GUESSING MODELS AND THE APPROACHABILITY IDEAL

RAHMAN MOHAMMADPOUR AND BOBAN VELIČKOVIĆ

Abstract. Starting with two supercompact cardinals we produce a generic extension of the universe in which the principles $\mathrm{ISP}(\omega_2)$ and $\mathrm{ISP}(\omega_3)$ hold simultaneously, and the restriction of the approachability ideal $I[\omega_2]$ to the set of ordinals of cofinality ω_1 is the non stationary ideal on this set.

Introduction

In [26] C. Weiß formulated combinatorial principles that capture the essence of some large cardinal properties, but can hold at small cardinals. These principles usually have two parameters, a regular uncountable cardinal κ and a cardinal $\lambda \geq \kappa$. Among them there are, in increasing strength, the principles $\mathrm{TP}(\kappa, \lambda)$, $\mathrm{ITP}(\kappa, \lambda)$, and $\mathrm{ISP}(\kappa, \lambda)$. We will write $\mathrm{P}(\kappa)$, if the property $\mathrm{P}(\kappa, \lambda)$ holds, for all $\lambda \geq \kappa$. The study of these principles was continued by M. Viale and C. Weiß in [25]. Using them they obtained a striking result saying that any standard forcing construction of a model of the Proper Forcing Axiom (PFA) requires at least a strongly compact cardinal. One important concept that emerged from this work is that of a guessing model. These models have generated considerable interest and have a number of interesting applications, see for instance [24], [2], [3], and [21].

Given the interest of these principles, it is natural to ask if they can hold simultaneously at several successive regular cardinals. In this direction, L. Fontanella [5] extended the previous work of U. Abraham [1] to obtain, modulo two supercompact cardinals, a model of ZFC in which $\mathrm{ITP}(\omega_2)$ and $\mathrm{ITP}(\omega_3)$ hold simultaneously. Now, it was shown in [26] that $\mathrm{ISP}(\omega_2)$ is strictly stronger that $\mathrm{ITP}(\omega_2)$. In fact, in the model constructed by B. König in [9], the principle $\mathrm{ITP}(\omega_2)$ holds, but $\mathrm{ISP}(\omega_2)$ fails. One can then ask if $\mathrm{ISP}(\omega_2)$ and $\mathrm{ISP}(\omega_3)$ can hold simultaneously. Let us point out that in [21] Trang showed the consistency of $\mathrm{ISP}(\omega_3)$. However, in his model CH holds, and therefore the principle $\mathrm{ISP}(\omega_2)$ fails. It is one of the goals of the current paper to show that it is consistent with ZFC, modulo two supercompact cardinals, that the principles $\mathrm{ISP}(\omega_2)$ and $\mathrm{ISP}(\omega_3)$ hold simultaneously.

One concept closely related to the above principles is that of the approachability property on a regular uncountable cardinal λ and the associated ideal $I[\lambda]$. These notions were introduced by Shelah implicitly in [17], and studied by him extensively over the past 40 years. For instance, in [18] he showed that if λ is a regular cardinal then $S^{<\lambda}_{\lambda^+} \in I[\lambda^+]$, and in [19] he showed that if κ is regular and $\kappa^+ < \lambda$ then $I[\lambda]$ contains a stationary subset of S^{κ}_{λ}. Shelah then asked in [18] if it is consistent to have a regular λ such that $I[\lambda^+] \restriction S^{\lambda}_{\lambda^+}$ is the non stationary ideal on $S^{\lambda}_{\lambda^+}$. This major question was finally answered

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by W. Mitchell [14]. He started with a cardinal κ that is κ^+-Mahlo, and built an involved forcing construction yielding a model in which $I[\omega_2] \upharpoonright S^{\omega_1}_{\omega_2}$ is the non stationary ideal on $S^{\omega_1}_{\omega_2}$. One feature of this construction is that it uses $\square_\kappa$ in the ground model, and so $\omega_3 \in I[\omega_3]$ in the extension. It is therefore unclear if Mitchell's method can be adapted to obtain a model in which both $I[\omega_2] \upharpoonright S^{\omega_1}_{\omega_2}$ and $I[\omega_3] \upharpoonright S^{\omega_2}_{\omega_3}$ contain only non stationary sets. The connection with the principles introduced by Weiß is the following. If κ is a regular uncountable cardinal then $\mathrm{ISP}(\kappa^+)$ implies that there is a stationary subset of $S^{\kappa}_{\kappa^+}$ that is not in $I[\kappa^+]$, but it does not imply Mitchell's result. We formulate a principle $\mathrm{FS}(\kappa)$ that does imply it, and show that $\mathrm{FS}(\omega_2)$ holds in our model. However, the principle $\mathrm{FS}(\omega_3)$ does not hold there, so it is still open if one can have a version of Mitchell's result simultaneously for ω_2 and ω_3.

The origin of this paper is as follows. Inspired by Mitchell's breakthrough, I. Neeman [15] introduced a method for iterating proper forcing notions using finite chains of elementary submodels as side conditions. This allowed him to give a new proof of the consistency of PFA using this type of iteration. More importantly, this opened a possibility of iterating forcing while preserving two successive cardinals and potentially getting strong forcing axioms at ω_2 and higher cardinals. The second author then extended Neeman style iteration to more general classes of forcing. This lead to the notion of a *virtual model*. Using this type of models as side conditions allows us not only to generalize Neeman's iteration theory to semiproper forcing, but also to formulate and prove iteration theorems for large classes of forcing notions preserving two uncountable cardinals, such as ω_1 and ω_2. This theory is presented in [22] and [23]. In fact, our main poset is an adaptation of the pure side condition forcing from [23] to two types of models, but replacing models of size ω_1 by models having a strong closure property that we call Magidor models.

The paper is organized as follows. In §1 we present the preliminaries and fix some notation. In §2 we review the theory of virtual models from [22] and [23] and adapt it to the context of Magidor models. In §3, we introduce our main forcing notion $\mathbb{P}^{\kappa}_{\lambda}$ and establish some of its properties. Finally, in §4 we study guessing models in the generic extension by $\mathbb{P}^{\kappa}_{\lambda}$, and show that $\mathrm{ISP}(\omega_2)$, $\mathrm{ISP}(\omega_3)$, and $\mathrm{FS}(\omega_2)$ hold there.

1. Preliminaries

Throughout this paper by a model M we mean a set or a class such that $(M, \in)$ satisfies a sufficient fragment of ZFC. For a model M, we let $\overline{M}$ denote its transitive collapse. For a set X and a cardinal θ, we let $\mathcal{P}_\theta(X)$ denote the set of all subsets of X of size less than θ. We let H_θ denote the collection of all sets whose transitive closure has size less than θ. We say that a subset $\mathcal{S}$ of $\mathcal{P}_\theta(X)$ is *stationary* if, for every function $F : [X]^{<\omega} \to X$, there exists $A \in \mathcal{S}$ that is closed under F. For regular cardinal $\kappa < \lambda$, we let S^{κ}_{λ} denote the set $\{\alpha < \lambda : \mathrm{cof}(\alpha) = \kappa\}$.

In order to state $\mathrm{ISP}(\kappa^+)$ in a precise way we need to give some basic definitions that will be useful to us later. The central one is the notion of a κ-guessing model, originally introduced by Viale and Weiß in [25], see also [24]. We start by recalling the κ-approximation property, introduced by Hamkins in [8].

Definition 1.1. *Let κ be a cardinal. Suppose M and N are models and $M \subseteq N$. We say the pair (M, N) satisfies the κ-*approximation property*, if whenever α is an ordinal in M, and $X \in N$ with $X \subseteq \alpha$ such that $X \cap Z \in M$, for all $Z \in M$ with $|Z|^M < \kappa$, then $X \in M$.*

The following is a reformulation due to Cox and Krueger [2] of the original definition from [25].

Definition 1.2. *Let κ be a cardinal and M a model such that $M \cap \kappa^+ \in \kappa^+$. We say M is a κ-*guessing model*, if the pair $(\overline{M}, V)$ satisfies the κ-approximation property.*

Definition 1.3 (ISP(κ^+))**.** *Let κ be a regular cardinal. The principle* ISP(κ^+) *asserts that, for every sufficiently large regular cardinal θ, the set of κ-guessing elementary submodels of H_κ is stationary in $\mathcal{P}_{\kappa^+}(H_\theta)$.*

Definition 1.4 (FS(κ^+))**.** *Let κ be a regular cardinal. The principle* FS(κ^+) *asserts that, for every $X \in H_{\kappa^+}$, there is a collection $\mathcal{G}$ of κ-guessing models all containing X such that $\{M \cap \kappa^+ : M \in \mathcal{G}\}$ is κ-closed (closed under κ-sequences) and unbounded in κ^+.*

We now recall the definition of the approachability ideal from [18].

