_{i_0}\cdot\varepsilon) + (\varepsilon \textstyle\sum_{j<n+1} 1/2^{j+2} - \varepsilon) \\
&< 0 - \varepsilon/2.
\end{aligned}$$

From this contradiction we conclude that (a) can occur at most at $(\sum_{i<k_0} m_i)$ many steps.

Now assume that n_0 is such that (a) does not hold at step n, for all $n \geq n_0$. Suppose the recursion does not stop at step $n \geq n_0$. Assume without loss of generality that $\varphi^T_\varepsilon(y_{n+1}) \cap V^*_{n,0} = \emptyset$. Therefore $\varphi^T_\varepsilon(y_{n+1}) \subseteq \bigcup_{0<i<k_n} V^*_{n,i}$. Since we also have $|T^\star(\delta_{y_{n+1}})|(V^*_{n,i}) \in I^n_i$, for each $i < k_n$, we obtain

$$\begin{aligned}
|T^\star(\delta_{y_{n+1}})|(\{x^{n+1}_i : i < k_{n+1}\}) &\leq \textstyle\sum_{i<k_n} |T^\star(\delta_{y_{n+1}})|(V^*_{n,i}) - |T^\star(\delta_{y_{n+1}})|(V^*_{n,0}) \\
&< \textstyle\sum_{i<k_n} \sup I^n_i - \inf I^n_0 \\
&\leq \textstyle\sum_{i<k_n} |T^\star(\delta_{y_n})(\{x^n_i\})| + k_n\varepsilon/(2^{n+2}k_n) \\
&\qquad -(\varepsilon - \varepsilon/2k_n) \\
&\leq |T^\star(\delta_{y_n})|(\{x^n_i : i < k_n\}) - \varepsilon/2.
\end{aligned}$$

The last inequality holds because $k_n \geq 1$. Since $|T^\star(\delta_{y_n})|(\{x^n_i : i < k_n\}) \geq \varepsilon$, for every n, we conclude that (b) cannot occur indefinitely. Hence the recursion must stop after finitely many steps.

CLAIM 2: There exists $i_0 < k$ and an infinite $B_1 \subseteq_* B'_0$ such that $V^*_{i_0} \cap \bigcup\{\varphi^T_\varepsilon(y) : y \in D^*\}$ has nonempty interior for every infinite $D \subseteq_* B_1$.

Suppose this is not the case. Then, we may find a sequence on infinite sets $B'_0 = A_0 \supseteq_* A_1 \supseteq_* \cdots \supseteq A_k$ such that $V^*_i \cap \bigcup\{\varphi^T_\varepsilon(y) : y \in A^*_{i+1}\}$ is nowhere dense, for $i < k$. By Claim 1, we know that $\bigcup\{\varphi^T_\varepsilon(y) : y \in A^*_k\} \subseteq \bigcup_{i<k} V^*_i$, and so $\bigcup\{\varphi^T_\varepsilon(y) : y \in A^*_k\} = \bigcup_{i<k} V^*_i \cap \bigcup\{\varphi^T_\varepsilon(y) : y \in A^*_k\}$ is also nowhere dense. But this contradicts the choice of B_0.

Now we may define $\psi : B^*_1 \to \mathbb{N}^*$ by $\{\psi(y)\} = V^*_{i_0} \cap \varphi^T_\varepsilon(y)$. It is clear that ψ is quasi-open by Claim 2, so we conclude the proof by showing that ψ is continuous. Let $U \subseteq V^*_{i_0}$ be any open set. Since $V^*_{i_0}$ is clopen and φ^T_ε is upper semicontinuous, we have that $\psi^{-1}[U] = B^*_1 \cap \{y \in \mathbb{N}^* : \varphi^T_\varepsilon(y) \subseteq U \cup \mathbb{N}^* \setminus V^*_{i_0}\}$ is open.

□

Theorem 1.5.6. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is a fountainless, everywhere present operator. Then, T is left-locally canonizable along a quasi-open map.*

Proof. Fix an infinite $B \subseteq \mathbb{N}$ and find using 1.5.5 an infinite $B_1 \subseteq B$ and a quasi-open $\psi : B_1^* \to \mathbb{N}^*$ such that

$$T^\star(\delta_y)(\{\psi(y)\}) \neq 0$$

for all $y \in B_1^*$. Now use 1.5.1 to find clopen sets $A^* \subseteq \psi[B_1^*]$ and $B_2^* \subseteq \psi^{-1}[A^*]$ and a real $r \in \mathbb{R}$ such that $T_{B_2,A}(f^*) = r(f^* \circ \psi)|B_2$ for every $f \in \ell_\infty(A)$. It follows that $T^\star(\delta_y)|A^* = r\delta_{\psi(y)}$ for each $y \in B_2^*$, and so $0 \neq T^\star(\delta_y)(\{\psi(y)\}) = r$. Hence $T_{B_2,A}$ is canonizable along ψ. □

If we weaken the hypothesis on an operator from being everywhere present to simply nonzero, we still get that it is somewhere canonizable along a quasi-open map:

Corollary 1.5.7. *If $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is a fountainless nonzero operator, then T is somewhere canonizable along a quasi-open map.*

Proof. Since T is nonzero, there exists $f_0 \in C(\mathbb{N}^*)$ such that $T(f_0) \neq 0$. By A.1.2, there exists an infinite $B \subseteq \mathbb{N}$ such that $(P_B \circ T)(f_0)$ is constantly nonzero, which implies that $P_B \circ T$ is everywhere present. Notice on the other hand that the fact that T is fountainless implies that $P_B \circ T$ is also fountainless. So by Theorem 1.5.6 we know there exist infinite $B_1 \subseteq B$ and $A \subseteq \mathbb{N}$ such that $T_{B_1,A}$ is canonizable along a quasi-open mapping.

□

1.5.2 Right-local canonization of funnelless automorphisms

The main result of this section is a consequence of our generalization 1.5.1 of the Drewnowski-Roberts canonization lemma and the following result of Plebanek which is implicitly proved in Theorem 6.1 of [45].

Theorem 1.5.8. *Suppose that $T : C(K) \to C(K)$ is an automorphism. Then, there is a π-base $\mathcal{U}$ of K such that for every $U \in \mathcal{U}$ there is a closed $F \subseteq K$ and a continuous surjection $\psi : F \to \overline{U}$ such that*

$$|T^\star(\delta_y)|(\{\psi(y)\}) \neq 0$$

for all $y \in F$.

Theorem 1.5.9. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is a funnelless automorphism. Then, T is right-locally canonizable along a quasi-open map.*

Proof. Fix an infinite $A \subseteq \mathbb{N}$. Let $\mathcal{U}$ be as in 1.5.8 for T and find $U \in \mathcal{U}$ such that $U \subseteq A^*$. Let F and ψ be as in 1.5.8 for U. Let $A_1 \subseteq \mathbb{N}$ be infinite such that $A_1^* \subseteq U$ and put $F_1 = \psi^{-1}[A_1^*] \subseteq F$. Let $F_2 \subseteq F_1$ be closed such that $\psi|F_2$ is irreducible and onto A_1^* and hence quasi-open (relative to the subspace topology on F_2) by lemma A.1.6.

As T is funnelless (Definition 1.3.18) and A_1^* is open, F_2 cannot be nowhere dense, so let B^* be a nonempty clopen set included in F_2. Now $\psi : B^* \to A_1^*$ is a continuous, quasi-open map (as a restriction of a quasi-open map to a clopen subset of F_2) satisfying $T^\star(\delta_y)(\{\psi(y)\}) \neq 0$ for each y in B^*. Therefore, we can apply 1.5.1 to obtain an infinite $A_2 \subseteq A_1$, a clopen $B_1^* \subseteq \psi^{-1}[A_2^*]$ and a real $r \in \mathbb{R}$ such that $T_{B_1,A_2}(f^*) = r(f^* \circ \psi)|B_1^*$, for every $f \in \ell_\infty(A_2)$. In particular we have that

$$T^\star(\delta_y)(E^*) = T_{B_1,A_2}(\chi_{E^*})(y) = r(\chi_{E^*} \circ \psi)(y) = r\delta_{\psi(y)}(E^*)$$

for every infinite $E \subseteq A_2$ and every $y \in B_1^*$. It follows that for each $y \in B_1^*$ we have $T^\star(\delta_y)|A_2^* = r\delta_{\psi(y)}$, and so $0 \neq T^\star(\delta_y)(\{\psi(y)\}) = r$. Having $r \neq 0$, we conclude that T_{B_1,A_2} is canonizable along ψ.

□

1.6 The impact of combinatorics on the canonization and trivialization of operators on ℓ_∞/c_0

As expected based on the study of $\mathbb{N}^*$ (e.g., [54], [58], [64], [20], [17], [25]), the impact of additional set-theoretic assumptions on the structure of operators on ℓ_∞/c_0 is also very dramatic.

1.6.1 Canonization and trivialization of operators on ℓ_∞/c_0 under OCA + MA

Recall from [25] that an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$ is called c.c.c. over Fin if, and only if, there are no uncountable almost disjoint families of $\mathcal{I}$-positive sets. Dually a closed subset $F \subseteq \mathbb{N}^*$ is called c.c.c. over Fin if $A_\xi^* \cap F = \emptyset$ for some $\xi < \omega_1$ whenever $\{A_\xi : \xi < \omega_1\}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$. The following theorem by I. Farah (3.3.3. and 3.8.1. from [25]) will be crucial in this subsection:

Theorem 1.6.1 (OCA + MA ([25])). *Let $h : \wp(\mathbb{N})/\mathrm{Fin} \longrightarrow \wp(\mathbb{N})/\mathrm{Fin}$ be a homomorphism. Then, there is an infinite $B \subseteq \mathbb{N}$, a function $\sigma : B \to \mathbb{N}$ and a homomorphism $h_2 : \wp(\mathbb{N})/\mathrm{Fin} \to \wp(\mathbb{N} \setminus B)/\mathrm{Fin}$ such that $h([A]) = [\sigma^{-1}[A]] \cup h_2([A])$ for every $A \subseteq \mathbb{N}$, and $Ker(h_2)$ is c.c.c. over Fin.*

The following is a topological reformulation of the above theorem:

Theorem 1.6.2 (Corollary 7, [20](OCA + MA)). *Suppose $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a continuous mapping. Then, there exist an infinite $B \subseteq \mathbb{N}$ and a function $\sigma : B \to \mathbb{N}$ such that*

$$\psi(x) = \sigma^*(x)$$

for all $x \in B^*$ and $F = \psi[(\mathbb{N} \setminus B)^*]$ is a nowhere dense closed c.c.c. over Fin set.

Proof. By the Stone duality every continuous $\psi : \mathbb{N}^* \to \mathbb{N}^*$ corresponds to a homomorphism $h : \wp(\mathbb{N})/\mathit{Fin} \to \wp(\mathbb{N})/\mathit{Fin}$ given by $h([A]) = [D]$, where $D^* = \psi^{-1}[A^*]$, for every $A \subseteq \mathbb{N}$. Let $B \subseteq \mathbb{N}$, $\sigma : B \to \mathbb{N}$ and h_2 be as in 1.6.1 for the homomorphism h. For every infinite $A \subseteq \mathbb{N}$ we have $h([A]) \cap [B] = [\sigma^{-1}[A]]$ by 1.6.1 and so $\psi^{-1}[A^*] \cap B^* = (\sigma^{-1}[A])^*$. Therefore, $\sigma^* = \psi|B^*$. For every infinite $A \subseteq \mathbb{N}$ we have $h([A]) \cap [\mathbb{N} \setminus B] = h_2([A])$ by 1.6.1, so for every $x \in (\mathbb{N} \setminus B)^*$ we have $\psi(x) = h_2^{-1}[\{[A] : A \in x\}]$ by the Stone duality. The set $F = \psi[(\mathbb{N} \setminus B)^*]$ is closed and for every $x \in (\mathbb{N} \setminus B)^*$, the set $h_2^{-1}[\{[A] : A \in x\}]$ is disjoint from $Ker(h_2)$, which is c.c.c. over Fin. Therefore, F is c.c.c. over Fin and c.c.c. over Fin sets are nowhere dense. □

It turns out that continuous surjective maps and quasi-open maps $\psi : B^* \to \mathbb{N}^*$ can be reduced to bijections between subsets of $\mathbb{N}$ assuming OCA + MA.

Lemma 1.6.3 (OCA + MA). *Let $B \subseteq \mathbb{N}$ be infinite. Suppose that $\psi : B^* \to \mathbb{N}^*$ is continuous such that $\psi[B^*]$ is not nowhere dense. Then, there are infinite $B_1 \subseteq B$, $A \subseteq \mathbb{N}$ and a bijection $\sigma : B_1 \to A$ such that $\psi|B_1^* = \sigma^*$. In particular, $\psi|B_1^*$ is a homeomorphism onto A^*.*

Proof. By 1.6.2 there exist an infinite $B_0 \subseteq B$ and a function $\sigma : B_0 \to \mathbb{N}$ such that $\psi(x) = \sigma^*(x)$ for all $x \in B_0^*$, and such that $\psi[(B \setminus B_0)^*]$ is a nowhere dense. Since the set $\psi[B^*] = \psi[(B^* \setminus B_0)^*] \cup \psi[B_0^*]$ is not nowhere dense, we have that $\psi[B_0^*] = \sigma^*[B_0^*]$ must have nonempty interior. In particular, there exists an infinite $E \subseteq \mathbb{N}$ such that $E \subseteq^* \sigma[B_0]$ and so the image of σ is infinite. Hence, there is an infinite $B_1 \subseteq B_0$ such that $\sigma|B_1$ is a bijection onto its image A.

□

Theorem 1.6.4 (OCA + MA). *If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a fountainless everywhere present operator, then it is left-locally trivial. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a funnelless automorphism, then it is right-locally trivial.*

Proof. Apply 1.5.6 and 1.5.9 to obtain left-local or right-local canonization along a quasi-open mapping, respectively. Now use 1.6.3 to conclude that this mapping is somewhere induced by a bijection. □

Corollary 1.6.5 (OCA + MA). *If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a fountainless nonzero operator, then it is somewhere trivial.*

Proof. By 1.5.7 we know that T is somewhere canonizable along a quasi-open mapping. Then, by 1.6.3 it is somewhere induced by a bijection. □

1.6.2 Operators on ℓ_∞/c_0 under CH

The continuum hypothesis is a strong tool allowing transfinite induction constructions in $\wp(\mathbb{N})/Fin$ which induce objects in ℓ_∞/c_0. Actually, a considerable part of this strength is included in a powerful consequence of Parovičenko's theorem: if X is zero-dimensional, locally compact, σ-compact, noncompact Hausdorff space of weight at most continuum, then $X^* = \beta X \setminus X$ is homeomorphic to $\mathbb{N}^*$ (1.2.6 of [61]). In this section we will often be using this result combined with the universal property of βX for locally compact X, that every continuous function on X into a compact space extends to βX

Theorem 1.6.6 (CH). *There is an automorphism $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ which is nowhere canonizable along a quasi-open map on an open set, in particular along a homeomorphism.*

Proof. Let K be an uncountable compact zero-dimensional metrizable space with countably many isolated points $\{x_m : m \in \mathbb{N}\}$ which form a dense open subspace of K. By the classical classification of separable spaces of continuous functions there is an isomorphism $S : C(2^{\mathbb{N}}) \to C(K)$.

Let $X = \mathbb{N} \times K$ and $Y = \mathbb{N} \times 2^{\mathbb{N}}$. Note that X, Y satisfy the hypothesis of the topological consequence of Parovičenko's theorem (1.2.6 of [61]) mentioned above, hence there are homeomorphisms $\pi : \mathbb{N}^* \to X^*$ and $\rho : Y^* \to \mathbb{N}^*$. Define $\tilde{\tau} : K \to \|S\| B_{C(2^{\mathbb{N}})^\star}$ by $\tilde{\tau}(x) = S^\star(\delta_x)$ for each $x \in K$, where the dual ball is considered with the weak* topology and identified with the Radon measures on $2^{\mathbb{N}}$. Define $\tau : X \to \|S\| B_{C(\beta Y)^\star}$ by putting $\tau(n, x)$ to be the measure on βY which is zero on the complement of $\{n\} \times 2^{\mathbb{N}}$ and is equal to the measure $\tilde{\tau}(x)$ on $\{n\} \times 2^{\mathbb{N}}$. By the universal property of βX there is an extension $\beta\tau : \beta X \to \|S\| B_{C(\beta Y)^\star}$

CLAIM: For each $t \in X^*$ the measure $\beta\tau(t)$ is concentrated on Y^*.

