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CONSTRUCTIONS PAR FORCING D'ESPACES LCS ET DE STRUCTURES PCF.

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La voie que l'on peut définir n'est pas le Tao, la Voie éternelle. Le nom que l'on peut prononcer n'est pas le Nom éternel. Ce qui ne porte pas de nom, le non-être, est l'origine du ciel et de la terre. Ce qui porte un nom est la mère de tout ce que nous percevons, choses et êtres. Ainsi à celui qui est sans passion se révèle l'inconnaissable, le mystère sans nom. Celui qui est habité par le feu de la passion a une vision bornée. Désir et non désir, ces deux états procèdent d'une même origine. Seuls leurs noms diffèrent. Ils sont l'Obscurité et le Mystère. Mais en vérité c'est au plus profond de cette obscurité que se trouve la porte. La porte de l'absolu du merveilleux. Le Tao.

Lao-Tseu

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Introduction générale

L'arithmétique des cardinaux se résume principalement à l'étude de la fonction puissance $\kappa \to 2^\kappa$. En effet, pour deux cardinaux infinis λ et μ on a $\lambda + \mu = \max\{\lambda, \mu\}$ et $\lambda \times \mu = \max\{\lambda, \mu\}$. Ainsi, ces deux opérations sont élémentaires en ce qui concernent les cardinaux infinis, et la seule opération pertinente est l'opération puissance. Le premier problème (et le plus célèbre) concernant cette opération est l'hypothèse du continu CH, formulée par Cantor, et qui affirme que $2^{\aleph_0} = \aleph_1$. Plus de quatre-vingt années de recherche ont été nécessaires pour apporter une réponse à cette question. En 1939, Gödel [1] a apporté un premier élément de réponse en introduisant le modèle L des ensembles constructibles, modèle minimal pour ZFC, et en montrant que CH était vraie dans ce modèle. Bien que Gödel ait immédiatement conjecturé que CH était indécidable, il a fallu attendre les travaux de Cohen [2] en 1963 qui introduisit la notion cruciale de forcing pour construire un modèle de ZFC où CH était fausse. Par la suite, le forcing est devenu une notion indispensable au développement de la théorie des ensembles en permettant de montrer de nombreux résultats d'indépendance.

En ce qui concerne l'arithmétique des cardinaux, après la découverte du forcing, une étude plus poussée a été menée sur les comportements de la fonction puissance. En particulier, Easton [3] a généralisé le résultat de Cohen en montrant qu'il n'y a pratiquement aucune restriction sur le comportement de la fonction puissance restreinte à la classe des cardinaux réguliers, la seule limitation résidant dans une des conséquences du théorème de König qui affirme que $\mathrm{cof}\,(2^\kappa) > \kappa$. Cependant, la classe des cardinaux singuliers offre une situation plus complexe, et reste finalement le seul sujet d'étude actuel dans l'arithmétique des cardinaux. A l'instar de GCH(l'hypothèse généralisée du continu), l'hypothèse majeure sur le comportement des puissances de cardinaux singuliers, notée SCH, affirme que $\kappa^{\mathrm{cof}\,\kappa} = \kappa^+ + 2^{\mathrm{cof}\,\kappa}$ pour tous les cardinaux singuliers κ. Solovay [4] a tout d'abord montré que SCH est vraie au dessus d'un cardinal fortement compact. Peu après, Silver [5] a montré que SCH ne peut pas être en défaut en premier pour un cardinal singulier de cofinalité non dénombrable et dernièrement, Viale [6] a démontré que l'axiome de forcing propre PFA impliquait SCH. Cependant, des travaux ont également démontré que SCH pouvait être mise en défaut.

A la fin des années quatre-vingt, Shelah [7] a développé la théorie PCF et démontré son fameux théorème, $\aleph_\omega^{\aleph_0} < \max\{(2^{\aleph_0})^+, \aleph_{\omega_4}\}$, qui est un résultat absolu dans ZFC. A l'heure actuelle on ne sait toujours pas si cette borne peut être améliorée. D'un côté, le mieux que l'on puisse faire est de rendre $2^{\aleph_\omega}$ égal à $\aleph_{\alpha+1}$ où α est un ordinal dénombrable quelconque et de l'autre, personne n'a encore réussi à réduire cette mystérieuse borne $\aleph_{\omega_4}$. Il reste donc un grand fossé à combler et cette question est devenu un problème central en arithmétique des cardinaux. Dans sa preuve, Shelah a établi certaines propriétés topologiques de l'espace PCF qu'il introduit et les a utilisées pour borner la taille des espaces PCF. Précisément, si A est un intervalle de cardinaux réguliers tel que $|A| < \min A$ alors $|\text{pcf }(A)| < |A|^{+4}$. L'hypothèse naturelle associée à cette propriété est la conjecture PCF et affirme que $|\text{pcf }(A)| = |A|$ pour tout intervalle A de cardinaux réguliers tel que $|A| < \min A$. Cette conjecture est le problème ouvert le plus important de la théorie PCF. Les propriétés topologiques de l'espace PCF pour un intervalle dénombrable A de cardinaux réguliers ont été isolées par Magidor. N'importe quel espace possèdant ces propriétés sera appelé structure PCF. Ainsi, le théorème de Shelah affirme en fait que la taille d'une structure PCF est strictement inférieure à $\aleph_4$. Il est naturel de se demander si en creusant ces propriétés topologiques on pourra réussir à améliorer la borne de Shelah. Dans ce sens, Jech et Shelah [8] ont construit dans ZFC une structure PCF de taille ω_1, ce qui indique déjà que cette méthode ne pourra pas réduire la borne en dessous de $\aleph_2$. Plus tard, en développant des idées de Velickovic, Ruyle [9] a montré que l'existence de structures PCF de taille ω_2 n'est pas contradictoire avec ZFC. Bien sûr, on ne sait pas si de telles structures peuvent correspondre à de vrais espaces PCF, puisqu'alors on pourrait répondre négativement à la conjecture PCF. Le principal résultat de cette thèse est d'étendre le résultat de Ruyle à tout ordinal $\alpha < \omega_3$. Le cas ω_3 résiste encore car toutes les techniques utilisées jusque là se révèlent inopérantes sur une cardinalité strictement supérieure à $\aleph_2$.

Cette thèse s'organise en trois chapitres. Dans le premier chapitre, on présente les bases de la théorie PCF et on démontre le théorème de Shelah. L'attention du lecteur est attirée sur le fait que les hypothèses qu'on utilise dans ce chapitre ne sont pas optimales. En effet, on suppose ici que $\aleph_\omega$ est fortement limite, ce qui n'est en fait pas nécessaire pour la conclusion du théorème. Cependant, cette hypothèse simplifie grandement la présentation de la preuve qui est ainsi facilement lisible. Le lecteur se convaincra que le point crucial où cette hypothèse intervient réside dans le résultat sur l'existence de bornes supérieures minimales, résultat qui intervient très tôt dans le chapitre mais qui nécessite beaucoup plus de travail si l'on veut se passer de l'hypothèse $\lambda > 2^{|A|}$. Cette présentation, inspirée de celle de Jech [10] permet donc de présenter les principales propriétés topologiques des espaces PCF qui seront utilisées par la suite et également une preuve que ces propriétés suffisent à la démonstration du théorème de Shelah.

Dans le deuxième chapitre, on présente un outil théorique important, la ρ-fonction. Cet outil, développé par Todorcevic dans sa théorie des marches sur les ordinaux se révèle très puissant pour de nombreux problèmes de forcing. Là encore, on ne développera que le minimum requis de la théorie pour arriver rapidement à la construction de la Δ-fonction qui est utilisée dans le troisième chapitre. Le lecteur intéressé par ce type de construction pourra se référer à l'ouvrage très complet de Todorcevic [11] pour plus de précisions.

Le troisième chapitre est le véritable coeur de la thèse et est consacré à la construction de structures PCF de taille α pour $\alpha < \omega_3$. Pour réaliser ceci, on regarde attentivement les propriétés d'une structure PCF. En fait, une structure PCF est simplement un espace localement compact et totalement dispersé, avec une propriété supplémentaire qu'on appelera propriété de club. Le cas des espaces localement compacts totalement dispersés a déjà été étudié depuis de nombreuses années. Le premier gros résultat vient de Baumgartner-Shelah [12] qui ont prouvé que l'existence d'espaces localement compacts totalement disperséssde taille ω_2 n'était pas contradictoire avec ZFC. Plus tard, Martinez [13] et Soukup [14] ont étendu ce résultat de deux manières différentes pour n'importe quel ordinal $\alpha < \omega_3$. On présentera tout d'abord une troisième preuve de ce résultat puis on s'intéressera au cas des structures PCF. La preuve sur les espaces localement compacts totalement dispersés s'avèrera alors très utile pour construire une structure PCF de taille α pour $\alpha < \omega_3$.

Introduction (english version)

The main result of this thesis is the relative consistency with ZFC of PCF structures of height δ for every $\delta < \omega_3$. The thesis is divided into three chapters. In chapter one we present the basis of PCF theory and a demonstration of Shelah's Theorem, following the presentation of Jech in [10]. One should note that the hypothesis we use here are not optimal. Indeed, we suppose that $\aleph_\omega$ is strong limit but in fact it is not necessary for the conclusion of the theorem. The purpose of making such an hypothesis is to simplify the proof and make it more understandable for the reader. The crucial proposition where this hypothesis makes a simplification is the existence of least upper bounds. As this proposition happens in the very beginning, the reader should be convinced that if we weaken the hypothesis to $\lambda > |A|$ then the rest of the proof would be the same. So the main purpose of this chapter is to present the topological properties of the PCF spaces we use later, and to convince to reader that these properties alone are enough to show Shelah's Theorem.

In chapter two we present a key tool, the ρ function which was introduced by Todorcevic in his theory of walks on ordinals. This function appear to be a crucial tool for many forcing constructions. As before, we will only present enough theory to construct a Δ function which is used in the third chapter. So the reader interested in such constructions should read Todorcevic's paper [11] in which he exposes a more general theory.

The purpose of chapter three is to construct PCF structures of height δ for all $\delta < \omega_3$. In order to do that, we look carefully at the properties of a PCF structure. In fact, a PCF structure is merely a locally compact scattered space, with an additional property we will call the club property. The case of the LCS spaces (locally compact scattered) has been studied for many years now. The first main result came from Baumgartner-Shelah

[12] who proved that it is consistent with ZFC to have LCS spaces of height ω_2. Later, Martinez [13] and Soukup in [14] proved in two different ways that is consistent to have LCS spaces of height δ for all $\delta < \omega_3$. We will first present another way to prove this result. Then, we will consider the case of PCF structures, and it will appear that the result of the second section will be very useful to show the main result of this thesis.

The reader is supposed to be familiar with set theory. In particular, the notions of ordinals, cardinals, filter, clubs will not be defined. A knowledge of forcing theory is also a prerequisite. The notations we use are standard and everything undefined can be found in [15] or in [16].

Chapitre I

A Short proof of Shelah's Theorem

The purpose of this chapter is to show Shelah's Theorem. In order to do that we will suppose that $\aleph_\omega$ is strong limit. As we noticed in the introduction, this hypothesis is not necessary but it considerably simplifies the proof of the main theorem. The reader interested in PCF theory should read [7], [17] or [18] for a complete overview of this theory.

I.1 Partial order on reduced products.

Let A be a set of regular cardinals. We define $\prod A$ as the set of all functions f from A to $\bigcup A$ such that $f(a) < a$ for all $a \in A$. Let F be a filter on A and F^- the set of all $X \subset A$ that are not in F. For ordinals functions f, g on A we define

$$\begin{aligned}
f =_F g \quad & \text{if} \quad \{a \in A : f(a) = g(a)\} \in F \\
f \leq_F g \quad & \text{if} \quad \{a \in A : f(a) \leq g(a)\} \in F \\
f \lneq_F g \quad & \text{if} \quad f \leq_F g \text{ and } \{a \in A : f(a) \geq g(a)\} \in F^- \\
f <_F g \quad & \text{if} \quad \{a \in A : f(a) < g(a)\} \in F
\end{aligned}$$

The relation $\leq_F$ is a partial ordering of equivalence classes and if F is an ultrafilter then it is a total ordering.

We say that a subset C of $\prod A$ is cofinal (in relation to F) if for all $f \in \prod A$ there exists $g \in C$ such that $f \leq_F g$. The *cofinality* of F is the minimum cardinality of cofinal sets.

A sequence $\langle f_\alpha : \alpha < \theta \rangle$ in $\leq_F$ is an *increasing* (resp. *decreasing*) sequence if $f_\alpha \lneq_F f_\beta$ (resp. $f_\beta \lneq_F f_\alpha$) whenever $\alpha < \beta$. If S is a set of ordinals functions on A then g is an *upper bound* of S in $\leq_F$ $f \leq_F g$ for all $f \in S$, and g is a *least upper bound* if it is an upper bound and if $g \leq_F h$ for every upper bound h.

Proposition I.1. *Let $\langle f_\alpha : \alpha < \theta \rangle$ be a decreasing sequence in $\leq_F$. Then $Card(\theta) \leq 2^{|A|}$.*

Proof :

Suppose $Card(\theta) \geq (2^{|A|})^+$. We define a coloring P of $[(2^{|A|})^+]^2$ in $|A|$ colors by :

$$P(\{\alpha, \beta\}) = \min\{a \in A : f_\beta(a) < f_\alpha(a)\} \quad (\alpha < \beta).$$

By Erdös-Rado Theorem, P has an infinite homogeneous set X. There exists $a \in A$ such that $P(\{\alpha, \beta\}) = a$ for every $\{\alpha, \beta\} \in [X]^2$. Let $\langle \beta_k : k \in \omega \rangle$ be an increasing sequence of ordinals in X. Then $\langle f_{\beta_k} : k \in \omega \rangle$ is a strictly decreasing infinite sequence of ordinals which is a contradiction.

□

Proposition I.2. *Let λ be a regular cardinal, $\lambda > 2^{|A|}$. Every increasing sequence $\langle f_\alpha : \alpha < \lambda \rangle$ in $\leq_F$ has a least upper bound.*

Proof :

Let $\langle g_\xi : \xi < \theta \rangle$ be a maximal decreasing sequence in $\leq_F$ of upper bounds of $\mathcal{F} = \{f_\alpha : \alpha < \lambda\}$. If we show that θ is a successor ordinal then we are done. By Proposition I.1, $Card(\theta) < 2^{|A|}$.

Let $\eta < \theta$ be a limit ordinal. For every $a \in A$, let $S_a = \{g_\xi(a) : \xi < \eta\}$, and let $H = \prod_{a \in A} S_a$. We have $|H| \leq 2^{|A|} < \lambda$. For every $h \in H$ which is not an upper bound of $\mathcal{F}$, let $\alpha_h = \min\{\alpha < \lambda : f_\alpha \not\leq_F h\}$. If h is an upper bound of $\mathcal{F}$ let $\alpha_h = 0$. Let $\alpha = sup\{\alpha_h : h \in H\}$. We define $g \in H$ as follows :

$$g(a) = \min(S_a \setminus f_\alpha(a)).$$

The function g is an upper bound of $\mathcal{F}$ because $g \geq_F f_\alpha \geq_F f_{\alpha_g}$, and if g were not an upper bound we would have $g \not\geq_F f_{\alpha_g}$.

Let $\xi < \eta$. Then by definition of g, $g(a) > g_\xi(a)$ if and only if $g_\xi(a) < f_\alpha(a)$, which implies $g \leq_F g_\xi$.

□

Proposition I.3. *Let g be the least upper bound of $\{f_\alpha : \alpha < \lambda\}$ $(\lambda > 2^{|A|})$ and let $h <_F g$. Then there exists $\alpha < \lambda$ such that $h <_F f_\alpha$. We say that $\{f_\alpha : \alpha < \lambda\}$ is cofinal in g.*

Proof : For each $\alpha < \lambda$ let $X_\alpha = \{a \in A : h(a) < f_\alpha(a)\}$. Since $\lambda > 2^{|A|}$, there exists an $X \subset A$ and a set $K \subset \lambda$ such that $|K| = \lambda$ and for all $\alpha \in K$, $X_\alpha = X$. We claim $X \in F$ wich completes the proof.
If not, let g' such that $g' = g$ on X and $g' = h$ on $A \setminus X$.
We have $g' \lneq_F g$ and g' is an upper bound of $\{f_\alpha : \alpha \in K\}$ and therefore of $\{f_\alpha : \alpha < \lambda\}$. Thus g is not a least upper bound, a contradiction.

□

Let S be a set of ordinal functions on A and let g be an $\leq_F$-upper bound of S. We say that S is *bounded below* g if there is an $h <_F g$ that is an upper bound of S.
If $X \subset A$ and $A \setminus X \notin F$ then we relativize the various concepts discussed above to X. We let $F[X]$ be the filter generated by $F \cup \{X\}$ and then we say $f \leq_F g$ on X if $f \leq_{F[X]} g$. We relativize as well to X the concepts of cofinal, upper bound, etc.

Lemma I.4. *Let F be a filter on A, let $\lambda > 2^{|A|}$ be a regular cardinal, and let $\langle f_\alpha : \alpha < \lambda \rangle$ be an increasing sequence in $\leq_F$. Let g be an upper bound of $\{f_\alpha : \alpha < \lambda\}$. Then either $\{f_\alpha : \alpha < \lambda\}$ is bounded below g, or $\{f_\alpha : \alpha < \lambda\}$ is cofinal in g, or A splits in two sets $A = X \cup Y$ such that $\{f_\alpha : \alpha < \lambda\}$ is bounded below g on X and is cofinal in g on Y.*

Proof : Suppose we are not in the first two cases and let f be a least upper bound of $\{f_\alpha : \alpha < \lambda\}$ and $X = \{a \in A : f(a) < g(a)\}$. By definition of X we have $f <_F g$ on X thus $\{f_\alpha : \alpha < \lambda\}$ is bounded below g on X.
Let $h <_F g$ on $A \setminus X$. We have $g \leq_F f$ on $A \setminus X$. Then by definition of f there exists $\alpha < \lambda$ such that $h < f_\alpha$ on $A \setminus X$ and $\{f_\alpha : \alpha < \lambda\}$ is cofinal in g on $A \setminus X$.

□

Let λ be a regular cardinal. A λ-*scale* (for F) on a set $X \subset A$, $A \setminus X \notin F$, is a sequence $\langle f_\alpha : \alpha < \lambda \rangle$ in $\prod A$ that is $<_F$ strictly increasing on X and cofinal on X. A λ-*scale* is a λ-scale on A. If there exists a λ-scale we say that $\prod A/F$ has *true cofinality* λ.
We say that F is λ-*directed* if every subset S of $\prod A$ of size $< \lambda$ is bounded.