Definition 1.5. *Let λ be a regular cardinal. A λ-*approaching sequence *is a sequence of bounded subsets of λ. If $\bar{a} = (a_\xi : \xi < \lambda)$ is a λ-approaching sequence, we let $B(\bar{a})$ denote the set of all $\delta < \lambda$ such that there is a cofinal subset $c \subseteq \delta$ such that:*

(1) $\mathrm{otp}(c) < \delta$, *in particular δ is singular,*
(2) for all $\gamma < \delta$, there exists $\eta < \delta$ such that $c \cap \gamma = a_\eta$.

Definition 1.6. *Suppose λ is a regular cardinal. Let $I[\lambda]$ be the ideal generated by the sets $B(\bar{a})$, for all λ-approaching sequences $\bar{a}$, and the non stationary ideal* NS_λ.

Remark 1.7. It is straightforward to check that $I[\lambda]$ is a normal ideal on λ, but it may be non proper. $I[\lambda]$ is called the *approachability ideal* on λ.

Shelah asked if it is consistent that $I[\kappa^+] \restriction S^\kappa_{\kappa^+} = \mathrm{NS}_{\kappa^+} \restriction S^\kappa_{\kappa^+}$, for a regular cardinal κ. Mitchell in [14] answered this question affirmatively by showing the following.

Theorem 1.8 (Mitchell, [14])**.** *Assume that κ is a κ^+-Mahlo cardinal. Then there is a generic extension in which $I[\omega_2] \restriction S^{\omega_1}_{\omega_2} = \mathrm{NS}_{\omega_2} \restriction S^{\omega_1}_{\omega_2}$.*

Remark 1.9. In his paper [14], Mitchell mentions that the large cardinal assumption used in his result is necessary by an unpublished theorem of Shelah; a proof can be found in [11], Theorem 13. It is also mentioned at the end of [14] that one can prove the same result for λ^{++} where λ is a regular cardinal, more precisely let λ regular be given, then under the same assumption (with $\kappa > \lambda$) of his theorem there is a generic extension of the universe satisfying $I[\lambda^{++}] \restriction S^{\lambda^+}_{\lambda^{++}} = \mathrm{NS}_{\lambda^{++}} \restriction S^{\lambda^+}_{\lambda^{++}}$.

Proposition 1.10. *Suppose κ is a regular cardinal and* FS(κ^+) *holds. Then we have that $I[\kappa^+] \restriction S^\kappa_{\kappa^+} = \mathrm{NS}_{\kappa^+} \restriction S^\kappa_{\kappa^+}$.*

Proof. Suppose that FS(κ^+) holds. Let $\bar{a} = (a_\xi : \xi < \kappa^+)$ be a κ^+-approaching sequence. Let $\mathcal{G}$ be the family of κ-guessing models all containing $\bar{a}$ whose existence is guaranteed by FS(κ^+). We show that $M \cap \kappa^+ \notin B(\bar{a})$, for any $M \in \mathcal{G}$ such that $\mathrm{cof}(M \cap \kappa^+) = \kappa$.

Fix one such $M \in \mathcal{G}$. Let $\delta = M \cap \kappa^+$ and suppose $c \subseteq \delta$ satisfies (1) and (2) of Definition 1.5. Let $\mu = \mathrm{otp}(c)$. Note that $\mu < \delta$, hence $\mu \in M$. Since $\bar{a} \in M$, we have that $c \cap \gamma \in M$, for all $\gamma < \delta$, and hence $c \cap Z \in M$, for all $Z \in M$ with $|Z| < \kappa$. Since M is a κ-guessing model, there must be $d \in M$ such that $c \cap \delta = d \cap \delta$. We may assume $d \subseteq \kappa^+$. Then c is an initial segment of d, so if ρ is the μ-th element of d then $d \cap \rho = c$. Since $\mu, d \in M$, we have $\rho \in M$ as well, and hence $c = d \cap \rho \in M$. But then $\delta = \sup(c)$ belongs to M, a contradiction. $\boxed{1.10}$

The main result of this paper is the following.

Theorem 1.11. *Suppose that κ is supercompact and $\lambda > \kappa$ is inaccessible. Then there is a forcing notion such that in the generic extension* $\mathrm{ISP}(\omega_2)$ *and* $\mathrm{FS}(\omega_2)$ *hold. If, in addition, λ is supercompact, then* $\mathrm{ISP}(\omega_3)$ *holds as well.*

The notion of strong properness, introduced by Mitchell in [12], plays the key role in our construction. We start with the following.

Definition 1.12. *Let $\mathbb{P}$ be a forcing notion and A a set. We say that $p \in \mathbb{P}$ is $(A, \mathbb{P})$-*strongly generic *if for all $q \leq p$ there is a condition $q \restriction A \in A$ such that any $r \leq q \restriction A$ with $r \in A$ is compatible with q.*

Definition 1.13 (Strong properness)**.** *Let $\mathbb{P}$ be a forcing notion, and $\mathcal{S}$ a collection of sets. We say that $\mathbb{P}$ is $\mathcal{S}$-*strongly proper *if, for every $A \in \mathcal{S}$ and $p \in A \cap \mathbb{P}$, there is $q \leq p$ that is $(A, \mathbb{P})$-strongly generic.*

The following proposition connects the approximation property with strong properness.

Proposition 1.14. *Let $\mathbb{P}$ be a forcing notion and κ an uncountable regular cardinal. Suppose $\mathbb{P}$ is $\mathcal{S}$-strongly proper, for some stationary subset $\mathcal{S}$ of $\mathcal{P}_\kappa(\mathbb{P})$. If G is V-generic over $\mathbb{P}$, then $(V, V[G])$ has the κ-approximation property.*

Proof. Work in V. Let α be an ordinal, $\dot{X}$ a $\mathbb{P}$-name, and suppose some condition $p \in \mathbb{P}$ forces that $\dot{X} \subseteq \alpha$ and $\dot{X} \cap \check{Z} \in V$, for all $Z \in V$ with $|Z|^V < \kappa$. Fix a sufficiently large regular cardinal θ. By the stationarity of $\mathcal{S}$, we can find $M \prec H_\theta$ containing $p, \mathbb{P}, \dot{A}$, and such that $M \cap \mathbb{P} \in \mathcal{S}$. Let $q \leq p$ be $(M \cap \mathbb{P})$-strongly generic. Since $M \cap \mathbb{P}$ is of size $< \kappa$, by strengthening q if necessary, we may assume that q decides $\dot{X} \cap M$. Since $q \restriction (M \cap \mathbb{P})$ and p are compatible, and M is elementary, they are compatible in M. Therefore, by replacing $q \restriction (M \cap \mathbb{P})$ by a stronger condition in M, we may assume that it extends p. We now argue that $q \restriction (M \cap \mathbb{P})$ decides $\dot{X}$. Otherwise, by elementary of M, we can find $\xi \in \alpha \cap M$ and $r_0, r_1 \in M$ with $r_0, r_1 \leq q \restriction (M \cap \mathbb{P})$ such that r_0 forces $\xi \in \dot{X}$ and r_1 forces $\xi \notin \dot{X}$. Now, by strong genericity of q, we have that r_0 and r_1 are both compatible with q. Let s_0 be a common extension of q and r_0, and s_1 a common extension of q and r_1. Then $s_0, s_1 \leq q$ and force contradictory information about $\xi \in \dot{X}$. This contradicts the fact that q decides $\dot{X} \cap M$. $\boxed{1.14}$

We will also need the following well-known theorem due to Magidor.

Theorem 1.15 (Magidor, [10])**.** *The following are equivalent for a regular cardinal κ.*

(1) κ is supercompact.

(2) *For every $\gamma > \kappa$ and $x \in V_\gamma$ there exist $\bar\kappa < \bar\gamma < \kappa$, and an elementary embedding $j : V_{\bar\gamma} \to V_\gamma$ with critical point $\bar\kappa$ such that $j(\bar\kappa) = \kappa$ and $x \in j[V_{\bar\gamma}]$.*

1.15

2. Virtual Models

In this section we review the notion of virtual models introduced in [22] and [23]. In [23] we used virtual models of two types: countable and internally club (I.C.) models of size $\aleph_1$. In the current situation we replace the I.C. models by models that have a much stronger closure property that we call Magidor models.

We shall consider the language $\mathcal{L}$ obtained by adding a single constant symbol c to the standard language $\mathcal{L}_\epsilon$ of set theory. Let us say that a structure $\mathcal{A}$ of the form $(A, \in, \kappa)$ is *admissible* if A is a transitive, satisfies ZFC and the interpretation κ of the constant symbol c is an inaccessible cardinal in A. When κ is clear from the context we omit it and write simple A for $\mathcal{A}$. Suppose A is an admissible structure. If α is an ordinal in A, we let A_α denote $A \cap V_\alpha$. Finally, we let

$$E_A = \{\alpha \in A : A_\alpha \prec A\}.$$

Note that E_A is a closed, possibly empty, subset of ORD^A. It is not definable in A, but $E_A \cap \alpha$ is uniformly definable in A with parameter α, for each $\alpha \in E_A$. If $\alpha \in E_A$ we let $\mathrm{next}_A(\alpha)$ be the least ordinal in E_A above α, if such an ordinal exists. Otherwise, we leave $\mathrm{next}_A(\alpha)$ undefined. We start with a simple technical lemma.