Fix $t \in X^*$. Note that for every $n \in \mathbb{N}$ the set $\beta X \setminus \{k \in \mathbb{N} : k \leq n\} \times K$ is a neighbourhood of t. Also, for $k > n$ if $x \in \{k\} \times K$, then $\tau(x)(U) = 0$ for every Borel subset U of $\{n\} \times 2^{\mathbb{N}}$. This completes the proof of the claim by the weak* continuity of $\beta\tau$.

Now we can define $T : C(Y^*) \to C(X^*)$ by $T(f)(t) = \int f \, \mathrm{d}(\beta\tau(t))$ for every $t \in X^*$. It is a well defined bounded linear operator by Theorem 1 in VI.7 of [23]. We will show that $T_\pi \circ T \circ T_\rho$ is an automorphism of $\mathbb{N}^*$ which is nowhere canonizable along a quasi-open map. For the former we need to prove that T is an isomorphism and for the latter we need to prove that for every nonempty clopen sets $U \subseteq X^*$, $O \subseteq Y^*$ there is no quasi-open $\phi : U \to O$ such that $(\beta\tau(t))|O = r\delta_{\phi(t)}$ for every t in U and some nonzero $r \in \mathbb{R}$.

To prove that T is an isomorphism, note that one can define $R : C(\beta Y) \to C(\beta X)$ by $R(f)(x) = \int f \, \mathrm{d}(\beta\tau(x))$ for every $x \in X$, and that $C(\beta Y)$ can be identified with the ℓ_∞-sum of $C(2^{\mathbb{N}})$ while $C(\beta X)$ can be identified with the ℓ_∞-sum of $C(K)$. R sends the subspace corresponding to the c_0-sum of $C(2^{\mathbb{N}})$ into the subspace corresponding to the c_0-sum of $C(K)$ since the original S is an isomorphism and R is the ℓ_∞-sum of the operator S. It follows that T is induced by R modulo the subspaces corresponding to the c_0-sums. Moreover, one can note using the fact that S is bounded below that elements outside the subspace corresponding to the c_0-sum of $C(K)$ are send by R onto elements outside the subspace corresponding to the c_0-sum of $C(K)$. It follows that T is nonzero on every nonzero element, i.e., is injective. The surjectivity of T follows from the surjectivity of R which follows from the surjectivity of S.

Now let us prove that T is nowhere canonizable along a quasi-open mapping. Fix U, O clopen subsets of X^* and Y^* respectively, and suppose ϕ is as above and quasi-open. Fix a clopen $V \subseteq \phi[U]$. Let U' be a clopen subset of X such that $\beta U' \cap X^* = U$. The set E of integers n such that $U_n = U' \cap (\{n\} \times K) \neq \emptyset$ must be infinite. Since the isolated points $\{x_m : m \in \mathbb{N}\}$ are dense in K, we may assume, by going to a subset of U, that $U_n = \{x_{k_n}\}$ for all $n \in E$ and some $k_n \in \mathbb{N}$. Therefore,

$$U' = (\bigcup_{n \in E} \{n\} \times \{x_{k_n}\}).$$

Let $V_n = V' \cap \{n\} \times 2^{\mathbb{N}}$ for $n \in E$, where $\beta V' \cap Y^* = V$. Let $W_n \subseteq V_n$ be a nonempty clopen such that $\tilde{\tau}(x_{k_n})|W_n$ has its total variation less than $|r|/2$ which can be found since $2^{\mathbb{N}}$ has no isolated points. Consider

$$W = (\bigcup_{n \in E} \{n\} \times W_n).$$

Then, $|\beta\tau(n, x_{k_n})(W')| < |r|/2$ for any $W' \subseteq W$ and any $n \in E$. By the weak* continuity of $\beta\tau$ we have that $|\beta\tau(t)(W')| \leq |r|/2$ for any $t \in U$, but this shows that $\beta\tau(t)$ is not $r\delta_{\phi(t)}$ as required.

□

One concrete construction using the methods as above due to E. van Douwen and J. van Mill is a nowhere dense retract $F \subseteq \mathbb{N}^*$ which is homeomorphic to $\mathbb{N}^*$ and which is a P-set (see 1.4.3. and 1.8.1. of [61]). We will require the following:

Lemma 1.6.7. *(CH) Let $F \subseteq \mathbb{N}^*$ be a nowhere dense P-set. The space $\{f \in C(\mathbb{N}^*) : f|F = 0\}$ is isomorphic to $C(\mathbb{N}^*)$.*

Proof. Fix a P-point $p \in \mathbb{N}^*$ which exists assuming CH by the results of [50]. Let $(A^*_\alpha : \alpha < \omega_1)$ and $(B^*_\alpha : \alpha < \omega_1)$ be sequences of strictly increasing clopen sets such that $\bigcap_{\alpha<\omega_1}(\mathbb{N}^* \setminus A^*_\alpha) = F$ and $\bigcap_{\alpha<\omega_1}(\mathbb{N}^* \setminus B^*_\alpha) = \{p\}$ (they exists because F is a P-set and p is a P-point).

Using the standard argument construct recursively one-to-one, onto functions $\sigma_\alpha : B_\alpha \to A_\alpha$ such that $\sigma_\alpha =^* \sigma_\beta|B_\alpha$ for all $\alpha < \beta < \omega_1$. Put $\psi_\beta = \sigma^*_\beta : B^*_\beta \to A^*_\beta$ which is the corresponding homeomorphism.

Note that if $f \in C(\mathbb{N}^*)$ is such that $f|F = 0$, then for every $n \in \mathbb{N}$ there exists $\alpha < \omega_1$ such that $\mathbb{N}^* \setminus A^*_\alpha \subseteq f^{-1}[\{t \in \mathbb{R} : |t| < 1/(n+1)\}]$. Therefore, for each such f there exists $\alpha < \omega_1$ such that $f|(\mathbb{N}^* \setminus A^*_\alpha) = 0$. So define

$$S : \{f \in C(\mathbb{N}^*) : f|F = 0\} \to \{f \in C(\mathbb{N}^*) : f(p) = 0\}$$

by putting $S(f) = (f \circ \psi_\alpha) \cup 0_{\mathbb{N}\setminus B^*_\alpha}$, where α is any countable ordinal such that $f|(\mathbb{N}^* \setminus A^*_\alpha) = 0$. It is well defined because the homeomorphisms extend each other, and it is clearly a linear isometry. Now it is enough to note that $\{f \in C(\mathbb{N}^*) : f(p) = 0\}$ is isomorphic to $C(\mathbb{N}^*)$. To see that this is the case, notice that this space is a hyperplane, and recall that all hyperplanes are isomorphic to each other in any Banach space (see exercises 2.6 and 2.7 of [24]). In the case of $C(\mathbb{N}^*)$ we have

$$C(\mathbb{N}^*) \sim C(\mathbb{N}^*) \oplus \ell_\infty \sim C(\mathbb{N}^*) \oplus \ell_\infty \oplus \mathbb{R} \sim C(\mathbb{N}^*) \oplus \mathbb{R}$$

and so all hyperplanes are isomorphic to the entire $C(\mathbb{N}^*)$. This completes the proof. □

Proposition 1.6.8. *(CH) The collection of locally null operators is not a right ideal. Moreover, the right ideal generated by locally null operators is improper.*

Proof. Let $F \subseteq \mathbb{N}^*$ be a nowhere dense retract of $\mathbb{N}^*$ homeomorphic to $\mathbb{N}^*$, $\psi_1 : \mathbb{N}^* \to F$ the witnessing retraction and $\psi_2 : \mathbb{N}^* \to F$ the homeomorphism. Note that $\psi_2^{-1} \circ \psi_1$ is a well defined continuous map from $\mathbb{N}^*$ onto itself, and so $T_{\psi_2^{-1}\circ\psi_1}$ is a well defined operator from $C(\mathbb{N}^*)$ into itself. T_{ψ_2} is locally null because F is nowhere dense and hence $T_{\psi_2}(f^*) = f^* \circ \psi_2$ is zero for every $f \in \ell_\infty$ such that $f^*|F$ is zero. But for every $f \in \ell_\infty$ we have

$$T_{\psi_2} \circ T_{\psi_2^{-1} \circ \psi_1}(f^*) = f^* \circ \psi_2^{-1} \circ \psi_1 \circ \psi_2 = f^*,$$

because $Im(\psi_2) = F$ and $\psi_1|F = Id_F$. This means that $T_{\psi_2} \circ T_{\psi_2^{-1} \circ \psi_1} = Id$, which is not locally null. Moreover, for any operator $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ we have that $S = (T_{\psi_2} \circ T_{\psi_2^{-1} \circ \psi_1}) \circ S$, which is in the right ideal generated by locally null operators. □

Nowhere dense P-sets homeomorphic to $\mathbb{N}^*$ which are retracts give also more concrete (compared to 1.6.6) examples of automorphisms failing canonizability like in 1.6.4.

Example 1.6.9 (CH). There is an automorphism T of ℓ_∞/c_0 with the following properties:

1. T is not fountainless
2. T is not left-locally canonizable along any continuous map.
3. T^{-1} is not funnelless
4. T^{-1} is not right-locally canonizable along any continuous map.

Proof. Let F be a nowhere dense retract of $\mathbb{N}^*$ which is a P-set and is homeomorphic to $\mathbb{N}^*$. Let $\psi_1 : \mathbb{N}^* \to F$ be the witnessing retraction. We will need one more additional property of F, namely that ψ_1 is not one-to-one while restricted to any nonempty clopen set. This can be obtained by modifying the construction of 1.4.3. of [61] by replacing $W(\omega_1 + 1)$ with the "zero-dimensional long line", i.e., the space K obtained by gluing Cantor sets inside every ordinal interval $[\alpha, \alpha+1)$ for $\alpha < \omega_1$, obtaining a nonmetrizable subspace K of the long line which contains $W(\omega_1 + 1)$ and has no isolated points. One takes $\tilde{\pi} : K \to K$ which collapses the entire K to the point ω_1, $X = \mathbb{N} \times K$, and $\pi : X \to X$ given by $\pi(n, x) = (n, \tilde{\pi}(x))$. As in 1.4.3. of [61] one proves that $\beta\pi[X^*] \subseteq X^*$ and $\psi_1 = \beta\pi|X^*$ is the required retraction. The argument why ψ_1 is not a one-to-one while restricted to any clopen set is similar to the one from the proof of Theorem 2.1 from [62]: if $U \subseteq X^*$ clopen, it is of the form $\beta U' \cap X^*$ where

$$U' = \bigcup_{n \in E} \{n\} \times U_n$$

for some infinite $E \subseteq \mathbb{N}$ and nonempty clopen sets $U_n \subseteq K$ (consider χ_U and the relation of X to βX). But these nonempty open sets have at least two points x_n, y_n as K has no isolated points. Of course $\pi(n, x_n) = (n, \omega_1) = \pi(n, y_n)$. Consider $x = \lim_{n \in u} x_n$ and $y = \lim_{n \in u} y_n$ (u is a nonprincipal ultrafilter in $\wp(\mathbb{N})$) which can be easily separated, so $x \neq y$ and $x, y \in U$. However $\psi_1(x) = \lim_{n \in u} \pi(n, x_n) = \lim_{n \in u} \pi(n, y_n) = \psi_1(y)$.

We can decompose $C(\mathbb{N}^*) = X \oplus Y$ where

$$X = \{g \circ \psi_1 : g \in C(F)\}, \quad Y = \{f \in C(\mathbb{N}^*) : f|F = 0\}.$$

The first factor is isometric to $C(F)$ (the isometry is defined by restricting to F), which in turn is isometric to $C(\mathbb{N}^*)$ because of the homeomorphism between F and $\mathbb{N}^*$. By Lemma 1.6.7, the second factor is also isomorphic $C(\mathbb{N}^*)$.

Fix an infinite, coinfinite $A \subseteq \mathbb{N}$. Let $S : Y \to C(\mathbb{N}^* \setminus A^*)$ be an isomorphism. Let $\psi_2 : A^* \to F$ be a homeomorphism. Finally, let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be the isomorphism defined by

$$T = I_A \circ T_{\psi_2} + I_{\mathbb{N}^* \setminus A^*} \circ S \circ (Id - T_{\psi_1}).$$

That is, roughly speaking, T sends X to $C(A^*)$ and Y to $C(\mathbb{N}^* \setminus A^*)$. For $y \in A^*$ we have $T^\star(\delta_y) = (I_A \circ T_{\psi_2})^\star(\delta_y) = \delta_{\psi_2(y)}$, i.e., $T^\star(\delta_y)$ is concentrated on F, so (F, A) is a fountain for T. This also implies that T cannot be canonized along a homeomorphism onto a clopen set below A^* because F is nowhere dense.

On the other hand,

$$T^{-1} = T_{\psi_1} \circ T_{\psi_2^{-1}} \circ P_A + S^{-1} \circ P_{\mathbb{N} \setminus A}$$

So $(T^{-1})^\star(\delta_x)|A^* = \delta_{\psi_2^{-1}(\psi_1(x))}$ for every $x \in \mathbb{N}^*$. In particular (A^*, F) is a funnel for T^{-1} and T^{-1} cannot be canonized on a pair (A_0, B_0) for infinite $A_0 \subseteq A$, $B_0 \subseteq \mathbb{N}$ because $\psi_2^{-1} \circ \psi_1$ is not one-to-one on any clopen set $B_0 \subseteq \mathbb{N}$ by the choice of ψ_1.

□

Theorem 1.6.10 (CH). *There is an automorphism of ℓ_∞/c_0 with no fountains and no funnels which is nowhere trivial.*

Proof. Let $\psi : \mathbb{N}^* \to \mathbb{N}^*$ be nowhere trivial homeomorphism of $\mathbb{N}^*$. The existence of such a homeomorphism is a folklore result, its first construction is implicitly included in [50]. By Propositions 1.3.21 and 1.3.22, T_ψ has no fountains nor funnels. It is not locally trivial because ψ is not trivial on any clopen set. □

Theorem 1.6.11 (CH). *There is a quasi-open surjective map $\psi : \mathbb{N}^* \to \mathbb{N}^*$ such that the images of nowhere dense sets under ψ are nowhere dense and it is not a bijection while restricted to any clopen set. Therefore, T_ψ is an everywhere present isomorphic embedding of ℓ_∞/c_0 into itself with no fountains and with no funnels which is nowhere canonizable along a homeomorphism.*

Proof. It is enough to construct a quasi-open irreducible surjection $\psi : \mathbb{N}^* \to \mathbb{N}^*$ which is not a homeomorphism when restricted to any clopen set and consider T_ψ by A.1.6, 1.3.21 and 1.3.22. Let $\tilde{\phi} : 2^\mathbb{N} \to 2^\mathbb{N}$ be an irreducible surjection which is not a bijection while restricted to any clopen subset of $2^\mathbb{N}$ (e.g., obtained via the Stone duality by taking a dense atomless subalgebra of the free countable algebra which is proper below any element, see A.1.5). Consider $X = \mathbb{N} \times 2^\mathbb{N}$ and $\phi : X \to X$, given by $\phi(n, x) = (n, \tilde{\phi}(x))$. By a topological consequence of Parovičenko's theorem (see Theorem 1.2.6. of [61]) $X^* = \beta X \setminus X$ is homeomorphic to $\mathbb{N}^*$. Moreover $\beta\phi : \beta X \to \beta X$ sends X^* into X^*.

To check that $\psi = \beta\phi|X^*$ is irreducible take any clopen $U \subseteq X^*$, which must be of the form $\beta U' \cap X^*$, where

$$U' = \bigcup_{n \in E} \{n\} \times U_n$$

for some infinite $E \subseteq \mathbb{N}$ and nonempty clopen sets $U_n \subseteq K$ (consider χ_U and the relation of X to βX). By the irreducibility of $\tilde{\phi}$, there are clopen $V_n \subseteq 2^{\mathbb{N}}$ such that $\tilde{\phi}[2^{\mathbb{N}} \setminus U_n] \cap V_n = \emptyset$. So

$$\beta\phi[U] \cap \beta(\bigcup_{n\in E}\{n\} \times V_n) = \emptyset,$$

which completes the proof of the irreducibility of ψ. The argument why ψ is not a one-to-one while restricted to any clopen set is similar to the one from the proof of Example 1.6.9. □

A similar example as above is constructed in the proof of Theorem 2.1 from [62] however it does not have the property of preserving nowhere dense sets.

1.7 Open problems and final remarks

In this section we mention some open problems and some observations related to them. We focus on problems related to the analysis carried out in the present chapter, ignoring other important open problems related to the space ℓ_∞/c_0.

Problem 1.7.1. *Is it consistent (does it follow from PFA or OCA + MA) that every automorphism $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ can be lifted modulo a locally null operator? That is, is every such operator of the form $T = [R] + S$, where $R : \ell_\infty \to \ell_\infty$ and S is locally null?*

This is related to the fact that our ZFC nonliftable operator (see 1.4.16) is of the above form.