Lemma I.5. *Let λ be a regular cardinal, $\lambda > 2^{|A|}$, and assume that F is λ-directed. Then either F is λ^+-directed, or there exists a set $B \notin F$ with $A \setminus B \notin F$ such that F has a λ-scale on B and that $F[A \setminus B]$ is λ^+-directed.*

Proof : Assume that F is not λ^+-directed and let S be a set of size λ that is not bounded. By using λ-directedness of F we construct a srictly increasing sequence $\langle f_\alpha : \alpha < \lambda\rangle$ such that for every $f \in S$ there exists $\alpha < \lambda$ such that $f \leq_F f_\alpha$. As $\langle f_\alpha : \alpha < \lambda\rangle$ is unbounded, by Lemma I.4 there exists some Y with $A \setminus Y \notin F$ such that $\langle f_\alpha : \alpha < \lambda\rangle$ is a scale on Y. Now consider the collection $\mathcal{S}$ of all sets Y with $A \setminus Y \notin F$ that have a λ-scale and for every $Y \in \mathcal{S}$ let $\langle f^Y_\alpha : \alpha < \lambda\rangle$ be a λ-scale on Y. Let $S = \{f^Y_\alpha : Y \in \mathcal{S}\}$. Since $\lambda > 2^{|A|}$ we have $|S| = \lambda$. Again we construct a srictly increasing sequence $\langle f_\alpha : \alpha < \lambda\rangle$ such that for every $f \in S$ there exists $\alpha < \lambda$ such that $f \leq_F f_\alpha$.
By Lemma I.4 either $\langle f_\alpha : \alpha < \lambda\rangle$ is a scale, or A splits into $A = X \cup B$ such that $\langle f_\alpha : \alpha < \lambda\rangle$ is bounded on X and cofinal on B. The set B has a λ-scale and we claim that $F[X]$ is λ-directed. If not, we repeat the argument above and find $Y \subset X$ that has a λ-scale. But S is bounded on X and then for every $Z \subset X$, S is bounded on Z which contradicts the fact that Y has a λ-scale.

□

Theorem I.6. *Let κ be a regular uncountable cardinal and $\aleph_\eta$ be a singular cardinal of cofinality κ such that $2^\kappa < \aleph_\eta$. Let F be the set of all subsets of η that contains a club. Then F is a filter on η and F naturally defines a filter on $A = \{\aleph_{\xi+1} : \xi < \eta\}$ that we will also call F. Then $\prod A/F$ has true cofinality $\aleph_{\eta+1}$.*

Proof : Let $\langle \eta(\xi) : \xi < \kappa\rangle$ be a club in η such that $2^\kappa < \aleph_{\eta(0)}$. Without loss of generality, we can consider $\prod B/F$ where $B = \{\aleph_{\eta(\xi)+1} : \xi < \kappa\}$ and F is the filter of all subsets of κ that contains a club. As η is of cofinality κ and $\kappa < 2^\kappa < \aleph_\eta$, F is $\aleph_{\eta+1}$-directed. If there is no $\aleph_{\eta+1}$-scale then by Lemma I.5 there exists a stationnary set S such that $F[S]$ is $\aleph_{\eta+2}$-directed. Without loss of generality, we can suppose that for every $\xi \in S$, ξ is a limit ordinal. For each limit ordinal $\beta < \aleph_{\eta+1}$ choose a club C_β of β of order type $cof(\beta)$ and for every $\alpha < \aleph_{\eta+1}$ let $\varepsilon_\alpha = \{C_\beta \cap \alpha : \beta < \aleph_{\eta+1}\}$. Each ε_α has size less than $\aleph_{\eta+1}$ and consists of sets of size $< \aleph_\eta$.
Now, we construct by induction on α a strictly increasing sequence $\langle f_\alpha : \alpha < \aleph_{\eta+1}\rangle$ on S.

Assume $\langle f_\nu : \nu < \alpha \rangle$ has been constructed. For each $E \in \varepsilon_\alpha$ let g^α_E be such that for every ξ such that $\aleph_{\xi+1} > |E|$, $g^\alpha_E = \sup\{f_\nu(\xi) : \nu \in E\}$. We define f_α as an upper bound (in $\prod_{\xi \in S} \aleph_{\eta+1}$) of $\{g^\alpha_E : E \in \varepsilon_\alpha\} \cup \{f_\nu : \nu < \alpha\}$.
Now let $h \in \prod_{\xi \in S} \aleph_{\eta+1}$ be the least upper bound of $\langle f_\alpha : \alpha < \aleph_{\eta+1} \rangle$. Since $h(\xi) < \aleph_{\eta(\xi)}$ for every $\xi \in S$ and $\aleph_{\eta(\xi)}$ is singular for every ξ limit, we have $cof(h(\xi)) < \aleph_{\eta(\xi)}$ for every $\xi \in S$. Then by Fodor's theorem there exists a stationnary set $S_0 \subset S$ and $\gamma < \eta$ such that $cof(h(\xi)) \leq \aleph_\gamma$ for every $\xi \in S_0$ (and such that $\gamma < \eta(\xi)$ for every $\xi \in S_0$). For every $\xi \in S_0$ we choose a cofinal set $D_\xi \subset h(\xi)$ of cardinality less than $\aleph_\gamma$.
Now, we construct by induction on $\nu < \aleph_{\gamma+1}$ a strictly increasing sequence on S_0 of fonctions $h_\nu \in \prod_{\xi \in S_0} D_\xi / F$ and a continuous increasing sequence $\langle \alpha(\nu) : \nu < \aleph_{\gamma+1} \rangle$ such that for every $\nu < \aleph\gamma + 1$ $f_{\alpha(\nu)} <_F h_\nu <_F f_{\alpha(\nu+1)}$ on S_0. Let $\beta = \lim_{\nu \to \aleph_{\gamma+1}} \alpha(\nu)$. The previously chosen club C_β has cardinality $\aleph_{\gamma+1}$. For every $\nu < \aleph_{\gamma+1}$ let ν' be the least $\nu' > \nu$ such that $alpha(\nu') \in C_\beta$. Let $\xi_\nu \in S_0$ such that

$$g^{\alpha(\nu)}_{C_\beta \cap \alpha(\nu)}(\xi_\nu) \leq f_{\alpha(\nu)}(\xi_\nu) < h_\nu(\xi_\nu) < f_{\alpha(\nu')}(\xi_\nu).$$

Since $|S| < \aleph_{\gamma+1}$ there exists $Z \subset \aleph_{\gamma+1}$ of cardinality $\aleph_{\gamma+1}$ and $xi \in S_0$ such that $\xi_\nu = \xi$ for every $\nu \in Z$. Moreover, we can suppose that if $\nu_1 < \nu_2$ are in Z then $\nu'_1 < \nu_2$.
If $\nu_1 < \nu_2$ are in Z then $\alpha(\nu'_1) \in C \cap \alpha(\nu_2)$ and so

$$f_{\alpha(\nu'_1)}(\xi) \leq g^{\alpha(\nu_2)}_{C \cap \alpha(\nu_2)}(\xi).$$

Therefore

$$h_{\nu_1}(\xi) < f_{\alpha(\nu'_1)}(\xi) \leq g^{\alpha(\nu_2)}_{C \cap \alpha(\nu_2)}(\xi) \leq f_{\alpha(\nu_2)}(\xi) < h_{\nu_2}(\xi).$$

Then $\{h_\nu(\xi) : \nu \in Z\}$ is a subset of D_ξ of cardinality $\aleph_{\gamma+1}$ which is a contradiction.

□

I.2 PCF theory

Let A be a set of regular cardinals. We define

$$\text{pcf } A = \{cof(\prod A/D) : D \text{ ultrafilter on A}\},$$

the set of all *possible cofinalities* of $\prod A$. It is a set of regular cardinals which includes A (via principal ultrafilters) and has cardinality at most $2^{2^{|A|}}$. If A_1 and A_2 are sets of regular cardinals, we have

$$\mathrm{pcf}(A_1 \cup A_2) = \mathrm{pcf}\ A_1 \cup \mathrm{pcf}\ A_2.$$

If pcf A has a largest element, we call it the *maximal cofinality* of $\prod A$. We say that A is an *interval* if it contains all regular cardinals λ such that $\min A \leq \lambda < \sup A$.

Lemma I.7. *If A is an interval and if $\min A = (2^{|A|})^+$ then pcf A is an interval.*

Proof : Let D be an ultrafilter on A and λ be a regular cardinal such that $\min A \leq \lambda < cof(\prod A/D)$.

Let $\langle f_\alpha : \alpha < cof(\prod A/D)\rangle$ be a $\leq_D$ increasing sequence in $\prod A$. Since $\lambda > 2^{|A|}$ the sequence $\langle f_\alpha : \alpha < \lambda\rangle$ has a least upper bound g. For every $a \in A$ let $h(a) = cof(g(a))$ and let S_a be a cofinal set in $g(a)$ of order type $h(a)$. Then $\prod_{a\in A} S_a/D$ has true cofinality λ and therefore there exists a $\leq_D$ increasing sequence $\langle h_\alpha : \alpha < \lambda\rangle$ cofinal in $\prod_{a\in A} h(a)/D$. Since the number of fonctions from A into $2^{|A|}$ is less than λ we have $h(a) > 2^{|A|}$ for D-almost all a. Thus we can suppose $h(a) \in A$ for every $a \in A$. Let $E = \{X \subset A : h^{-1}(X) \in D\}$. E is an ultrafilter on A.

We can now construct by induction a sequence $\langle g_\alpha : \alpha < \lambda\rangle$ such that the sequence $\langle g_\alpha \circ h : \alpha < \lambda\rangle$ is $\leq_D$ increasing and cofinal in h. Then the sequence $\langle g_\alpha : \alpha < \lambda\rangle$ is $\leq_E$ increasing and cofinal in $\prod A$.

□

Corollary I.8. *If $\aleph_\omega$ is a strong limit cardinal then pcf $\{\aleph_n : n \in \omega\}$ is an interval and* $\sup$ *pcf* $\{\aleph_n : n \in \omega\} < \aleph_{\aleph_\omega}$.

Proof : The first statement is a direct consequence of the previous lemma applied to the interval $A = [(2^{\aleph_0})^+, \aleph_\omega]$.

The second follows from $|\mathrm{pcf}\ \{\aleph_n : n \in \omega\}| < \aleph_\omega$.

□

We are now ready to show some consequences of PCF Theory on cardinal arithmetic. Our purpose is to show the following :

Theorem I.9.

$$2^{\aleph_\omega} = \max pcf\,\{\aleph_n : n \in \omega\}.$$

Since $cof(2^{\aleph_\omega}) > \aleph_\omega$ by König's theorem and $\sup \mathrm{pcf}\,\{\aleph_n : n \in \omega\} < \aleph_{\aleph_\omega}$, if we show $2^{\aleph_\omega} = \sup \mathrm{pcf}\,\{\aleph_n : n \in \omega\}$ then $2^{\aleph_\omega}$ is a successor cardinal and then $2^{\aleph_\omega} = \max \mathrm{pcf}\,\{\aleph_n : n \in \omega\}$. Let $\lambda = \sup \mathrm{pcf}\,\{\aleph_n : n \in \omega\}$.

Lemma I.10. *Let $k < \omega$ and let μ be a cardinal such that $2^{\aleph_k} \leq \mu < \aleph_{\aleph_k}$. Then there exists a family $\mathcal{F}_\mu \subset \mathcal{P}(\mu)$ of cardinality μ such that every $X \in \mathcal{F}_\mu$ is of cardinality $\aleph_k$ and for every $Z \subset \mu$ of cardinality $\aleph_k$ there exists $X \in \mathcal{F}_\mu$ such that $X \subset Z$.*

Proof : We show by induction on μ that for every *ordinal* μ such that $2^{\aleph_k} \leq \mu < \aleph_{\aleph_k}$ there exists a family $\mathcal{F}_\mu$ that has the desired properties. The result is immediate for $\mu = 2^{\aleph_k}$. If μ is not a cardinal then $\mathcal{F}_\mu$ is constructed naturally with $\mathcal{F}_{|\mu|}$. If μ is a cardinal and $2^{\aleph_k} < \mu < \aleph_{\aleph_k}$ then $cof(\mu) \neq \aleph_k$ and we let $\mathcal{F}_\mu = \bigcup_{\alpha<\mu} \mathcal{F}_\alpha$.

□

Lemma I.11. *There exists a family $\mathcal{F}$ of functions in $\prod_{n=0}^{\infty}$ with $|\mathcal{F}| = \lambda$ such that for every $g \in \prod_{n=0}^{\infty}$ there exists $f \in \mathcal{F}$ such that $g(n) \leq f(n)$ for every n.*

Proof : For every ultrafilter D on ω we choose a cofinal sequence $\langle f_\alpha^D : \alpha < \mathrm{cof}\ D\rangle$ in $\prod_{n=0}^{\infty}/D$. Let $\mathcal{F}$ be the set of all $f = \max\{f_{\alpha_1}^{D_1}, \ldots, f_{\alpha_m}^{D_m}\}$ where m is finite. Since $\lambda > \aleph_\omega > 2^{2^{\aleph_0}}$ we have $|\mathcal{F}| = \lambda$. Suppose there exists g that is not bounded by any $f \in \mathcal{F}$. We let $X_\alpha^D = \{n : g(n) > f_\alpha^D\}$. The family $\{X_\alpha^D\}_{\alpha,D}$ has the finite intersection property and then can be extended to an ultrafilter U. But then there exists α such that $g \leq_U f_\alpha^U$ which is a contradiction.

□

Let $\mathcal{A}$ be the structure $\langle \aleph_\omega, \in, e\rangle$ where e is such that $e(\alpha, .) \restriction \alpha$ is an injection from α to $Card(\alpha)$ for every α.

For $X \subset \aleph_\omega$ we let $Hull(X)$ be the smallest elementary submodel of $\mathcal{A}$ which contains X.

Proposition I.12. *Let M be an elementary submodel of $\mathcal{A}$ and let X be a cofinal set in $M \cap \aleph_{n+1}$. Then,*

$$M \cap \aleph_{n+1} \subset Hull((M \cap \aleph_n) \cup X).$$

Proof : Let $Y \subset \aleph_\omega$. Then $Hull(Y)$ contains all inverse images of Y under the functions $e(\alpha, .)$. Since X is cofinal in $M \cap \aleph_{n+1}$ we are done.

□

Proposition I.13. *Let M be an elementary submodel of $\mathcal{A}$ and X_n be a cofinal set in $M \cap \aleph_n$ for every $n \in \omega$. Then,*

$$M = Hull(\bigcup_n X_n).$$

Proof : We show by induction on $n < \omega$ that

$$M \cap \aleph_n \subset Hull(\bigcup_{k \leq n} X_k).$$

Since every elementary submodel contains ω the property is true for $n = 0$.
By the previous proposition and by induction hypothesis we have

$$M \cap \aleph_{n+1} \subset Hull((M \cap \aleph_n) \cup X_{n+1}) \subset Hull(Hull(\bigcup_{k \leq n} X_k) \cup X_{n+1}).$$

But for every Y, Z we have $Hull(Y) \cup Z \subset Hull(Y \cup Z)$ and $Hull(Hull(Y)) = Hull(Y)$ so we are done.

□

We say that an elementary submodel M is *good* if $\sup(M \cap \aleph_n)$ has uncountable cofinality and M contains a club in $\sup(M \cap \aleph_n)$ for every $n > 0$. We let then $\chi_M(n) = \sup(M \cap \aleph_n)$.

Lemma I.14. *If M, N are good elementary submodels such that $\chi_M = \chi_N$ then $M = N$.*

Proof : Since M and N are good, for every n there exists two clubs A_n and B_n in $\sup(M \cap \aleph_n) = \sup(N \cap \aleph_n)$. Then $C_n = A_n \cap B_n$ is also a club. By the previous proposition we have $M = Hull(\bigcup_n C_n) = N$.

□

Let $k < \omega$ be large enough so that $\aleph_k \geq 2^{\aleph_0}$ and $\aleph_{\aleph_k} > \lambda$ and let $\mathcal{F}$ be a family which satisfies the properties in Proposition I.11.

We say that an elementary submodel M is *excellent* if there exists an increasing sequence of elementary submodels $\langle M_\xi : \xi \leq \aleph_k \rangle$ with $M_{\aleph_k} = M$ and an increasing sequence $\langle f_\xi : \xi < \aleph_k \rangle$ of elements of $\mathcal{F}$ such that $\chi_{M_\xi} \lneq f_\xi \lneq \chi_{M_{\xi+1}}$ for every $\xi < \aleph_k$ and for every $\gamma \leq \aleph_k$ limit $M_\gamma = \bigcup\{M_\xi : \xi < \gamma\}$ and $M_{\gamma+1}$ contains the image of χ_{M_γ}.

An excellent elementary submodel M is also good. Indeed, $\sup M \cap \aleph_n = \aleph_n$ if $n < k$ and have cofinality $\aleph_k$ if $n \geq k$. Moreover, $\{\chi_{M_\xi}(n) : \xi < \aleph_k\}$ is a club in $\sup M \cap \aleph_n$ for every n.

Proposition I.15. *There are at most λ excellent elementary submodels.*

Proof : We note $\mathcal{F} = \{f_\xi : \xi < \lambda\}$ and we let $\mathcal{F}_\lambda$ be a family that satisfies the properties of Lemma I.10.

Let M be an excellent elementary submodel and let $Z \subset \lambda$ with $|Z| = \aleph_k$ such that $\chi_M(n) = \sup\{f_\xi : \xi \in Z\}$ for every n. Then there exists $X \in \mathcal{F}_\lambda$ (with $|X| = \aleph_k$) such that $X \subset Z$. We have $\chi_M(n) = \sup\{f_\xi : \xi \in X\}$. Since $|\mathcal{F}_\lambda| = \lambda$ and M is completely determined by χ_M, there are at most λ excellent elementary submodels.

□

Proof of Theorem I.9 : Let D be a countable subset of $\aleph_\omega$. We construct a chain of elementary submodels such that $M_0 = Hull(D)$ and $M = M_{\aleph_k}$ is an excellent elementary submodel. Then we have $D \subset M$.

But every excellent elementary submodel is of cardinality $\aleph_k$ and then have at most $(\aleph_k)^{\aleph_0} = \aleph_k$ countable subsets. By the previous proposition $\aleph_\omega$ have at most $\aleph_k \times \lambda = \lambda$ countable subsets. But then we have $(\aleph_\omega)^{\aleph_0} = \lambda$.

On the other hand we have $(\aleph_\omega)^{\aleph_0} = 2^{\aleph_\omega}$. This is because the function Γ from $\mathcal{P}(\aleph_\omega)$ to $\prod_{n<\omega} \aleph_n$ such that $\Gamma(A) = (A \cap \aleph_n)_{n<\omega}$ is one to one.

□

We will now state some properties of the structure of PCF. Let $\{B_\lambda : \lambda \in \text{pcf } A\}$ be a family of subsets of A. We say that the B_λ's are *generators* for pcf A if the following properties hold for every $\lambda \in \text{pcf } A$:

1. $\max \operatorname{cof} B_\lambda = \lambda$

2. for every ultrafilter D on A such that $\operatorname{cof} D = \lambda$ then $B_\lambda \in D$.

If $\{B_\lambda : \lambda \in \operatorname{pcf} A\}$ are generators for pcf A we define J_κ for every $\kappa \leq \max \operatorname{pcf} A$ as the ideal generated by $\{B_\lambda : \lambda \in \kappa \cap \operatorname{pcf} A\}$. This makes sense since for $X \in J_\kappa$ we have $X \subset B_{\lambda_1} \cup \ldots \cup B_{\lambda_n}$ (for some n and some $\lambda_1, \ldots, \lambda_n$) and then $\max \operatorname{cof} X < \kappa$. Thus $X \neq A$ and J_κ is an ideal.

Theorem I.16. *If $2^{|A|} < \min A$ then there exists generators for pcf A.*

It should be noted that the theorem is also true under the weaker assumption $|A| < \min A$.