Lemma 2.1. *Suppose M is an elementary submodel of an admissible structure A. Then*

(1) *If $\alpha \in E_A$ and $(M \cap \mathrm{ORD}^A) \setminus \alpha \neq \varnothing$, then $\min(M \cap \mathrm{ORD}^A \setminus \alpha) \in E_A$.*
(2) $\sup(E_A \cap M) = \sup(E_A \cap \sup(M \cap \mathrm{ORD}^A))$.

Proof. The second item follows from the first one, so we only give the proof of (1). Let β be the least ordinal in $M \setminus \alpha$. We need to show that A_β is an elementary submodel of A. Suppose otherwise, then by the Tarski-Vaught criterion, there is a tuple $\bar{x} \in A_\beta$ and a formula $\varphi(y, \bar{x})$ such that $A \models \exists y \varphi(y, \bar{x})$, but there is no $y \in A_\beta$ such that $A \models \exists y \varphi(y, \bar{x})$. Since $\beta \in M$ and M is an elementary submodel of A, there is such a tuple $\bar{x} \in A_\beta \cap M$. Now, β is the least ordinal in M above α, therefore $\bar{x} \in M \cap A_\alpha$. Since A_α is an elementary submodel of A, there is $y' \in A_\alpha$ witnessing that $A_\alpha \models \varphi(y', \bar{x})$ and so $A \models \varphi(y', \bar{x})$. Since $\alpha \leq \beta$, it follows that $y' \in A_\beta$, a contradiction. 2.1

Definition 2.2. *Suppose M is a submodel of an admissible structure A and X is a subset of A. Let*

$$\mathrm{Hull}(M, X) = \{f(\bar{x}) : f \in M, \bar{x} \in X^{<\omega}, f \text{ is a function, and } \bar{x} \in \mathrm{dom}(f)\}.$$

The main reason we have defined the Hull operation in this way is that it allows us to define the Skolem hull of M and X without referring explicitly to the ambient model A.

Lemma 2.3. *Suppose A is an admissible structure, M is an elementary submodel of A and X is a subset of A. Let δ be $\sup(M \cap \mathrm{ORD}^A)$, and suppose $X \cap A_\delta$ is nonempty. Then $\mathrm{Hull}(M, X)$ is the least elementary submodel of A containing M and $X \cap A_\delta$.*

Proof. For each $\gamma \in A$, let id_γ be the identity function on A_γ. Clearly, if $\gamma \in M$ then $\mathrm{id}_\gamma \in M$. Therefore, $X \cap A_\delta$ is a subset of $\mathrm{Hull}(M, X)$. Let $\gamma \in M$ be such that $X \cap A_\gamma$ is nonempty. For each $z \in M$, the constant function c_z defined on A_γ is in M, therefore M is a subset of $\mathrm{Hull}(M, X)$. The minimality of $\mathrm{Hull}(M, X)$ is clear from the definition. It remains to show that $\mathrm{Hull}(M, X)$ is an elementary submodel of A. We check the Tarski-Vaught criterion for $\mathrm{Hull}(M, X)$ and A. Let φ be a formula and $a_1, \ldots, a_n \in \mathrm{Hull}(M, X)$ such that $A \models \exists u \varphi(u, a_1, \ldots, a_n)$. Then we can find functions $f_1, \ldots, f_n \in M$ and tuples $\bar{x}_1, \ldots, \bar{x}_n \in X^{<\omega}$ such that $a_i = f_i(\bar{x}_i)$, for all i. If D_i is the domain of f_i, this implies that $\bar{x}_i \in D_i$. By regularity and the axiom of choice in A we can find a function g defined on $D_1 \times \ldots \times D_n$ such that for every $\bar{y}_1 \in D_1, \ldots, \bar{y}_n \in D_n$, if there is u such that $A \models \varphi(u, f_1(\bar{y}_1), \ldots, f_n(\bar{y}_n))$ then $g(\bar{y}_1, \ldots, \bar{y}_n)$ is such a u. Moreover, by elementarity of M, we may assume that $g \in M$. Let $a = g(\bar{x}_1, \ldots, \bar{x}_n)$. It follows that $a \in \mathrm{Hull}(M, X)$ and $A \models \varphi(a, a_1, \ldots, a_n)$. Therefore, $\mathrm{Hull}(M, X)$ is an elementary submodel of A. 2.3

Now, let us fix an inaccessible cardinal κ, a cardinal $\lambda > \kappa$ such that V_λ satisfies ZFC. We shall write E instead of E_{V_λ} and $\mathrm{next}(\alpha)$ instead of $\mathrm{next}_{V_\lambda}(\alpha)$. For each $\alpha \in E$, we shall define certain families $\mathscr{F}_\alpha \in V_\lambda$, as well as relations R_α and operations O_α on V_λ. Being a member of $\mathscr{F}_\alpha$ will be expressed by a Σ_1-formula with parameter V_α and similarly for R_α and O_α. If A is another admissible structure we can interpret these formulas in A and obtain families $\mathscr{F}^A_\alpha$, relations R^A_α and operations O^A_α. In this section we shall only consider admissible A such that the interpretation of the constant symbol c is κ and $A \subseteq V_\lambda$. Note that if we have such an A and $\alpha \in E_A \cap E$ with $A_\alpha = V_\alpha$ then $\mathscr{F}^A_\alpha \subseteq \mathscr{F}_\alpha$. Similarly, if $x, y \in A$ are such that $x R^A_\alpha y$ then $x R_\alpha y$, and if $x \in A$ then $O^A_\alpha(x) = O_\alpha(x)$. If M is an elementary submodel of an admissible A and $\alpha \in E_A \cap M$ we shall write $\mathscr{F}^M_\alpha$ for $\mathscr{F}^A_\alpha \cap M$, and R^M_α and O^M_α for restrictions of R^A_α and O^A_α to M.

Definition 2.4. *Suppose $\alpha \in E$. We let $\mathscr{A}_\alpha$ denote the set of all transitive A that are elementary extensions of V_α and have the same cardinality as V_α.*

Note that if $A \in \mathscr{A}_\alpha$ and $\alpha \in A$ then $E_A \cap \alpha = E \cap \alpha$. If $A \in \mathscr{A}_\alpha$ we will refer to V_α as the *standard part* of A. Note that if A has nonstandard elements then $\alpha \in E_A$.

Definition 2.5. *Suppose $\alpha \in E$. We let $\mathscr{V}_\alpha$ denote the collection of all substructures M of V_λ of size less than κ such that, if we let $A = \mathrm{Hull}(M, V_\alpha)$, then $A \in \mathscr{A}_\alpha$ and M is an elementary submodel of A.*

Definition 2.6. *We refer to the members of $\mathscr{V}_\alpha$ as the* α-models. *We write $\mathscr{V}_{<\alpha}$ for $\bigcup\{\mathscr{V}_\gamma : \gamma \in E \cap \alpha\}$. Collections $\mathscr{V}_{\leq\alpha}$ and $\mathscr{V}_{\geq\alpha}$ are defined in the obvious way. We will write $\mathscr{V}$ for $\mathscr{V}_{<\lambda}$. If $M \in \mathscr{V}$, we then write $\eta(M)$ for the unique ordinal α such that $M \in \mathscr{V}_\alpha$.*

Remark 2.7. Note that if $M \in \mathscr{V}_\alpha$ then $\sup(M \cap \mathrm{ORD}) \geq \alpha$. In general, M is not elementary in V_λ, in fact, this only happens if $M \subseteq V_\alpha$. In this case we will say that M is a *standard α-model.*

Convention 2.8. *We refer to members of $\mathscr{V}$ as* virtual models. *We also refer to members of $\mathscr{V}^A$, for some admissible A with $A \subseteq V_\lambda$, as* general virtual models.

Definition 2.9. *Suppose $M, N \in \mathscr{V}$ and $\alpha \in E$. We say that an isomorphism $\sigma : M \to N$ is an* α-isomorphism *if it has an extension to an isomorphism $\bar{\sigma}$: $\mathrm{Hull}(M, V_\alpha) \to$*

$\mathrm{Hull}(N, V_\alpha)$. *We say that M and N are* α-isomorphic *and write $M \cong_\alpha N$ if there is an α-isomorphism between them. Note that if σ and $\bar{\sigma}$ exist, they are unique.*

Clearly, $\cong_\alpha$ is an equivalence relation, for every $\alpha \in E$. Note that if $M \in \mathscr{V}_\gamma$, for some $\gamma < \alpha$, then the only model α-isomorphic to M is M itself. Suppose $\alpha, \beta \in E$ and $\alpha \leq \beta$. It is easy to see that, if $M, N \in \mathscr{V}$ are β-isomorphic, then they are α-isomorphic. We will now see that, if $\alpha < \beta$, then for every β-model M there is a canonical representative of the $\cong_\alpha$-equivalence class of M which is an α-model.