A ZFC possibility of somewhere canonizing every isomorphic embedding is excluded by 1.6.11 or 1.6.6. But as under PFA or OCA + MA canonization along a homeomorphism gives trivialization we may ask:

Problem 1.7.2. *Is it consistent (does it follows from PFA or OCA + MA) that every isomorphic embedding $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is somewhere trivial?*

Problem 1.7.3. *Is it true in ZFC that for every isomorphic embedding $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ there is an infinite $A \subseteq \mathbb{N}$, a closed $F \subseteq \mathbb{N}^*$ and a homeomorphism $\psi : F \to A^*$ such that $T(f^*)|F = r(f^* \circ \psi)$ for A-supported f's and some nonzero $r \in \mathbb{R}$?*

The positive solution of this problem would give the positive solution to Problem 1.7.6 (see 0.0.9).

Problem 1.7.4. *Is it consistent (does it follow from PFA, or OCA + MA) that every automorphism of ℓ_∞/c_0 is somewhere trivial?*

In other words we ask here if the hypothesis in 1.6.4 of T being funnelless or fountainless is needed under PFA or OCA + MA. In principle there may not be any fountains or funnels of automorphisms of ℓ_∞/c_0 under these assumptions, as the only examples we have of such phenomena are for automorphisms under CH (1.6.9).

The last couple of problems are related to possible applications of canonizations of embeddings.

Problem 1.7.5. *Is it consistent that every copy of ℓ_∞/c_0 inside ℓ_∞/c_0 is complemented?*

Problem 1.7.6. *Is it true or consistent that every copy of ℓ_∞/c_0 inside ℓ_∞/c_0 contains a further copy of ℓ_∞/c_0 which is complemented in the entire space?*

One should note that under CH examples of uncomplemented copies of ℓ_∞/c_0 inside ℓ_∞/c_0 were constructed in [11]. They can also be obtained under CH from a superspace of ℓ_∞/c_0 obtained in [5] in which ℓ_∞/c_0 is not complemented.

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Chapter 2

ℓ_∞-sums of ℓ_∞/c_0

2.1 Introduction

The subject matter of this chapter is motivated by a question raised in [36] and partially answered in [22]. It pertains the concept of primary Banach space: a Banach space X is said to be primary if for every direct sum decomposition $X = A \oplus B$ at least one of the summands is isomorphic to X. Drewnowski and Roberts proved that under the Continuum Hypothesis ℓ_∞/c_0 is primary, but it remains an open question whether this holds in ZFC alone.

Here, our focus will be a related question, namely, whether the ℓ_∞-sum of ℓ_∞/c_0, here forth denoted $\ell_\infty(\ell_\infty/c_0)$, is isomorphically embeddable into ℓ_∞/c_0. The link to the former comes from the fact that the existence of such an embedding is essential to the argument of Drewnowski and Roberts which shows that ℓ_∞/c_0 is primary. However, this question is interesting in its own right, as it relates to the universality properties of ℓ_∞/c_0 [1].

The existence of an embedding $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ is a direct consequence of the corresponding topological statement, namely, that the closure of an open F_σ set in $\mathbb{N}^*$ is a retract of $\mathbb{N}^*$. Let us take a moment to comment on this last statement, as the development of our understanding of the topological setting has served as a guide to the research in the Banach space context. It is well-known that under CH the closure of an open F_σ set in $\mathbb{N}^*$ is a retract of $\mathbb{N}^*$ (see [39]), but this was proved to be false in the Cohen model in [19]. Moreover, thanks to the development of the theory of the Open Coloring Axiom we know that it does not hold in models of this axiom either: by Theorem 6 of [58] and by Stone duality it is enough to notice that $\wp(\mathbb{N})/(\emptyset \times \mathit{Fin})$ [2] is isomorphic to the Boolean algebra $\mathfrak{B}$ of clopen subsets of $\overline{\bigcup_{n\in\mathbb{N}} A_n^*}$, for some disjoint family $(A_n)_{n\in\mathbb{N}}$ such that $\mathbb{N} = \bigcup_{n\in\mathbb{N}} A_n$. Indeed, identify $\mathbb{N}\times\mathbb{N}$ with $\mathbb{N}$ in such a way that the columns $c_k = \{(k,n) \in \mathbb{N}^2 : n \in \mathbb{N}\}$ correspond with the A_n. Then it is routine to check that $\varphi : \mathfrak{B} \to \wp(\mathbb{N})/(\emptyset \times \mathit{Fin})$ defined by $\varphi(B^* \cap \overline{\bigcup_{n\in\mathbb{N}} A_n^*}) = [B]_{\emptyset\times \mathit{Fin}}$ is a Boolean isomorphism.

1. It is known that under CH this space is universal for the class of Banach spaces of density continuum, but this does not hold in ZFC alone (see [40, 53, 9]).

2. The ideal $\emptyset \times \mathit{Fin}$ is thus denoted in [25]. It is the ideal generated by $\{L_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$, where $L_\sigma = \{(m,n) \in \mathbb{N}^2 : n \leq \sigma(m)\}$ for every $\sigma \in \mathbb{N}^\mathbb{N}$.

Turning back to the question of the embeddability of $\ell_\infty(\ell_\infty/c_0)$ in ℓ_∞/c_0, it turns out that Brech and Koszmider were able to emulate and adapt the argument in [19] to obtain that in the Cohen model there is no such isomorphic embedding. Nowadays, attention is directed towards obtaining the same impossibility result from an axiomatic approach, which from a set-theoretic point of view would be interesting in itself. However, in the context of the question of the primariness of ℓ_∞/c_0 this gains interest as Koszmider has mapped in [33] (see questions 20 and 21) a possible route to the solution of this problem which could work in some forcing extensions of PFA, as suggested by a result in [18].

Dow has recently made a breakthrough development in this direction by proving in [21] that under PFA there is no linear extension operator from the space of continuous functions on the closure of an open F_σ subset on $\mathbb{N}^*$, i.e., if $E \subseteq \mathbb{N}^*$ is the closure of an open F_σ, then there is no linear bounded operator $T : C(E) \to C(\mathbb{N}^*)$ such that $T(f)|E = f$, for every $f \in C(E)$. Here, we build upon the work of Dow to show the impossibility under PFA of embedding $\ell_\infty(\ell_\infty/c_0)$ into ℓ_∞/c_0 by means of a larger class of operators.

We begin by listing some of the properties of the embedding of $\ell_\infty(\ell_\infty/c_0)$ into ℓ_∞/c_0 constructed in [22] using CH. This serves as a backdrop against which to contrast the impossibility results of Section 2.5.

The main result of the chapter is proved in Section 2.3, and in Section 2.4 we show why it is an improvement to Dow's result. In Section 2.5 we discuss some well-known classes of operators in the light of our main result. We end the chapter with some remarks and suggest some possible ways forward toward the complete resolution of the main question considered.

2.2 Embedding $\ell_\infty(C(\mathbb{N}^*))$ into $C(\mathbb{N}^*)$ under CH

A key step in the now classical result by L. Drewnowski and J. W. Roberts (see [22]) which proves the primariness of $C(\mathbb{N}^*)$ under CH is the construction of an isomorphic embedding of $\ell_\infty(C(\mathbb{N}^*))$ into $C(\mathbb{N}^*)$ such that the image has complemented range. This is achieved using the fact that under CH the closure of every open F_σ set in $\mathbb{N}^*$ is a retract of $\mathbb{N}^*$ (see [39]). Indeed, given an almost disjoint family $(A_n)_{n\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ and a retraction $\psi : \mathbb{N}^* \to E$, where $E = \overline{\bigcup_{n\in\mathbb{N}} A_n^*}$, we have that $\ell_\infty(C(\mathbb{N}^*))$ is isometrically isomorphic to $C(E)$ and $T_\psi : C(E) \to C(\mathbb{N}^*)$, given by $T_\psi(f) = f \circ \psi$, is an isomorphic embedding. The purpose of this section is to show some of the properties of this embedding. To this end, we start by giving some relevant definitions.

For the sake of simplifying some of the statements, in this chapter we will be using a variation of our terminology from Chapter 1:

Definition 2.2.1. Let $S : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a linear bounded operator and let $F \subseteq \mathbb{N}^*$ be closed and $A \subseteq \mathbb{N}$ be infinite. We will say that S is canonical on (A^*, F) if there exists a quasi-open mapping $\psi : F \to A^*$ and a nonzero real r such that $S(f)|F = rf \circ \psi$, for every A^*-supported $f \in C(\mathbb{N}^*)$.

If F is clopen we will simply say that S is canonical on a clopen set.

Since we are interested in taking advantage of our knowledge of operators on $C(\mathbb{N}^*)$ in the study of operators on $\ell_\infty(C(\mathbb{N}^*))$, we establish the following notation:

Definition 2.2.2. If $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ is a bounded linear operator, we will denote by $T_n : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ the n-th coordinate of T, that is, $T_n(f) = T((\delta_n(i)f)_{i\in\mathbb{N}})$, for every $f \in C(\mathbb{N}^*)$.

We summarize some of the properties of T_ψ in the following:

Remark 2.2.3. Let $(A_n)_{n\in\mathbb{N}}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$ and put $E = \overline{\bigcup_{n\in\mathbb{N}} A_n^*}$. If $\psi : \mathbb{N}^* \to E$ is a retraction, we have that $T_\psi : C(E) \to C(\mathbb{N}^*)$, given by $T_\psi(f) = f \circ \psi$, has the following properties:

1. it is an isomorphic embedding
2. it is positive
3. it is an isometry
4. its range is complemented in $C(\mathbb{N}^*)$
5. every $(T_\psi)_n$ preserves multiplication
6. every $(T_\psi)_n$ is funnelless and is canonical on a clopen set.

Proof. (1)–(4) are well-known or easy consequences of the fact that ψ is a retraction.

For (5), notice that $(T_\psi)_n : C(A_n^*) \to C(\mathbb{N}^*)$ is given by

$$(T_\psi)_n(f)(y) = \begin{cases} (f\circ\psi)(y), \text{ if } y \in \psi^{-1}[A_n^*] \\ 0, \text{ otherwise} \end{cases}$$

Therefore, $(T_\psi)_n(f \cdot g)(y) = (T_\psi)_n(f)(y) \cdot (T_\psi)_n(g)(y)$, for every $y \in \mathbb{N}^*$.

For (6), notice that $\psi|A_n^* : A_n^* \to A_n^*$, which is the identity, is a witness to the fact that $(T_\psi)_n$ is canonical on a clopen set. It is funnelless by 1.3.22. □

The fact that the only known embedding of $\ell_\infty(C(\mathbb{N}^*))$ into $C(\mathbb{N}^*)$ has these properties serves as motivation for considering the corresponding classes of operators in Section 2.5.

2.3 Embedding $\ell_\infty(C(\mathbb{N}^*))$ into $C(\mathbb{N}^*)$ under PFA

In this section we investigate what can be said about isomorphic embeddings of the ℓ_∞-sum of $C(\mathbb{N}^*)$ into $C(\mathbb{N}^*)$ under PFA, building on Dow's breakthrough result. For our main theorem we need the following lemma, which says that every isomorphic embedding $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ gives rise to another which can be nicely decomposed into the sum of the T_n.

Lemma 2.3.1. *Let $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ be a bounded linear operator and let $\psi_n : B_n^* \to A_n^*$ be a homeomorphism between clopen subsets of $\mathbb{N}^*$, for each $n \in \mathbb{N}$. Then, there exist an infinite $M \subseteq \mathbb{N}$, infinite $B_i' \subseteq_* B_i$ and infinite A_i' such that $\{B_i' : i \in M\}$ is a disjoint family, $\psi_i^{-1}[(A_i')^*] \subseteq (B_i')^*$ and for every $n \in M$ we have*

$$T((\chi_M(i)f_i)_{i\in\mathbb{N}})|(B_n')^* = T_n(f_n)|(B_n')^*$$

for every $(f_i)_{i\in\mathbb{N}} \in \ell_\infty(C(\mathbb{N}^*))$ with $supp(f_i) \subseteq (A_i')^*$.

Proof. We claim there are an infinite $M \subseteq \mathbb{N}$ and infinite $B'_i \subseteq_* B_i$ for $i \in M$ such that $\{B'_i : i \in M\}$ is a disjoint family. We choose these sets recursively as follows. Let $E_0 = B_0$ and $n_0 = 0$. Suppose that for some $k \in \mathbb{N}$ we have $E_i \subseteq_* B_{n_i}$ for every $i \leq k$ such that the E_i's are pairwise disjoint. If $\bigcup_{i \leq k} E_i \subseteq_* B_j$ for infinitely many $j > n_k$, then the recursion stops and we put $M = \{j \in \mathbb{N} : E_0 \subseteq_* B_j\}$ and take $(B'_i)_{i \in \mathbb{N}}$ to be a partition of E_0 into infinite sets. Otherwise, there exists $n_{k+1} > n_k$ and an infinite $E_{k+1} \subseteq B_{n_{k+1}} \setminus \bigcup_{i \leq k} E_i$. If the recursion does not stop after finitely many steps, then we put $B'_{n_i} = E_i$ for $i \in \mathbb{N}$, and $M = \{n_i : i \in \mathbb{N}\}$.

Therefore, we may assume that the B_i's are pairwise disjoint.

By the Stone-Weierstrass theorem and since T and T_n are norm continuous, it is enough to find infinite $B'_n \subseteq_* B_n$ and infinite $A'_n \subseteq \mathbb{N}$ such that $\psi_n^{-1}[(A'_n)^*] \subseteq (B'_n)^*$ and such that for every sequence of infinite sets $(E_i)_{i \in \mathbb{N}}$ with $E_i \subseteq_* A'_i$ we have

$$T((\chi_M(i)\chi_{E_i^*})_{i \in \mathbb{N}})|(B'_n)^* = T_n(\chi_{E_n^*})|(B'_n)^*$$

for every $n \in M$.

So suppose this is not the case. We carry out an inductive construction of length ω_1 as follows. Fix $\gamma < \omega_1$ and suppose for every $\alpha < \gamma$ we have constructed

1. $B_i^\alpha \subseteq_* B_i$ for each $i \in \mathbb{N}$ such that $B_i^\beta \subseteq_* B_i^\alpha$ if $\alpha < \beta < \gamma$
2. $A_i^\alpha \subseteq_* A_i$ for each $i \in \mathbb{N}$ such that $A_i^\alpha \cap A_i^\beta =_* \emptyset$ if $\alpha \neq \beta$, $\alpha < \beta < \gamma$, and such that $(B_i^\beta)^* \cap \psi_i^{-1}[(A_i^\alpha)^*] = \emptyset$, if $\alpha \leq \beta < \gamma$.
3. $n_\alpha \in M$, $m_\alpha \in \mathbb{N}$ such that for every $x \in (B_{n_\alpha}^\alpha)^*$ we have

$$|T((\chi_M(i)\chi_{(A_i^\alpha)^*})_{i \in \mathbb{N}})(x) - T_{n_\alpha}(\chi_{(A_{n_\alpha}^\alpha)^*})(x)| > 1/m_\alpha.$$

For each $i \in \mathbb{N}$ take an infinite D_i such that $D_i \subseteq_* B_i^\alpha$ for every $\alpha < \gamma$ (take $D_i = B_i$ if $\gamma = 0$). Take a sequence of infinite E_i's such that $\psi_i^{-1}[E_i^*] \subseteq D_i^*$. By our assumption there exist infinite $E'_i \subseteq_* E_i$, $n_\gamma \in M$ and $m \in \mathbb{N}$ such that

$$|T((\chi_M(i)\chi_{(E'_i)^*})_{i \in \mathbb{N}})(x) - T_{n_\gamma}(\chi_{(E'_{n_\gamma})^*})(x)| > 1/m \tag{2.1}$$

for some $x \in D_{n_\gamma}^*$. Then, there exists $B'_{n_\gamma} \subseteq_* D_{n_\gamma}$ such that 2.1 holds for every $x \in B'_{n_\gamma}$. Let $E_{n_\gamma}^1$, $E_{n_\gamma}^2$ be infinite such that $E_{n_\gamma}^1 \cup E_{n_\gamma}^2 = E'_{n_\gamma}$ and $(B'_{n_\gamma})^*$ is not included in $\psi^{-1}[(E_{n_\gamma}^1)^*]$ nor in $\psi^{-1}[(E_{n_\gamma}^2)^*]$, and for every $i \neq n_\gamma$ take an arbitrary partition into infinite sets $E_i^1 \cup E_i^2 = E'_i$. Then, for some $j \in \{1, 2\}$ we have

$$|T((\chi_M(i)\chi_{(E_i^j)^*})_{i \in \mathbb{N}})(x) - T_{n_\gamma}(\chi_{(E_{n_\gamma}^j)^*})(x)| > 1/(2m) \tag{2.2}$$

for every $x \in (B'_{n_\gamma})^*$. So let $m_\gamma = 2m$; let $A_i^\gamma = E_i^j$ for every $i \in \mathbb{N}$; let $B_{n_\gamma}^\gamma$ be such that $(B_{n_\gamma}^\gamma)^* = (B'_{n_\gamma})^* \setminus \psi_{n_\gamma}^{-1}[(A_{n_\gamma}^\gamma)^*]$; and for every $i \neq n_\gamma$ take B_i^γ such that $(B_i^\gamma)^* = D_i^* \setminus \psi_i^{-1}[(A_i^\gamma)^*]$. This ends the inductive construction.