Proof : We construct the B_λ's by induction such that the following properties hold at each step :

1. The ideal J_κ generated by $\{B_\lambda : \lambda \in \kappa \cap \operatorname{pcf} A\}$ is κ-directed.

2. If $\kappa \notin \operatorname{pcf} A$ then J_κ is κ^+-directed.

3. If $\kappa \in \operatorname{pcf} A$ then there exists $B_\kappa \notin J_\kappa$ such that J_κ has a κ-scale on B_κ.

4. If $\kappa = \max \operatorname{pcf} A$ then $B_\kappa = A$ and if not then $J[B_\kappa]$ is a κ^+-directed ideal.

Formally, the notions of directeness, scale, etc, refers to the dual filter of J_κ.
First note that if the conditions above are satisfied then the B_λ's are generators. Let $\lambda \in \operatorname{pcf} A$ and let D be an ultrafilter on B_λ which extends the dual filter of J_λ (i.e. $D \cap J_\lambda = \emptyset$). Any λ-scale for $\leq_{J_\lambda}$ is a scale for $\leq_D$ and so we have $\operatorname{cof} D = \lambda$. If D is an ultrafilter on B_λ such that $D \cap J_\lambda \neq \emptyset$, let ν be the least ν such that $B_\nu \in D$. Then D is an ultrafilter on B_ν such that $D \cap J_\nu = \emptyset$ and since B_ν has a ν-scale for J_ν we have cof $D = \nu$. Thus $\max \operatorname{pcf} B_\lambda = \lambda$.
If D is an ultrafilter on A such that $B_\lambda \notin D$ then either $B_\nu \in D$ for some $\nu < \lambda$ in which case $\operatorname{cof} D < \lambda$, or else $D \cap J_\lambda[B_\lambda]$, and since $J_\lambda[B_\lambda]$ is λ^+-directed, D is also λ^+-directed and we have $\operatorname{cof} D > \lambda$.
We shall now prove the properties by induction on κ. We use Lemma I.5 and the assumption that $2^{|A|} < \min A$.

1. If $\kappa \leq \min A$ then $J_\kappa = \{\emptyset\}$ is κ-directed. If κ is a limit cardinal then $J_\kappa = \bigcup_{\lambda<\kappa} J_\lambda$ and the claim follows easily. If $\kappa = \lambda^+$ then the statement follow either from property 2 or 4.
2. If κ is a singular cardinal then κ-directed implies κ^+-directed and so J_κ is κ^+-directed. If κ is regular then by Lemma I.5 either J_κ is κ^+-directed of J_κ has a scale on some $B \notin J_\kappa$. But if there is a scale on some $B \notin J_\kappa$ then let D be any ultrafilter on B such that $D \cap J_\kappa = \emptyset$. The κ-scale on B is a κ-scale for $\leq_D$ and so $\operatorname{cof} D = \kappa$ which means $\kappa \in \operatorname{pcf} A$.
3. We prove that if J_κ is κ^+-directed then $\kappa \notin \operatorname{pcf} A$. So let D be any ultrafilter on A. Either $B_\lambda \in D$ for some $\lambda < \kappa$ in which case $\operatorname{cof} D < \kappa$, or else $D \cap J_\kappa = \emptyset$ in which case D is κ^+-directed and $\operatorname{cof} D > \kappa$. Hence $\kappa \notin \operatorname{pcf} A$.
4. We show that if $\kappa \in \operatorname{pcf} A$ then $\kappa = \max \operatorname{pcf} A$ if and only if there exists a κ-scale on A. The property is then a consequence from Lemma I.5.

   If there exists a κ-scale then for every ultrafilter D on A either $B_\lambda \in D$ for some $\lambda < \kappa$ in which case $\operatorname{cof} D < \kappa$, or else $D \cap J_\kappa = \emptyset$ and D has a κ-scale, and so $\kappa = \max \operatorname{cof} A$. If there is no κ-scale then $J_\kappa[B_\kappa]$ is a κ^+-directed ideal and if D is any ultrafilter such that $D \cap J_\kappa[B_\kappa] = \emptyset$ then D is κ^+-directed and so $\kappa < \operatorname{cof} D$.

□

Corollary I.17. *If $2^{|A|} < \min A$ then $|pcf\ A| \leq 2^{|A|}$.*

Proof : The generators corresponding to distinct λ's are distinct.

□

Corollary I.18. *If $\aleph_\omega$ is a strong limit cardinal then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.*

Proof : This is a consequence of Theorem I.9.

□

Corollary I.19. *If $2^{|A|} < \min A$ then* $\max$ *cof A exists.*

Proof : Assume that pcf A does not have a largest element. Then the set $\{A \setminus B_\lambda : \lambda \in \operatorname{pcf} A\}$ has the finite intersection property and so extends to an ultrafilter D. But we have $B_{\operatorname{cof} D} \in D$ which is a contradiction.

□

Corollary I.20. *For every $X \subset A$ we have $X \in J_\kappa$ if and only if cof $D < \kappa$ for every ultrafilter D on X.*

Proof : We know that if $X \in J_\kappa$ then $\max \mathrm{cof}\, X < \kappa$. If X is not in J_κ then the set $\{X \setminus B_\lambda : \lambda < \kappa\}$ has the finite intersection property and so extends to an ultrafilter D. Then $B_\lambda \notin D$ for every $\lambda < \kappa$ and cof $D \geq \kappa$.

□

Lemma I.21. *For every $X \subset A$ there exists a finite set $\{\lambda_1, \ldots, \lambda_n\} \subset$ pcf X such that $X \subset B_{\lambda_1} \cup \ldots \cup B_{\lambda_n}$.*

Proof : Assume the contrary. Then the set $\{X \setminus B_\nu : \nu \in \mathrm{pcf}\ X\}$ has the finite intersection property and thus extends to an ultrafilter D such that $B_\nu \notin D$ for every $\nu \in \mathrm{pcf}\ X$. But we have $B_{\mathrm{cof}\ D} \in D$ which is a contradiction.

□

We shall now try to produce better generators for pcf A. Let $\{B_\lambda : \lambda \in \mathrm{pcf}\ A\}$ be generators for pcf A. We say that the B_λ's are *good* if for every $\lambda \in \mathrm{pcf}\ A$

$$\max \mathrm{cof}\ (\bigcup \{B_\mu : \mu \in B_\lambda\}) \leq \lambda.$$

We say that they are *excellent* if they are good and if $B_\mu \subset B_\lambda$ whenever $\mu \in B_\lambda$.

Theorem I.22. *If $2^{|A|} < \min A$, there exists good generators for pcf A.*

Proof : Let $\{B_\lambda : \lambda \in \mathrm{pcf}\ A\}$ be generators for pcf A. As we showed in Theorem I.16, for each λ there exists an $\leq_{J_\lambda}$-increasing sequence $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ of functions on A that is cofinal in B_λ. Moreover, by Proposition I.2 we may assume that f^λ_α is an upper bound of $\langle f^\lambda_\beta : \beta < \alpha\rangle$ for each λ and each α of cofinality $> 2^{|A|}$.
Let $\kappa = (2^{|A|})^+$ and assume that $\kappa < \min A$. If not, then replace A by $A \setminus \min A$ which will not change the result. We consider an elementary chain $\langle M_\xi : \xi \leq \kappa\rangle$ of models of size κ with the property that M_0 contains (as elements) A, pcf A, the generators B_λ, the scales $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$, and every function $\phi : A \to A$. Moreover, we assume $\langle M_\xi : \xi \leq \eta\rangle \in M_{\eta+1}$

for every η. Let $M = M_\kappa$.

For each $\xi \le \kappa$ we define $\chi_\xi \in \prod A$ as follows :

$$\chi_\xi(\nu) = \sup(M_\xi \cap \nu) \qquad (\nu \in A).$$

Let $\chi = \chi_\kappa$. We extend χ to $\prod \mathrm{pcf}\, A$ by letting $\chi(\lambda) = \sup(M \cap \lambda)$ for every $\lambda \in \mathrm{pcf}\, A$. Each χ_ξ belongs to M_ξ and therefore to M, and if $\xi < \eta$ then $\chi_\xi(\nu) < \chi_\eta(\nu)$ for every $\nu \in A$. The function χ is the pointwise least upper bound of $\{\chi_\xi : \xi < \kappa\}$.

Let $\lambda \in \mathrm{pcf}\, A$ and consider the function $f^\lambda_{\chi(\lambda)}$. We have $\mathrm{cof}\ \chi(\lambda) = \kappa$ and therefore $f^\lambda_{\chi(\lambda)}$ is a least upper bound (in $\le_{J_\lambda}$) of $\{f^\lambda_\alpha : \alpha \in M \cap \lambda\}$. Now if $\alpha \in M \cap \lambda$ then $f^\lambda_\alpha \in M$ and so $f^\lambda_\alpha(\nu) < \chi(\nu)$ for every $\nu \in A$. On the other hand, if $\xi < \kappa$ then there exists an α such that $\chi_\xi \le_{J_\lambda} f^\lambda_\alpha$ on B_λ, and since M is an elementary submodel, there exists such an α in M. It follows that the function χ is a least upper bound (in J_λ) of $\{f^\lambda_\alpha : \alpha \in M \cap \lambda\}$ on B_λ, and consequently $f^\lambda_{\chi(\lambda)} = \chi$ almost everywhere (mod J_λ) on B_λ.

Thus we replace each generator B_λ by the generator

$$B^*_\lambda = \{\nu \in B_\lambda : f^\lambda_{\chi(\lambda)}(\nu) = \chi(\nu)\}$$

and we proceed to show that the B^*_λ are good.

Let $E = \bigcup\{B^*_\mu : \mu \in B^*_\lambda\}$. We claim that $E \in J_{\lambda^+}$. For each $\nu \in E$ let $\mu = \varphi(\nu)$ be such that $\nu \in B^*_\mu$ and $\mu \in B^*_\lambda$ (and $\varphi(\nu) \in A$ arbitrary if $\nu \notin E$). By our assumption on M we have that $\varphi \in M$. For each $\alpha < \lambda$ let $g_\alpha \in \prod A$ be the function defined as follows :

$$g_\alpha(\nu) = f^\mu_\beta(\nu), \text{ where } \mu = \varphi(\nu) \text{ and } \beta = f^\lambda_\alpha(\mu) \quad (\nu \in A).$$

The set $\{g_\alpha : \alpha < \lambda\}$ is in M and since J_{λ^+} is λ^+-directed there exists a $g \in M$ such that $g_\alpha <_{J_{\lambda^+}} g$ for every $\alpha < \lambda$. Since $g \in M$ we have $g(\nu) < \chi(\nu)$ for every ν.

Now let $\alpha = \chi(\lambda)$. Since $g_\alpha <_{J_{\lambda^+}} \chi$, we complete the proof by showing $g_\alpha < \chi$ on E. Thus let $\nu \in E$ be arbitrary, and let $\mu = \varphi(\nu)$ and $\beta = f^\lambda_\alpha(\mu)$. Since $\mu \in B^*_\lambda$ we have $\beta = f^\lambda_\alpha(\mu) = f^\lambda_{\chi(\lambda)}(\mu) = \chi(\mu)$ and since $\nu \in B^*_\lambda$, it follows that $g_\alpha(\nu) = f^\mu_\beta(\nu) = f_{\chi(\nu)}(\nu) = \chi(\nu)$.

□

The next two theorems are used in the proof of Shelah's theorem and give crucial informations about the structure of pcf A.

Theorem I.23. *Let κ be an uncountable regular cardinal and let $\aleph_\eta$ be a singular cardinal of cofinality κ such that $2^\kappa < \aleph_\eta$. Then there exists a club $C \subset \eta$ such that* $\max$ *cof* $\{\aleph_{\alpha+1} : \alpha \in C\} = \aleph_{\eta+1}$.

Proof : Let C_0 be a club in η of order type κ and let $A = \{\aleph_{\alpha+1} : \alpha \in C_0\}$. Let $\lambda = \aleph_{\eta+1}$ and let B_λ be a generator of pcf A for λ. Let $X = \{\alpha \in C_0 : \aleph_{\alpha+1} \in B_\lambda\}$. If D is an ultrafilter on C_0 which extends the filter of sets which contains a club then by Theorem I.6 we have cof $\prod_{\alpha \in C_0} \aleph_{\alpha+1}/D = \lambda$ and then $X \in D$. Thus X contains a club C and then $\max \mathrm{cof}\, \{\aleph_{\alpha+1} : \alpha \in C\} \leq \lambda$ which implies the equality.

□

Theorem I.24. *Let $C \subset$ pcf A be such that $|A| < |C|$ and $2^{|C|} < \min A$. Then there exists $B \subset C$ such that $|B| \leq |A|$ and* $\max$ *cof* $B \geq \sup C$

Proof : Let $\{B_\lambda : \lambda \in \mathrm{pcf}\, (A \cup C)\}$ be good generators for pcf $A \cup C$. For every $\lambda \in C$ let $B^A_\lambda = A \cap B_\lambda$. Since $\lambda \in$ pcf A there exists an ultrafilter D on A such that cof $D = \lambda$. We have $B_\lambda \in D$ and so $B^A_\lambda \in D$. Thus $\lambda \in$ pcf B^A_λ.
Let $E = \bigcup\{B^A_\lambda : \lambda \in C\}$. We have $C \subset E$ and therefore $\max \mathrm{cof}\, E \geq \sup C$. Now let $B \subset C$ a set of cardinality at most $|A|$ such that $E = \bigcup\{B^A_\lambda : \lambda \in B\}$.
By Lemma I.21 there exists $\lambda_1, \ldots, \lambda_n \in$ pcf B such that $B \subset B_{\lambda_1} \cup \ldots \cup B_{\lambda_n}$, and thus

$$E \subset \bigcup\{B_\mu : \mu \in B_{\lambda_1}\} \cup \ldots \cup \bigcup\{B_\mu : \mu \in B_{\lambda_n}\}.$$

Since the B_λ's are good we have

$$\max \mathrm{cof}\, E \leq \max\{\lambda_1, \ldots, \lambda_n\} \leq \max \mathrm{cof}\, B$$

and so $\max \mathrm{cof}\, B \geq \sup C$.

□

I.3 Shelah's Theorem

In this section we will give a proof of Shelah's theorem. First, let summarize the relevant results of the pcf theory we will use here.

If A is an interval of regular cardinals such that $2^{|A|} < \min(A)$ then (A) is also an interval of regular cardinals and the pcf operator has the following properties for any X, $Y \subset (A)$.

(a) $X \subset (X)$, $(X) \cup (Y) = (X \cup Y)$, $((X)) = (X)$.

(b) If $\gamma \in (X)$, then there exists $X' \subset X$ with $|X'| = |A|$ such that $\gamma \in (X')$.

(c) (X) has a maximal element.

(d) If $\nu < \max(A)$ is a singular cardinal of uncountable cofinality then there exists a club C in ν such that $\max(\{\lambda^+ : \lambda \in C\}) = \nu^+$.

Theorem I.25. *Assume $\aleph_\omega$ is strong limit and let Θ be the ordinal such that $2^{\aleph_\omega} = \aleph_{\Theta+1}$. Then there exists an ordinal function F on $\mathcal{P}(\Theta)$ which has the following properties :*

1. *If $X \subset Y$ then $F(X) \leq F(Y)$*
2. *If $\nu < \Theta$ is a limit ordinal of uncountable cofinality then there exists a club $C \subset \nu$ such that $F(C) = \nu$*
3. *If $X \subset \Theta$ is a set of order type ω_1 then there exists a $\gamma \in X$ such that $F(X \cap \gamma) \geq$* $\sup X$

Proof : Let $X \subset \Theta$ and let $B = \{\aleph_{\xi+1} : \xi \in X\}$. Since $\aleph_\omega$ is strong limit we have $2^{|B|} < \aleph_\omega$. Then pcf B has a greatest element $\aleph_{\eta+1}$. We let $F(X) = \eta$.
By definition of F, property 1 holds. Property 2 is a consequence of Theorem I.23. Indeed if $\kappa = \text{cof}\,(\eta)$ then $\kappa < \aleph_\omega$ and thus $2^\kappa < \aleph_\omega < \aleph_\eta$ and we can apply the theorem. Finally, property 3 is a consequence of Theorem I.24. Indeed, if $X \subset \Theta$ then $\{\aleph_{\xi+1} : \xi \in X\} \subset \text{pcf}\,\{\aleph_n\}_{n<\omega}$ and since $2^{|X|} < \aleph_\omega$ we can apply the theorem and then X contains a countable subset Y such that $F(Y) \geq \sup X$.

□

Lemma I.26. *Let $E^3_1 = \{\alpha < \omega_3 : cof\, \alpha = \omega_1\}$. Then there exists a family $\{C_\alpha : \alpha \in E^3_1\}$ such that for every $\alpha \in E^3_1$ we have that C_α is a club in α and for every club $C \subset \omega_3$ the set $\{\alpha \in E^3_1 : C_\alpha \subset C\}$ is stationnary.*

Proof : We want to find a family $\{C_\alpha : \alpha \in E^3_1\}$ such that every C_α is a subset of α and for every club $C \subset \omega_3$ the set $\{\alpha \in E^3_1 : C_\alpha$ is a club in α and $C_\alpha \subset C\}$ is stationnary.

Assume that there is no such family and let $\{C^0_\alpha : \alpha \in E^3_1\}$ be a family of clubs in α such that $C^0_\alpha = \aleph_1$. We construct by induction on $\nu < \omega_2$ clubs $E_\nu \subset \omega_3$ and collections $\{C^\nu_\alpha : \alpha \in E^3_1\}$ as follows : $C^\nu_\alpha = C^0_\alpha \cap \bigcap_{\xi\in\nu} E_\xi$ and E_ν is such that the set $\{\alpha \in E^3_1 :$ C^ν_α is a club in α and $C^\nu_\alpha \subset E_\nu\}$ is not stationnary.
Let $E = \bigcap_{\nu<\omega_2} E_\nu$ and for every α let $C_\alpha = C^0_\alpha \cap E$. The set $S = \{\alpha \in E^3_1 : E \cap$ α is a club in $\alpha\}$ is stationnary and for every $\alpha \in S$ there exists a $\nu(\alpha) < \omega_2$ such that $C_\alpha = C^{\nu(\alpha)}_\alpha$.
There exists a $\nu < \omega_2$ and a stationnary set $T \subset S$ such that $C_\alpha = C^\nu_\alpha$ for every $\alpha \in T$. If $\alpha \in T$ then $C^\nu_\alpha = C^{\nu+1}_\alpha = C^\nu_\alpha \cap E_\nu$ and so $C^\nu_\alpha \subset E_\nu$ which is in contradiction with the choice of E_ν.

□

Lemma I.27. *Let F be an ordinal function on $\mathcal{P}$ which satisfies the properties of Theorem I.25. Then $\Theta < \omega_4$.*

Proof : Assume $\Theta \geq \omega_4$. Let $\{C_\alpha : \alpha \in E^3_1\}$ be a family which satisifies the condition of Lemma I.26. Let $\langle M_\alpha : \alpha < \omega_3\rangle$ be a chain of elementary submodels such that $|M_\alpha| = \aleph_3$, contain the family $\{C_\alpha\}_\alpha$, are closed under the function F, such that $\langle M_\xi : \xi \leq \alpha\rangle \in M_{\alpha+1}$ for every α and such that $\eta_\alpha = M_\alpha \cap \omega_4$ is an ordinal. Let $\eta : \omega_3 \to \omega_4$ be the continuous function defined by $\eta(\alpha) = \eta_\alpha$. Let $\alpha \in E^3_1$ such that $C_\alpha \subset C$. By property 3 (in Theorem I.25) there exists $\beta < \alpha$ such that $F(\eta[C_\alpha \cap \beta]) \geq \eta(\alpha)$.
Let $X = \eta[C_\alpha \cap \beta]$. Since $C_\alpha \in M_\alpha$ and $\eta \restriction \beta \in M_\alpha$ we have $X \in M_\alpha$. But $X \subset \eta[C]$ then by property 1 (still in Theorem I.25) we have $F(X) \leq F(\eta[C]) < \omega_4$. Since M_α is closed under F we have $F(X) \in M_\alpha$. But $\omega_4 \cap M_\alpha = \eta(\alpha)$ which implies $F(X) < \eta(\alpha)$ which is a contradiction.