Definition 2.10. *Suppose α and β are members of E and M is a β-model. Let $\overline{\mathrm{Hull}(M, V_\alpha)}$ be the transitive collapse of $\mathrm{Hull}(M, V_\alpha)$, and let π be the collapsing map. We define $M \restriction \alpha$ to be $\pi[M]$, i.e. the image of M under the collapsing map of $\mathrm{Hull}(M, V_\alpha)$.*

Remark 2.11. Note that if $\beta < \alpha$ then $M \restriction \alpha = M$. If $\beta \geq \alpha$, then $\overline{\mathrm{Hull}(M, V_\alpha)}$ belongs to $\mathscr{A}_\alpha$, so $M \restriction \alpha$ is an α-model which is α-isomorphic to M. Note also that if $\beta = \alpha$, then $M \restriction \alpha = M$ since $\mathrm{Hull}(M, V_\alpha)$ is already transitive.

Note also that if $A \in \mathscr{A}_\alpha$ then $\mathscr{V}_\alpha^A \subseteq \mathscr{V}_\alpha$. Therefore, if $A, B \in \mathscr{A}_\alpha$, $M \in \mathscr{V}^A$, and $N \in \mathscr{V}^B$, we can still write $M \cong_\alpha N$ if $M \restriction \alpha = N \restriction \alpha$. This is of course equivalent to the existence of an α-isomorphism between M and N.

The following is straightforward.

Proposition 2.12. *Suppose $\alpha, \beta \in E$ and $\alpha \leq \beta$. Let $M \in \mathscr{V}$. Then $(M \restriction \beta) \restriction \alpha = M \restriction \alpha$.*

2.12

We also need to define a version of the membership relation, for every α in E.

Definition 2.13. *Suppose $M, N \in \mathscr{V}$ and $\alpha \in E$. We write $M \in_\alpha N$ if there is $M' \in N$ with $M' \in \mathscr{V}^N$ such that $M' \cong_\alpha M$. If this happens, we say that M is α-in N.*

Note that if $M \subseteq V_\alpha$, this simply means that $M \in N$. However, in general, we may have $M \in_\alpha N$ even if the rank of M is higher than the rank of N. We shall often use the following simple facts without mentioning them.

Proposition 2.14. *Suppose $M, N \in \mathscr{V}$ with $M \in N$. Let $\alpha \in E$, and suppose $N' \in \mathscr{V}^A$, for some $A \in \mathscr{A}_\alpha$, and $\sigma : N \to N'$ is an α-isomorphism. Then M and $\sigma(M)$ are α-isomorphic.*

Proof. Since $|M| < \kappa < |V_\alpha|$, we conclude that $M \subseteq \mathrm{Hull}(N, V_\alpha)$, and hence $\mathrm{Hull}(M, V_\alpha) \subseteq \mathrm{Hull}(N, V_\alpha)$. Let $\bar{\sigma}$ be the extension of σ to $\mathrm{Hull}(N, V_\alpha)$. It follows that $\bar{\sigma} \restriction \mathrm{Hull}(M, V_\alpha)$ is an isomorphism between $\mathrm{Hull}(M, V_\alpha)$ and $\mathrm{Hull}(\sigma(M), V_\alpha)$. Hence $\bar{\sigma} \restriction M$ is an α-isomorphism between M and $\sigma(M)$. 2.14

Proposition 2.15. *Let $\alpha, \beta \in E$ with $\alpha \leq \beta$. Suppose $M, N \in \mathscr{V}_{\geq \beta}$ and $M \in_\beta N$. Then $M \restriction \alpha \in_\alpha N \restriction \alpha$.*

Proof. Fix some $M' \in N$ such that $M \cong_\beta M'$. Since $\alpha \leq \beta$, we have that $M \cong_\alpha M'$. If π is the Mostowski collapse map of $\mathrm{Hull}(N, V_\alpha)$, then $\pi(M') \in N \restriction \alpha$. On the other hand, since $|M| < \kappa < |V_\alpha|$, we have that $\mathrm{Hull}(M', V_\alpha) \subseteq \mathrm{Hull}(N, V_\alpha)$ and $\pi[M'] = \pi(M')$. It follows that $\pi \restriction \mathrm{Hull}(M', V_\alpha)$ is an isomorphism between $\mathrm{Hull}(M', V_\alpha)$ and $\mathrm{Hull}(\pi(M'), V_\alpha)$. Therefore, $M \cong_\alpha \pi(M') \in N \restriction \alpha$. 2.15

We refer to the following proposition as the continuity of the α-isomorphism.

Proposition 2.16. *Let α be a limit point of E. Suppose $N, M \in \mathscr{V}$ and $M \cong_\gamma N$ for unboundedly many γ below α. Then $M \cong_\alpha N$.*

Proof. For each $\gamma \in E \cap \alpha$, let σ_γ be the unique isomorphism between $\mathrm{Hull}(M, V_\gamma)$ and $\mathrm{Hull}(N, V_\gamma)$ such that $\sigma_\gamma[M] = N$. If $\gamma < \gamma'$, we have that $\sigma_{\gamma'} \restriction \mathrm{Hull}(M, V_\gamma) = \sigma_\gamma$. Let $\sigma = \bigcup\{\sigma_\gamma : \gamma \in E \cap \alpha\}$. Then σ witnesses that M and N are α-isomorphic. $\boxed{2.16}$

Proposition 2.17. *Let α be a limit point of E of uncountable cofinality. Assume that $M, N \in \mathscr{V}$ and N is countable. Suppose that $M \in_\delta N$ for unboundedly many $\gamma < \alpha$. Then $M \in_\alpha N$.*

Proof. Since N is countable and α is of uncountable cofinality, there is $M' \in N$ with $M' \in \mathscr{V}^N$ such that $M \cong_\gamma M'$, for unboundedly many $\gamma \in E \cap \alpha$. By Proposition 2.16 we have that $M \cong_\alpha M'$, and hence $M \in_\alpha N$. $\boxed{2.17}$

In our forcing we will use two types of virtual models, the countable ones and some nice models of size less than κ defined below.

Definition 2.18. *For $\alpha \in E$, we let $\mathscr{C}_\alpha$ denote the collection of countable models in $\mathscr{V}_\alpha$. We define similarly $\mathscr{C}_{<\alpha}$, $\mathscr{C}_{\leq\alpha}$ and $\mathscr{C}_{\geq\alpha}$. We write $\mathscr{C}$ for $\mathscr{C}_{<\lambda}$, and $\mathscr{C}_{\mathrm{st}}$ for the collection of standard models in $\mathscr{C}$.*

Proposition 2.19. *Suppose λ is of uncountable cofinality. Then $\mathscr{C}_{\mathrm{st}}$ contains a club in $\mathcal{P}_{\omega_1}(V_\lambda)$.*

Proof. First note that since λ is of uncountable cofinality E is unbounded and thus club in λ. Suppose M is a countable elementary submodel of $(V_\lambda, \in, E)$. Let $\alpha = \sup(M \cap E)$. Note that $M \cap \mathrm{ORD}$ is unbounded in α. Hence M is a standard α-model. $\boxed{2.19}$

The following definition is motivated by Magidor's reformulation of supercompactness Theorem 1.15.

Definition 2.20. *We say that $M \in \mathscr{V}$ is a* Magidor model *if, letting $\overline{M}$ be the transitive collapse of M and π the collapsing map, $\overline{M} = V_{\bar\gamma}$, for some $\bar\gamma < \kappa$ with $\mathrm{cof}(\bar\gamma) \geq \pi(\kappa)$, and $V_{\pi(\kappa)} \subseteq M$.*

Remark 2.21. Suppose M is a Magidor α-model. Let $V_{\bar\gamma}$ be its transitive collapse, and let j be the inverse of the collapsing map π. Let $\bar\kappa = \pi(\kappa)$, and let $A = \mathrm{Hull}(M, V_\alpha)$. Note that $j : V_{\bar\gamma} \to A$ is an elementary embedding with critical point $\bar\kappa$ and $j(\bar\kappa) = \kappa$.