Let $n, m \in \mathbb{N}$ be such that $(n_\alpha, m_\alpha) = (n, m)$ for infinitely many α's. Let $k \in \mathbb{N}$ be such that $k/2m > \|T\|$ and take $\alpha_0 < \cdots < \alpha_{k-1} < \omega_1$ such that $(n_{\alpha_j}, m_{\alpha_j}) = (n, m)$, for every $j < k$. Fix $x \in (B_n^{\alpha_{k-1}})^*$ and let $\overline{f_j} = (\chi_M(i)\chi_{(A_i^{\alpha_j})^*})_{i\in\mathbb{N}}$, for $j < k$. Then, there exists $F \subseteq k$ with at least $k/2$ elements such that

$$\begin{aligned} |T(\sum_{j\in F}\overline{f_j} - (\delta_n(i)\chi_{(A_i^{\alpha_j})^*})_{i\in\mathbb{N}})(x)| &= \sum_{j\in F}|T(\overline{f_j})(x) - T_n(\chi_{(A_i^{\alpha_j})^*})(x)| \\ &> (k/2)(1/m) \\ &> \|T\|. \end{aligned}$$

But since for each $i \in \mathbb{N}$ we have that the $A_i^{\alpha_j}$'s are almost disjoint, we obtain $\|\sum_{j\in F}\chi_{(A_i^{\alpha_j})^*}\| = 1$. So,

$$|T(\sum_{j\in F}\overline{f_j} - (\delta_n(i)\chi_{(A_i^{\alpha_j})^*})_{i\in\mathbb{N}})(x)| \leq \|T\|\|\sum_{j\in F}\overline{f_j} - (\delta_n(i)\chi_{(A_i^{\alpha_j})^*})_{i\in\mathbb{N}}\| = \|T\|.$$

A contradiction. □

With this we are ready to prove the main result of the chapter.

Theorem 2.3.2 (PFA). *There is no isomorphic embedding $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ such that every T_n is canonical on a clopen set.*

Proof. For each $n \in \mathbb{N}$ consider a nonzero real r_n, infinite $A_n,\ B_n \subseteq \mathbb{N}$ and a continuous quasi-open map $\psi_n : B_n^* \to A_n^*$ such that $T_n(f)|B_n^* = r_n(f\circ\psi_n)$, for every A_n^*-supported f.

By Lemma 1.6.3 we may assume that ψ_n is actually a homeomorphism. By Lemma 2.3.1, there exist an infinite $M \subseteq \mathbb{N}$, infinite $B_n' \subseteq_* B_n$ and infinite A_n' such that $\{B_n' : n \in M\}$ is a disjoint family, $\psi_n^{-1}[(A_n')^*] \subseteq (B_n')^*$ and for every $n \in M$ we have

$$T((\chi_M(i)f_i)_{i\in\mathbb{N}})|(B_n')^* = T_n(f_n)|(B_n')^*$$

for every $(f_i)_{i\in\mathbb{N}} \in \ell_\infty(C(\mathbb{N}^*))$ with $supp(f_i) \subseteq (A_i')^*$. Since $\psi_n^{-1}[(A_n')^*] \subseteq (B_n')^*$, we may assume that ψ_n is a homeomorphism from $(B_n')^*$ onto $(A_n')^*$.

Since T is an embedding and is bounded, there exist $m_0, m_1 > 0$ such that $m_0 \leq \|T(f)\| \leq m_1$, for every f in the unit ball of $\ell_\infty(C(\mathbb{N}^*))$. In particular we have $m_0 \leq \|T_n(\chi_{(A_n')^*})\| \leq m_1$, for every $n \in \mathbb{N}$. But since $|r_n| = \|r_n(\chi_{(A_n')^*}\circ\psi_n)\| = \|T_n(\chi_{(A_n')^*})\|$, we have that $m_0 \leq |r_n| \leq m_1$, for every $n \in \mathbb{N}$.

Now define $S : C(\overline{\bigcup_{n\in M}(B_n')^*}) \to \ell_\infty(C(\mathbb{N}^*))$ by

$$S(f)(n) = \begin{cases} \frac{1}{r_n}(f\circ\psi_n^{-1})|(A_n')^* \cup 0_{\mathbb{N}^*\setminus(A_n')^*}, & \text{if } n \in M \\ 0, & \text{otherwise} \end{cases}$$

It is clear that S is a linear operator. Since $|r_n| \geq m_0$ holds for every $n \in \mathbb{N}$, we have that S is bounded. Moreover, it is easy to see that $\|S(f)\| \geq 1/m_1$ whenever $f \in C(\overline{\bigcup_{n\in M}(B_n')^*})$ is such that $\|f\| = 1$. So S is an isomorphic embedding. Then, $T\circ S$ is an isomorphic embedding from $C(\overline{\bigcup_{n\in M}(B_n')^*})$ into $C(\mathbb{N}^*)$ such that $(T\circ S)(f)|\overline{\bigcup_{n\in M}(B_n')^*} = f$. But this contradicts Theorem 1 of [21].

□

Corollary 2.3.3 (PFA). *There is no isomorphic embedding $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ such that every T_n is fountainless.*

Proof. See Theorem 1.6.5. □

2.4 Linear extension operators

A. Dow has proved in [21] that there is no linear extension operator from the space of continuous functions on the closure of an open F_σ subset on $\mathbb{N}^*$. Such linear extension operators give rise to isomorphic embeddings of $\ell_\infty(C(\mathbb{N}^*))$ into $C(\mathbb{N}^*)$, for if $E \subseteq \mathbb{N}^*$ is the closure of an open F_σ then $\ell_\infty(C(\mathbb{N}^*))$ is isometrically isomorphic to $C(E)$. For the sake of the following proposition, let us say that an embedding $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ is given by a linear extension if it is constructed in this manner.

With the following proposition we show that Theorem 2.3.2 deals with a larger class of operators than Dow's original result.

Proposition 2.4.1. *If $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ is an embedding given by an linear extension, then every T_n is canonical on a clopen set. Moreover, if there exists a retraction $\varphi : \mathbb{N}^* \to \overline{\bigcup_{n\in\mathbb{N}} A_n^*}$, then there exists an embedding $R : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ which is not given by a linear extension and is such that every R_n is canonical on a clopen set.*

Proof. By hypothesis, there exist an almost disjoint family $(A_i)_{i\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}^*$, a sequence of homeomorphisms $\psi_i : A_i^* \to \mathbb{N}^*$, and a linear extension operator $T' : C(E) \to C(\mathbb{N}^*)$, where $E = \overline{\bigcup_{i\in\mathbb{N}} A_i^*}$, such that $T((f_i)_{i\in\mathbb{N}}) = T'(\bigcup_{i\in\mathbb{N}} f_i \circ \psi_i)$. Here there is mild abuse justified by the fact that $\bigcup_{i\in\mathbb{N}} A_i^*$ is C^*-embedded in $\overline{\bigcup_{i\in\mathbb{N}} A_i^*}$ (see 1.2 of [61]).

Notice that for every $n \in \mathbb{N}$ and every $f \in C(\mathbb{N}^*)$ we have that $T_n(f) = T((\delta_n(i)f)_{i\in\mathbb{N}}) = T'(f \circ \psi_n \cup 0)$, and so $T_n(f)|A_n^* = T'(f \circ \psi_n \cup 0)|A_n^* = f \circ \psi_n$. Therefore, T_n is canonical on $(\mathbb{N}^*, A_n^*)$.

For the second part of the proposition we adapt the argument of 1.2.16. Let $\mathbb{N} = \bigcup_{i\in\mathbb{N}} B_i$ be a partition into infinite sets and fix $x_i \in \overline{\bigcup_{n\in B_i} A_n^*} \setminus \bigcup_{n\in B_i} A_n^*$ for each $i \in \mathbb{N}$. Let σ be a permutation of $\mathbb{N}$ which does not fix any infinite set. To lighten notation, let $\overline{f} \in C(E)$ stand for the extension of $\bigcup_{n\in\mathbb{N}} f_n \circ \psi_n$. Now define

$$R((f_n)_{n\in\mathbb{N}})(y) = (\overline{f} \circ \varphi)(y) - \overline{f}(x_i) + \overline{f}(x_{\sigma(i)}),$$

for every $(f_n)_{n\in\mathbb{N}} \in \ell_\infty(C(\mathbb{N}^*))$ and for every $y \in \varphi^{-1}[\bigcup_{n\in B_i} A_n^*]$. Notice that since $\mathbb{N}^*$ is an F-space (see 1.2 from [61]), we have that the collection $\{\bigcup_{n\in B_i} A_n^* : i \in \mathbb{N}\}$ is pairwise disjoint and covers $\bigcup_{n\in\mathbb{N}} A_n^*$. Therefore, R is well defined. It is clear that it is linear and bounded, and it is an embedding because if $R((f_n)_{n\in\mathbb{N}})(y) = 0$ for some $(f_n)_{n\in\mathbb{N}} \in \ell_\infty(C(\mathbb{N}^*))$ and every $y \in \mathbb{N}^*$, then in particular $0 = R((f_n)_{n\in\mathbb{N}})(x_i) = \overline{f}(x_{\sigma(i)})$ for every $i \in \mathbb{N}$, which implies that $(f_n)_{n\in\mathbb{N}} = 0$. Moreover, R is not an extension operator because if $(f_n)_{n\in\mathbb{N}}$ is such that $\overline{f}(x_i) \neq \overline{f}(x_{\sigma(i)})$, then $R((f_n)_{n\in\mathbb{N}})(x_i) = \overline{f}(x_{\sigma(i)}) \neq \overline{f}(x_i)$. But every R_n is canonical because $R_n(f) = R((\delta_n(i)f)_{i\in\mathbb{N}})|A_n^* = f \circ \psi_n$.

□

2.5 Impossible embeddings under PFA

In this section we use Theorem 2.3.2 to prove the non-existence under PFA of certain well-known classes of isomorphic embeddings. Therefore, we are interested in conditions on an embedding $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ which imply that every T_n is canonical on a clopen set.

We begin by recalling some known results and by calling our attention upon some easy facts

Proposition 2.5.1. *Let $S : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator. Then, S is canonical on (A^*, F) for some infinite $A \subseteq \mathbb{N}$ and some closed $F \subseteq \mathbb{N}^*$ if any one of the following holds*

1. *S is a positive embedding*
2. *S is a norm-preserving isomorphic embedding.*

Proof.
1. An easy consequence of Theorem 4.4 from [46] together with lemmas A.1.4 and A.1.6 is that every positive embedding from $C(\mathbb{N}^*)$ into itself is canonical on some (A^*, F).
2. By the results of Holsztyński ([29]) and of Amir and Arbel ([2]), we know that there exists a continuous mapping ψ from a closed subset $F \subseteq \mathbb{N}^*$ onto $\mathbb{N}^*$ such that for every $f \in C(\mathbb{N}^*)$ we have $S(f)|F = \alpha(f \circ \psi)$, where $\alpha : F \to \{-1, 1\}$ is continuous. Once more by lemmas A.1.4 and A.1.6 and by further restricting to a subset if necessary, we conclude that S canonical on (A^*, F), for some infinite $A \subseteq \mathbb{N}$ and some closed $F \subseteq \mathbb{N}^*$.

□

The following remark reduces the problem of finding a canonization on a clopen set for the operators in the previous proposition to the problem of proving that the closed set F is not nowhere dense.

Remark 2.5.2. Let $S : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator and suppose that it is canonical on (A^*, F), for some infinite $A \subseteq \mathbb{N}$ and some closed $F \subseteq \mathbb{N}^*$. If F has nonempty interior, then S is canonical on a clopen set.

Another easy remark is that every T_n inherits important properties of T:

Remark 2.5.3. If $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ is an isomorphic embedding, then every T_n is also an isomorphic embedding. If in addition T has complemented range, then every T_n has complemented range.

Proof. Since T is an isomorphic embedding, there exists a positive real m such that $\|T(\overline{g})\| \geq m$, for every norm one $\overline{g}$ in $\ell_\infty(C(\mathbb{N}^*))$. Then, for every norm one f in $C(\mathbb{N}^*)$ we have $\|T_n(f)\| = \|T((\delta_n(i)f)_{i\in\mathbb{N}})\| \geq m$, which means that T_n is an isomorphic embedding.

Assume that T has complemented range. Since T is an isomorphism onto its image and each factor of the ℓ_∞-sum of $C(\mathbb{N}^*)$ is complemented in $\ell_\infty(C(\mathbb{N}^*))$, we have that the range of T restricted to each factor, i.e. the range of T_n, is complemented in the range of T. Therefore, the range of T_n is complemented in $C(\mathbb{N}^*)$.

□

So we are able to conclude the following

Corollary 2.5.4 (PFA). *There is no isomorphic embedding $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ such that every T_n is funnelless and any one of the following holds*

1. *T is positive*
2. *T preserves the norm.*

Proof. 1. Assume T is positive. By the definition of T_n it is clear that each T_n must be positive as well. By 2.5.3 and 2.5.1 we know that each T_n is canonical on some (A^*, F), and since T_n is funnelless F cannot be nowhere dense.

2. Assume T is an isometry. By the definition of T_n we have that $\|T_n(f)\| = \|T((\delta_n(i)f)_{i\in\mathbb{N}})\| = \|(\delta_n(i)f)_{i\in\mathbb{N}}\| = \|f\|$. So T_n is also an isometric embedding. As before, by 2.5.3 and 2.5.1 we know that each T_n is canonical on some (A^*, F), and since T_n is funnelless F cannot be nowhere dense.

□

We now turn our attention towards another type of operator, namely, multiplicative operators.

Lemma 2.5.5 (OCA+MA). *Let $S : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be an isomorphic embedding which preserves multiplication. Then, S is canonical on a clopen set.*

Proof. It is well-known (see for example Theorem 4.27 from [1]) that a linear bounded operator between function spaces which preserves multiplication is somewhere a composition operator. Therefore, for some infinite $A \subseteq \mathbb{N}$ there exists a continuous map $\psi : A^* \to \mathbb{N}^*$ such that for every $f \in C(\mathbb{N}^*)$ we have

$$S(f)(x) = \begin{cases} (f\circ\psi)(x), & \text{if } x \in A^*, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, since S is an embedding we have that ψ must be onto $\mathbb{N}^*$, and so by Lemma 1.6.3 ψ is a homeomorphism when restricted to some clopen set. Hence, S is canonical on a clopen set. □

So we obtain the following

Corollary 2.5.6 (PFA). *There is no isomorphic embedding $T : \ell_\infty(C(\mathbb{N}^*)) \to C(\mathbb{N}^*)$ such that every T_n preserves multiplication.*

2.6 Open problems and final remarks

To start, we would like to point out that 2.5.4 could be improved if we could prove that the fact that T is positive or norm-preserving implies that in the canonizations for each T_n obtained thanks to 2.5.1, the closed part cannot be nowhere dense. An approach towards this goal where success seems plausible is suggested by the following proposition, which is a consequence of results of Farah 3.3.3 and 3.8.1 of [25] (see Corollary 7 of [20]) and further reduces the problem of finding a canonization on a clopen set:

Proposition 2.6.1 (OCA+MA). *If $F \subseteq \mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$ and is not c.c.c. over Fin, then it is not nowhere dense.*

Half of the problem is solved by the following

Proposition 2.6.2. *Let $S : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be an isomorphic embedding and suppose it is canonical on (A^*, F). Then, F is not c.c.c. over Fin if any one of the following holds*

1. *S is positive*
2. *S preserves the norm.*

Proof. 1. Let $\psi : F \to A^*$ be a witness to the fact that S is canonical on (A^*, F).
CLAIM: If A_1, A_2 are infinite almost disjoint subsets of A, then there is an infinite $B \subseteq \mathbb{N}$ such that $B^* \cap F = \psi^{-1}[A_1]$ and for every infinite $A_3 \subseteq A_2$ we have

$$\{x \in B^* : S(\chi_{A_3})(x) \geq 1/2\} = \emptyset.$$

As $\{x \in F : S(\chi_{A_1^*})(x) \geq 1/2\} = \psi^{-1}[A_1^*]$ and $\{x \in F : S(\chi_{A_2^*})(x) \geq 1/2\} = \psi^{-1}[A_2^*]$ are disjoint, by the normality of $\mathbb{N}^*$ we find an infinite $B \subseteq \mathbb{N}$ such that $B^* \cap F = \psi^{-1}[A_1]$ and $\{x \in B^* : S(\chi_{A_2})(x) \geq 1/2\} = \emptyset$. Now, for any infinite $A_3 \subseteq A_2$ we have $0 \leq \chi_{A_3^*} \leq \chi_{A_2^*}$ and so by the positivity of S we have $0 \leq S(\chi_{A_3^*}) \leq S(\chi_{A_2^*})$, which gives the claim.
Using the claim we can construct by induction a sequence $\{B_\xi : \xi < \omega_1\}$ of infinite subsets of $\mathbb{N}$ and a sequence $\{A_\xi : \xi < \omega_1\}$ of almost disjoint infinite subsets of A such that for every $\xi < \eta < \omega_1$ we have $B_\xi^* \cap F = \psi^{-1}[A_\xi]$ and

$$\{x \in B_\xi^* : S(\chi_{A_\eta})(x) \geq 1/2\} = \emptyset.$$

Now consider infinite $C_\xi \subseteq \mathbb{N}$ such that

$$\psi^{-1}[A_\xi^*] \subseteq C_\xi^* \subseteq \{x \in B_\xi^* : S(\chi_{A_\xi})(x) > 2/3\}.$$

This is possible by the normality of $\mathbb{N}^*$ and the fact that $S(\chi_{A_\xi})(x) = (\chi_{A_\xi^*} \circ \psi)(x) = 1$ if $x \in \psi^{-1}[A_\xi^*]$. Note that if $\xi < \eta$, a point in $C_\xi^* \cap C_\eta^*$ would give rise to a point in $\{x \in B_\xi^* : S(\chi_{A_\eta})(x) \geq 1/2\}$ which is impossible, so the C_ξ's are almost disjoint. On the other hand, they are F-positive since $\psi^{-1}[A_\xi^*] \subseteq C_\xi^*$. Therefore, F is not c.c.c. over Fin.