□

Chapitre II

Construction of a strong Δ-function

In this chapter, we build a strong Δ-function from the $\square_{\omega_1}$ principle.

II.1 The ρ function

A $\square_{\omega_1}$-sequence is a sequence $\langle C_\alpha : \alpha < \omega_2 \text{ and } \lim(\alpha)\rangle$ such that :

1. C_α is a club in α, for all α.
2. If α is a limit point of C_β then $C_\alpha = C_\beta \cap \alpha$.
3. $o.t.(C_\alpha) \leq \omega_1$, for all α.

If α is a successor ordinal, say $\alpha = \beta + 1$, we let $C_\alpha = \{\beta\}$. Note that it is consistent with ZFC that there exists a $\square_{\omega_1}$-sequence We now recall the definition of Todorcevic's ρ function (see [11], page 204). First, let

$$\Lambda(\alpha, \beta) = \text{ maximal limit point of } C_\beta \cap (\alpha + 1)$$

when such a limit point exists ; otherwise let $\Lambda(\alpha, \beta) = 0$. We define the function $\rho : [\omega_2]^2 \to \omega_1$ recursively by the following formula.

$$\rho(\alpha, \beta) = \max\{o.t.(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\}$$

where we define by convention $\rho(\alpha, \alpha) = 0$.
We will now state some properties of the ρ function.

Lemma II.1. *If α is a limit point of C_β then $\rho(\xi, \alpha) = \rho(\xi, \beta)$ for every $\xi < \alpha$.*

Proof : Since α is a limit point of C_β we have $C_\beta \cap \alpha = C_\alpha$. Then for every $\xi < \alpha$ we have $o.t.(C_\beta \cap \xi) = o.t.(C_\alpha \cap \xi)$, $\min(C_\beta \setminus \xi) = min(C_\alpha \setminus \xi)$ and $\Lambda(\xi, \beta) = \Lambda(\xi, \alpha)$.

□

Lemma II.2. *If* $\alpha < \beta < \gamma < \omega_2$ *then,*

(a) $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$

(b) $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$

Proof : The proof is by induction on γ and the inductive hypothesis is "for every α, β such that $\alpha < \beta < \gamma$ we have (a) and (b)".
Let $\gamma < \omega_2$ and let $\alpha < \beta < \gamma$ and suppose the inductive hypothesis is satisfied for every $\eta < \gamma$. Let $\gamma_\alpha = \min(C_\gamma \setminus \alpha)$ and $\gamma_\beta = \min(C_\gamma \setminus \beta)$. We first proceed to show (a).
Let $\nu = \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$, we need to show $\rho(\alpha, \gamma) \leq \nu$.

Case 1^a : Suppose $\alpha < \Lambda(\beta, \gamma)$.

Then by Lemma II.1 we have $\rho(\alpha, \Lambda(\beta, \gamma)) = \rho(\alpha, \gamma)$ and by definition of $\rho(\beta, \gamma)$, we have $\rho(\Lambda(\beta, \gamma), \beta) \leq \rho(\beta, \gamma) \leq \nu$. We apply the inductive hypothesis (b) for $\alpha < \Lambda(\beta, \gamma) \leq \beta$ and we get :

$$\rho(\alpha, \gamma) = \rho(\alpha, \Lambda(\beta, \gamma)) \leq \max\{\rho(\alpha, \beta), \rho(\Lambda(\beta, \gamma), \beta)\} \leq \nu$$

Case 2^a : Suppose $\alpha \geq \Lambda(\beta, \gamma)$. Then $\Lambda(\beta, \gamma) = \Lambda(\alpha, \gamma) = \lambda$.

Subcase $2^a.1$: Suppose $\gamma_\alpha = \gamma_\beta = \bar{\gamma}$. We apply the inductive hypothesis (a) for $\alpha < \beta < \bar{\gamma}$ and we get :

$$\rho(\alpha, \bar{\gamma}) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \bar{\gamma})\} \leq \nu$$

since $\rho(\beta, \bar{\gamma}) \leq \rho(\beta, \gamma)$ by definition.
Now let $\xi \in C_\gamma \cap [\lambda, \alpha)$. We apply the inductive hypothesis (b) for $\xi < \alpha < \beta$ and we get :

$$\rho(\xi, \alpha) \leq \max\{\rho(\xi, \beta), \rho(\alpha, \beta)\}.$$

But $\xi \in C_\gamma \cap [\lambda, \alpha) \subset C_\gamma \cap [\lambda, \beta)$ and so $\rho(\xi, \beta) \leq \rho(\beta, \gamma) \leq \nu$. Then $\rho(\xi, \alpha) \leq \nu$.
Since

$$o.t.(C_\gamma \cap \alpha) \leq o.t.(C_\gamma \cap \beta) \leq \rho(\beta, \gamma) \leq \nu$$

every ordinal appearing in the definition of $\rho(\alpha,\gamma)$ is bounded by ν, and so is $\rho(\alpha,\gamma)$.

Subcase $2^a.2$: Suppose $\gamma_\alpha < \gamma_\beta$. Then $\gamma_\alpha \in C_\gamma \cap [\lambda,\beta)$ and so $\rho(\gamma_\alpha,\beta) \leq \rho(\beta,\gamma) \leq \nu$. We apply the inductive hypothesis (b) for $\alpha \leq \gamma_\alpha < \beta$ and we get :

$$\rho(\alpha,\gamma_\alpha) \leq \max\{\rho(\alpha,\beta),\rho(\gamma_\alpha,\beta)\} \leq \nu.$$

Now let $\xi \in C_\gamma \cap [\lambda,\alpha)$. We apply the inductive hypothesis (b) for $\xi < \alpha < \beta$ and we get :

$$\rho(\xi,\alpha) \leq \max\{\rho(\xi,\beta),\rho(\alpha,\beta)\}$$

and again we have $\rho(\xi,\beta) \leq \rho(\beta,\gamma) \leq \nu$. So $\rho(\xi,\alpha) \leq \nu$.
As before we have $o.t.(C_\gamma \cap \alpha) \leq \nu$ and then every ordinal appearing in the definition of $\rho(\alpha,\gamma)$ is bounded by γ and so is $\rho(\alpha,\gamma)$.

Now we proceed to show (b). This time let $\nu = \max\{\rho(\alpha,\gamma),\rho(\beta,\gamma)\}$ and we need to show $\rho(\alpha,\beta) \leq \nu$.

Case 1^b : Suppose $\alpha < \Lambda(\beta,\gamma)$.

By Lemma II.1 we have $\rho(\alpha,\Lambda(\beta,\gamma)) = \rho(\alpha,\gamma) \leq \nu$ and we have $\rho(\Lambda(\beta,\gamma),\beta) \leq \rho(\beta,\gamma) \leq \nu$ by definition. We apply the inductive hypothesis (a) for $\alpha < \Lambda(\beta,\gamma) \leq \beta$ and we get :

$$\rho(\alpha,\beta) \leq \max\{\rho(\alpha,\Lambda(\beta,\gamma)),\rho(\Lambda(\beta,\gamma),\beta)\} \leq \nu.$$

Case 2^b : Suppose $\alpha \geq \Lambda(\beta,\gamma)$. Then $\Lambda(\beta,\gamma) = \Lambda(\alpha,\gamma) = \lambda$.

Subcase $2^b.1$: Suppose $\gamma_\alpha = \gamma_\beta = \bar{\gamma}$. Then $\rho(\alpha,\bar{\gamma}) \leq \rho(\alpha,\gamma) \leq \nu$ and $\rho(\beta,\bar{\gamma}) \leq \rho(\beta,\gamma) \leq \nu$. We apply the inductive hypothesis (b) for $\alpha < \beta < \bar{\gamma}$ and we get

$$\rho(\alpha,\beta) \leq \max\{\rho(\alpha,\bar{\gamma}),\rho(\beta,\bar{\gamma})\} \leq \nu.$$

Subcase $2^b.2$: Suppose $\gamma_\alpha < \gamma_\beta$. Then $\gamma_\alpha \in C_\gamma \cap [\lambda,\beta)$ and so $\rho(\gamma_\alpha,\beta) \leq \rho(\beta,\gamma) \leq \nu$. Similarly, $\rho(\alpha,\gamma_\alpha) \leq \rho(\alpha,\gamma) \leq \nu$. We apply the inductive hypothesis (a) for $\alpha \leq \gamma_\alpha < \beta$ and we get :

$$\rho(\alpha,\beta) \leq \max\{\rho(\alpha,\gamma_\alpha),\rho(\gamma_\alpha,\beta)\} \leq \nu.$$

□

Lemma II.3. $|\{\xi \leq \alpha : \rho(\xi, \alpha) \leq \nu\}| \leq \aleph_0$ *for all* $\alpha < \omega_2$ *and* $\nu < \omega_1$

Proof : The proof is by induction on $\alpha < \omega_2$. If $\alpha < \omega_1$ then the result is immediate. If not, let $\Gamma \subset \alpha$ be a given set of order type ω_1. We need to find $\xi \in \Gamma$ such that $\rho(\xi, \alpha) > \nu$. If there exits some $\xi \in \Gamma$ such that $o.t.(C_\alpha \cap \xi) > \nu$ then we are done. So, suppose that $o.t.(C_\alpha \cap \xi) \leq \nu$ for all $\xi \in \Gamma$. Then there must be an ordinal $\alpha_1 \in C_\alpha$ such that the set

$$\Gamma_1 = \{\xi \in \Gamma : \alpha_1 = \min(C_\alpha \setminus \xi)\}$$

has size ω_1. By the inductive hypothesis there is $\xi \in \Gamma_1$ such that

$$\rho(\xi, \alpha_1) > \nu \geq o.t.(C_\alpha \cap \xi).$$

It follows that

$$\rho(\xi, \alpha) \geq \max\{o.t.(C_\alpha \cap \xi), \rho(\xi, \alpha_1)\} = \rho(\xi, \alpha_1) > \nu.$$

□

Let $\alpha < \beta < \omega_2$. We define the *trace of the walk* from β to α, and we note $Tr(\alpha, \beta)$ by letting

$$Tr(\alpha, \beta) = \{\beta_0 > \beta_1 > \ldots > \beta_n\}$$

where the β_i's are defined recursively by $\beta_0 = \beta$ and $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$. Since we construct a decreasing sequence the induction stops after n steps (where n is finite) and we have $\beta_n = \alpha$.
The following lemma is an immediate consequence of the definition of the walk.

Lemma II.4. *Let* $\alpha \leq \beta \leq \gamma < \omega_2$. *If* β *belongs to the trace of the walk from* γ *to* α *then* $\rho(\alpha, \gamma) \geq \rho(\alpha, \beta)$.

Proof : The proof is immediate by finite induction on the trace of the walk.

□

Lemma II.5. *If* $0 < \beta \leq \gamma$ *and if* β *is a limit ordinal, then there exists* $\bar{\beta} < \beta$ *such that* $\rho(\alpha, \gamma) \geq \rho(\alpha, \beta)$ *for all* α *in the interval* $[\bar{\beta}, \beta)$.

Proof : Let $\gamma = \gamma_0 > \gamma_1 > \ldots > \gamma_n = \beta$ be the trace of the walk from γ to β. Let $\bar{\gamma} = \gamma_{n-1}$ if β is a limit point of $C_{\gamma_{n-1}}$ and $\bar{\gamma} = \beta$ otherwise. By Lemma II.1 we have that (in any case) $\rho(\alpha, \beta) = \rho(\alpha, \bar{\gamma})$ for every $\alpha < \beta$.
Let $\bar{\beta} < \beta$ be an upper bound of all $C_{\gamma_i} \cap \beta$ which are bounded in β. Then for every $\alpha \in [\bar{\beta}, \beta)$ we have that $\bar{\gamma}$ belongs to the trace of the walk from γ to α. By Lemma II.4 we get :

$$\rho(\alpha, \gamma) \geq \rho(\alpha, \bar{\gamma}) \text{ for every } \alpha \in [\bar{\beta}, \beta).$$

But since $\rho(\alpha, \bar{\gamma}) = \rho(\alpha, \beta)$ for every $\alpha < \beta$, the proof is finished.

□

If α and β are ordinals less than ω_2 we define $\rho\{\alpha, \beta\} = \rho(\alpha, \beta)$ if $\alpha < \beta$ and $\rho\{\alpha, \beta\} = \rho(\beta, \alpha)$ if $\beta \leq \alpha$.

Lemma II.6. *Let $\gamma < \omega_2$ and let $\{\alpha_\xi, \beta_\xi\}$ $(\xi < \omega_1)$ be a sequence of pairwise-disjoint elements of $[\omega_2]^{\leq 2}$. Then there exists an unbounded set $\Gamma \subset \omega_1$ such that $\rho\{\alpha_\xi, \beta_\eta\} \geq \min\{\rho\{\alpha_\xi, \gamma\}, \rho\{\beta_\eta, \gamma\}\}$ for every $\xi \neq \eta$ in Γ.*

Proof : By shrinking if necessary we may assume that the sequences α_ξ $(\xi < \omega_1)$ and β_ξ $(\xi < \omega_1)$ are strictly increasing. Moreover, we assume that $\alpha_\xi \leq \beta_\xi$ for every ξ. The case $\alpha_\xi > \beta_\xi$ for every ξ is similar.
Let $\alpha = \sup \alpha_\xi$ and $\beta = \sup \beta_\xi$. Then α and β are ordinals of cofinality ω_1, so the order types of C_α and C_β are both equal to ω_1. Thus we may assume that the two sequences $\{\alpha_\xi\}$ and $\{\beta_\xi\}$ are indexed by C_α rather than ω_1 and then we may assume that

$$\beta_\xi \geq \alpha_\xi \geq \xi \text{ for every } \xi \in C_\alpha.$$

Case 1 : Suppose $\alpha = \beta$.
By Lemma II.5, for each limit point ν of C_α, there exists $f(\nu) < \nu$ in C_α such that

$$\rho(\xi, \alpha_\nu), \rho(\xi, \beta_\nu) \geq \rho(\xi, \nu) = \rho(\xi, \alpha) \text{ for every } \xi \in [f(\nu), \nu).$$

By Fodor's theorem there exists a stationnary $\Gamma \subset \lim\ (C_\alpha)$ such that f is constant on Γ. Then

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) \text{ for every } \xi < \eta \text{ in } \Gamma, \text{ and}$$

$$\rho(\beta_\eta, \alpha_\xi) \geq \rho(\beta_\eta, \alpha) \text{ for every } \eta < \xi \text{ in } \Gamma.$$

Subcase 1^a : Suppose $\gamma \geq \alpha$.

Using the two subadditive properties of ρ in Lemma II.2 we can conclude that $\rho(\xi, \alpha) = \rho(\xi, \gamma)$ for any $\xi < \alpha$ such that $\rho(\xi, \alpha) > \rho(\alpha, \gamma)$. So, going to a tail of Γ we may assume that

$$\rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma) \text{ and } \rho(\beta_\xi, \alpha) = \rho(\beta_\xi, \gamma) \text{ for every } \xi \in \Gamma.$$

Let ξ, η in Γ. If $\xi < \eta$ we have

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma).$$

If $\xi > \eta$ we have

$$\rho(\beta_\eta, \alpha_\xi) \geq \rho(\beta_\eta, \gamma) = \rho(\beta_\eta, \gamma).$$

In any case, the conclusion of the lemma is satisfied.

Subcase 1^b : Suppose $\gamma < \alpha$.

Using Lemma II.3 and going to a tail of the set Γ we may assume that Γ lies above γ and that

$$\rho(\alpha_\xi, \alpha), \rho(\beta_\xi, \alpha) > \rho(\gamma, \alpha) \text{ for every } \xi \in \Gamma.$$

Applying the subadditive property (b) of Lemma II.2 we get for every $\xi \in \Gamma$:

$$\rho(\gamma, \alpha_\xi) \leq \max\{\rho(\gamma, \alpha), \rho(\alpha_\xi, \alpha)\} = \rho(\alpha_\xi, \alpha), \text{ and}$$

$$\rho(\gamma, \beta_\xi) \leq \max\{\rho(\gamma, \alpha), \rho(\beta_\xi, \alpha)\} = \rho(\beta_\xi, \alpha).$$

Let ξ, η in Γ. If $\xi < \eta$ we have

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) \geq \rho(\gamma, \alpha_\xi).$$

If $\xi > \eta$ we have

$$\rho(\beta_\eta, \alpha_\xi) \geq \rho(\beta_\eta, \alpha) \geq \rho(\gamma, \beta_\eta).$$

In any case, the conclusion of the lemma is satisfied.

Case 2 : Suppose $\alpha < \beta$.

We may assume that $\beta_\xi > \alpha$ for every ξ and working as in Case 1 we find a stationnary set Γ of limit points of C_α such that

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) \text{ for every } \xi < \eta \text{ in } \Gamma.$$

By Lemma II.3, for each $\eta \in \Gamma$ there exists $g(\eta) \in C_\alpha$ such that $\rho(\xi, \alpha) > \rho(\alpha, \beta_\eta)$ for every $\xi \in [g(\eta), \alpha)$. Using the two subadditive properties of ρ from Lemma II.2 we get that $\rho(\xi, \alpha) = \rho(\alpha, \beta_\eta)$ for every $\xi \in [g(\eta), \alpha)$. Intersecting Γ with the closed and unbounded subset of C_α of all ordinals that are closed under the mapping g we may assume

$$\rho(\alpha_\xi, \beta_\eta) = \rho(\alpha_\xi, \alpha) \text{ for every } \xi > \eta \text{ in } \Gamma.$$

Subcase 2^a : Suppose $\gamma < \alpha$.
Apllying Lemma II.2 again and going to a tail of Γ we may assume that Γ lies above γ and

$$\rho(\alpha_\xi, \alpha) > \rho(\gamma, \alpha) \text{ for every } \xi \in \Gamma.$$

Applying the subadditive properties of ρ we get

$$\rho(\gamma, \alpha_\xi) \leq \max\{\rho(\gamma, \alpha), \rho(\alpha_\xi, \alpha)\} = \rho(\alpha_\xi, \alpha) \text{ for every } \xi \in \Gamma.$$

Let ξ, η in Γ. If $\xi < \eta$ we have

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) \geq \rho(\gamma, \alpha_\xi).$$

If $\xi > \eta$ we have

$$\rho(\alpha_\xi, \beta_\eta) = \rho(\alpha_\xi, \alpha) \geq \rho(\gamma, \alpha_\xi).$$

In any case the conclusion of the lemma is satisfied.
Subcase 2^b : Suppose $\gamma \geq \alpha$.
Going to a tail of Γ we may assume that $\rho(\alpha_\xi, \alpha) > \rho(\alpha, \gamma)$ for every $\xi \in \Gamma$, so as before the subadditive properties give us that

$$\rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma) \text{ for every } \xi \in \Gamma.$$

Let ξ, η in Γ. If $\xi < \eta$ we have

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma).$$

If $\xi > \eta$ we have

$$\rho(\alpha_\xi, \beta_\eta) = \rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma).$$

In any case the conclusion of the lemma is satisfied.