Definition 2.22. *Let $\mathscr{U}^\kappa_\alpha$ be the collection of all $M \in \mathscr{V}_\alpha$ that are Magidor models. We define $\mathscr{U}^\kappa_{<\alpha}$, $\mathscr{U}^\kappa_{\leq\alpha}$, and $\mathscr{U}^\kappa_{\geq\alpha}$ in the obvious way. We write $\mathscr{U}^\kappa$ for $\mathscr{U}^\kappa_{<\lambda}$. When κ is clear from the context, we omit it. We also write $\mathscr{U}_{\mathrm{st}}$ for the standard models in $\mathscr{U}$.*

Proposition 2.23. *Suppose κ is supercompact and λ is inaccessible. Then $\mathscr{U}_{\mathrm{st}}$ is stationary in $\mathcal{P}_\kappa(V_\lambda)$.*

Proof. Fix a function $F : [V_\lambda]^{<\omega} \to V_\lambda$. We have to find a standard Magidor model closed under F. Since λ is inaccessible, we can find γ which is a limit point of E such that $\mathrm{cof}(\gamma) \geq \kappa$, and V_γ is closed under F. Let $\delta = \mathrm{next}(\gamma)$. Since κ is supercompact, by Theorem 1.15 we can find $\bar\kappa < \bar\delta < \kappa$ and an elementary embedding $j : V_{\bar\delta} \to V_\delta$ with critical point $\bar\kappa$ such that $j(\bar\kappa)=\kappa$ and such that $F\restriction[V_\gamma]^{<\omega}\in j[V_{\bar\delta}]$. Note that this implies that $V_\gamma\in j[V_{\bar\delta}]$. Let $\bar\gamma$ be such that $j(\bar\gamma)=\gamma$. Since $\mathrm{cof}(\gamma)\geq\kappa$, by elementarity we must have that $\mathrm{cof}(\bar\gamma)\geq\bar\kappa$. Let $N=j[V_{\bar\gamma}]$ and $\alpha=\sup(N\cap\gamma)$. Since $E\cap\gamma$ is definable in V_δ from parameter V_γ, we have that $E\cap\gamma\in j[V_{\bar\delta}]$, and hence α is a limit point of E. Moreover, since $j[\bar\gamma]$ is cofinal in α, we have that $\mathrm{cof}(\alpha)=\mathrm{cof}(\bar\gamma)$. Since $F\restriction[V_\gamma]^{<\omega}\in j[V_{\bar\delta}]$ and V_γ is closed under F, we must have that N is also closed under F. Moreover, we have that N is elementary in V_γ, and hence also in V_α. It follows that N is a standard Magidor α-model that is closed under F, as required. $\boxed{2.23}$

Note that both classes $\mathscr{C}$ and $\mathscr{U}$ of virtual models are closed under projections. We shall study some particular finite collections of these two types of models. We start by establishing the following easy fact.

Proposition 2.24. *Let $\alpha\in E$. Suppose $M,N,P\in\mathscr{V}$ and $M\in_\alpha N\in_\alpha P$. If either N is countable or P is a Magidor model then $M\in_\alpha P$.*

Proof. Pick $N'\in P$ with $N'\in\mathscr{V}^P$ which is α-isomorphic to N. We first establish that $N'\subseteq P$. If N is countable this is immediate. Suppose both N and P are Magidor models. Let $\overline{N'}$ be the transitive collapse of N', and let π be the collapsing map. Then $\overline{N'}\in V_\kappa\cap P$ since $|N'|<\kappa$. Since P is a Magidor model, we know that $V_\kappa\cap P$ is transitive, and hence $\overline{N'}\subseteq P$, but then also $N'\subseteq P$. Let σ be an α-isomorphism between N and N', and let $M'\in N$ with $M'\in\mathscr{V}^N$ be a model that is α-isomorphic to M. By Proposition 2.14 we know that $\sigma(M')$ is α-isomorphic to M', and also to M by the transitivity of $\cong_\alpha$. On the other hand $\sigma(M')\in N'\subseteq P$ and thus $M\in_\alpha P$, as desired. $\boxed{2.24}$

Our next goal is to define when a virtual model M is active at some $\alpha\in E$.

Definition 2.25. *Let $M\in\mathscr{V}$. We say that M is* active *at $\alpha\in E$ if $\eta(M)\geq\alpha$ and* $\mathrm{Hull}(M,V_{\kappa_M})\cap E\cap\alpha$ *is unbounded in $E\cap\alpha$, where $\kappa_M=\sup(M\cap\kappa)$. We say that M is* strongly active *at α if $\eta(M)\geq\alpha$ and $M\cap E\cap\alpha$ is unbounded in $E\cap\alpha$.*

Remark 2.26. We are primarily interested in the case $M\in\mathscr{C}\cup\mathscr{U}$. First note that if M is a Magidor model, then $V_{\kappa_M}\subseteq M$, hence M is active at some $\alpha\in E$ if and only if it is strongly active at α. The situation is quite different for countable models. If M is countable, then the set of $\alpha\in E$ at which M is strongly active is at most countable, while the set of $\alpha\in E$ at which M is active can be of size $|V_{\kappa_M}|$. On feature of our definition is that if $N\in_\alpha M$, then for all $\gamma\in E\cap\alpha$ in which N is active at, M is active as well.

Let us also remark what happens at levels α that are successor points of E. Suppose $\alpha=\mathrm{next}(\beta)$, for some $\beta\in E$, and M is active at α. We must have $\beta\in M$ as $\beta=\max(E\cap\alpha)$. we must also have $\sup(M\cap\mathrm{ORD})\geq\alpha$ since $\eta(M)\geq\alpha$,. If $\sup(M\cap\mathrm{ORD})=\alpha$ then M is a countable standard model. If $\sup(M\cap\mathrm{ORD})>\alpha$, let $\gamma=\min(M\cap\mathrm{ORD}\setminus\alpha)$, and let $A=\mathrm{Hull}(M,V_\gamma)$. Then by Lemma 2.1 $\gamma\in E_A$. Since $\gamma\in M$, we have that $E_A\cap(\gamma+1)\in M$ and therefore we can compute α in M as the the next element of $E_A\cap(\gamma+1)$ above β. Thus, in this case we have $\alpha\in M$.

It will be convenient to also have the following definition.

Definition 2.27. *Suppose $M\in\mathscr{V}$. Let $a(M)=\{\alpha\in E : M$ is active at $\alpha\}$ and $\alpha(M)=\max a(M)$.*

Note that $a(M)$ is a closed subset of E of size at most $|\mathrm{Hull}(M, V_{\kappa_M})|$.

Proposition 2.28. *Let $M \in \mathscr{V}$ and $N \in \mathscr{U}$. Suppose $\alpha \in E$, M and N are active at α, and $M \in_\alpha N$. Then $\alpha \in N$.*

Proof. We may assume that M and N are α-models. Let $A = \mathrm{Hull}(N, V_\alpha)$. Then $A \in \mathscr{A}_\alpha$. Fix $M^* \in N$ with $M^* \in \mathscr{V}^A$ which is α-isomorphic to M. Since M^* is α-equivalent to M, we have that $\alpha \in a^A(M^*)$. On the other hand, $a^A(M^*) \in N$ and has size $< \kappa_N$, hence $a^A(M^*) \subseteq N$. It follows that $\alpha \in N$. 2.28

Proposition 2.29. *Let $M \in \mathscr{V}$ and $N \in \mathscr{U}$. Suppose $\alpha \in a(M)$ is a limit point of E and $M \in_\gamma N$, for all $\gamma \in E \cap \alpha$. Then $\alpha \in N$ and $M \in_\alpha N$.*

Proof. Let $a = a(M) \cap N \cap \alpha$. Note that A is unbounded in α and has size $< \kappa_N$. Since N is closed under $< \kappa_N$-sequences, it follows that $a \in N$, and hence $\alpha = \sup(a) \in N$. For $\gamma < \alpha$, let $M_\gamma = M \restriction \gamma$. For $\gamma \in a$, we have that $M \in_\gamma N$, and hence $M_\gamma \in N$. Let $A_\gamma = \mathrm{Hull}(M_\gamma, V_\gamma)$. For $\gamma, \delta \in a$ with $\gamma < \delta$, we have that $M_\delta \restriction \gamma = M_\gamma$. In other words, A_γ is the transitive collapse of $\mathrm{Hull}(M_\delta, V_\gamma)$, and if $\sigma_{\gamma,\delta}$ is the inverse of the collapsing map, we have $\sigma_{\gamma,\delta}[M_\gamma] = M_\delta$. Each of the maps $\sigma_{\gamma,\delta}$ is definable from M_δ and γ, and hence it belongs to N. Now, N is closed under $< \kappa_N$-sequences and therefore the whole system $(A_\gamma, \sigma_{\gamma,\delta} : \gamma \leq \delta \in a)$ belongs to N. Let A be the direct limit of this system, and let σ_γ be the canonical embedding of A_γ to A. If we let π_γ be the collapsing map of $\mathrm{Hull}(M, V_\gamma)$ to A_γ, we then have that, for every $\gamma < \delta$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hull}(M, V_\gamma) & \xrightarrow{\ \mathrm{id}\ } & \mathrm{Hull}(M, V_\delta) \\ \downarrow{\scriptstyle \pi_\gamma} & & \downarrow{\scriptstyle \pi_\delta} \\ A_\gamma & \xrightarrow{\ \sigma_{\gamma,\delta}\ } & A_\delta \end{array}$$