2. Let $\psi : F \to A^*$ be a witness to the fact that S is canonical on (A^*, F). Let $(A_\xi)_{\xi<\omega_1}$ be an almost disjoint family of infinite subsets of A. Notice that for each $\xi < \omega_1$, $\psi^{-1}[A_\xi^*]$ is compact and is a subset of the open set $U_\xi = \{y \in \mathbb{N}^* : |S(\chi_{A_\xi^*})(y)| > 2/3\}$. So let B_ξ be infinite such that $\psi^{-1}[A_\xi^*] \subseteq B_\xi^* \subseteq U_\xi$. Since S preserves the norm, we have that the sets U_ξ are pairwise disjoint. Therefore, $(B_\xi)_{\xi<\omega_1}$ is an almost disjoint family such that $B_\xi^* \cap F \neq \emptyset$ for every $\xi < \omega_1$, i.e., F is not c.c.c. over Fin.

□

In this line of thought it is natural to ask two more general questions:

Problem 2.6.3. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is an automorphism (an isomorphic embedding) which is canonical on (A^*, F), for some infinite $A \subseteq \mathbb{N}$ and some closed $F \subseteq \mathbb{N}^*$. Is it consistent (under OCA + MA, or PFA) that F cannot be c.c.c. over Fin?*

Problem 2.6.4. *Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is an automorphism and $F \subseteq \mathbb{N}^*$ is closed nowhere dense and $A \subseteq \mathbb{N}$ infinite such that (F, A^*) is a fountain for T or (A^*, F) is a funnel for T. Is it true or consistent (under OCA+MA, or PFA) that F cannot be c.c.c. over Fin?*

Notice that these ideas are related to problem 1.7.3.

Turning to the general question of embeddability of $\ell_\infty(C(\mathbb{N}^*)$ into $C(\mathbb{N}^*)$, we should note that Theorem 2.3.2 reduces this problem to finding a canonization for any isomorphic embedding $S : \ell_\infty/c_0 \to \ell_\infty/c_0$. Therefore a positive answer to 1.7.2 would imply that $\ell_\infty(C(\mathbb{N}^*)$ is not embeddable into $C(\mathbb{N}^*)$ under PFA. This would have the interesting consequence that ℓ_∞/c_0 is not universal for the class of Banach spaces of density continuum under PFA. Also, it would further support the conjecture that ℓ_∞/c_0 is not primary under PFA or some forcing extensions of it.

Other unexplored routes which may lead to a solution of this general problem are suggested by some representations of $\ell_\infty(\ell_\infty/c_0)$ which relate it more directly to $\wp(\mathbb{N})/(\emptyset \times \textit{Fin})$, thus opening the possibility of profiting from the well developed knowledge on analytic quotients existing today (see e.g. [25, 58]). In order to obtain the representations we are referring to, we start by the following

Definition 2.6.5. If $\mathcal{I}$ is an ideal over $\mathbb{N}$, we define $D_\mathcal{I} \subseteq \ell_\infty$ by stating that $f \in D_\mathcal{I}$ if, and only if, for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |f(n)| \geq \varepsilon\}$ is in the ideal $\mathcal{I}$.

Remark 2.6.6. $D_\mathcal{I}$ is a closed subspace of ℓ_∞.

Proof. To see that it is a subspace it is sufficient to note that for every $f, g \in \ell_\infty$, every nonzero $r \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\begin{aligned}\{n \in \mathbb{N} : |rf(n) + g(n)| \geq \varepsilon\} &\subseteq \{n \in \mathbb{N} : |r||f(n)| + |g(n)| \geq \varepsilon\} \\ &\subseteq \{n \in \mathbb{N} : |f(n)| \geq \frac{\varepsilon}{2|r|}\} \cup \{n \in \mathbb{N} : |g(n)| \geq \varepsilon/2\}.\end{aligned}$$

To see that it is closed, take $f \in \ell_\infty \setminus D_\mathcal{I}$. Then there exists $\varepsilon > 0$ such that $\{n \in \mathbb{N} : |f(n)| \geq \varepsilon\}$ is not in $\mathcal{I}$. It is easy to check that $\{g \in \ell_\infty : \|f - g\| < \varepsilon/2\} \cap D_\mathcal{I} = \emptyset$. □

Finally, if we denote by $K_\mathcal{I}$ the Stone dual of $\wp(\mathbb{N})/\mathcal{I}$ for every ideal $\mathcal{I}$ over $\mathbb{N}$, then we have the following

Proposition 2.6.7. $\ell_\infty(\ell_\infty/c_0) \cong C(K_{\emptyset \times \mathrm{Fin}}) \cong \ell_\infty/D_{\emptyset \times \mathrm{Fin}}$

Proof. Let $\mathbb{N} = \bigcup_{n \in \mathbb{N}} A_n$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite sets. Since $\ell_\infty/c_0 \cong C(\mathbb{N}^*)$, it is clear that $\ell_\infty(\ell_\infty/c_0) \cong C(\overline{\bigcup_{n \in \mathbb{N}} A_n^*})$. So to show $\ell_\infty(\ell_\infty/c_0) \cong C(K_{\emptyset \times \mathit{Fin}})$ it suffices to show that $\wp(\mathbb{N})/(\emptyset \times \mathit{Fin})$ is isomorphic to the Boolean algebra of clopen subsets of $\bigcup_{n \in \mathbb{N}} A_n^*$, but this was already noted in the introduction to the present chapter.

We now turn to the isometry between $\ell_\infty(\ell_\infty/c_0)$ and $\ell_\infty/D_{\emptyset \times \mathit{Fin}}$. Identify $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$ in such a way that the columns $c_k = \{(k, n) \in \mathbb{N}^2 : n \in \mathbb{N}\}$ correspond with the A_n. Then we see that the ideal $\emptyset \times \mathit{Fin}$ corresponds to the family of subsets of $\mathbb{N}$ whose intersection with each A_n is finite. Define $T : \ell_\infty/D_{\emptyset \times \mathit{Fin}} \to \ell_\infty(\ell_\infty/c_0)$ by $T([f]_{D_{\emptyset \times \mathit{Fin}}}) = ([f \upharpoonright A_n]_{c_0})_{n \in \mathbb{N}}$. It is easy to check that this is a linear isometry. □

It is natural to ask:

Question 2.6.8. Are there conditions on an ideal $\mathcal{I}$ such that $\ell_\infty/D_{\mathcal{J}}$ is consistently (under PFA or OCA + MA) not isomorphically embeddable into ℓ_∞/c_0?

The study of quotients $\wp(\mathbb{N})/\mathcal{I}$ ([25]), and in particular Theorem 6 of [58], suggests nonatomic analytic ideals is the natural class of ideals to consider.

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Chapter 3

Sequences of Radon measures and non-weakly compact operators on ℓ_∞/c_0

3.1 Introduction

The main part of this chapter is devoted to giving a "modern" proof of a result which appears in its most general form as Lemma 2 in [55] and is stated below as Lemma 3.2.7. We came upon this result in our study of liftable operators $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ (see 1.4.1) when we noticed that a lifting of such an operator is determined by a sequence in $M(\beta\mathbb{N})$. Indeed, for any operator $R : C(\beta\mathbb{N}) \to C(\beta\mathbb{N})$ there is a weak* continuous function associated to it which sends each $x \in \beta\mathbb{N}$ to $R^\star(\delta_x) \in M(\beta\mathbb{N})$. Moreover, since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, such a function is determined by its values on $\mathbb{N}$ (see 1.2.13).

Despite the fact that in the end we did not make use of the result in the context of the previous chapters, we deemed it worthwhile to present a proof because on the one hand we were able to make a very slight improvement to the already powerful result, and on the other hand, because its existing proof is scattered between several papers: it must be traced all the way back to the work of Lebesgue ([35]), first going through an article by Kadec and Pełczyński ([30]) and Banach's book ([7]). Our proof follows the general guidelines of the original in [30], but nevertheless makes recourse to the Dieudonné-Grothendieck Theorem, reason why we dare qualify it as "modern". In the last section of the chapter we give an application of this result in the context of operators from ℓ_∞/c_0 into itself. It provides a different kind of canonization for non-weakly compact operators on ℓ_∞/c_0.

Throughout this chapter $B(K)$ will denote the σ-algebra of Borel subsets of K for any compact Hausdorff K, while $M(K)$ will denote the Banach space of Radon measures over K with the total variation norm. By the Riesz representation theorem, we view $M(K)$ as the dual of $C(K)$ with the the w^*-topology. We will say that a measure $\mu \in M(K)$ is supported by a set $A \subseteq K$ whenever A has full measure, i.e. whenever $|\mu|(A) = \|\mu\|$. We will say that a signed measure μ in $M(K)$ is positive if $\mu(E) \geq 0$ for all measurable $E \subseteq K$.

3.2 An extracting principle

We will need the following, which is Lemma 1 from [41]:

Lemma 3.2.1 (Pełczyński). *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a bounded sequence in $M(K)$, and let $E_n \subseteq K$ be a sequence of pairwise disjoint Borel sets. If $\lambda_n(E_n) > \delta$ for some $\delta > 0$ and for every $n \in \mathbb{N}$, then there exist a subsequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ and a sequence of disjoint open sets $(U_k)_{k\in\mathbb{N}}$ such that $\lambda_{n_k}(U_k) > \delta/2$, for all $k \in \mathbb{N}$.*

Now recall the following

Definition 3.2.2. Let (X, Σ) be a measurable space and let μ, ν be signed measures on Σ. We say that μ is absolutely continuous with respect to ν if for every measurable set A we have $\mu(A) = 0$ whenever $|\nu|(A) = 0$. This relation will be denoted by $\mu \ll \nu$.

We will need the following easy consequence of Theorem B from section 30 of [27].

Lemma 3.2.3. *Let $\lambda, \mu \in M(K)$ be such that μ is positive and λ is absolutely continuous with respect to μ. Then, for any sequence $(E_n)_{n\in\mathbb{N}}$ of subsets of K, $\mu(E_n) \to 0$ implies $\lambda(E_n) \to 0$.*

We introduce the following definition which will serve to characterize weakly relatively compact subsets of $M(K)$.

Definition 3.2.4. Let $\mu \in M(K)$ be a positive measure. For every $\lambda \in M(K)$ and every real $0 \leq r \leq \|\mu\|$ we define

$$\eta_\mu(\lambda, r) = \sup\{|\lambda|(E) : E \in B(K) \wedge \mu(E) \leq r\}.$$

For every bounded set $M \subseteq M(K)$ and for every real $0 \leq r \leq \|\mu\|$ define

$$\eta_\mu(M, r) = \sup\{\eta_\mu(\lambda, r) : \lambda \in M\}.$$

Finally, let $\eta_\mu(M) = \lim_{r\to 0^+} \eta_\mu(M, r)$.

To see that these are well defined, fix a positive $\mu \in M(K)$. For a fixed $r \in [0, \|\mu\|]$ and any $\lambda \in M(K)$ we have that the set $\{|\lambda|(E) : E \in B(K) \wedge \mu(E) \leq r\}$ is nonempty simply because $\mu(\emptyset) = 0$, and it is bounded because $|\lambda|(E) \leq \|\lambda\|$ for every Borel $E \subseteq K$. This means that $\eta_\mu(\lambda, r)$ exists for every $\lambda \in M(K)$ and it is at most $\|\lambda\|$. So if $M \subseteq M(K)$ is bounded, then $\eta_\mu(M, r)$ is well defined. To see that the limit exists, it is enough to notice that $\eta_\mu(M, r)$ is non-negative and that $\eta_\mu(\lambda, r) \leq \eta_\mu(\lambda, r')$ whenever $r < r'$.

The following is a technical lemma which will be useful in the sequel:

Lemma 3.2.5. *Let $\mu \in M(K)$ be a positive measure and let $M \subseteq M(K)$ be a bounded set such that every element of M is absolutely continuous with respect to μ. If $\eta_\mu(M) > 0$, then there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of pairwise disjoint Borel sets and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of distinct measures in M such that for every $n \in \mathbb{N}$ we have*

$$|\lambda_n|(F_n) > \eta_\mu(M)\left(1 - \frac{1}{2(n+1)}\right).$$

Proof. Choose a sequence $(r_n)_{n\in\mathbb{N}}$ of positive reals converging to zero such that for every $n \in \mathbb{N}$ we have $|\eta_\mu(M) - \eta_\mu(M, r_n)| < \frac{\eta_\mu(M)}{4(n+1)}$. Then, by definition 3.2.4 we may find for each $n \in \mathbb{N}$ a measure $\lambda_n \in M$ and a Borel set $E_n \subseteq K$ such that $\mu(E_n) \le r_n$ and $|\lambda_n|(E_n) > \eta_\mu(M) - \frac{\eta_\mu(M)}{4(n+1)}$.

Notice that since $|\lambda_n| \ll \mu$ (see Theorem A from section 30 of [27]) and $\mu(E_n) \to 0$, we may apply Lemma 3.2.3 to obtain that $(|\lambda_k|(E_n))_{n\in\mathbb{N}}$ converges to zero, for each $k \in \mathbb{N}$. Therefore, there is no $\lambda \in M$ such that $\lambda = \lambda_n$ for infinitely many $n \in \mathbb{N}$, for otherwise there would exist a subsequence $(E_{n_k})_{k\in\mathbb{N}}$ such that $|\lambda|(E_{n_k}) > \eta_\mu(M) - \frac{\eta_\mu(M)}{4(n_k+1)}$ for every $k \in \mathbb{N}$, while at the same time $|\lambda|(E_{n_k}) \to 0$. So we may assume that the measures λ_n are all distinct.

CLAIM: There exists a strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$

$$\sum_{j>i} |\lambda_{n_i}|(E_{n_j}) < \frac{\eta_\mu(M)}{4(n_i+1)}.$$

We construct such a sequence by induction. Let $n_0 = 0$ and assume that for some $k \in \mathbb{N}$ and every $i \le k$ we have constructed n_i such that $|\lambda_{n_i}|(E_{n_j}) < \eta_\mu(M)2^{-n_{j-1}-4}$, for every $i < j \le k$. Since $(|\lambda_{n_i}|(E_m))_{m\in\mathbb{N}}$ converges to zero for every $i \le k$, we may choose $n_{k+1} > n_k$ such that $|\lambda_{n_i}|(E_{n_{k+1}}) < \eta_\mu(M)2^{-n_k-4}$, for every $i \le k$.