□

II.2 The Δ-function from the ρ function

We are now almost ready to construct a strong Δ-function but we need one more lemma first.

Lemma II.7. *Let A be a familiy of size ω_1 of finite subsets of ω_2. Then there exists a subfamily B of A of size ω_1 such that for every a and b in B we have*

$$\rho\{\alpha,\beta\} \geq \min\{\rho\{\alpha,\gamma\},\rho\{\beta,\gamma\}\} \text{ for every } \alpha \in a\setminus b, \beta \in b \setminus a \text{ and } \gamma \in a \cap b.$$

Proof : By shrinking if necessary we can assume that A forms a Δ-system with root r and that for some $n < \omega$ we have $|a \setminus r| = n$ for every $a \in A$.
For $a \in A$ let $a(0), \dots, a(n-1)$ be the increasing enumeration of $A \setminus r$. By shrinking again we may assume that A can be enumerated as $\{a_\xi : \xi < \omega_1\}$ in such a way that $\langle a_\xi(i) : \xi < \omega_1\rangle$ is strictly increasing for every $i < n$. By $n^2 \times |r|$ successive applications of Lemma II.6 we find a single unbounded set $\Gamma \subset \omega_1$ such that for every $\gamma \in r, (i,j) \in n \times n$ and $\xi \neq \eta$ in Γ we have

$$\rho\{a_\xi(i), a_\eta(j)\} \geq \min\{\rho\{a_\xi(i),\gamma\},\rho\{a_\eta(j),\gamma\}\}.$$

This gives us the conclusion of the lemma.

□

Let $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ be a function with $f\{\alpha,\beta\} \subseteq \alpha \cap \beta$ for $\{\alpha,\beta\} \in [\omega_2]^2$. We say that two finite subsets a and b of ω_2 are *good* for f if for $\gamma \in a \cap b$, $\alpha \in a \setminus b$ and $\beta \in b \setminus a$ we have :

(a) if $\gamma < \alpha$, β then $\gamma \in f\{\alpha,\beta\}$
(b) if $\gamma < \beta$ then $f\{\alpha,\gamma\} \subseteq f\{\alpha,\beta\}$
(c) if $\gamma < \alpha$ then $f\{\beta,\gamma\} \subseteq f\{\alpha,\beta\}$

We say that f is a Δ-*function* if every uncountable family of finite subsets of ω_2 contains two elements which are good for f. We say that f is a *strong* Δ-*function* if every uncountable family A of finite subsets of ω_2 contains an uncountable subfamily B such that any two elements of B are good for f.

Theorem II.8. *Assume $\square_{\omega_1}$. Then there exists a strong Δ-function.*

Proof : We define $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ by the following formula.

$$f\{\alpha,\beta\} = \{\xi < \min\{\alpha,\beta\} : \rho(\xi,\alpha) \leq \rho\{\alpha,\beta\}\}$$

where ρ is the function defined previously with the square sequence. Note that by the subadditive properties of ρ we have also

$$f\{\alpha,\beta\} = \{\xi < \min\{\alpha,\beta\} : \rho(\xi,\beta) \leq \rho\{\alpha,\beta\}\}.$$

Let A be an uncountable family of finite subsets of ω_2. Then by Lemma II.7 there exists $B \subset A$ such that for every a and b in B we have

$$\rho\{\alpha,\beta\} \geq \min\{\rho\{\alpha,\gamma\},\rho\{\beta,\gamma\}\} \text{ for every } \alpha \in a \setminus b, \beta \in b \setminus a \text{ and } \gamma \in a \cap b.$$

This and the subadditive properties of ρ give us that if $\gamma < \alpha,\beta$ then $\rho\{\alpha,\beta\} \geq \rho(\gamma,\alpha),\rho(\gamma,\beta)$. So f satisifies property (a) of the definition of the strong Δ-function. Suppose $\beta > \gamma > \alpha$. If $\rho(\alpha,\beta) \geq \rho(\gamma,\beta)$ then by the subadditivity of ρ we have that $\rho(\alpha,\gamma) \leq \rho(\alpha,\beta)$. Since $\rho\{\alpha,\beta\} \geq \min\{\rho\{\alpha,\gamma\},\rho\{\beta,\gamma\}\}$ we have in any case $\rho(\alpha,\gamma) \leq \rho(\alpha,\beta)$ and then $f\{\alpha,\gamma\} \subset f\{\alpha,\beta\}$. This proves that f satisifies property (b). By symmetry, f satisifies property (c). Thus, f is a strong Δ-function.

□

The strong Δ-function has the following property which will be useful in the next chapter.

Proposition II.9. *Let $\langle C_\alpha : \alpha < \omega_2\rangle$ be a $\square_{\omega_1}$-sequence and let f be the strong Δ-function constructed from this sequence. Then for every $\eta < \alpha < \beta < \omega_2$, if η is a limit point of both C_α and C_β then $\eta \in f\{\alpha,\beta\}$.*

Proof : Recall that $f\{\alpha,\beta\} = \{\xi < \alpha : \rho(\xi,\alpha) \leq \rho(\alpha,\beta)\}$ where ρ is as defined as before. Since η is a limit point of C_α it is straightforward to check that $\rho(\eta,\alpha) = o.t.(C_\eta)$. On the other hand, we have $\rho(\alpha,\beta) \geq o.t.(C_\beta \cap \alpha)$. But since η is also a limit point of C_β we have $C_\beta \cap \eta = C_\eta$ and thus $o.t.(C_\beta \cap \alpha) \geq o.t.(C_\eta)$ that is $\rho(\eta,\alpha) \leq \rho(\alpha,\beta)$

□

Chapitre III

LCS Spaces and PCF Structures

The purpose of this chapter is to expose the proof of the main result of the thesis. In the first section we will be interested only in LCS spaces and we will give another proof of the relative consistency with ZFC of the existence of LCS spaces of height δ for every $\delta < \omega_3$. In the second section, we will give a precise definition of the concept of PCF structures and we will show the relative consistency with ZFC of the existence of PCF structures of height δ for every $\delta < \omega_3$.

III.1 About LCS spaces

A topological space X is called *scattered* if every non-empty subspace of X has an isolated point. In the following, a LCS space will denote a locally-compact scattered space. For $Y \subseteq X$, let $I(Y)$ the set of isolated points of Y. For every ordinal α, we define the α-th *Cantor-Bendixson* level of X as :

$$I_\alpha(X) = I(X \setminus \cup\{I_\beta(X) : \beta < \alpha\})$$

If X is scattered, then there exists α such that $I_\alpha = \emptyset$. The minimal such α is called the *height* of X and is denoted by $ht(X)$. The *cardinal sequence* of X is the sequence of the cardinalities of its levels in the Cantor-Bendixson process. More precisely, we have :

$$CS(X) = \langle |I_\alpha(X)| \ ; \ \alpha < ht(X)\rangle$$

The *width* of X is the maximum cardinality of its Cantor-Bendixson levels. The following definition is a useful tool for the purpose of forcing such spaces. In fact, it arises from the result of Baumgartner-Shelah in [12] which says it is consitent with ZFC to have a LCS space of height ω_2 and width ω. This is exactly the same definition as in [19].

Definition III.1. *Given a cardinal sequence $\theta = \langle \kappa_\alpha : \alpha < \lambda \rangle$, where each κ_α is an infinite cardinal, we say that a poset $(T, \leq, i)$ is a θ-poset if*

1. *$T = \bigcup\{T_\alpha : \alpha < \lambda\}$, where each T_α is of the form $\{\alpha\} \times Y_\alpha$, and Y_α is a set of cardinality κ_α.*
2. *i is a function from $[T]^2$ to $[T]^{<\omega}$ with the following properties :*
   (a) *If $u \in i\{s,t\}$, then $u \leq s, t$*
   (b) *If $u \leq s, t$, then there exists $v \in i\{s,t\}$ such that $u \leq v$.*
3. *If $s \in T_\alpha$, $t \in T_\beta$ and $s < t$, then $\alpha < \beta$.*
4. *For every $\alpha < \beta < \lambda$, if $t \in T_\beta$ then the set $\{s \in T_\alpha : s < t\}$ is infinite.*

Let $\theta = \langle \kappa_\alpha : \alpha < \lambda \rangle$ be a sequence of infinite cardinals and $(T, \leq, i)$ is a θ poset. Then λ is called the *height* of T and is denoted by $ht(T)$. The *width* of T is $wd(T) = \sup(\kappa_\alpha : \alpha < ht(T))$. For $\alpha < ht(T)$, T_α is called the α-th level of T. For every $t \in T$, we denote by $C(t)$ the cone $\{s \in T : s < t\}$ of t and by $C_\alpha(t)$ the intersection of $C(t)$ with T_α, the α-th level of T. If $s = (\alpha, x)$ is an element of T then we call α the *height* of s and denote it by $\pi_1(s)$ and we denote x by $\pi_2(s)$. A θ-poset is called *thin* if $wd(T) = \omega$. Thus, in this case θ is simply a constant sequence of length λ with all entries equal to ω. We denote such a sequence by $\Omega(\lambda)$.

The following proposition is implicitly due to Baumgartner. We reproduce the simple proof for completeness, see also [19].

Proposition III.2. *Let $\theta = \langle \kappa_\alpha : \alpha < \lambda \rangle$ be a sequence of cardinals. If there exists a θ-poset, then there exists a locally compact scattered, Hausdorff, space X with $CS(X) = \theta$.*

Proof : Suppose $(T, \leq, i)$ is a θ-poset. For each t let

$$\mathcal{B}_t = \{C(t) \setminus (C(s_1) \cup \ldots C(s_n)) : n < \omega, s_1, \ldots, s_n \in T, s_1, \ldots, s_n < t\}.$$

Using (2) it is easy to check that $\mathcal{B} = \bigcup\{\mathcal{B}_t : t \in T\}$ is a clopen basis for a Hausdorff topology $\mathcal{T}$ on T. Let $X = (T, \mathcal{T})$. For each $t \in T$, $C(t)$ is a compact neighborhood of t

(use (3)). Hence, X is locally-compact. Further, if Z is a non-empty closed subspace of X, by (3) we can always find $t \in Z$ with $C(t) \cap Z = \{t\}$. i.e., t is an isolated point of Z. Hence, X is scattered. Finally, using (4), it is clear that for each $\alpha < \lambda$, $I_\alpha(X) = T_\alpha$. So, X has height λ and $CS(X) = \theta$.

□

The idea is to extend an $\Omega(\omega_2)$-poset to an $\Omega(\delta)$-poset of height δ, for every $\delta < \omega_3$. Our plan is to show this by induction on δ. For the successor case, the idea is to add another level above all the previous ones. The next theorem will show how to do this. For the limit case, things are more complicated since for technical reasons we can't take the union of what we constructed before without disturbing the size of the levels. The idea is to construct it in two steps. Imagine what we try to obtain as a body. To build it, we obviously first need a skeleton. And then, we can add some flesh between the bones to obtain the full body. In case of a limit ordinal α, the skeleton will naturally be a club of order type cof (α). And the flesh is all the missing ordinals between two consecutive points of the club. So, we intend to have an $\Omega($ cof $(\alpha))$-poset with some additional properties, and we want to add the missing levels between each consecutive levels of the skeleton. We will now begin with a simple proposition about how to extend an $\Omega(\lambda)$-poset to an $\Omega(\lambda+1)$-poset.

Proposition III.3. *For every ordinal λ, if there exists an $\Omega(\lambda)$-poset then there exists an $\Omega(\lambda+1)$-poset.*

Proof : Let $(T, \leq, i)$ be an $\Omega(\lambda)$-poset. The α-th level T_α of T is of the form $\{\alpha\} \times Y_\alpha$, for some Y_α. We define a $\Omega(\lambda+1)$-poset T' as follows. We first define the Y'_α, for all $\alpha \leq \lambda$, by

$$Y'_\alpha = \begin{cases} \omega \times Y_\alpha & \text{if } \alpha < \lambda \\ \omega \times \{0\} & \text{if } \alpha = \lambda \end{cases}$$

Let $T'_\alpha = \{\alpha\} \times Y'_\alpha$, for all α, and let $T' = \bigcup\{T'_\alpha : \alpha \leq \lambda\}$. We define the ordering $\leq'$ on T' as follows. Let s and t be two elements of T' and suppose $s = (\alpha, (m, x))$ and $t = (\beta, (n, y))$. If $\alpha < \lambda$ and $\beta = \lambda$, say $s \leq' t$ if and only if $m = n$. If $\alpha,\ \beta < \lambda$, say $s \leq' t$ if and only if $m = n$ and $(\alpha, x) \leq (\beta, y)$.

So, what we did here is simply put a single point above all the points in T and then make ω copies of what we obtain. Now, it is obvious how to define the function i'. Suppose s and t are two elements of T', say $s = (\alpha, (m, x))$ and $t = (\beta, (n, y))$. If $m \neq n$ we let $i'\{s, t\} = \emptyset$. If $m = n$ and $\alpha < \lambda$ while $\beta = \lambda$ we let $i'\{s, t\} = \{s\}$. Finally, if $m = n$ and $\alpha, \beta < \lambda$ we let

$$i'\{s, t\} = \{(\xi, (m, u)) : (\xi, u) \in i\{(\alpha, x), (\beta, y)\}\}.$$

It should be clear now that $(T', \leq', i')$ is an $\Omega(\lambda + 1)$-poset.

□

So, this theorem allows us to handle the successor case easily. But, one should note that we cannot hope to extend this construction up to a limit ordinal since if we deal with an uncountable cofinality the size of the levels will no longer be countable. Thus, we need something more. Let us begin with a new definition.

Definition III.4. *Let $(T^1, \leq_1, i_1)$ be an $\Omega(\alpha+1+\beta)$-poset and $(T^2, \leq_2, i_2)$ an $\Omega(\rho)$-poset. We assume that ξ-th level, T^j_ξ, of T^j is of the form $\{\xi\} \times Y^j_\xi$, for $j = 0, 1$ and all ξ. We define a new poset $(T, \leq, i)$, denoted by $T^2 \hookrightarrow^\alpha T^1$ as follows. Let height of T will be $\lambda = \alpha + 1 + \rho + \beta$. We first define*

$$Y_\xi = \begin{cases} Y^1_\xi & \text{if } \xi \leq \alpha \\ \omega \times Y^2_\eta & \text{if } \xi = \alpha + 1 + \eta \text{ for } \eta < \rho \\ Y^1_{\alpha+1+\eta} & \text{if } \xi = \alpha + 1 + \rho + \eta \text{ for } \eta < \beta \end{cases}$$

Let $T_\xi = \{\xi\} \times Y_\xi$ and $T = \bigcup\{T_\xi : \xi < \lambda\}$.

So basically, $T^2 \hookrightarrow^\alpha T^1$ is T^1 with ω copies of T^2 added between the α-th and $\alpha + 1$-th level of T^1. The partial order on T will be defined naturally, but first we need some preparation. Let $\{y_n\}_n$ be an enumeration of $Y^1_{\alpha+1}$. For each n, the set C_n of predecessors of $(\alpha + 1, y_n)$ on level α in T^1 is infinite, so we can find infinite sets $C_{n,k}$, for $n, k < \omega$, such that $C_{n,k}$ is a subset of C_n, and $C_{m,k} \cap C_{n,l} = \emptyset$, whenever $(m, k) \neq (n, l)$. Finally, let $\{x_k\}_k$ be an enumeration of Y^2_0. We put the n-th element of $(\alpha + 1)$-th level of T^1 above the n-th copy of T^2 and the k-th element of the 0-th level of the n-th copy of T^2 above the elements of $C_{n,k}$ and then extend the ordering by transitivity. More precisely, for an ordinal $\xi \in (\alpha + 1) \cup [\alpha + 1 + \rho, \lambda)$ let $\varphi(\xi)$ be defined by :

$$\varphi(\xi) = \begin{cases} \xi & \text{if } \xi \leq \alpha \\ \alpha + 1 + \eta & \text{if } \xi = \alpha + 1 + \rho + \eta \text{ for } \eta < \beta \end{cases}$$

Let s and t be two elements of T and suppose $s = (\nu, x)$ and $t = (\xi, y)$. We have several cases.

Case 1 : If $\nu, \xi \in (\alpha + 1) \cup [\alpha + 1 + \rho, \lambda)$ we let $s \leq t$ if and only if $(\varphi(\nu), x) \leq_1 (\varphi(\xi), y)$.

Case 2 : Suppose $\nu, \xi \in [\alpha + 1, \alpha + 1 + \rho)$. Then we have $\nu = \alpha + 1 + \eta$ and $\xi = \alpha + 1 + \theta$, for some $\eta, \theta < \rho$. We have that x is of the form (n, u) for some $u \in Y^2_\eta$ and y is of the form (m, v) for some $v \in Y^2_\theta$. In this case we let $s \leq t$ if and only if $n = m$ and $(\eta, u) \leq_2 (\theta, v)$.

Case 3 : Suppose $\nu \leq \alpha + 1$ and $\xi \in [\alpha + 1, \alpha + 1 + \rho)$. We have that $\xi = \alpha + 1 + \theta$, for some $\theta < \rho$, and $y = (n, v)$ for some n and $v \in Y^2_\theta$. In this case we let $s \leq t$ if and only if there is k such that $(0, x_k) \leq_2 (\theta, v)$ and there is $z \in C_{n,k}$ such that $s \leq_1 (\alpha, z)$.

Case 4 : Suppose $\nu \in [\alpha + 1, \alpha + 1 + \rho)$ and $\xi \in [\alpha + 1 + \rho, \lambda)$. Then we have $\nu = \alpha + 1 + \eta$, for some $\eta < \rho$ and $x = (n, u)$ for some $u \in Y^2_\eta$. We also have that $y \in Y_{\varphi(\xi)}$. We let $s \leq t$ if and only $(\alpha + 1, y_n) \leq_1 (\varphi(\xi), y)$.

Let $e : T^1 \to T$ be defined by setting $e((\eta, x)) = (\varphi^{-1}(\eta), x)$, for all $(\eta, x) \in T^1$. For each n let $f_n : T^2 \to T$ be defined by $f_n((\theta, y)) = (\alpha + 1 + \theta, (n, y))$, for $(\theta, y) \in T^2$. It is clear that e and the f_n are embeddings and

$$T = e[T^1] \cup \bigcup_n f_n[T^2].$$

We now define the function i on $[T]^2$. If $s, t \in e[T^1]$ let $i\{s, t\} = \{e(u) : u \in i^1\{e^{-1}(s), e^{-1}(t)\}$. If $s, t \in f_n[T^2]$ let $i\{s, t\} = \{f_n(u) : u \in i^2\{f_n^{-1}(s), f_n^{-1}(t)\}\}$. If $s \in f_m[T^2]$ and $t \in f_n[T^2]$, for $m \neq n$, let $i\{s, t\} = \emptyset$. Suppose now $s \in f_n[T^2]$, for some n, and $t \in e[T^1]$. If $s \leq t$ we let $i\{s, t\} = \{s\}$ and if $t \leq s$ we let $i\{s, t\} = \{t\}$. Finally, suppose s and t are incomparable. Let $C_\alpha(s)$ be the set of predecessors of s on level α of T. Then we let

$$i\{s, t\} = \bigcup\{i_1\{u, t\} : u \in C_\alpha(s)\}.$$

Notice that we cannot hope that $i\{s,t\}$ will always be finite, because of the last clause of the definition. It is at least clear that $i\{s,t\}$ is a basis, e.g it verifies properties $2(a)$ and $2(b)$ of Definition III.1. We now isolate a condition which guarantees that $i\{s,t\}$ will be finite, for all $s,t \in T$.