Since $\mathrm{Hull}(M, V_\alpha) = \bigcup\{\mathrm{Hull}(M, V_\gamma) : \gamma \in a\}$, we have that A is isomorphic to $\mathrm{Hull}(M, V_\alpha)$. Therefore, its transitive collapse is $A_\alpha = \mathrm{Hull}(M \restriction \alpha, V_\alpha)$, and if we let π be the collapsing map, $\pi[M] = M \restriction \alpha$. We can therefore identify A with A_α, and we get that $\sigma_\gamma[M_\gamma] = M \restriction \alpha$, for any $\gamma \in a$. Thus $M \restriction \alpha \in N$, as required. 2.29

We now define an operation that will play the role of intersection for virtual models. We call it the *meet*. We only define the meet of two models of different types. Suppose $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Let $\overline{N}$ be the transitive collapse of N, and let π be the collapsing map. Note that if $\overline{N} \in M$, then $\overline{N} \cap M$ is a countable elementary submodel of $\overline{N}$. Then $\overline{N} \cap M \in \overline{N}$ since $\overline{N}$ is closed under countable sequence. Note that $\pi^{-1}(\overline{N} \cap M) = \pi^{-1}[\overline{N} \cap M]$, and this model is elementary in N.

Definition 2.30. *Suppose $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Let $\alpha = \max(a(N) \cap a(M))$. We will define $N \wedge M$ if $N \in_\alpha M$. Let $\overline{N}$ be the transitive collapse of N, and let π be the collapsing map. Set*

$$\eta = \sup(\sup(\pi^{-1}[\overline{N} \cap M] \cap \mathrm{ORD}) \cap E \cap (\alpha + 1)).$$

We define the meet of N and M to be $N \wedge M = \pi^{-1}[\overline{N} \cap M] \restriction \eta$.

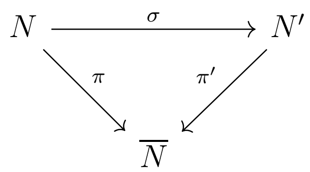
To make sense of the above definition, we need to prove the following.

Proposition 2.31. *Under the assumptions of the above definition, $N \wedge M \in \mathscr{C}_\eta$.*

Proof. Since $\eta(N) \geq \alpha$ we can form the model $A = \mathrm{Hull}(N, V_\alpha)$ and, we therefore have $N \prec A$ and $V_\alpha \prec A$. Since $\overline{N} \in M$, we have that $\overline{N} \cap M \prec \overline{N}$. Therefore, we have $\pi^{-1}[\overline{N} \cap M] \prec N$. Now, $\eta \in E \cap (\alpha + 1)$ and so $V_\eta \prec V_\alpha \prec A$. Moreover, $\sup(\pi^{-1}[\overline{N} \cap M] \cap \mathrm{ORD}) \geq \eta$. By Lemma 2.3 we have that $\mathrm{Hull}(\pi^{-1}[\overline{N} \cap M], V_\eta) \prec A$ and $V_\eta \subseteq \mathrm{Hull}(\pi^{-1}[\overline{N} \cap M], V_\eta)$. It follows that the transitive collapse of $\mathrm{Hull}(\pi^{-1}[\overline{N} \cap M], V_\eta)$ belongs to $\mathscr{A}_\eta$ and thus the image of $\pi^{-1}[\overline{N} \cap M]$ under the collapsing map belongs to $\mathscr{C}_\eta$. 2.31

Proposition 2.32. *Let $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Suppose $\alpha \in E$ and the meet $N \wedge M$ is defined and active at α. Then $(N \wedge M) \cap V_\alpha = N \cap M \cap V_\alpha$.*

Proof. Let $\beta = \max(a(N) \cap a(M))$. Since the meet of N and M is defined we must have $N \in_\beta M$. Since $N \wedge M$ is active at α, we must have $\alpha \leq \beta$. Let $N' \in \mathscr{V}^M$ be such that $N' \cong_\beta N$. Let σ be the β-isomorphism between N and N'. Notice that σ is the identity on $N \cap V_\beta$ and thus also on $N \cap V_\alpha$. Let $\overline{N}$ denote the common transitive collapse of N and N', and let π and π' be the collapsing maps. Then the following diagram commutes.



Note that $(N \wedge M) \cap V_\alpha = \pi^{-1}[\overline{N} \cap M] \cap V_\alpha = \sigma^{-1}[N \cap M \cap V_\alpha]$. Since σ is the identity on $N \cap V_\alpha$, it follows that $(N \wedge M) \cap V_\alpha = N \cap M \cap V_\alpha$. 2.32

Proposition 2.33. *Let $\alpha \in E$. Suppose $N \in \mathscr{U}$ and $\mathcal{M} \in \mathscr{C}$, the meet $N \wedge M$ is defined, and N and M strongly active at α. Then $N \wedge M$ is strongly active at α.*

Proof. Let $\beta = \max(a(N) \cap a(M))$. Since both N and M are active at α, we must have $\alpha \leq \beta$. Let $N' \in M$ with $N' \in \mathscr{V}^M$ be such that $N' \cong_\beta N$. Let σ be the β-isomorphism between N' and N. Then $\sigma \restriction N' \cap V_\beta$ is the identity. Note that $N' \cap M \cap E \cap \alpha$ is unbounded in $E \cap \alpha$. Since $N' \cap V_\alpha = N \cap V_\alpha$, we must have that $N \cap M \cap E \cap \alpha$ is also unbounded in $E \cap \alpha$. By Proposition 2.32, $N \wedge M$ is strongly active at α. 2.33

The next proposition states the meet operation commutes with projections.

Proposition 2.34. *Let $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Suppose $\alpha \in E$ and the meet $N \wedge M$ is defined and active at α. Then $(N \wedge M) \restriction \alpha = N \restriction \alpha \wedge M \restriction \alpha$.*

Proof. First note that if $N \wedge M$ is active at α, then $\alpha \in a(N) \cap a(M)$. It follows that α is the maximum of $a(N \restriction \alpha) \cap a(M \restriction \alpha)$. Then note that $N \wedge M$ depends only on $\max(a(N) \cap a(M))$, N, and $M \cap \overline{N}$, where $\overline{N}$ is the transitive collapse of N. Now, $\overline{N}$ is also the transitive collapse of $N \restriction \alpha$. In fact, if σ is the α-isomorphism between N and $N \restriction \alpha$, and π and π' are the collapsing maps of N and N' respectively, then $\pi = \pi' \circ \sigma$. Therefore, $\sigma \restriction \pi^{-1}[\overline{N} \cap M]$ is an an α-isomorphism between $\pi^{-1}[\overline{N} \cap M]$ and $\pi'^{-1}[\overline{N} \cap M \restriction \alpha]$. It follows that $(N \wedge M) \restriction \alpha = N \restriction \alpha \wedge M \restriction \alpha$. 2.34

Proposition 2.35. *Let $\alpha \in E$. Suppose $N \in \mathscr{U}$, $M \in \mathscr{C}$, both are active at α and $N \in_\alpha M$. Let P be another virtual model also active at α. Then $P \in_\alpha N \wedge M$ if only if $P \in_\alpha N$ and $P \in_\alpha M$.*

Proof. By Proposition 2.34 we may assume that N, M and P are all α-models. Assume first that $P \in_\alpha N \wedge M$. In particular this means that $N \wedge M$ is active at α. In particular we have that $N \wedge M \subseteq N$, and hence $P \in_\alpha N$. Fix $N' \in \mathscr{V}^M$ which is α-isomorphic to N. Let $\overline{N}$ be the transitive collapse of both N and N' and let π and π' be the respective collapsing maps. Note that $\sigma = \pi' \circ \pi^{-1}$ is the α-isomorphism between N and N'. Then $\sigma[N \wedge M] = N' \cap M$. Pick also $P' \in N \wedge M$ which is α-isomorphic to P. By Proposition 2.14 P' and $\sigma(P')$ are also α-isomorphic. Since $\sigma(P') \in M$, by the transitivity of $\cong_\alpha$ we get that P is α-isomorphic to $\sigma(P')$. This implies that $P \in_\alpha M$.