Clearly, this construction satisfies the claim, as for any $i \in \mathbb{N}$ we have

$$\sum_{j>i} |\lambda_{n_i}|(E_{n_j}) < \sum_{j>i} \frac{\eta_\mu(M)}{2^{n_{j-1}+4}} \le \sum_{n\ge n_i} \frac{\eta_\mu(M)}{2^{n+4}} = \frac{\eta_\mu(M)}{2^{n_i+3}} < \frac{\eta_\mu(M)}{4(n_i+1)}.$$

Set $F_k = (E_{n_k} \setminus \bigcup_{m>k} E_{n_m})$. Then, $(F_k)_{k\in\mathbb{N}}$ is clearly a sequence of pairwise disjoint Borel sets, and for every $k \in \mathbb{N}$ we have that

$$\begin{aligned}
|\lambda_{n_k}|(F_k) &\ge |\lambda_{n_k}|(E_{n_k}) - \sum_{m>k} |\lambda_{n_k}|(E_{n_m}) \\
&> \eta_\mu(M) - \frac{\eta_\mu(M)}{4(n_k+1)} - \frac{\eta_\mu(M)}{4(n_k+1)} \\
&= \eta_\mu(M)\left(1 - \frac{1}{2(n_k+1)}\right).
\end{aligned}$$

□

We now turn to a characterization of weakly relatively compact sets of measures which will be key in the proof of 3.2.7.

Lemma 3.2.6. *Let $\mu \in M(K)$ be a positive measure and let $M \subseteq M(K)$ be a bounded set such that every element of M is absolutely continuous with respect to μ. M is weakly relatively compact if, and only if, $\eta_\mu(M) = 0$.*

Proof. Suppose M is not weakly relatively compact. Then, the Dieudonné-Grothendieck Theorem (Theorem VII.14 of [15]) implies that there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint open subsets of K, a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of measures in M, and $\varepsilon > 0$ such that $|\lambda_n|(U_n) \geq \lambda_n(U_n) > \varepsilon$, for every $n \in \mathbb{N}$.

On the other hand, notice that $\mu(U_n) \to 0$ because the U_n's are pairwise disjoint. Let $r_n = \mu(U_n)$. Then, it is clear that $\eta_\mu(M, r_n) > \varepsilon$, for every $n \in \mathbb{N}$. Hence, $\eta_\mu(M) \neq 0$.

Conversely, suppose $\eta_\mu(M) > 0$. We will show that M is not weakly relatively compact using the Dieudonné-Grothendieck characterization. By Lemma 3.2.5 we have a sequence $(F_n)_{n\in\mathbb{N}}$ of pairwise disjoint Borel sets and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of measures in M such that for every $n \in \mathbb{N}$ we have

$$|\lambda_n|(F_n) > \eta_\mu(M)\left(1 - \frac{1}{2(n+1)}\right) \geq \frac{\eta_\mu(M)}{2}.$$

For each $n \in \mathbb{N}$, let $K = A_n^0 \cup A_n^1$ be a Hahn decomposition with respect to λ_n where A_n^0 is the positive part. Then, since $|\lambda_n|(F_n) = \lambda_n(F_n \cap A_n^0) - \lambda_n(F_n \cap A_n^1)$, there exists $i \in \{0, 1\}$ such that $(-1)^i\lambda_n(F_n \cap A_n^i) > \eta_\mu(M)/4$ holds for infinitely many $n \in \mathbb{N}$. Since $F_n \cap A_n^i$ is Borel, we may use Lemma 3.2.1 to obtain a subsequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ and disjoint open sets $U_k \subseteq K$ such that $|\lambda_{n_k}(U_k)| = (-1)^i\lambda_{n_k}(U_k) > \eta_\mu(M)/8$, for every $k \in \mathbb{N}$. This implies that for each $k \in \mathbb{N}$ we have $\sup\{|\nu(U_k)| : \nu \in M\} > \eta_\mu(M)/8$. Hence, by the Dieudonné-Grothendieck Theorem we conclude that M is not weakly relatively compact.

□

Lemma 3.2.7. *Let $(\nu_n)_{n\in\mathbb{N}}$ be a bounded sequence in $M(K)$. Then, there exists a non-negative real r and a subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$ each element of which may be decomposed into the sum of two measures $\nu_{n_k} = \lambda_k + \theta_k$ satisfying*

1. *the measures λ_k are supported by pairwise disjoint measurable sets,*
2. *$(\theta_k)_{k\in\mathbb{N}}$ is weakly convergent, and*
3. *$\|\lambda_k\| = r$, for every $k \in \mathbb{N}$.*

Moreover, if $\{\nu_n : n \in \mathbb{N}\}$ is not weakly relatively compact, then $r > 0$.

Proof. Let $M = \{\nu_n : n \in \mathbb{N}\}$. If M is weakly relatively compact, then by the Eberlein-Šmulian Theorem (see Theorem 4.47 in [24]) there is a weakly convergent subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$. So we may set $\lambda_k = 0$ and $\theta_k = \nu_{n_k}$, for every $k \in \mathbb{N}$ (notice that the empty set is a supporting set for the zero measure).

So suppose M is not weakly relatively compact. Define $\mu = \sum_{n\in\mathbb{N}} \frac{|\nu_n|}{2^n}$ and notice that μ is a positive measure such that every ν_n is absolutely continuous with respect to μ.

By Lemma 3.2.6 we know that $\eta_\mu(M) > 0$. By Lemma 3.2.5 and by going to a subsequence if necessary, we may assume that there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of pairwise disjoint Borel subsets of K such that for every $n \in \mathbb{N}$ we have

$$|\nu_n|(F_n) > \eta_\mu(M)\left(1 - \frac{1}{2(n+1)}\right).$$

Let $\theta_n' = \nu_n|(K \setminus F_n)$, for every $n \in \mathbb{N}$, and let $N = \{\theta_n' : n \in \mathbb{N}\}$.

We claim that N is weakly relatively compact. Suppose this is not the case. Notice that $\theta'_n \ll \mu$, as for every $E \subseteq K$ and every $n \in \mathbb{N}$ we have $0 \leq |\theta'_n(E)| = |\nu_n(E \setminus F_n)| \leq |\nu_n|(E \setminus F_n) \leq |\nu_n|(E)$ and $|\nu_n| \ll \mu$ (see Theorem A from section 30 of [27]). N is bounded because M is bounded, and therefore by Lemma 3.2.6 we have that $\eta_\mu(N) > 0$. Then, by the definition of $\eta_\mu(N)$ we know that for each $k \in \mathbb{N}$ there exist $\sigma(k) \in \mathbb{N}$ and $E_k \in B(K)$ such that $\mu(E_k) \to 0$ and $|\theta'_{\sigma(k)}|(E_k) > \eta_\mu(N) - 1/(k+1)$.

Notice that σ may be assumed to be a strictly increasing sequence. Indeed, σ must be finite-to-one because otherwise there would be $\theta \in N$ such that $|\theta|(E_k) > \eta_\mu(N) - 1/(k+1)$, for infinitely many $k \in \mathbb{N}$. But this is a contradiction since $\mu(E_k) \to 0$ implies $|\theta|(E_k) \to 0$, by Lemma 3.2.3 and because $|\theta| \ll \mu$. Therefore, we may choose $\sigma' \subseteq \sigma$ strictly increasing and by re-enumerating we obtain a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $|\nu_{n_k}|(E_k \setminus F_{n_k}) = |\theta'_{n_k}|(E_k) > \eta_\mu(N) - 1/(k+1)$.

Since $\mu(E_k \cup F_{n_k}) \leq \mu(E_k) + \mu(F_{n_k})$, we have that $\mu(E_k \cup F_{n_k}) \to 0$. So let $s_k = \mu(E_k \cup F_{n_k})$ and consider the following:

$$\begin{aligned}\eta_\mu(M, s_k) &\geq |\nu_{n_k}|(E_k \cup F_{n_k}) \\ &= |\nu_{n_k}|(F_{n_k}) + |\nu_{n_k}|(E_k \setminus F_{n_k}) \\ &> \eta_\mu(M)\left(1 - \frac{1}{2(n_k+1)}\right) + \eta_\mu(N) - 1/(k+1).\end{aligned}$$

Therefore, $\eta_\mu(M) \geq \eta_\mu(M) + \eta_\mu(N)$. However, we were assuming that $\eta_\mu(N) > 0$. With this contradiction we conclude that N is weakly relatively compact.

Once again by the Eberlein-Šmulian Theorem, and by going to a subsequence if necessary, we may assume that $(\theta'_n)_{n\in\mathbb{N}}$ is weakly convergent.

Now let $\lambda'_n = \nu_n|F_n$. Observe that since $(\nu_n)_{n\in\mathbb{N}}$ is bounded we have that

$$t \geq \|\nu_n\| \geq |\nu_n|(F_n) > \eta_\mu(M)\left(1 - \frac{1}{2(n+1)}\right) \geq \frac{\eta_\mu(M)}{2} > 0,$$

for some real t and for every $n \in \mathbb{N}$. So since $\|\lambda'_n\| = |\nu_n|(F_n)$, we have that $(\|\lambda'_n\|)_{n\in\mathbb{N}}$ is a bounded sequence of reals which is separated from zero. Let $(\lambda'_{n_k})_{k\in\mathbb{N}}$ be a subsequence such that such that $\|\lambda'_{n_k}\| \to r$, for some $r > 0$.

Finally, define $\lambda_k = \frac{r}{\|\lambda'_{n_k}\|}\lambda'_{n_k}$ and $\theta_k = \theta'_{n_k} - \left(\frac{r}{\|\lambda'_{n_k}\|} - 1\right)\lambda'_{n_k}$, and observe that this choice satisfies the properties we need.

Indeed,

1. $\lambda_k + \theta_k = \lambda'_{n_k} + \theta'_{n_k} = \nu_{n_k}$, for every $k \in \mathbb{N}$.
2. λ_k is supported by F_{n_k}, for every $k \in \mathbb{N}$.
3. Since $\left(\left(\frac{r}{\|\lambda'_{n_k}\|} - 1\right)\lambda'_{n_k}\right)_{k\in\mathbb{N}}$ converges to zero in the norm topology, we have that $(\theta_k)_{k\in\mathbb{N}}$ is still weakly convergent.
4. $\|\lambda_k\| = \frac{r}{\|\lambda'_{n_k}\|}\|\lambda'_{n_k}\| = r$, for every $k \in \mathbb{N}$.

□

3.3 Non-weakly compact operators on ℓ_∞/c_0

To end the chapter, we will give an application of Lemma 3.2.7 in the context of operators from ℓ_∞/c_0 into itself. This result (3.3.3) corresponds to another type of "canonization" of operators on ℓ_∞/c_0 in the sense that instead of finding copies of ℓ_∞/c_0 where the operator acts canonically, we look for copies of ℓ_∞ where the operator acts canonically. It relates to now classical results of A. Pełczyński ([42]) and H. Rosenthal ([49]), the first of which characterizes non-weakly compact operators on a $C(K)$ space as those operators that preserve an isomorphic copy of c_0, while the second states that a non-weakly compact operator on ℓ_∞ preserves an isomorphic copy of ℓ_∞.

We begin with two easy lemmas.

Lemma 3.3.1. *Let (A_n^*) and (B_n^*) be two sequences of disjoint clopen subsets of $\mathbb{N}^*$ such that $A_n^* \cap B_m^* = \emptyset$, for every $n, m \in \mathbb{N}$. Then, there exists an isometrically isomorphic embedding $H : C(\beta\mathbb{N}) \to C(\mathbb{N}^*)$ such that $H(\beta f)|A_n^* \equiv f(n)$ and $H(\beta f)|B_n^* \equiv 0$, for every $n \in \mathbb{N}$.*

Proof. We may assume the A_n to be pairwise disjoint and such that $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{N}$.

Define $h : \wp(\mathbb{N}) \to \wp(\mathbb{N})/\mathit{Fin}$ by $h(B) = [\bigcup_{n\in B} A_n]$. Clearly, h is a Boolean embedding since $B_0 \subseteq B_1$ if, and only if, $h(B_0) \leq h(B_1)$. Now, by the Stone duality there exists a continuous onto map $\psi_h : \mathbb{N}^* \to \beta\mathbb{N}$. Note that for every $y \in A_n^*$ we have $\psi_h(y) = n$ (because $h(\{n\}) = [A_n]$).

Let $D \subseteq \mathbb{N}$ be such that $B_n \subseteq_* D$ and $D \cap A_n =_* \emptyset$, for all $n \in \mathbb{N}$. Define $H : C(\beta\mathbb{N}) \to C(\mathbb{N}^*)$ by $H(\beta f) = (\beta f \circ \psi_h) \cdot \chi_{\mathbb{N}^*\setminus D^*}$. It is easy to check that H is an isometrically isomorphic embedding which satisfies what we wanted. □

Lemma 3.3.2. *If $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is not weakly compact, then there exists a discrete sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^*$ such that $\{T^\star(\delta_{y_n}) : n \in \mathbb{N}\}$ is not weakly relatively compact.*

Proof. The fact that T is not weakly compact implies, by Corollary 17-VI of [13], that there exists $\varepsilon > 0$ and a bounded sequence of pairwise disjoint functions $(f_n)_{n\in\mathbb{N}}$ in $C(\mathbb{N}^*)$ such that $\|T(f_n)\| > \varepsilon$, for every $n \in \mathbb{N}$. Put $U_n = supp(f_n)$ and let $M > 0$ be such that $\|f_n\| \leq M$, for every $n \in \mathbb{N}$. By A.1.2, there exists infinite sets $B_n \subseteq \mathbb{N}$ such that $\varepsilon < |T(f_n)(y)|$, for every $y \in B_n^*$ and every $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$ and every $y \in B_n^*$ we have

$$\varepsilon < |T^\star(\delta_y)(f_n)| = |\int_{U_n} f_n \, \mathrm{d}T^\star(\delta_y)| \leq M|T^\star(\delta_y)|(U_n). \tag{3.1}$$

Now choose $y_n \in B_n^*$ for each $n \in \mathbb{N}$. Notice that each y_n may lie inside at most finitely many B_m's, because if $x \in \bigcap_{i\in I} B_i^*$ for some infinite $I \subseteq \mathbb{N}$, then since the U_n's are pairwise disjoint we would have $\|T^\star(\delta_x)\| \geq |T^\star(\delta_x)|(\bigcup_{i\in I} U_i) = \sum_{i\in\mathbb{N}} |T^\star(\delta_x)|(U_i)$, but this last series diverges by 3.1. Therefore, by going to a subsequence if necessary, we may assume that $y_n \in B_m^*$ if, and only if, $n = m$. Then, $(y_n)_{n\in\mathbb{N}}$ is discrete because $y_n \in B_n^* \setminus (\bigcup_{i<n} B_i^*)$.

Fix a Hahn decomposition $\mathbb{N}^* = D_n^0 \cup D_n^1$ with respect to the measure $T^\star(\delta_{y_n})$, where D_n^0 is the positive part. Then, since $\varepsilon/M < |T^\star(\delta_{y_n})|(U_n) = T^\star(\delta_{y_n})(U_n \cap D_n^0) - T^\star(\delta_{y_n})(U_n \cap D_n^1)$ we have that

$$\varepsilon/(2M) < (-1)^i T^\star(\delta_{y_n})(U_n \cap D_n^i),$$

for some $i \in \{0,1\}$ and for infinitely many $n \in \mathbb{N}$. So by applying Lemma 3.2.1 we know there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and disjoint open sets V_n such that

$$\varepsilon/(4M) < (-1)^i T^\star(\delta_{x_{n_k}})(V_k) = |T^\star(\delta_{x_{n_k}})(V_k)| \leq \sup\{|T^\star(\delta_{x_{n_i}})(V_k)| : i \in \mathbb{N}\},$$

for every $k \in \mathbb{N}$. From this we conclude, using the Dieudonné-Grothendieck characterization of weakly relatively compact sets (see Theorem 14.VII of [15]), that $\{T^\star(\delta_{x_{n_k}}) : k \in \mathbb{N}\}$ is not weakly relatively compact.

□

By Pełczyński's characterization ([42]), we know that a non-weakly compact $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ fixes an isomorphic copy of c_0. Then, by results in [4] (see section 4.3) the copy of c_0 in the domain must be contained in a subspace X of ℓ_∞/c_0 which is isometrically isomorphic to ℓ_∞. So by applying Theorem 1.3 from [49] to $T|X$ we obtain an isomorphic copy of ℓ_∞ inside ℓ_∞/c_0 which is preserved by T. The following is a similar result in which we are able to find for every non-weakly compact $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ an isometrically isomorphic copy of ℓ_∞ inside ℓ_∞/c_0 where this operator is the identity, modulo a weakly compact operator and an ε-error. The second part of the theorem says that this weakly compact part is quite constrained.