Definition III.5. *Let T be a $\Omega(\lambda)$-poset for some λ and let $\gamma < \lambda$. We say that T_γ, the γ-th level of T, is a* bone level *if :*

1. *If $s,t \in T_\gamma$ and $s \neq t$ then $i\{s,t\} = \emptyset$,*
2. *If $t \in T_{\gamma+1}$ and $s < t$ then there exists $u \in T_\gamma$ such that $s \leq u < t$*

*T is called an $ht(T)$-*skeleton *if every level of T is a bone level.*

Now we show that a bone-level can be used to fix the previous problem and thus to obtain a new $\Omega(\lambda)$-poset.

Proposition III.6. *Let $(T, \leq, i)$ be an $\Omega(\lambda)$ poset and let $\gamma < \lambda$. If T_γ is a bone level, then for every $s \in T_{\gamma+1}$ and every t incomparable with s, the set $I(s,t) = \{u \in T_\gamma : u < s \text{ and } i\{u,t\} \neq \emptyset\}$ is finite.*

Proof : Note that if $x \leq s,t$ then by (2) of Definition III.5 there is $u \in I(s,t)$ such that $x \leq u$. On the other hand, if $u,v \in I(s,t)$ then by (1) of Definition III.1 $i\{u,v\} = \emptyset$. It follows that the sets $i\{u,t\}$, for $u \in I(s,t)$, are pairwise disjoint and $Card(i\{s,t\}) \geq Card(I(s,t))$. Since $i\{s,t\}$ is finite it follows that $I(s,t)$ is finite, as well.

□

We now have the following immediate consequence.

Lemma III.7. *Let $(T^1, \leq_1, i_1)$ be an $\Omega(\alpha+1+\beta)$-poset and $(T^2, \leq_2, i_2)$ an $\Omega(\rho)$-poset. Assume T^1_α, the α-th level of T^1, is a bone level. Then $T^2 \hookrightarrow^\alpha T^1$ is an $\Omega(\alpha+1+\rho+\beta)$-poset.*

□

In the next theorem we show that a skeleton of height κ can be stretched to a $\Omega(\delta)$-poset, for any $\delta < \kappa^+$.

Theorem III.8. *Let κ be an infinite cardinal. Assume there is a κ-skeleton. Then there is an $\Omega(\delta)$-poset, for any $\delta < \kappa^+$.*

Proof : We show by induction that for every $\delta < \kappa^+$ there is an $\Omega(\delta)$-poset $(T^\delta, \leq_\delta, i_\delta)$. We start the induction at $\delta = \kappa$ for which by the assumption of the theorem we know that there is a δ-skeleton, which is of course an $\Omega(\delta)$-poset. If $\delta = \gamma + 1$ is a successor, Proposition III.3 allows us to build an $\Omega(\delta)$-poset from an $\Omega(\gamma)$-poset. Assume now δ is a limit ordinal and let $\mu = \text{ cof }(\delta)$. Then $\mu \leq \kappa$ and we know that there is an μ-skeleton S. We could, for instance, take the first μ levels of a κ-skeleton. Let $C = \{\gamma_\nu : \nu < \mu\}$ be a club in δ of order type μ and let $\gamma_\mu = \delta$ by convention. Let $\delta_\nu = o.t.(\gamma_{\nu+1} \setminus (\gamma_\nu + 1))$. The idea is to simply insert an $\Omega(\delta_\nu)$-poset between the ν-th and the $\nu + 1$-st level of S as in Proposition III.7. More precisely, let $E_\nu = \gamma_\nu \cup (C \setminus \gamma_\nu)$. Then the order type of E_ν is $\gamma_\nu + (\mu - \nu)$. Let us define the function $e_\nu : \mu \to \gamma_\nu + (\mu - \nu)$ by

$$e_\nu(\eta) = \begin{cases} \gamma_\eta & \text{if } \eta \leq \nu \\ \gamma_\nu + \xi & \text{if } \eta = \nu + \xi \text{ for some } \xi < \mu - \eta \end{cases}$$

and let φ_ν be defined on S by $\varphi_\nu(\eta, y) = (e_\nu(\eta), y)$. We will construct by induction on $\nu \leq \mu$ an $\Omega(\gamma_\nu + (\mu - \nu))$-poset R^ν with the following properties.

1. $R^0 = S$.
2. If $\nu < \xi$ the $R^\xi \restriction \gamma_\nu = R^\nu \restriction \gamma_\nu$.
3. φ_ν is an isomorphism between S and $R^\nu \restriction e_\nu[\mu]$.
4. $R^{\nu+1} = T^{\delta_\nu} \hookrightarrow^{\gamma_\nu} R^\nu$.

If $\nu = \eta + 1$, then γ_η is a bone-level of R^η so we can let $R^\nu = T^{\delta_\eta} \hookrightarrow^{\gamma_\eta} R^\eta$. If ν is a limit ordinal we let

$$R^\nu = \bigcup \{R^\eta \restriction \gamma_\eta : \eta < \nu\} \cup \varphi_\nu[S].$$

The ordering $\leq_\nu$ is defined in the natural way. On $\bigcup\{T^\eta \restriction \gamma_\eta : \eta < \nu\}$ we take $\bigcup\{\leq_\eta : \eta < \nu\}$. On $\varphi_\nu[S]$ we copy the ordering of S and then extend to an ordering of all of T^ν by transitivity. The definition of the function i_ν is similar. Now, it should be clear that R^ν has the required properties. Then T^δ is just R^μ.

□

Our next goal is to show that it is relatively consistent with ZFC to have an ω_2-skeleton. Since the proof is a mild modification of an argument of Baumgartner and Shelah from [12] we will be rather sketchy. We first recall the definition and the basic properties of a Δ-function.

Let $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ be a function with $f\{\alpha,\beta\} \subseteq \alpha \cap \beta$ for $\{\alpha,\beta\} \in [\omega_2]^2$. We say that two finite subsets a and b of ω_2 are *good* for f if for $\gamma \in a \cap b$, $\alpha \in a \setminus b$ and $\beta \in b \setminus a$ we have :

(a) if $\gamma < \alpha$, β then $\gamma \in f\{\alpha,\beta\}$

(b) if $\gamma < \beta$ then $f\{\alpha,\gamma\} \subseteq f\{\alpha,\beta\}$

(c) if $\gamma < \alpha$ then $f\{\beta,\gamma\} \subseteq f\{\alpha,\beta\}$

We say that f is a Δ-*function* if every uncountable family of finite subsets of ω_2 contains two elements which are good for f. We say that f is a *strong* Δ-*function* if every uncountable family A of finite subsets of ω_2 contains an uncountable subfamily B such that any two elements of B are good for f.

A Δ-function is the key technical tool used in [12] to generically add a thin very tall LCS space by a ccc forcing notion. Originally Baumgartner and Shelah first added a Δ-function by Shelah's method of historical forcing. It was later shown by Velickovic in [20] that a Δ-function can be obtained from Jensen's principle $\square_{\omega_1}$ using Todorcevic's method of minimal walks. In fact, we have shown in chapter two that we can obtain a strong Δ-function (see also [11], Lemma 7.4.9.) We now have the following.

Proposition III.9. *Assume $\square_{\omega_1}$. Then there exists a strong Δ-function.*

We recall that we have also the following property.

Proposition III.10. *Let $\langle C_\alpha : \alpha < \omega_2\rangle$ be a $\square_{\omega_1}$-sequence and let f be the strong Δ-function constructed from this sequence. Then for every $\eta < \alpha < \beta < \omega_2$, if η is a limit point of both C_α and C_β then $\eta \in f\{\alpha,\beta\}$.*

The proof of the following theorem is a slight modification of an argument from [12]. We present the proof since it will be used later.

Theorem III.11. *Assume $\square_{\omega_1}$. Then there is a property K forcing notion which adds an ω_2-skeleton.*

Proof : Let us fix a strong Δ-function f as in Proposition III.9. We define a forcing notion $\mathcal{P}$ which adds a required partial ordering on $\omega_2 \times \omega$. Recall that if x is of the form (α, n) then we denote α by $\pi_1(x)$ and n by $\pi_2(x)$. We say that $p \in \mathcal{P}$ if $p = \langle x_p, \leq_p, i_p \rangle$ where x_p is a finite subset of $\omega_2 \times \omega$, $\leq_p$ is a partial ordering of x_p and $i_p : [x_p]^2 \to [x_p]^{<\omega}$ and the following conditions are satisfied.

1. If $s, t \in x_p$ and $s <_p t$ then $\pi_1(x) < \pi_1(t)$.
2. If $s, t \in x_p$ with $s <_p t$ and $\pi_1(t)$ is a successor ordinal then there is $u \in x_p$ such that $s \leq_p u <_p t$ and $\pi_1(t) = \pi_1(u) + 1$.
3. If s and t are incomparable then $i_p\{s, t\} \subseteq (f\{\pi_1(s), \pi_1(t)\} \times \omega) \cap x_p$.
4. If $s \leq_p t$ then $i_p\{s, t\} = \{s\}$.
5. If $s \neq t$ and $\pi_1(s) = \pi_1(t)$ then $i_p\{s, t\} = \emptyset$.
6. If $u \in i_p\{s, t\}$ then $u \leq_p s, t$.
7. For every $u \leq_p s, t$ there is $v \in i_p\{s, t\}$ such that $u \leq_p v$.

We let $p \leq q$ if and only if $x_p \supseteq x_q$, $\leq_p \restriction x_q = \leq_q$ and $i_p \restriction [x_q]^2 = i_q$. In order to verify that $\mathcal{P}$ satisfies Knaster's chain condition, suppose $\mathcal{A}$ is an uncountable subset of $\mathcal{P}$. By extending if necessary, we can assume that the domain x_p of each $p \in \mathcal{A}$ is of the form $E_p \times n_p$. By shrinking $\mathcal{A}$ we may assume that the sets E_p, for $p \in \mathcal{A}$, form a Δ-system with root R, that there is an integer n such that $n_p = n$, for all $p \in \mathcal{A}$, the conditions in $\mathcal{A}$ generate isomorphic structures over $R \times n$ and that if $p, q \in \mathcal{A}$ are distinct and $\alpha \in E_p \setminus R$ and $\beta \in E_q \setminus R$ and $\alpha < \beta$ then $\beta - \alpha \geq \omega$.

Using the fact that f is a strong Δ-function we can find an uncountable subset $\mathcal{B}$ of $\mathcal{A}$ such that if $p, q \in \mathcal{B}$ and $p \neq q$ then E_p and E_q are good for f. Consider now two conditions p and q from $\mathcal{B}$. We will show that they are compatible. Let $r = \langle x_r, \leq_r, i_r \rangle$ be defined as follows : $x_r = x_p \cup x_q$ and $s \leq_r t$ if and only if $s \leq_p t$, or $s \leq_q t$, or there is $u \in R \times n$ such that $s \leq_p u \leq_q t$ or $s \leq_q u \leq_p t$. One can verify easily that $\leq_r$ is a partial ordering on x_r and $\leq_r \restriction x_p = \leq_p$ and $\leq_r \restriction x_q = \leq_q$. We define $i_r : [x_r]^2 \to [x_r]^{<\omega}$ by letting $i_r \restriction [x_p]^2 = i_p$, $i_r \restriction [x_q]^2 = i_q$, $i_r\{s, t\} = \{s\}$ if $s \leq_r t$ and if $s \in x_p \setminus R \times n$ and $t \in x_q \setminus R \times n$ are $\leq_r$-incomparable then

$$i_r\{s, t\} = \{u \in (f\{\pi_1(s), \pi_1(t)\} \times n) \cap x_r : u \leq_r s, t\}.$$

We need to check that $r \in \mathcal{P}$. Conditions (1) and (2) follow from the way we have defined the ordering $\leq_r$.

It is nontrivial to check that r satisfies condition (7). To see this, assume $u \leq_r s, t$. If $s, t \in x_p \setminus x_q$ and $u \in x_q \setminus x_p$ then there are $z_s, z_t \in R \times n$ such that $u \leq_q z_s \leq_p s$ and $u \leq_q z_t \leq_p t$. By (7) for q there is $v \in i_q\{z_s, z_t\}$ such that $u \leq_q v$. since $i_q\{z_s, z_t\} \subseteq R \times n$ we conclude that $v \in R \times n$. Thus $v \leq_p s, t$ so by (7) for p there is $w \in i_p\{s, t\}$ such that $v \leq_p w$ and therefore $u \leq_r w$. The case $s \in R \times n$, $\beta \in x_p \setminus x_q$ and $u \in x_q \setminus x_p$ is similar.

Suppose now $s \in x_p \setminus x_q$, $t \in x_q \setminus x_p$ and they are $\leq_r$-incomparable. Suppose now $u \leq s, t$ and for concreteness $u \in x_p$. Then $u \leq_p s$ and $u \leq_p v \leq_q t$, for some $v \in R \times n$. By (7) for p there is $w \in i_p\{s, v\}$ such that $u \leq_p w \leq_p s$. We need to check that w was put in $i_r\{s, t\}$. Let $\alpha = \pi_1(s)$, $\beta = \pi_1(t)$, $\tau = \pi_1(v)$ and $\xi = \pi_1(w)$. We need to check that $\xi \in f\{\alpha, \beta\}$. Note that by (1) for p and q we must have $\tau < \beta$. If $v \leq_p s$ then $w = v$ and we must have $\tau < \alpha, \beta$ and since E_p and E_q are good for f then $\tau \in f\{\alpha, \beta\}$. If v and s are incomparable then by property (3) for p we must have $i_p\{s, v\} \subseteq f\{\alpha, \tau\} \times n$. Now, again since E_p and E_q are good for f we must have that $f\{\alpha, \tau\} \subseteq f\{\alpha, \beta\}$, i.e. $\xi \in f\{\alpha, \beta\}$. It follows that $w \in i_r\{s, t\}$. The remaining cases are similar. We should point out that condition (2) does not pose a problem since we have that if $\alpha \in E_p \setminus R$ and $\beta \in E_q \setminus R$ then the distance between α and β is infinite.

A simple density argument shows that if p is any condition in $\mathcal{P}$, $t \in \omega_2 \times \omega$, $\alpha < \pi_1(t)$ and n is an integer then there is a condition $q \leq p$ such that $t \in x_q$ and the set $\{s \in x_q : \pi_1(s) = \alpha \text{ and } s \leq_q t\}$ has at least n elements.

Let now G be a V-generic filter on $\mathcal{P}$ and let $\leq_G$ be the union of $\leq_p$, for $p \in G$, and let i_G be the union of i_p, for $p \in G$. It is now straightforward to check that $(\omega_2 \times \omega, \leq_G, i_G)$ is an ω_2-skeleton.

□

Now, putting Theorem III.8 and Theorem III.11 we have the following immediate corollary.

Corollary III.12. *It is relatively consistent with ZFC that there are $\Omega(\delta)$-posets, for all $\delta < \omega_3$.*

□

III.2 PCF structures

In this section, we deal with the main topic of this thesis, that is PCF structures. The idea is to isolate the properties of the PCF spaces Shelah used to prove his celebrated theorem in [7], and see how far can we go with these. The definition we give here is a simplified one, since we only deal with thin PCF structures and don't investigate all possible cardinal sequences.

We first recall the basic properties of the PCF operator. If A is a set of regular cardinals and if U is an ultrafilter over A, the set $\prod A/U$ is linearly ordered, and so it has some cofinality κ. Define

$$\text{pcf}\,(A) = \{\ \text{cof}\,(\prod A/U)\ :\ U \text{ ultrafilter on } A\}$$

If A is an interval of regular cardinals such that $|A| < \min(A)$ (in chapter one we supposed $2^{|A|} < \min(A)$ for simplification) then pcf (A) is also an interval of regular cardinals and the pcf operator has the following properties for any $X,\ Y \subseteq$ pcf (A).

(a) $X \subseteq \text{pcf}\,(X)$, $\text{pcf}\,(X) \cup \text{pcf}\,(Y) = \text{pcf}\,(X \cup Y)$, $\text{pcf}\,(\text{pcf}\,(X)) = \text{pcf}\,(X)$.

(b) If $\gamma \in \text{pcf}\,(X)$, then there exists $X' \subseteq X$ with $|X'| = |A|$ such that $\gamma \in \text{pcf}\,(X')$.

(c) pcf (X) has a maximal element.

(d) If $\nu < \max\ \text{pcf}\,(A)$ is a singular cardinal of uncountable cofinality then there exists a club C in ν such that $\max\ \text{pcf}\,(\{\lambda^+ : \lambda \in C\}) = \nu^+$.

All these properties together implies that $|\ \text{pcf}\,(A)| < |A|^{+4}$. A proof of this fact can be found in chapter one, in Shelah's book [7] or in [17].

We remark that by the first property we can view the pcf operator as a topological closure operator.

In our case we will work with $A = \{\aleph_{n+1} : n \in \omega\}$. Then $\max(\text{pcf}\,(A))$ exists and is equal to some $\aleph_{\rho+1}$ with $\rho < \omega_4$. Since pcf (A) is an interval of regular cardinals the map $\alpha \mapsto \aleph_{\alpha+1}$ is a bijection from $\rho + 1$ to pcf (A). Then we might as well define our topological structure which satisifies all the properties above on $\rho + 1$. We give here a simpler definition of a PCF structure for the purpose of forcing :

Definition III.13. *A* thin PCF structure *of height λ is a quadruple $(T, \leq_T, i_T, \lhd_T)$ such that $(T, \leq_T, i_T)$ is an $\Omega(\lambda)$-poset and $\lhd_T$ is a well-ordering on T such that :*

1. *If* $\pi_1(s) < \pi_1(t)$ *then* $s \lhd_T t$.

2. T_α *is ordered by* $\lhd_T$ *in order type* ω*, for every* $\alpha < ht(T)$.

3. *For every* $s \in T$ *such that cof* $(s) > \omega$*, there exists a club* C *of* s *such that for every* $x \in C$, $x \leq_T s$.

In this definition, the notions of order type, cofinality or club are defined relative to the well-ordering $\lhd_T$. Later on, when we will refer to these notions, we will talk about $\lhd_T$-cofinality or $\lhd_T$-club so that there will not be any confusion with the well-ordering on ordinals.

We remark here that the $C(s)$ for $s \in T$ (recall that $C(s) = \{t \in T \ : \ t \leq_T s\}$) are closely related to the notion of generators introduced in [7]. So, the next proposition is in fact a slight modification of arguments from [7] or in [17].

Proposition III.14. *If there exists a thin PCF structure* $(T, \leq_T, i, \lhd_T)$ *of height* λ *with cof* $(\lambda) > \omega$*, then there exists a closure operator on* $\lambda + 1$ *with properties (a)-(d) above.*

Proof : First notice that since cof $(\lambda) > \omega$ and all levels of T are of order type ω under $\lhd_T$, $(T, \lhd_T)$ is isomorphic to λ with the usual ordering. We can therefore identify T with λ via this isomorphism. We let $T^* = T \cup \{\lambda\}$, i.e. $T^* = \lambda + 1$, and extend the ordering $\leq_T$ to $\leq_{T^*}$ by letting $t \leq_{T^*} \lambda$, for every $t \in T$. The ordering $\lhd_{T^*}$ is just the usual well ordering on $\lambda + 1$. By Proposition III.2 we know that $\{C(s) \setminus \bigcup_{1 \leq j \leq n} C(s_j) : s \in T^*, n \in \omega, s_j <_T s\}$ is a clopen basis for a locally compact scattered topology. Moreover, since T^* has a maximal point this topology is actually compact. We then consider the closure operator relative to this topology. So if we show Properties (a)-(d) for this closure operator, then we are done. Property (a) is obvious as it is a closure operator.