Now assume $P \in_\alpha N$ and $P \in_\alpha M$. By Proposition 2.28 we know that $\alpha \in N$. Since P is an α-model, we conclude that $P \in N$. If also $\alpha \in M$, we have that $N, P \in M$ and $N \wedge M = N \cap M$. Therefore, $P \in N \wedge M$. Assume now that $\alpha \notin M$ and let $\alpha^* = \min(M \cap \lambda \setminus \alpha)$. Let $A = \mathrm{Hull}(M, V_\alpha)$. Since we assumed that M is an α-model, we have that $A \in \mathscr{A}_\alpha$ and $\alpha \in E_A$. By Lemma 2.1 we also have that $\alpha^* \in E_A$. Fix $P^*, N^* \in M$ that are α-isomorphic to P and N respectively. By projecting them to α^* if necessary, we may assume $P^*, N^* \in \mathscr{V}^A_{\alpha^*}$. Moreover, N^* is a Magidor model from the point of view of A. Since $P^* \in_\alpha N^*$ and α^* is the least ordinal in M above α we have

$$M \models \forall \delta \in E_A \cap \alpha^* P^* \in_\delta N^*.$$

Moreover, $M \models$ "P^* is active at α^*". Since α^* is a limit point of E_A, we can apply Proposition 2.29 in A and conclude that $\alpha^* \in N^*$ and $P^* \in N^*$. Hence $P^* \in N^* \cap M$. Let σ be the α-isomorphism between N^* and N. Then $\sigma[N^* \cap M] = N \wedge M$. Hence $\sigma(P^*) \in N \wedge M$ and is α-isomorphic to P. It follows that $P \in_\alpha N \wedge M$. 2.35

One feature of the meet is the following absorption property.

Proposition 2.36. *Suppose $N \in \mathscr{U}$, $M \in \mathscr{C}$, and the meet $N \wedge M$ is defined. Let $\alpha \in E$, and suppose P is a Magidor α-model active at α such that $P \in_\alpha N \wedge M$. Then $P \wedge M = P \wedge (N \wedge M)$.*

Proof. Since $P \in_\alpha N \wedge M$ and P is active at α, so is $N \wedge M$, and hence both N and M are active at α as well. Let $\overline{P}$ be the transitive collapse of P. Then $\overline{P} \in N \cap V_\kappa$, and since $N \cap V_\kappa$ is transitive, we have $\overline{P} \subseteq N$. Hence $\overline{P} \cap (N \wedge M) = \overline{P} \cap M$. It follows that $P \wedge M = P \wedge (N \wedge M)$. 2.36

Proposition 2.37. *Let $\alpha \in E$. Suppose $N \in \mathscr{U}$, $M \in \mathscr{C}$ and the meet $N \wedge M$ is defined and active at α. Suppose $P \in \mathscr{V}$ and $N, M \in_\alpha P$. Then $N \wedge M \in_\alpha P$.*

Proof. We may assume M, N and P are all α-models. If $\alpha \in P$ then $N, M \in P$, and hence also $N \wedge M \in P$. Suppose now $\alpha \notin P$. Let $A = \mathrm{Hull}(P, V_\alpha)$ and let $\alpha^* = \min(P \cap \mathrm{ORD} \setminus \alpha)$. Note that α^* has uncountable cofinality in A. By Lemma 2.1 we have $\alpha^* \in E_A$. We can find $N^*, M^* \in P$ such that $N^* \cong_\alpha N$ and $M^* \cong_\alpha M$. We may assume that $N^* \in \mathscr{U}^A_{\alpha^*}$ and $M^* \in \mathscr{C}^A_{\alpha^*}$. Work for a moment in A. Since $N^* \in_\alpha M^*$, α^* is the least ordinal of P above α, and $N^*, M^* \in P$, we have

$$A \models \forall \gamma \in E_A \cap \alpha^* N^* \in_\gamma M^*.$$

By applying Proposition 2.17 inside A we have that $N^* \in_{\alpha^*} M^*$, and hence A can compute the meet, say Q, of N^* and M^*. Then $Q \in P$, and by applying Proposition 2.34 inside A, we get $Q \restriction \alpha = N^* \restriction \alpha \wedge M^* \restriction \alpha$. Hence $Q \cong_\alpha N \wedge M$. $\boxed{2.37}$

Definition 2.38. *Let $\alpha \in E$ and let $\mathcal{M}$ be a set of virtual models. We let $\mathcal{M} \restriction \alpha = \{M \restriction \alpha : M \in \mathcal{M}\}$ and $\mathcal{M}^\alpha = \{M \restriction \alpha : M \in \mathcal{M}$ is active at $\alpha\}$.*

We can now define what we mean by an α-chain.

Definition 2.39. *Let $\alpha \in E$ and let $\mathcal{M}$ be a subset of $\mathscr{U} \cup \mathscr{C}$. We say $\mathcal{M}$ is an α-*chain *if for all distinct $M, N \in \mathcal{M}$, either $M \in_\alpha N$ or $N \in_\alpha M$, or there is a $P \in \mathcal{M}$ such that either $M \in_\alpha P \in_\alpha N$ or $N \in_\alpha P \in_\alpha M$.*

Proposition 2.40. *Suppose $\alpha \in E$ and $\mathcal{M}$ is a finite subset of $\mathscr{U} \cup \mathscr{C}$. Then $\mathcal{M}$ is an α-chain if and only if there is an enumeration $\{M_i : i < n\}$ of $\mathcal{M}$ such that $M_0 \in_\alpha M_1 \in_\alpha \cdots \in_\alpha M_{n-1}$.*

Proof. Suppose first $\mathcal{M}$ is an α-chain. Define the relation $<$ on $\mathcal{M}$ by letting $M < N$ iff $\kappa_M < \kappa_N$. It is straightforward to see that $<$ is a total ordering on $\mathcal{M}$. We can then let $\{M_i : i < n\}$ be the $<$-increasing enumeration of $\mathcal{M}$. Conversely, suppose $\mathcal{M} = \{M_i : i < n\}$ is the enumeration such that $M_0 \in_\alpha M_1 \in_\alpha \cdots \in_\alpha M_{n-1}$. Let $i < j < n$. If $j = i + 1$ then $M_i \in_\alpha M_j$. Suppose $j > i + 1$. If M_j is a Magidor model or if there are no Magidor models between M_i and M_j by Proposition 2.24 we conclude that $M_i \in_\alpha M_j$. Otherwise let $k < j$ be the largest such that M_k is a Magidor model. Then again by Proposition 2.24, we conclude that $M_i \in_\alpha M_k \in_\alpha M_j$. $\boxed{2.40}$

Let $\alpha \in E$ and let $\mathcal{M}$ be an α-chain. Let $\in_\alpha^*$ be the transitive closure of $\in_\alpha$. Then $\in_\alpha^*$ is a total ordering on $\mathcal{M}$. For $M, N \in \mathcal{M}$, we say M is α-*below* N in $\mathcal{M}$, or equivalently N is α-*above* M in $\mathcal{M}$, if $M \in_\alpha^* N$ in $\mathcal{M}$. Now using the transitivity of $\in_\alpha^*$ we can form intervals in $\mathcal{M}$. Let

$$(M, N)_{\mathcal{M}}^\alpha = \{P \in \mathcal{M} : M \in_\alpha^* P \in_\alpha^* N\}.$$

Similarly we can define $[M, N]_{\mathcal{M}}^\alpha$, $[M, N)_{\mathcal{M}}^\alpha$, etc. For convenience we also allow that the endpoints of the intervals to be $\emptyset$ or V_λ; let $(\emptyset, N)_{\mathcal{M}}^\alpha$ be $\{P \in \mathcal{M}^\alpha : P \in_\alpha^* N\}$ in the first case, and let $(N, V_\lambda)_{\mathcal{M}}^\alpha$ be $\{P \in \mathcal{M} : N \in_\alpha^* P\}$ in the second.

3. Main Forcing

We fix an inaccessible cardinal κ and a cardinal $\lambda > \kappa$ with $\mathrm{cof}(\lambda) \geq \kappa$ such that $(V_\lambda, \in, \kappa)$ is admissible. We start by defining the forcing notions $\mathbb{M}_\alpha^\kappa$, for all $\alpha \in E \cup \{\lambda\}$.

Definition 3.1. *Suppose $\alpha \in E$. We say that $p = \mathcal{M}_p$ belongs to $\mathbb{M}_\alpha^\kappa$ if:*

(1) $\mathcal{M}_p$ is a finite subset of $\mathscr{C}_{\leq\alpha} \cup \mathscr{U}_{\leq\alpha}^\kappa$ that is closed under meets,

(2) $\mathcal{M}_p^\delta$ is a δ-chain, for all $\delta \in E \cap (\alpha + 1)$.

We let $\mathcal{M}_q \leq \mathcal{M}_p$ if for all $M \in \mathcal{M}_p$ there is $N \in \mathcal{M}_q$ such that $N \restriction \eta(M) = M$. Finally, let $\mathbb{M}_\lambda^\kappa = \bigcup\{\mathbb{M}_\alpha^\kappa : \alpha \in E\}$ with the same ordering.

Remark 3.2. Conditions (1) and (2) can be merged to a single condition. Let us say that a δ-chain $\mathcal{M}$ consisting of models active at δ is *closed under meets* if for every $M, N \in \mathcal{M}$, if the meet $M \wedge N$ is defined and active at δ then $M \wedge N \in \mathcal{M}$. Thus we can simply say that $\mathcal{M}_p^\delta$ is a δ-chain closed under meets, for all $\delta \in E \cap (\alpha+1)$. The order is natural since if $N \restriction \eta(M) = M$, then N carries all the information that M does.