Theorem 3.3.3. *Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator. If T is not weakly compact, then*

- *for every $\varepsilon > 0$ there exists an isometric isomorphism $H : \ell_\infty \to C(\mathbb{N}^*)$, a discrete sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^*$ and a nonzero real r such that*

$$\|J \circ T \circ H - (rId + S_0)\| < \varepsilon,$$

  where $J : C(\mathbb{N}^*) \to \ell_\infty$ is the operator sending f^* to $(f^*(y_n))_{n\in\mathbb{N}}$ and S_0 is weakly compact.
- *for every $\varepsilon > 0$ there exists an isometric isomorphism $H : \ell_\infty \to C(\mathbb{N}^*)$, a homeomorphic embedding $\psi : \mathbb{N}^* \to \mathbb{N}^*$, and a nonzero real r such that*

$$\|T_\psi \circ T \circ H - (rQ + S_1)\| < \varepsilon,$$

  where $Q : \ell_\infty \to C(\mathbb{N}^*)$ is the operator sending f to f^* and S_1 has rank one.

Proof. By Lemma 3.3.2 there exists a discrete sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^*$ such that the set $\{T^\star(\delta_{y_n}) : n \in \mathbb{N}\}$ is not weakly relatively compact. Then, by going to a subsequence if necessary using Lemma 3.2.7, we obtain a decomposition $T^\star(\delta_{y_n}) = \lambda_n + \theta_n$ such that

1. the measures λ_n are supported by pairwise disjoint measurable sets,
2. $(\theta_n)_{n\in\mathbb{N}}$ is weakly convergent to some $\theta \in M(K)$, and
3. $\|\lambda_n\| = r_0 > 0$, for every $n \in \mathbb{N}$.

Now fix $\varepsilon > 0$. Using the regularity of the measures and by a variation of Lemma 3.2.1 (see Lemma 1 from [55]) and by going to a subsequence if necessary, we may assume there exist pairwise disjoint clopen sets F_n^* such that $|\lambda_n|(F_n^*) > r_0 - \varepsilon$, for every $n \in \mathbb{N}$.

Fix a Hahn decomposition $\mathbb{N}^* = D_n^0 \cup D_n^1$ with respect to the measure λ_n, where D_n^0 is the positive part. Then, since $\lambda_n(F_n^* \cap D_n^0) - \lambda_n(F_n^* \cap D_n^1) = |\lambda_n|(F_n^*) > r_0 - \varepsilon$, we have that $(-1)^i\lambda_n(F_n^* \cap D_n^i) > r_0/2 - \varepsilon/2$ for infinitely many $n \in \mathbb{N}$ and for some $i \in \{0,1\}$. By the regularity of λ_n, there exist clopen sets $A_n^* \subseteq F_n^*$ such that $r_0 \geq |\lambda_n(A_n^*)| > r_0/4 + \varepsilon/4$, for some $i \in \{0,1\}$ and for infinitely many $n \in \mathbb{N}$. So by going to a subsequence if necessary, we may assume there exists a nonzero $r \in \mathbb{R}$ such that $\lambda_n(A_n^*)$ converges to r. Define $\lambda_n^0 = \frac{r}{\lambda_n(A_n^*)}\lambda_n$ $\lambda_n^1 = \left(1 - \frac{r}{\lambda_n(A_n^*)}\right)\lambda_n$.

Let $B_n = F_n \setminus A_n$ and apply Lemma 3.3.1 to obtain an isometrically isomorphic embedding $H : \ell_\infty \to C(\mathbb{N}^*)$ such that $H(f)|A_n^* \equiv f(n)$ and $H(f)|B_n^* \equiv 0$, for every $n \in \mathbb{N}$. Notice that λ_n and λ_n^0 have the same support, so since H preserves the norm we have

$$\begin{aligned}
|\int H(f)\,\mathrm{d}\lambda_n^0 - rf(n)| &= |\int_{A_n^*} H(f)\,\mathrm{d}\lambda_n^0 + \int_{B_n^*} H(f)\,\mathrm{d}\lambda_n^0 + \int_{\mathbb{N}^*\setminus F^*} H(f)\,\mathrm{d}\lambda_n^0 - rf(n)| \\
&= |f(n)\lambda_n^0(A_n^*) - rf(n) + \int_{\mathbb{N}^*\setminus F^*} H(f)\,\mathrm{d}\lambda_n^0| \\
&\leq \|H(f)\||\lambda_n^0|(\mathbb{N}^* \setminus F^*) \\
&< \varepsilon,
\end{aligned}$$

for every f in the unit ball of ℓ_∞ and every $n \in \mathbb{N}$.

Notice that since $(\lambda_n^1)_{n\in\mathbb{N}}$ is norm-null, it is also weakly null and so $(\theta_n + \lambda_n^1)_{n\in\mathbb{N}}$ is still weakly convergent to θ. In particular, the operator $S_0 : \ell_\infty \to \ell_\infty$ defined by $S_0(f) = (\int H(\beta f)\,\mathrm{d}(\theta_n + \lambda_n^1))_{n\in\mathbb{N}}$ is weakly compact. So if we call $J : C(\mathbb{N}^*) \to \ell_\infty$ the restriction operator $J(f^*) = (f^*(y_n))_{n\in\mathbb{N}}$, then we have $(J \circ T \circ H)(f)(n) = T^\star(\delta_{y_n})(H(f)) = \int H(f)\,\mathrm{d}\lambda_n^0 + \int H(f)\,\mathrm{d}(\theta_n + \lambda_n^1)$, and so

$$\|J \circ T \circ H - (rId + S_0)\| < \varepsilon.$$

Moreover, since θ_n converges weakly and hence pointwise to θ, if we put $M = \{y_n : n \in \mathbb{N}\}$ and if $\psi : \mathbb{N}^* \to M^* = \overline{M} \setminus M$ is the homeomorphism corresponding to the map $n \mapsto y_n$, we obtain $|(T \circ H)(f)(\psi(z)) - (rf^*(z) + \int H(f)\,\mathrm{d}\theta)| \leq \varepsilon$, for every $f \in B_{\ell_\infty}$ and every $z \in \mathbb{N}^*$. In other words, $\|T_\psi \circ T \circ H - (rQ + S_1)\| \leq \varepsilon$, where $Q : \ell_\infty \to C(\mathbb{N}^*)$ is the operator sending f to f^*, and $S_1 : \ell_\infty \to C(\mathbb{N}^*)$ is given by $S_1(f)(z) = (\theta \circ H)(f)$, for every $z \in \mathbb{N}^*$. Since the constant functions form a one dimensional subspace, we have that S_1 has rank one.

□

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Appendix A

Auxiliary results and extra proofs

A.1 Topological facts about $\mathbb{N}^*$

Lemma A.1.1. *Every nonempty G_δ set in $\mathbb{N}^*$ has a nonempty interior.*

Proof. See section 1.2 of [61]. □

Lemma A.1.2. *Suppose $f : \mathbb{N}^* \to \mathbb{R}$ is continuous and $r \in \mathbb{R}$ is a value of f at some point. Then there is a clopen $A^* \subseteq \mathbb{N}^*$ such that $f|A^* \equiv r$.*

Proof. $f^{-1}[\{r\}] = \bigcap_{n\in\mathbb{N}} f^{-1}[\{t \in \mathbb{R} : r - 1/n < t < r + 1/n\}]$ is a nonempty G_δ set. By A.1.1, we obtain an infinite $A \subseteq \mathbb{N}$ such that $A^* \subseteq f^{-1}[\{r\}]$. □

Definition A.1.3. A surjective map is called irreducible if, and only if, it is not surjective when restricted to any proper closed subset.

Lemma A.1.4. *If $\psi : K \to L$ is surjective and K and L are compact, then there is a closed $F \subseteq K$ such that $\psi|F : F \to L$ is irreducible.*

Lemma A.1.5. *Suppose that $\psi : F \to \mathbb{N}^*$ is a continuous surjection, where $F \subseteq \mathbb{N}^*$ is a closed subset of $\mathbb{N}^*$. ψ is irreducible if, and only if, $\{\psi^{-1}[A^*] : A \subseteq \mathbb{N}^*\}$ is a dense subalgebra of clopen subsets of F.*

Proof. If $U \subseteq F$ were a clopen subset of F such that $\psi^{-1}[A^*] \subseteq U$ does not hold for any infinte $A \subseteq \mathbb{N}$, then $\psi|(F \setminus U)$ is onto $\mathbb{N}^*$ contradicting the irreducibility. If $U \subseteq F$ were a clopen subset of F such that $\psi|(F \setminus U)$ is onto $\mathbb{N}^*$, then $\psi^{-1}[A^*] \subseteq U$ cannot hold for any infinite $A \subseteq \mathbb{N}$. □

Lemma A.1.6. *Irreducible maps are quasi-open and map nowhere dense sets onto nowhere dense sets.*

Proof. Suppose that $\psi : F \to G$ is irreducible. If the interior of $\psi[U]$ is empty for some open $U \subseteq F$, then it means that $\psi[F \setminus U]$ is dense in G, but $\psi[F \setminus U]$ is compact, and so is equal to G contradicting the irreducibility of ψ.

Now suppose that $K \subseteq F$ is nowhere dense whose image contains an open $U \subseteq G$. As $\psi^{-1}[U]$ is open, there is $V \subseteq \psi^{-1}[U]$ such that $V \cap K = \emptyset$ and so $\psi[V] \subseteq \psi[K]$. Note that $\psi[F \setminus V] = G$ contradicting the irreducibility of ψ.

□

Lemma A.1.7. *Suppose $f : F \to \mathbb{R}$ is continuous, $F \subseteq \mathbb{N}^*$ is compact and there is an irreducible map $\psi : F \to \mathbb{N}^*$. Then, there is an infinite $A \subseteq \mathbb{N}$ such that $f|\psi^{-1}[A^*]$ is constant.*

Proof. Construct infinite $A_n \subseteq \mathbb{N}$ such that $A_{n+1} \subseteq_* A_n$ and intervals $I_n \subseteq \mathbb{R}$ such that the diameter of I_n is less than $1/n$ and such that $f|\psi^{-1}[A_n^*] \subseteq I_n$. The irreducibility guarantees the recursive step through Lemma A.1.5. If $A \subseteq_* A_n$ for all n, then $f|\psi^{-1}[A^*]$ is constant. □

Lemma A.1.8. *Countable unions of nowhere dense sets in $\mathbb{N}^*$ are nowhere dense.*

Proof. Let $F_n \subseteq \mathbb{N}^*$ be nowhere dense for every $n \in \mathbb{N}$. We may assume each F_n to be closed. Fix an open set $U \subseteq \mathbb{N}^*$. Let $B_0 \subseteq U \setminus F_0$ be a nonempty clopen and choose by induction $B_{n+1} \subseteq B_n \setminus F_{n+1}$ nonempty clopen. Since there exists a nonempty clopen $V \subseteq \bigcap_{n\in\mathbb{N}} B_n \subseteq U \setminus \bigcup_{i\in\mathbb{N}} F_i$, we know that $U \not\subseteq \overline{\bigcup_{n\in\omega} F_n}$.

□

A.2 Operators on ℓ_∞ preserving c_0

Lemma A.2.1. *Let $(b_{ij})_{i,j\in\mathbb{N}}$ be a c_0-matrix. If $J \subseteq \mathbb{N}$ is such that $(\sum_{j\in J} |b_{ij}|)_i \notin c_0$, then there exist $\varepsilon > 0$, an infinite set B and finite $F_n \subseteq J$ for each $n \in B$, such that*

1. *$F_n \cap F_k = \emptyset$, for distinct $n, k \in B$,*
2. *$\sum_{j\in F_i} |b_{ij}| = |\sum_{j\in F_i} b_{ij}| > \varepsilon/4$, for all $i \in B$, and*
3. *$\lim_{\substack{i\to\infty \\ i\in B}} \sum_{j\in\cup_{k\neq i} F_k} |b_{ij}| = 0$*

Proof. Let $(b_{ij})_{i,j\in\mathbb{N}}$ be a c_0-matrix and fix a $J \subseteq \mathbb{N}$ as in the hypothesis. Since $(b_{ij})_{i,j\in\mathbb{N}}$ is a c_0-matrix, we know that for every $k \in \mathbb{N}$ the sequence $(\sum_{j\leq k} |b_{ij}|)_i$ converges to zero. Hence, J must be infinite. Let $J = \{j_n : n \in \mathbb{N}\}$ be the increasing enumeration of J. By hypothesis, there exist $\varepsilon > 0$ and an infinite $\tilde{B} \subseteq \mathbb{N}$ such that $\sum_{n\in\mathbb{N}} |b_{ij_n}| > \varepsilon$, for all $i \in \tilde{B}$.

We will carry out an inductive construction from where we will obtain the sequence (F_n) and the set B. Let i_0 be the first element of $\tilde{B}$ and $m_0 = 0$. Since $\sum_{n\in\mathbb{N}} |b_{i_0 j_n}|$ converges, we may choose $m_1 > m_0$ such that $\sum_{n\geq m_1} |b_{i_0 j_n}| < \varepsilon/2$. Suppose for every $l \leq k$ we have chosen i_l and m_{l+1} satisfying

(a) $m_l < m_{l+1}$, $i_l \in \tilde{B}$, and $i_{l-1} < i_l$,

(b) $\sum_{n<m_l} |b_{i_l j_n}| < \frac{\varepsilon}{4(l+1)}$, and

(c) $\sum_{n\geq m_{l+1}} |b_{i_l j_n}| < \frac{\varepsilon}{4(l+1)}$.

Since $(\sum_{j<m_{k+1}} |b_{ij}|)_i$ converges to zero, we may choose $i_{k+1} \in \tilde{B}$, such that $i_{k+1} > i_k$ and for all $i \geq i_{k+1}$ we have $\sum_{n<m_{k+1}} |b_{ij_n}| < \frac{\varepsilon}{4(k+2)}$. Furthermore, since $\sum_{n\in\mathbb{N}} |b_{i_{k+1} j_n}|$ converges, we may choose $m_{k+2} > m_{k+1}$ such that $\sum_{n\geq m_{k+2}} |b_{i_{k+1} j_n}| < \frac{\varepsilon}{4(k+2)}$. This finishes the inductive construction.

Notice that for every $k \in \mathbb{N}$ we have

$$\begin{aligned} \varepsilon < \sum_{n\in\mathbb{N}} |b_{i_k j_n}| &= \sum_{n<m_k} |b_{i_k j_n}| + \sum_{m_k\leq n<m_{k+1}} |b_{i_k j_n}| + \sum_{n\geq m_{k+1}} |b_{i_k j_n}| \\ &< \frac{\varepsilon}{4(k+1)} + \sum_{m_k\leq n<m_{k+1}} |b_{i_k j_n}| + \frac{\varepsilon}{4(k+1)} \\ &\leq \sum_{m_k\leq n<m_{k+1}} |b_{i_k j_n}| + \varepsilon/2. \end{aligned}$$

Hence, $\sum_{m_k \leq n < m_{k+1}} |b_{i_k j_n}| > \varepsilon/2$. By splitting the sum $\sum_{m_k \leq n < m_{k+1}} b_{i_k j_n}$ into its positive and negative parts, we obtain $F_{i_k} \subseteq \{j_n \in \mathbb{N} : m_k \leq n < m_{k+1}\}$ such that $|\sum_{j \in F_{i_k}} b_{i_k j}| > \varepsilon/4$. So by letting $B = \{i_k : k \in \mathbb{N}\}$, we know that conditions (1) and (2) of the lemma are satisfied. To obtain (3), fix $\delta > 0$ and take $m \in \mathbb{N}$ such that $\frac{\varepsilon}{2(m+1)} < \delta$. By construction we have that $\bigcup_{l \neq k} F_{i_l} \subseteq \{j_n \in \mathbb{N} : n < m_k \text{ or } n \geq m_{k+1}\}$, for every $k \in \mathbb{N}$. So, in particular for every $k > m$, we have

$$\sum_{j \in \cup_{l \neq k} F_{i_l}} |b_{i_k j}| \leq \sum_{n < m_k} |b_{i_k j_n}| + \sum_{n \geq m_{k+1}} |b_{i_k j_n}| < \frac{\varepsilon}{2(k+1)} < \delta.$$

□

In the following propositions we list some facts leading to the proof of Theorem 1.2.15.

We start by noting that the two conditions that characterize matrices which induce operators from ℓ_∞ into ℓ_∞, also characterize matrices whose transpose induces operators from ℓ_1 into ℓ_1.