For Property (b), first notice that T_0 (recall that T_0 is the first level of T^*) is dense in our topology. This is a basic result from LCS spaces, as any non empty set must contain an isolated point. Now, let $X \subseteq T^*$ and let $s \in \bar{X}$ ($\bar{X}$ is the closure of X). We define $C(Y) = \bigcup\{C(s) : s \in Y\}$ for any $Y \subseteq T^*$. Now, find a countable $Y \subseteq C(s) \cap X$ such that $C(Y) \cap T_0 = (C(s) \cap C(X)) \cap T_0$. Suppose $s \notin \bar{Y}$. Then, there exists s_1, ..., $s_n <_{T^*} s$ such that

$$[C(s) \setminus \bigcup_{1 \leq j \leq n} C(s_j)] \cap Y = \emptyset.$$

By transitivity of the ordering $\leq_{T^*}$, this implies that $C(Y) \subseteq \bigcup_{1\leq j\leq n} C(s_j)$ and so that

$$[C(s) \setminus \bigcup_{1\leq j\leq n} C(s_j)] \cap C(Y) \cap T_0 = \emptyset.$$

But since Y has been chosen so that $C(Y) \cap T_0 = (C(s) \cap C(X)) \cap T_0$ we have

$$[C(s) \setminus \bigcup_{1\leq j\leq n} C(s_j)] \cap C(X) \cap T_0 = \emptyset.$$

And this is a contradiction since $[C(s) \setminus \bigcup_{1\leq j\leq n} C(s_j)] \cap C(X)$ is an nonempty open set and T_0 is dense.

For Property (c) we already known T^* is compact since it has a maximal element. For $X \subseteq T^*$, $\bar{X}$ is then compact since it is a closed subset of T^*. By compactness there exist $s_1, \ldots, s_n \in X$ such that $\bar{X} \subseteq \bigcup_{1\leq j\leq n} C(s_j)$. If we let s be the maximum in the well ordering $\lhd_{T^*}$ of the s_j then by Property 1 of the previous definition, s is the $\lhd_T$-maximum of $\bar{X}$.

Finally Property (d) is an immediate consequence of Property 3 in the previous definition.

□

Our goal is to construct thin PCF structures of height δ for all $\delta < \omega_3$. The proof is quite similar to the one we saw in previously, as our construction gives the club property almost for free. But in order to do this, we need to have an ω_2-skeleton which is also a thin PCF structure. In §1, we built an ω_2-skeleton on $\omega_2 \times \omega$, so the well-ordering we will define is a natural one. We let $(\alpha, n) \lhd (\beta, m)$ if and only if $\alpha < \beta$ or $\alpha = \beta$ and $n < m$. Then for each $\alpha < \omega_2$ the α-th level of the skeleton can be identified with the interval $[\omega \cdot \alpha, \omega \cdot (\alpha + 1))$ and the whole skeleton is isomorphic to ω_2. With this ordering, it is easy to see that $(\alpha, 0)$ has $\lhd$-cofinality ω_1 if and only if α has cofinality ω_1 and that if C_α is a club in α then $C_\alpha \times \{0\}$ is a $\lhd$-club in $(\alpha, 0)$.

So, what we need to do is to find clubs C_α in ω_2 such that property 3 of Definition III.13 is satisfied for the $\lhd$-clubs $C_\alpha \times \{0\}$. Since we want to force our space with a ccc forcing notion, we need to find the clubs in the ground model. For technical reasons (namely because of the function i defined in §2) we will need that our clubs have finite intersection which is the purpose of the next two lemmas.

Lemma III.15. *Let $\mathcal{A}$ be a family of ω_1 closed countable subsets of ω_1 such that $A \cap B$ is finite, for all distinct $A, B \in \mathcal{A}$. Then there is a proper forcing notion $\mathcal{C}$ of size $\aleph_1$ which adds a club in ω_1 which has finite intersection with all members of $\mathcal{A}$.*

Proof : Let us fix a $1-1$ enumeration $\langle A_\alpha : \alpha < \omega_1 \rangle$ of $\mathcal{A}$. We define the forcing notion $\mathcal{C}$ as follows. Elements of $\mathcal{C}$ are of the form $p = \langle _p, F_p \rangle$ where $_p$ is a finite collection of countable closed disjoints intervals of ω_1 and F_p is a finite subset of ω_1. For $p \in \mathcal{C}$ let $C_p = \{\min I : I \in_p\}$. We say $p \leq q$ if $_p \supseteq_q$, $F_p \supseteq F_q$, and $(C_p \setminus C_q) \cap A_\alpha = \emptyset$, for all $\alpha \in F_q$. Now, if G is a V-generic filter over $\mathcal{C}$ then we claim that $C_G = \bigcup\{C_p : p \in G\}$ will be the required club. First note that C_G will be unbounded in ω_1. To see this note that for any $\alpha < \omega_1$ the set of $p \in \mathcal{C}$ such that $C_p \setminus \alpha \neq \emptyset$ is dense in $\mathcal{C}$. Next, notice that C_G will have finite intersection with A_α, for all $\alpha < \omega_1$. To see this note that the set of $p \in \mathcal{C}$ such that $\alpha \in F_p$ is dense in $\mathcal{C}$. Finally, note that C_G will be closed in ω_1. To see this suppose $p \in \mathcal{C}$, $\gamma < \omega_1$ and $p\gamma \notin C_{\dot{G}}$. Then in particular $\gamma \notin C_p$. If there is $I \in_p$ such that $\gamma \in I$ then let $J = I \setminus \{\min I\}$. Then J is an open interval containing γ and $pJ \cap C_{\dot{G}} = \emptyset$. If $\gamma \notin \bigcup_p$ then we can find a closed interval J such that $J \cap \bigcup_p = \emptyset$ and $\gamma \in J^* = J \setminus \{\min J\}$. Let $q = \langle_p \cup \{J\}, F_p$. Then $q \leq p$ and $qJ^* \cap C_{\dot{G}} = \emptyset$. Thus, we have show that for every γ the set of conditions p such that either p forces that γ is in $C_{\dot{G}}$ or there is an open interval U containing γ such that p forces that $U \cap C_{\dot{G}} = \emptyset$ is dense in $\mathcal{C}$. Therefore, C_G is closed in ω_1. It remains to establish the following.

Claim $\mathcal{C}$ is a proper forcing notion.

Proof : Let θ be a sufficiently large regular cardinal and fix a countable elementary submodel M of H_θ containing all the relevant information. Let $p \in \mathcal{C} \cap M$. We need to find a $q \leq p$ which is $(M, \mathcal{C})$-generic. Let $\delta = M \cap \omega_1$. First notice that $A_\alpha \subseteq M$, for all $\alpha \in F_p$ therefore $q = \langle_p \cup \{[\alpha, \alpha]\}, F_p \rangle \in \mathcal{C}$. We show that q is the required condition. To see this let D be a dense subset of $\mathcal{C}$ with $D \in M$. We would like to show that $D \cap M$ is predense below q. Fix $r \leq q$ and assume, without loss of generality, that $r \in D$. Let $r_0 = \langle_r \cap M, F_r \cap M \rangle$. Then $r_0 \in M$. If α is a countable ordinal and a finite collection of intervals let $\restriction \alpha$ denote the collection of all $I \in$ such that $\sup I < \alpha$. For a condition $s \in \mathcal{C}$ let $s \restriction \alpha$ denote $\langle_s \restriction \alpha, F_s \cap \alpha \rangle$. Let $\nu < \delta$ be sufficiently large so that $r_0 \restriction \gamma = r_0$. Let n be the number of intervals in ${}_r \setminus I_{r_0}$. Then for every ξ such that $\nu \leq \xi < \delta$ there exists $s \in D$ such that $s \restriction \xi = r_0$ and $|I_s \setminus I_{r_0}| = n$, namely we can take $s = r$. By elementarity such a condition s exists in M. Therefore, we can find a sequence $\langle s_i : i < \omega \rangle \in M$ such that $s_i \in D$, $s \restriction \nu = r_0$ and $|{}_{s_i} \setminus I_{r_0}| = n$, for all i, and such that $\max(C_{s_i} \setminus C_{r_0}) < \min(C_{s_j} \setminus C_{r_0})$, whenever $i < j$. We want to show that some s_i is compatible with r, for some i. If not then since $s_i \in M$ and $s \leq \nu = r_0$ then there there is $\xi_i \in F_r \setminus F_{r_0}$ and $J_i \in I_{s_i} \setminus I_{r_0}$ such that $\min J_i \in A_{\xi_i}$. Let $l_i < n$ be such that J_i is the l_i-th interval in the natural order of ${}_{s_i} \setminus {}_{r_0}$. Now let $\mathcal{U}$ be a nonprincipal ultrafilter on ω such that $\mathcal{U} \in M$. Since $F_r \setminus F_{r_0}$ is finite there must exist a fixed $\xi \in F_r \setminus F_{r_0}$ and $l < n$ such that $\xi_i = \xi$ and $l_i = l$, for $\mathcal{U}$-many i. Now $\xi > \delta$ and by elementarity of M there must exist $\eta < \delta$ such that the set of i such that A_η contains the minimum of the l-th interval of ${}_{s_i}$ is in $\mathcal{U}$. It follows that $A_\xi \cap A_\eta$ is infinite, which contradicts our assumption on $\mathcal{A}$. This finishes the proof of the claim and Lemma III.15.

□

The following lemma will be used to build a poset which adds a PCF structure of size ω_2.

Lemma III.16. *Assume GCH. Then there is a $\aleph_2$-cc proper forcing notion $\mathcal{Q}$ which adds a sequence $\langle C_\alpha : \alpha < \omega_2 \,\&\; cof\,(\alpha) = \omega_1 \rangle$ such that C_α is a club in α, for all α, and $C_\alpha \cap C_\beta$ is finite, for all $\alpha \neq \beta$.*

Proof : The forcing notion $\mathcal{Q}$ is obtained as a countable support iteration $\langle \mathcal{Q}_\xi; \dot{\mathcal{C}}_\xi : \xi < \omega_2 \rangle$ of forcing notions as constructed in Lemma III.15. At the ξ-th stage of the iteration we have already build C_η, for all $\eta < \xi$ such that cof $(\eta) = \omega_1$. If ξ is not of cofinality ω_1 let $\dot{\mathcal{C}}_\xi$ be the $\mathcal{Q}_\xi$-name for a trivial forcing notion. Otherwise, fix a club E in ξ of order type ω_1. Let $A_\eta = C_\eta \cap E$, for $\eta < \xi$. Then the A_η are countable, closed and have pairwise finite intersections. Let $\mathcal{A} = \{A_\eta : \eta < \xi \,\&\, cof(\eta) = \omega_1\}$. By Lemma III.15 there is a proper forcing notion $\mathcal{C}$ of size $\aleph_1$ which adds a club C in ω_1 which has finite intersection with A_η, for all $\eta < \xi$. Let then $\dot{\mathcal{C}}_\xi$ be a $\mathcal{Q}_\xi$-name for such a forcing notion and let $\dot{C}_\xi$ be a $\mathcal{Q}_{\xi+1}$-name for such a C. The fact that the iteration is proper and satisfies the $\aleph_2$-cc is standard and can be found, for instance, in [21].

□

We now state a simple property that the previous sequence of clubs satisfies. It will be useful to prove the ccc for our forcing notion.

Proposition III.17. *Let A be an uncountable set of finite pairwise disjoint subsets of ω_2 and let $\langle C_\alpha : \alpha < \omega_2 \rangle$ be a sequence such that C_α is a subset of α, for all α, and $C_\alpha \cap C_\beta$ is finite, for all $\alpha \neq \beta$. Then there exist distinct $x, y \in A$ such that for every $\alpha \in x$ and $\beta \in y$, $C_\alpha \cap y = C_\beta \cap x = \emptyset$.*

Proof : Since A is uncountable, we may assume that there exists $n \in \omega$ such that $|x| = n$, for all $x \in A$. For $x \in A$ let $C_x = \bigcup\{C_\alpha : \alpha \in x\}$. Note that the sets C_x, for $x \in A$, have pairwise finite intersection.

We claim that if $x_0, ..., x_n \in A$ are distinct then the set

$$Y = \{y \in A : y \cap C_{x_i} \neq \emptyset, \text{ for all } i \leq n\}$$

is finite. To see this, note that for every $y \in Y$ there are distinct $i_y, j_y \leq n$ such that $y \cap C_{x_{i_y}} \cap C_{x_{j_y}} \neq \emptyset$. Since Y is infinite there are $i, j \leq n$ and an infinite subset Y_0 of Y such that $i_y = i$ and $j_y = j$, for all $y \in Y_0$. Since the elements of Y_0 are pairwise disjoint it follows that $C_{x_i} \cap C_{x_j}$ is infinite, a contradiction.

Now, we claim that there exists an uncountable $B \subseteq A$ such that for all $x \in B$,

$$\{y \in B : y \cap C_x \neq \emptyset\}$$

is countable. If not, let $A_0 = A$, and given an uncountable $A_i \subseteq A$, let $x_i \in A_i$ be distinct from the x_j for $j < i$ and such that

$$A_{i+1} = \{y \in A_i : y \cap C_{x_i} \neq \emptyset\}$$

is uncountable. But then $x_0, \ldots, x_n$ contradict the previous claim.

We can now construct a sequence $\langle x_\xi : \xi < \omega_1 \rangle$ of distinct elements of B such that if $\xi < \eta < \omega_1$ then $x_\eta \cap C_{x_\xi} = \emptyset$. We claim that there are $\xi < \eta$ such that $x_\xi \cap C_{x_\eta} = \emptyset$. Otherwise, we have that $x_\xi \cap C_{x_\eta} \neq \emptyset$, for all $\xi < \eta$. But then $\{x_i : i \in \omega\}$ and $x_\omega, ..., x_{\omega+n}$ contradict the previous claim.

□

We are now almost ready to define our forcing notion. Before we start we introduce some notations. Recall that $\lhd$ is the well-ordering on $\omega_2 \times \omega$ such that $(\alpha, n) \lhd (\beta, m)$ if and only if $\alpha < \beta$ or $\alpha = \beta$ and $n < m$. So $\lhd$ is simply the lexicographic ordering on $\omega_2 \times \omega$. The notions of $\lhd$-cofinality and $\lhd$-club are the natural ones related to the well-ordering $\lhd$. An ω_2-skeleton $(T, \leq_T, i_T)$ such that $T = \omega_2 \times \omega$ will be called a *candidate*. Given a candidate $U = (\omega_2 \times \omega, \leq_U, i_U)$ and $\alpha \in \omega_2$ of uncountable cofinality, α is said to be *good for* U if there exists a club C in α such that for all $\gamma \in C$ $(\gamma, 0) \leq_U (\alpha, 0)$. Obviously if every ordinal in ω_2 of uncountable cofinality is good for U then U is a PCF structure which is also an ω_2-skeleton. This is because $x \in \omega_2 \times \omega$ is of uncountable $\lhd$-cofinality if and only if $x = (\alpha, 0)$ for some $\alpha \in \omega_2$ of uncountable cofinality. In a more general way the set of good points for a candidate U will be denoted by S_U. In the following S^κ_λ will be the set of ordinals less than λ of cofinality κ for any λ, κ infinite cardinals. So, our hope is to define a ccc forcing notion which adds a PCF structure but unfortunately the best we can hope for is the following theorem :

Theorem III.18. *It is relatively consistent with ZFC that there exists a candidate U such that S_U is stationnary.*

Assume Theorem III.18 for now and we will show that this result is in fact enough to achieve our goal. The next lemma is a standard result and a proof of it can be found in [22].

Lemma III.19. *Assume $2^{\aleph_1} = \aleph_2$ and let $S \subseteq S^{\omega_1}_{\omega_2}$ be a stationnary subset of ω_2. Then there exists a forcing notion which preserves cardinals and such that in the generic extension $S \cup S^{\omega}_{\omega_2}$ contains a club.*

□

The following is due to Ruyle but we reproduce the proof here for completeness.

Lemma III.20. *If there exists a candidate $U = (\omega_2 \times \omega, \leq_U, i_U)$ such that $S_U \cup S^{\omega}_{\omega_2}$ contains a club of ω_2 then there exists a PCF-structure of height ω_2 whiwh is also an ω_2-skeleton.*

Proof : Fix a club $C \subseteq S_U \cup S^{\omega}_{\omega_2}$ such that no successor element of C is an ordinal of uncountable cofinality. For every α good for U, let C_α be a club witnessing it. Now, for $\alpha \in C \cap S_U$ let $D_\alpha = C_\alpha \cap C$. Obviously D_α is a club in α (that's because α must be a limit point of C) and since C_α witnesses that α is good for U, $(\beta, 0) \leq_U (\alpha, 0)$ for all $\beta \in D_\alpha$.

Let $Z = S^{\omega_1}_{\omega_2} \setminus C$. Choose a disjoint family $\{D_\alpha : \alpha \in Z\}$ such that $D_\alpha \cap C = \emptyset$ and D_α is a club in α, for all $\alpha \in Z$. To do this note that if γ and δ are two consecutive elements of C we can choose the D_α for $\alpha \in [\gamma, \delta] \cap Z$ to be subsets of $[\gamma, \delta]$. Since there are at most $\aleph_1$ such α we can arrange that the D_α are pairwise disjoint.

Now we define a bijection h from $\omega_2 \times \omega$ to itself. For each $\alpha \in Z$ and each $\beta \in D_\alpha$, find some $(\beta, n) \leq_U (\alpha, 0)$ and let $h(\beta, n) = (\beta, 0)$ and $h(\beta, 0) = (\beta, n)$. For any other points, let h be the identity.

Let $\leq^*_U$ be the partial order on $\omega_2 \times \omega$ defined by $x \leq^*_U y$ if and only if $h(x) \leq_U h(y)$ and let i^* be defined in the same obvious way. Now we claim that $U^* = (\omega_2 \times \omega, \leq^*_U, i^*, \lhd)$ is a PCF-structure which is also an ω_2-skeleton. Since h is a bijection and preserves the levels of U it follows $(\omega_2 \times \omega, \leq^*_U, i^*)$ is also an ω_2-skeleton. Now for each $\alpha \in \omega_2$ of uncountable cofinality observe that α is good for U^*. This is an immediate consequence of the definition of the bijection h and the fact that the D_α's for $\alpha \in Z$ are disjoint.

□

Now we are ready to prove Theorem III.18.

Proof : Starting with a model of ZFC+GCH+$\square_{\omega_1}$ we use Lemma III.16 to force a sequence $\langle D_\alpha : \alpha < \omega_2 \,\&\, \text{cof}(\alpha) = \omega_1 \rangle$ such that D_α is a club in α for all α and $D_\alpha \cap D_\beta$ is finite for all $\alpha \neq \beta$. Since the forcing in Lemma III.16 preserves cardinals $\square_{\omega_1}$ holds in the generic extension. So by Proposition III.9 there exists a strong Δ-function f in the generic extension. Let $\langle C_\alpha : \alpha < \omega_2 \,\&\, \lim(\alpha) \rangle$ be the $\square_{\omega_1}$-sequence from which f is constructed. Without loss of generality, we may assume that for every α of uncountable cofinality, $D_\alpha \subseteq \lim(C_\alpha)$ where $\lim(C_\alpha)$ is the set of limit points of C_α. If not then just replace D_α by $D_\alpha \cap \lim(C_\alpha)$. We also assume that for every $\delta < \omega_2$ the set $\{\alpha < \omega_2 : \text{cof}(\alpha) = \omega_1 \,\&\, \min(D_\alpha) > \delta\}$ is stationnary in ω_2. If not, then use Solovay's Theorem to partition $S^{\omega_1}_{\omega_2} = \{\alpha < \omega_2 : \text{cof}(\alpha) = \omega_1\}$ into ω_2 stationnary pieces, let's say $\langle S_\delta : \delta < \omega_2 \rangle$. Then, if $\alpha \in (S^{\omega_1}_{\omega_2} \cap S_\delta) \setminus (\delta + 1)$ replace D_α by $D_\alpha \setminus \delta$.

Observe that by Proposition III.10 for every $\alpha \neq \beta$ of uncountable cofinality, $D_\alpha \cap D_\beta \subseteq f\{\alpha,\beta\}$ and that cof $(\xi) \neq \omega_1$ for every $\xi \in D_\alpha$. We define a forcing notion $\mathcal{Q}$ which will add a candidate U such that S_U is stationnary. We let $p \in \mathcal{Q}$ if $p = \langle x_p, \leq_p, i_p \rangle$ where x_p is a finite subset of $\omega_2 \times \omega$, $\leq_p$ is a partial ordering of x_p and $i_p : [x_p]^2 \to [x_p]^{<\omega}$ and the following conditions are satisfied.

1. If $s, t \in x_p$ and $s <_p t$ then $\pi_1(s) < \pi_1(t)$.
2. If $s, t \in x_p$ with $s <_p t$ and $\pi_1(t)$ is a successor ordinal then there is $u \in x_p$ such that $s \leq_p u <_p t$ and $\pi_1(t) = \pi_1(u) + 1$.
3. If $s, t \in x_p$ are distinct and of $\lhd$-cofinality ω_1 then $[D_{\pi_1(s)} \cap D_{\pi_1(t)}] \times \{0\} \subseteq x_p$.
4. If $s \in x_p$ is of $\lhd$-cofinality ω_1 then there is no $t \neq s$ such that $s \leq_p t$.
5. If s and t are incomparable then $i_p\{s,t\} \subseteq [(f\{\pi_1(s), \pi_1(t)\} \times \omega) \cap x_p]$
6. If $s \leq_p t$ then $i_p\{s,t\} = \{s\}$.
7. If $s \neq t$ and $\pi_1(s) = \pi_1(t)$ then $i_p\{s,t\} = \emptyset$.
8. If $u \in i_p\{s,t\}$ then $u \leq_p s, t$.
9. For every $u \leq_p s, t$ there is $v \in i_p\{s,t\}$ such that $u \leq_p v$.

For $\alpha < \omega_2$ of uncountable cofinality and $p \in \mathcal{Q}$ say that α is *p-good* if $(\alpha, 0) \in x_p$ and for all $\beta \in D_\alpha$ if $(\beta, 0) \in x_p$ then $(\beta, 0) \leq_p (\alpha, 0)$. We let $p \leq q$ if and only if $x_p \supseteq x_q$, $\leq_p \restriction x_q = \leq_q$, $i_p \restriction [x_q]^2 = i_q$ and for all $\alpha < \omega_2$, if α is q-good then α is p-good.

Claim 1 *$\mathcal{Q}$ has the countable chain condition.*

Proof : Let $\mathcal{A} \subseteq \mathcal{Q}$ be uncountable. If we let $\mathcal{P}$ be the forcing defined in Theorem III.11 one can note that $\mathcal{Q} \subseteq \mathcal{P}$ (assuming that we used the same Δ-function). So by repeating the proof of Theorem III.11 and by shrinking $\mathcal{A}$ if necessary we can assume that every p, q in $\mathcal{A}$ are $\mathcal{P}$-compatible. For $p \in \mathcal{Q}$ we let $L_p = \{\alpha \in \omega_2 : (\alpha, 0) \in x_p\}$. By shrinking again we may assume that $\{L_p : p \in \mathcal{A}\}$ is a Δ-system with root, say R. By applying Proposition III.17 to $\{L_p \setminus R : p \in \mathcal{A}\}$ we can find distinct p and q in $\mathcal{A}$ such that for every $\alpha \in L_p \setminus R$ and $\beta \in L_q \setminus R$ of uncountable cofinality then

$$D_\alpha \cap (L_q \setminus R) = D_\beta \cap (L_p \setminus R) = \emptyset.$$

Now we claim that such p and q are compatible.

Let r_0 be an amalgamation of p and q in $\mathcal{P}$ such that $x_{r_0} = x_p \cup x_q$. We define $r = \langle x_r, \leq_r, i_r \rangle$ by letting

$$x_r = x_{r_0} \cup \bigcup \{(D_\alpha \cap D_\beta) \times \{0\} : \alpha \in (L_p \setminus R) \cap S^{\omega_1}_{\omega_2}; \beta \in (L_q \setminus R) \cap S^{\omega_1}_{\omega_2}\}.$$

Note that since the D_α contain no points of uncountable cofinality, r will satisfy (3) in the definition of $\mathcal{Q}$. We have to define the ordering $\leq_r$. First of all, set $\leq_r \restriction x_{r_0} = \leq_{r_0}$. For every $s \in x_r \setminus x_{r_0}$ we have $s \in (D_\alpha \cap D_\beta) \times \{0\}$ for some $\alpha \in (L_p \setminus R) \cap S^{\omega_1}_{\omega_2}$ and $\beta \in (L_q \setminus R) \cap S^{\omega_1}_{\omega_2}$. Then we set $s \leq_r (\alpha, 0)$ and $s \leq_r (\beta, 0)$. Note that by condition (4) of our forcing $\leq_r$ is a partial order and r obviously satisfies condition (1),(2) and (4) in the definition of $\mathcal{Q}$.

Now, we let $i_r \restriction [x_{r_0}]^2 = i_{r_0}$ and for every $\alpha \in (L_p \setminus R) \cap S^{\omega_1}_{\omega_2}$ and $\beta \in (L_q \setminus R) \cap S^{\omega_1}_{\omega_2}$, we let

$$i_r\{(\alpha, 0), (\beta, 0)\} = [(D_\alpha \cap D_\beta \setminus R) \times \{0\}] \cup i_{r_0}\{(\alpha, 0), (\beta, 0)\}.$$

Since by proposition III.10 $D_\alpha \cap D_\beta \subseteq f\{\alpha, \beta\}$, r will satisfy condition (5) in the definition of $\mathcal{Q}$.

Then, using condition (3) for p and q and the fact that r_0 is an amalgamation of p and q in $\mathcal{P}$ it is straightforward to check that $r \in \mathcal{Q}$ and that r extends both p and q. The key point is that for every $\gamma \in D_\alpha$, we have $\gamma \notin L_q \setminus L_p$ and so if α is p-good then α is r-good.

□

Our forcing will then preserve cardinals. We have to show now that if G is generic then $\bigcup_{p \in G} x_p = \omega_2 \times \omega$.

Claim 2 *For every $s \in \omega_2 \times \omega$ the set $F_s = \{p \in \mathcal{Q} : s \in x_p\}$ is dense.*

Proof : Let $s \in \omega_2 \times \omega$ and $p \in \mathcal{Q}$ such that $s \notin x_p$. We have to find $q \leq p$ such that $s \in x_q$. Several cases appear :

Case 1 : s is not of $\lhd$-cofinality ω_1 and $\pi_2(s) \neq 0$ or s is not of $\lhd$-cofinality ω_1, $\pi_2(s) = 0$ and for all α p-good $\pi_1(s) \notin D_\alpha$.

Then q can be defined in the obvious way. Let $x_q = x_p \cup \{s\}$; $\leq_q\restriction x_p =\leq_p$; $\leq_q \cap(x_p \times \{s\}) =\leq_q \cap(\{s\} \times x_p) = \emptyset$; $i_q \restriction [x_p]^2 = i_p$ and $i_q\{s,t\} = \emptyset$ for all $t \in x_p$. Then $q = \langle x_q, \leq_q, i_q\rangle$ is as wished.

Case 2 : s is not of $\lhd$-cofinality ω_1, $\pi_2(s) = 0$ and for some $\alpha < \omega_2$, α is p-good and $\pi_1(s) \in D_\alpha$. Note that by condition (3) of the definition of $\mathcal{Q}$, α must be unique.

Let $x_q = x_p \cup \{s\}$. Let $\leq_q$ extends $\leq_p$ and for α p-good such that $\pi_1(s) \in D_\alpha$ let $s \leq_q (\alpha, 0)$. The function i_q is defined in the obvious way. Then again $q = \langle x_q, \leq_q, i_q\rangle$ is as wished.

Case 3 : s is of $\lhd$-cofinality ω_1. Then s is of the form $(\alpha, 0)$, for some α of cofinality ω_1.

Let $L_p^{\omega_1} = \{\beta \in \omega_2 : (\beta, 0) \in x_p\} \cap S^{\omega_1}_{\omega_2}$. Then let

$$x_q = x_p \cup \{s\} \cup \bigcup\{(D_\beta \cap D_\alpha) \times \{0\} : \beta \in L_p^{\omega_1}\}.$$

Let $\leq_q$ extends $\leq_p$ and for all β p-good, for all $\gamma \in D_\beta \cap D_\alpha$ let $(\gamma, 0) \leq_q (\beta, 0)$. The function i_q extends i_p and is defined in an obvious way for what is left. Observe that it is possible since by clause (3) of the definition of $\mathcal{Q}$ if β_1, β_2 are p-good then $(D_{\beta_1} \cap D_{\beta_2}) \times \{0\} \subseteq x_p$ and so if $t \in x_q \setminus x_p$ we cannot have $t \leq_q (\beta_1, 0), (\beta_2, 0)$.

□

The two previous claims show that if G is $\mathcal{Q}$-generic then in $V[G]$, if we let $\leq_G = \bigcup_{p\in G} \leq_p$ and $i_G = \bigcup_{p\in G} i_p$, $G^* = (\omega_2 \times \omega, \leq_G, i_G)$ is a candidate. So what is left now to show Theorem III.18 is to prove that S_{G^*} (the set of good points of G^*) is stationnary in $V[G]$.

Claim 3 *S_{G^*} is stationnary in $V[G]$.*

Proof : Since our forcing is ccc, every club in the generic extension contains a club in the ground model. So we are done if we show that the set of good points meets every club in ω_2 in the ground model.

Let $C \subseteq \omega_2$ be a club in V and let $p = \langle x_p, \leq_p, i_p\rangle \in \mathcal{Q}$. Let $L_p = \{\gamma : (\gamma, 0) \in x_p\}$ and $L_p^{\omega_1} = L_p \cap S^{\omega_1}_{\omega_2}$. Since L_p is finite and $\{\alpha < \omega_2 : \text{ cof }(\alpha) = \omega_1 \ \& \ \min(D_\alpha) > \max(L_p)\}$ is stationnary, there exists $\alpha \in C$ of cofinality ω_1 with $D_\alpha \cap L_p = \emptyset$. Let

$$x_q = x_p \cup \{(\alpha, 0)\} \cup \bigcup \{(D_\beta \cap D_\alpha) \times \{0\} : \beta \in L_p^{\omega_1}\}.$$

Let $\leq_q$ extends $\leq_p$ and for all β p-good, for all $\gamma \in D_\beta \cap D_\alpha$ let $(\gamma, 0) \leq_q (\beta, 0)$ and $(\gamma, 0) \leq_q (\alpha, 0)$. Observe that by the same remark as in Case 3 of the previous claim, i_q can be defined in an obvious way. Let $q = \langle x_q, \leq_q, i_q \rangle$. Then by definition of $\leq_q$, α is q-good. So the set

$$F_C = \{p \in \mathcal{Q} : \text{ for some } \alpha \in C \text{ of uncountable cofinality } \alpha \text{ is } p\text{-good}\}$$

is dense and in the generic extension, the set of good points meets C. This finishes the proof of the claim and of Theorem III.18.

□

So finally we can state the following theorem which is an immediate consequence of Theorem III.18 and Lemmas III.19 and III.20.

Theorem III.21. *It is relatively consistent with ZFC that there exists a PCF structure which is also an ω_2-skeleton.*

□

Now using the ideas developed in §1 we can prove the following :

Theorem III.22. *If there exists a thin PCF structure which is also a κ-skeleton then there exists thin PCF structures of height α, for all $\alpha < \kappa^+$.*

Before we start the proof let us remark that any thin PCF structure of height, say λ, is isomorphic to a thin PCF structure T of the form $(\lambda \times \omega, \leq_T, i, \lhd_T)$ where $\lhd_T$ is just the lexicographic ordering on $\lambda \times \omega$ and T_α, the α-th level of T, is equal to $\{\alpha\} \times \omega$.
Proof of Theorem III.22 : As in Theorem III.8 we construct by induction on δ, a thin PCF structure T^δ of height δ, for all $\delta < \kappa^+$. For $\delta = \kappa$ this is the hypothesis of the theorem.

For $\delta = \gamma + 1$ let $(T^\gamma, \leq^\gamma, i^\gamma, \lhd^\gamma)$ be a thin PCF-structure of height γ. By the above remark we may assume that $T = \gamma \times \omega$ and $\lhd^\gamma$ is the lexicographic ordering on $\gamma \times \omega$.

Then Proposition III.3 provides an $\Omega(\gamma+1)$-poset $T^{\gamma+1}$. Recall that since $T^\gamma = \gamma \times \omega$ then

$$T^{\gamma+1} = \gamma \times (\omega \times \omega) \cup \{\gamma+1\} \times \omega.$$

All is left to do is to define a well ordering $\lhd^{\gamma+1}$ on $T^{\gamma+1}$. We can let $\lhd^{\gamma+1}$ be any well-ordering such that each level of $T^{\gamma+1}$ is of order type ω and for every $\alpha < \gamma$ and every $n, m \in \omega$ we have $(\alpha, 0, 0) \prec^{\gamma+1} (\alpha, n, m)$. That is, we let the first point of each level in the lexicographic order be the first point of that level in the well ordering.

Now if $\alpha < \gamma$ is of uncountable cofinality then by induction hypothesis there exists a club C_α in α such that for all $\xi \in C_\alpha$ we have $(\xi, 0) \leq^\gamma (\alpha, 0)$. Then by definition of $\leq^{\gamma+1}$ we have $(\xi, 0, 0) \leq^{\gamma+1} (\alpha, 0, 0)$. By noticing that for $s \in T^{\gamma+1}$, s has $\lhd^{\gamma+1}$-cofinality ω_1 if and only if $s = (\alpha, 0, 0)$ for some $\alpha < \gamma$ of uncountable cofinality we are done.

Assume now δ is a limit ordinal and let $\mu = \text{cof}(\delta)$. Then $\mu \leq \kappa$ and we know that there is a thin PCF structure S which is also a μ-skeleton. Let $C = \{\gamma_\nu : \nu < \mu\}$ be a club in δ of order type μ and let by convention $\gamma_\mu = \delta$. Let $\delta_\nu = o.t.(\gamma_{\nu+1} \setminus (\gamma_\nu + 1))$. As in Theorem III.8, let $E_\nu = \gamma_\nu \cup (C \setminus \gamma_\nu)$. Then the order type of E_ν is $\gamma_\nu + (\mu - \nu)$. Recall that $e_\nu : \mu \longrightarrow \gamma_\nu + (\mu - \nu)$ is defined by

$$e_\nu(\eta) = \begin{cases} \gamma_\eta & \text{if } \eta \leq \nu \\ \gamma_\nu + \xi & \text{if } \eta = \nu + \xi \text{ for some } \xi < \mu - \eta \end{cases}$$

and φ_ν is defined on S by $\varphi_\nu(\eta, y) = (e_\nu(\eta), y)$. We construct by induction on $\nu \leq \mu$ an $\Omega(\gamma_\nu + (\mu - \nu))$-poset R^ν with the following properties.

1. $R^0 = S$.
2. If $\nu < \xi$ then $R^\xi \restriction \gamma_\nu = R^\nu \restriction \gamma_\nu$.
3. φ_ν is an isomorphism between S and $R^\nu \restriction e_\nu[\mu]$.
4. $R^{\nu+1} = T^{\delta_\nu} \hookrightarrow^{\gamma_\nu} R^\nu$.

If $\nu = \eta + 1$, then γ_η is a bone-level of R^η so we can let $R^\nu = T^{\delta_\eta} \hookrightarrow^{\gamma_\eta} R^\eta$. where T^{δ_η} is a PCF structure of height δ_η. By the previous remark, we may assume that $T^{\delta_\eta} = \delta_\eta \times \omega$. If ν is a limit ordinal we let

$$R^\nu = \bigcup \{R^\eta \restriction \gamma_\eta : \eta < \nu\} \cup \varphi_\nu[S].$$

and we define $\leq^\nu$ and i^ν as in Theorem III.8. Then, $T^\delta = R^\mu$.

Note that $T^\delta_\alpha = \{\alpha\} \times \omega$ if $\alpha = \gamma_\nu$ for some $\nu < \mu$ and $T^\delta_\alpha = \{\alpha\} \times (\omega \times \omega)$ if not. Then the well ordering $\lhd^\delta$ is defined in a natural way. If $\alpha = \gamma_\nu$ for some $\nu < \mu$ then $(\alpha, n) \lhd^\delta (\alpha, m)$ if and only if $n < m$. Otherwise, we define it as we did in the successor case.

Now, let $\alpha < \delta$ be of uncountable cofinality. Then, we have 3 possibilities :

Case 1 : $\alpha \neq \gamma_\nu$, for all $\nu < \mu$.

Let $\nu < \mu$ be the first such that $\alpha < \gamma_{\nu+1}$. Then $T^\delta \restriction \gamma_{\nu+1} = T^{\delta_\nu} \hookrightarrow^{\gamma_\nu} R^\nu \restriction c_{\nu+1}$. Since T^{δ_ν} is a thin PCF structure, by an argument similar to the one used in the successor case there exists a club C_α such that for all $\xi \in C_\alpha$, $(\xi, 0, 0) \leq^\delta (\alpha, 0, 0)$.

Case 2 : $\alpha = \gamma_\nu$, for some limit $\nu < \mu$.

Since S is a thin PCF structure, there exists a club C_ν such that for all $\xi \in C_\nu$, $(\xi, 0) \leq_S (\nu, 0)$. But then if we let $C_\alpha = \{\gamma_\xi : \xi \in C_\nu\}$ we have $(\gamma_\xi, 0) \leq^\delta (\alpha, 0)$ for all $\xi \in C_\nu$, and C_α is a club in α. This is because in our induction we did not modify anything in the skeleton.

Case 3 : $\alpha = \gamma_{\nu+1}$, for some $\nu < \mu$.

We have $T^\delta \restriction \gamma_{\nu+1} = T^{\delta_\nu} \hookrightarrow^{\gamma_\nu} R^\nu \restriction \gamma_{\nu+1}$. Then, for all ξ such that $\gamma_\nu < \xi < \gamma_{\nu+1}$ we have $(\xi, 0, 0) \leq^\delta (\alpha, 0)$. (because of the definition of the $\hookrightarrow$ operation).

Finally, by noticing that s is of uncountable $\lhd^\delta$-cofinality if and only if s is of the form $(\alpha, 0)$ or $(\alpha, 0, 0)$ for α of uncountable cofinality we are done.

□

Now, putting Theorem III.21 and III.22 together we get the main theorem.

Theorem III.23. *It is relatively consistent with ZFC that there are PCF structures of height δ, for all $\delta < \omega_3$.*

□

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