Proposition A.2.2. *$S : \ell_1 \to \ell_1$ is a linear bounded operator if, and only if, there exists a matrix $(b_{ij})_{i,j \in \mathbb{N}}$ which induces S and is such that every column is in ℓ_1 and the set $\{\|(b_{ij})_{i \in \mathbb{N}}\|_{\ell_1} : j \in \mathbb{N}\}$ is bounded.*

Proof. Let $S : \ell_1 \to \ell_1$ be a linear bounded operator. Since $\ell_1^* = \ell_\infty$, for each $i \in \mathbb{N}$ there exists $(b_{ij})_{j \in \mathbb{N}} \in \ell_\infty$ such that $S(a)(i) = S^\star(\delta_i)(a) = \sum_{j \in \mathbb{N}} b_{ij} a_j$, for every $a = (a_k)_{k \in \mathbb{N}} \in \ell_1$. In other words, S is given by the matrix $(b_{ij})_{i,j \in \mathbb{N}}$.

Note that the j-th column of the matrix is equal to $S(\delta_j) \in \ell_1$. Moreover, since S is bounded, we have that $S(\{\delta_j : j \in \mathbb{N}\}) = \{(b_{ij})_{i \in \mathbb{N}} : j \in \mathbb{N}\}$ is bounded in ℓ_1.

Conversely, suppose $(b_{ij})_{i,j \in \mathbb{N}}$ is a matrix such that every column $(b_{ij})_{i \in \mathbb{N}}$ is in ℓ_1 and the set $\{\|(b_{ij})_{i \in \mathbb{N}}\|_{\ell_1} : j \in \mathbb{N}\}$ is bounded. We claim that this matrix induces a linear bounded operator $S : \ell_1 \to \ell_1$.

First, we shall prove that such an operator is well defined. Fix $a = (a_k)_{k \in \mathbb{N}} \in \ell_1$. Start by noting that for every $i \in \mathbb{N}$ we have that $(b_{ij})_{j \in \mathbb{N}}$ is bounded by hypothesis, and so is in ℓ_∞. Hence, $\sum_{j \in \mathbb{N}} b_{ij} a_j$ is convergent for every $i \in \mathbb{N}$. Now, we need to show that the sequence $(\sum_{j \in \mathbb{N}} b_{ij} a_j)_{i \in \mathbb{N}}$ is in ℓ_1.

In view of applying Theorem 8.43 of [3], note that by hypothesis $\sum_{i \in \mathbb{N}} b_{ij} a_j$ is absolutely convergent for every $j \in \mathbb{N}$, and there exists $M \in \mathbb{N}$ such that $\sum_{i \in \mathbb{N}} |b_{ij}| < M$ for every $j \in \mathbb{N}$. Therefore,

$$\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} |b_{ij} a_j| = \sum_{j \in \mathbb{N}} |a_j| \sum_{i \in \mathbb{N}} |b_{ij}| \leq \sum_{j \in \mathbb{N}} |a_j| M < \infty.$$

Hence, by the cited Theorem, we have that both iterated series $\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} b_{ij} a_j$ and $\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} b_{ij} a_j$ converge absolutely. In particular, we have that

$$\sum_{i \in \mathbb{N}} |\sum_{j \in \mathbb{N}} b_{ij} a_j| \leq \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |b_{ij} a_j| < \infty,$$

so that $S(a) \in \ell_1$.

Clearly, S is linear. Moreover, if we take $a = (a_k)_{k\in\mathbb{N}} \in \ell_1$ such that $\|a\|_{\ell_1} \leq 1$, then by a similar argument as above we have

$$\|S(a)\|_{\ell_1} \leq \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} |b_{ij}a_j| = \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} |b_{ij}a_j| \leq \sum_{j\in\mathbb{N}} |a_j|M \leq M.$$

Therefore, S is bounded.

□

In the following proposition we identify $\ell_1^\star$ with ℓ_∞.

Proposition A.2.3. *Let $R : \ell_\infty \to \ell_\infty$ be a linear bounded operator. Then $R = S^\star$ for some linear bounded operator $S : \ell_1 \to \ell_1$ if, and only if, R is given by a matrix. Moreover, the matrix corresponding to the operator S is the transpose of the matrix corresponding to R.*

Proof. Assume $R = S^\star$, for a linear bounded operator $S : \ell_1 \to \ell_1$. Let $M_S = (b_{ij})_{i,j\in\mathbb{N}}$ be the matrix corresponding to S. Since $S(\delta_j)$ is the j-th column of M_S, for every $f \in \ell_\infty$ we have that

$$R(f)(j) = S^\star(f)(j) = (f \circ S)(\delta_j) = \sum_{i\in\mathbb{N}} b_{ij} f(i).$$

In other words, R is given by the transpose of M_S.

Conversely, suppose R is given by a matrix $M_R = (b_{ij})_{i,j\in\mathbb{N}}$. By 1.2.2 and A.2.2 we have that the transpose of M_R defines a linear bounded operator $S : \ell_1 \to \ell_1$. It only remains to show that $S^\star = R$. So fix for this purpose $f \in \ell_\infty$. If we regard it as an element of $\ell_1^\star$, then for every $i \in \mathbb{N}$ we have

$$S^\star(f)(\delta_i) = f \circ S(\delta_i) = \sum_{j\in\mathbb{N}} f(j)b_{ij},$$

but on the other hand,

$$R(f)(i) = \sum_{j\in\mathbb{N}} f(j)b_{ij}.$$

□

In the following proposition we identify $c_0^\star$ with ℓ_1.

Proposition A.2.4. *Let $R : c_0 \to c_0$ be a linear bounded operator. Then, $R^\star$ is induced by the transpose of the matrix corresponding to R.*

Proof. Let $M_R = (b_{ij})_{i,j\in\mathbb{N}}$ and $M_{R^\star} = (b'_{ij})_{i,j\in\mathbb{N}}$ be the matrices corresponding to R and $R^\star$, respectively. Observe that for every $i, j \in \mathbb{N}$ we have that $b'_{ij} = R^\star(\delta_j)(\chi_{\{i\}}) = \delta_j(R(\chi_{\{i\}})) = b_{ji}$. Hence, $M_{R^\star}$ is the transpose of M_R. □

Proposition A.2.5. *A linear bounded operator $R : \ell_\infty \to \ell_\infty$ is given by a c_0-matrix if, and only if, $R = (R|c_0)^{\star\star}$.*

Proof. Assume R is given by a c_0-matrix M. Then, $R[c_0] \subseteq c_0$ and so $R|c_0 : c_0 \to c_0$ is well defined and is also induced by M. By propositions A.2.3 and A.2.4 we have that $(R|c_0)^{\star\star}$ is also induced by M. In other words, $R = (R|c_0)^{\star\star}$.

Conversely, assume $R = (R|c_0)^{\star\star}$. Let M be the matrix corresponding to $R|c_0 : c_0 \to c_0$. By propositions A.2.3 and A.2.4 we have that $R = (R|c_0)^{\star\star}$ is also induced by M, which is a c_0-matrix. □

Theorem A.2.6. *Let $R : c_0 \to c_0$ be a linear bounded operator and let $(b_{ij})_{i,j\in\mathbb{N}}$ be the corresponding matrix. The following are equivalent:*

1. *R is weakly compact.*
2. *$R^{\star\star}[\ell_\infty] \subseteq c_0$.*
3. *$\|b_i\|_{\ell_1} \to 0$, where $b_i = (b_{ij})_{j\in\mathbb{N}}$.*

Proof. The equivalence of of (1) and (2) is well-known (see exercise 3 of Chapter 3 in [15]).

By A.2.5, we have that $R^{\star\star}$ is given $(b_{ij})_{i,j\in\mathbb{N}}$. Therefore, for any $f \in \ell_\infty$ and any $i \in \mathbb{N}$ we have $|R^{\star\star}(f)(i)| = |\sum_{j\in\mathbb{N}} b_{ij} \cdot f(j)| \leq \sum_{j\in\mathbb{N}} |b_{ij}| \|f\|_{\ell_\infty}$, and it is clear that (3) implies (2).

For the converse, assume (3) does not hold. Then, by Lemma A.2.1 there exist $\varepsilon > 0$, an infinite set $A \subseteq \mathbb{N}$ and finite $F_n \subseteq \mathbb{N}$ for each $n \in A$, such that

(i) $F_n \cap F_k = \emptyset$, for distinct $n, k \in A$,

(ii) $\sum_{j\in F_i} |b_{ij}| = |\sum_{j\in F_i} b_{ij}| > \varepsilon/4$, for all $i \in A$, and

(iii) there is an $m \in \mathbb{N}$ such that

$$\sum_{j\in\cup_{k\neq i}F_k} |b_{ij}| < \varepsilon/8, \quad \text{for all } i > m.$$

Let $f \in \ell_\infty$ be such that $supp(f) \subseteq \bigcup_{n\in\mathbb{N}} F_n$ and $b_{ij} \cdot f(j) = |b_{ij}|$, for every $i \in A$ and every $j \in F_i$. Then, for every $i \in A \setminus m$ we have

$$\begin{aligned} |R^{\star\star}(f)(i)| &= |\textstyle\sum_{j\in\bigcup_{n\in\mathbb{N}} F_n} b_{ij} \cdot f(j)| \\ &\geq |\textstyle\sum_{j\in F_i} b_{ij} \cdot f(j)| - |\textstyle\sum_{j\in\bigcup_{k\neq i} F_k} b_{ij} \cdot f(j)| \\ &\geq \textstyle\sum_{j\in F_i} |b_{ij}| - \textstyle\sum_{j\in\bigcup_{k\neq i} F_k} |b_{ij}| \\ &> \varepsilon/4 - \varepsilon/8 = \varepsilon/8. \end{aligned}$$

Therefore, $R^{\star\star}(f) \notin c_0$.

□

Proposition A.2.7. *Let $T : X^{\star\star} \to X^{\star\star}$. Then, $T = R^{\star\star}$ for some $R : X \to X$, and only if, T is w^*-w^*-continuous and $T[X] \subseteq X$.*

Proof. Assume $T = R^{\star\star}$ for some $R : X \to X$. It is well known that the fact that T is a dual operator implies that it is w^*-w^*-continuous. Since $R[X] \subseteq X$ and $R \subseteq R^{\star\star}$, we have that $T[X] \subseteq X$.

Conversely, suppose T is w^*-w^*-continuous and $T[X] \subseteq X$. Then, T is a dual operator, say of $S : X^\star \to X^\star$. It is sufficient to show that S is w^*-w^*-continuous. So fix an open set $U \subseteq \mathbb{R}$ and an $x \in X$. We denote by $\tilde{x} \in X^{\star\star}$ the element corresponding to $x \in X$. Consider the preimage by S of the w^*-open subbasic $\{g \in X^\star : g(x) \in U\}$, for given $x \in X$ and $U \subseteq \mathbb{R}$ open:

$$\begin{aligned} S^{-1}[\{g \in X^\star : g(x) \in U\}] &= \{h \in X^\star : S(h)(x) \in U\} \\ &= \{h \in X^\star : \tilde{x}(S(h)) \in U\} \\ &= \{h \in X^\star : S^\star(\tilde{x})(h) \in U\}. \end{aligned}$$

Since $S^\star = T$ and $T[X] \subseteq X$, if we put $v = S^\star(\tilde{x}) \in X$, then we have

$$S^{-1}[\{g \in X^\star : g(x) \in U\}] = \{h \in X^\star : h(v) \in U\},$$

which is clearly w^*-open. □

Lemma A.2.8. *Let X be a Banach space.*

(a) For every $x \in X$ the functional $F_x : X^\star \to \mathbb{R}$ given by $F_x(f) = f(x)$ is w^*-continuous.

(b) $T : X^\star \to X^\star$ is w^*-w^*-continuous if, and only if, for every $x \in X$ the operator $T_x : X^\star \to \mathbb{R}$ given by $T_x(f) = T(f)(x)$ is w^*-continuous.

Proposition A.2.9. *In both ℓ_1 and ℓ_∞ the product topology is coarser than the w^*-topology. Moreover, in both cases the converse is only true when restricted to a bounded subspace.*

Proof. We prove the statement for ℓ_1, the argument being the same for ℓ_∞.

Let k be a positive integer and I_i be an open interval for every $i < k$. We will show that the following basic open of the product topology $U = \{(b_j)_{j\in\mathbb{N}} \in \ell_1 : b_i \in I_i, \forall i < k\}$ is also w^*-open. So let $(a_j)_{j\in\mathbb{N}} \in U$ and take $\varepsilon > 0$ such that $(a_i - \varepsilon, a_i + \varepsilon) \subseteq I_i$, for each $i < k$. Consider the following w^*-open set

$$\begin{aligned} O &= \{(b_j)_{j\in\mathbb{N}} \in \ell_1 : |\textstyle\sum_{j\in\mathbb{N}}(b_j - a_j)\chi_{\{i\}}| < \varepsilon/2, \forall i < k\} \\ &= \{(b_j)_{j\in\mathbb{N}} \in \ell_1 : |b_i - a_i| < \varepsilon/2, \forall i < k\}. \end{aligned}$$

Clearly, $(a_j)_{j\in\mathbb{N}} \in O$ and $O \subseteq U$. So the product topology is coarser than the w^*-topology.

Now fix $M \in \mathbb{R}$ and let $Id : (B_{\ell_1}(M), \tau_{w^*}) \to (B_{\ell_1}(M), \tau_p)$ be the identity map, where $B_{\ell_1}(M) = \{a \in \ell_1 : \|a\| \leq M\}$ and τ_{w^*}, τ_p are the weak* topology and the product topology, respectively. By the above, this is a continuous function, and since B_{ℓ_1} is w^*-compact, we know that it is actually a homeomorphism. So τ_{w^*} and τ_p coincide on every bounded set.

However, this is not the case everywhere. Indeed, fix any $(a_j)_{j\in\mathbb{N}} \in \ell_1$ and $\varepsilon > 0$, and let $f \in c_0$ be such that it is not eventually zero. We show that the w^*-open set $O = \{(b_j)_{j\in\mathbb{N}} \in \ell_1 : |\sum_{j\in\mathbb{N}}(b_j - a_j)f(j)| < \varepsilon\}$ is not open in the product topology. For every positive integer k and every $\delta > 0$, let $A_k^\delta = \{(b_j)_{j\in\mathbb{N}} \in \ell_1 : |b_j - a_j| < \delta, \forall i < k\}$. Note that the family $\{A_k^\delta : k \in \mathbb{N} \setminus \{0\}, \delta > 0\}$ is a local basis for $(a_i)_{j\in\mathbb{N}}$ in the product topology. Hence, it is enough to show that $A_k^\delta \nsubseteq O$, for every k and every δ. So fix $k \in \mathbb{N} \setminus \{0\}$ and $\delta > 0$. Take $m > k$ such that $f(m) \neq 0$ and define $(b_j)_{j\in\mathbb{N}} \in \ell_1$ by putting $b_j = a_j$ for all $j \neq m$ and choosing b_m such that $b_m \geq \frac{\varepsilon}{|f(m)|} + a_m$. Then, $\varepsilon \leq (b_m - a_m)|f(m)| = |\sum_{j\in\mathbb{N}}(b_j - a_j)f(j)|$. Therefore, $(b_j)_{j\in\mathbb{N}} \in A_k^\delta \setminus O$. □

Theorem A.2.10. *Let $R : \ell_\infty \to \ell_\infty$ be a linear bounded operator. The following are equivalent:*

1. $R = (R|c_0)^{\star\star}$.
2. R *is given by a* c_0*-matrix.*
3. R *is* w^**-*w^**-continuous and* $R[c_0] \subseteq c_0$.
4. $R|B_{\ell_\infty} : (B_{\ell_\infty}, \tau_p) \to (\ell_\infty, \tau_p)$ *is continuous and* $R[c_0] \subseteq c_0$.

Proof. **(1) ⇔ (2)** See Proposition A.2.5

(1) ⇔ (3) See Proposition A.2.7.

(3) ⇔ (4) Suppose R is w^*-w^*-continuous and $R[c_0] \subseteq c_0$. Fix a set $U \subseteq \ell_\infty$ open in the product topology. By Proposition A.2.9 and by the w^*-w^*-continuity of R, we know that $R^{-1}[U] \cap B_{\ell_\infty}$ is open in the product topology.

Conversely, assume R restricted to the unit ball is continuous in the product topology and $R[c_0] \subseteq c_0$. Then, by Proposition A.2.9 and since R is bounded, we have that R restricted to the unit ball is w^*-w^*-continuous. Now consider for each $a \in \ell_1$ the functional $R_a : \ell_\infty \to \mathbb{R}$ defined by $R_a(x) = R(x)(a)$. By Lemma A.2.8(a), we have that $R_a|B_{\ell_\infty} = F_a \circ R|B_{\ell_\infty}$ is w^*-w^*-continuous. Then, by Corollary 4.46 in [24] we have that R_a is w^*-continuous, and since this is true for every $a \in \ell_1$, we know by Lemma A.2.8(b) that R is w^*-w^*-continuous. □

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Index