If κ is supercompact and λ is inaccessible, the forcing notion $\mathbb{M}_\lambda^\kappa$ does many of the things we want to achieve. It turns κ to ω_2 and λ to ω_3. It forces the principle $\mathrm{ISP}(\omega_2)$ and if λ is weakly compact, respectively supercompact, then it also forces $\mathrm{TP}(\omega_3)$, respectively $\mathrm{ISP}(\omega_3)$. However, it may not force the principle $\mathrm{FS}(\omega_2)$ and thus we cannot say that $I[\omega_2] \restriction S^{\omega_1}_{\omega_2}$ is the non stationary ideal on $S^{\omega_1}_{\omega_2}$ in the generic extension. To see the problem, suppose G is generic over $\mathbb{M}_\lambda^\kappa$ and let $\mathcal{M}_G = \bigcup G$. For $\alpha \in E$, let $G_\alpha = G \cap \mathbb{M}^\kappa_\alpha$. We would like to fix some sufficiently large $\delta \in E$, and consider the models $M[G_\alpha]$, for Magidor models $M \in \mathcal{M}_G^\delta$ with $\alpha \in M$. These models will be ω_1-guessing models, but we do not know that the set of their intersections with ω_2 will be ω_1-closed. In order to arrange this we will have to modify our forcing by adding *decorations* to our conditions. This device, introduced by Neeman [15], consists of attaching to each model M of an $\in$-chain a finite set $d_p(M)$ which belongs to all models N of the chain such that $M \in N$. In a stronger condition this finite set is allowed to increase. The main point is that $d_p(M)$ controls what models can be added $\in$-above M in stronger conditions. In our situation there are some complications. First, we have not one chain, but a δ-chain, for each $\delta \in E$. It is therefore reasonable to have a decoration for each level $\delta \in E$. Now, models from a higher level project to lower levels at which they are active, but also in order to arrange strong properness for countable models, some models from lower levels will be *lifted* to higher levels and put on the chain. This imposes a subtle interplay between the decorations on different levels. In order to describe this precisely, we need to make some preliminary definitions.

Definition 3.3. *Suppose $\mathcal{M}_p \in \mathbb{M}_\lambda^\kappa$. Let $\mathcal{L}(\mathcal{M}_p) = \{M \restriction \alpha : M \in \mathcal{M}_p$ and $\alpha \in a(M)\}$.*

Definition 3.4. *Suppose $\mathcal{M}_p \in \mathbb{M}_\lambda^\kappa$. We say that $M \in \mathcal{L}(\mathcal{M}_p)$ is* $\mathcal{M}_p$-free *if every $N \in \mathcal{M}_p$ with $M \in_{\eta(M)} N$ is strongly active at $\eta(M)$. Let $\mathcal{F}(\mathcal{M}_p)$ denote the set of all $M \in \mathcal{L}(\mathcal{M}_p)$ that are $\mathcal{M}_p$-free.*

Note that if $\mathcal{M}_q \leq \mathcal{M}_p$ then $\mathcal{L}(\mathcal{M}_p) \subseteq \mathcal{L}(\mathcal{M}_q)$ and $\mathcal{F}(\mathcal{M}_q) \cap \mathcal{L}(\mathcal{M}_p) \subseteq \mathcal{F}(\mathcal{M}_p)$. In other words, a node $M \in \mathcal{L}(\mathcal{M}_p)$ that is not $\mathcal{M}_p$-free is not $\mathcal{M}_q$-free, for any $\mathcal{M}_q \leq \mathcal{M}_p$. We are now ready to define our main forcing notion.

Definition 3.5. *Suppose $\alpha \in E \cup \{\lambda\}$. We say that a pair $p = (\mathcal{M}_p, d_p)$ belongs to $\mathbb{P}_\alpha^\kappa$ if $\mathcal{M}_p \in \mathbb{M}_\alpha^\kappa$, d_p is a finite partial function from $\mathcal{F}(\mathcal{M}_p)$ to $\mathcal{P}_\omega(V_\kappa)$, and*

(∗) *if $M \in \mathrm{dom}(d_p)$, $N \in \mathcal{M}_p$, and $M \in_{\eta(M)} N$, then $d_p(M) \in N$.*

We say that $q \leq p$ if $\mathcal{M}_q \leq \mathcal{M}_p$, and for every $M \in \mathrm{dom}(d_p)$ there is $\gamma \in E \cap (\eta(M)+1)$ such that $M \restriction \gamma \in \mathrm{dom}(d_q)$ and $d_p(M) \subseteq d_q(M \restriction \gamma)$.

Remark 3.6. We refer to d_p as the decoration of p. The point is that if $M \in \mathrm{dom}(d_p)$ is a δ-model then $d_p(M)$ constraints what models N with $M \in_\delta N$ can be put on $\mathcal{M}_q^\delta$, for any $q \leq p$. In general, M may not be $\mathcal{M}_q$-free, in which case $M \notin \mathrm{dom}(d_q)$, but then we have some $\gamma \leq \delta$ such that $M \restriction \gamma$ is $\mathcal{M}_q$-free and $d_p(M) \subseteq d_q(M \restriction \gamma)$. Note that then we must have $d_p(M) \in N$, for any $N \in \mathcal{M}_q$ such that $M \in_\delta N$.

The ordering on $\mathbb{P}^\kappa_\lambda$ is clearly transitive. We will say that q is *stronger* than p if q forces that p belongs to the generic filter, in order words, any $r \leq q$ is compatible with p. We write $p \sim q$ if each of p and q is stronger than the other. We identify equivalent conditions, often without saying it. Our forcing does not have meets, but if p and q do have a weakest lower bound we will denote it by $p \wedge q$. To be precise we should refer to $p \wedge q$ as the $\sim$-equivalence class of a weakest lower bound, but we ignore this point since it should not cause any confusion. Note that if $p \in \mathbb{P}^\kappa_\alpha$ and $M \in \mathcal{M}_p$ is a δ-model that is not active at δ, we may replace M by $M \restriction \alpha(M)$ and we get an equivalent condition. Thus, if $\alpha \in E$ and $\mathrm{cof}(\alpha) \geq \kappa$, then $\mathbb{P}^\kappa_\alpha$ is forcing equivalent to $\bigcup\{\mathbb{P}^\kappa_\gamma : \gamma \in E \cap \alpha\}$.

Convention 3.7. *Suppose $p \in \mathbb{P}^\kappa_\lambda$ and $\delta \in E$. If $M, N \in \mathcal{M}^\delta_p$ with $M \in^*_\delta N$, we will write $(M,N)^\delta_p$ for the interval $(M,N)^\delta_{\mathcal{M}_p}$, and similarly, for $[M,N)^\delta_p$, $(M,N]^\delta_p$, etc.*

Suppose $\alpha, \beta \in E$ and $\alpha \leq \beta$. For every $p \in \mathbb{P}^\kappa_\beta$, we let $\mathcal{M}_{p\restriction\alpha} = \mathcal{M}_p \restriction \alpha$ and $d_{p\restriction\alpha} = d_p \restriction \mathcal{F}(\mathcal{M}_p \restriction \alpha)$. It is easily seen that $p = (\mathcal{M}_{p\restriction\alpha}, d_{p\restriction\alpha}) \in \mathbb{P}^\kappa_\alpha$. The following is straightforward.

Lemma 3.8. *Suppose $\alpha, \beta \in E$ with $\alpha \leq \beta$. Let $p \in \mathbb{P}^\kappa_\beta$ and let $q \in \mathbb{P}^\kappa_\alpha$ be such that $q \leq p \restriction \alpha$. Then there exists $r \in \mathbb{P}^\kappa_\beta$ such that $r \leq p, q$.*

Proof. We let $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$. Note that $\mathcal{M}_r$ is closed under meets. We define d_r by letting $d_r(M) = d_q(M)$ if $M \in \mathrm{dom}(d_q)$, and $d_r(M) = d_p(M)$ if $M \in \mathrm{dom}(d_p)$ with $\eta(M) > \alpha$. It is straightforward that r is as required. 3.8

Remark 3.9. The condition r from the previous lemma is the greatest lower bound of p and q, so we will write $r = p \wedge q$.

Corollary 3.10. *Suppose $\alpha \in E$. Then $\mathbb{P}^\kappa_\alpha$ is a complete suborder of $\mathbb{P}^\kappa_\lambda$.*

3.10

Our goal is to prove that our poset $\mathbb{P}^\kappa_\lambda$ is strongly proper for an appropriate class of models. We start by showing that if a condition p belongs to a model M we can always add M to $\mathcal{M}_p$ and form a new condition.

Lemma 3.11. *Let $p \in \mathbb{P}^\kappa_\lambda$ and $M \in \mathscr{C} \cup \mathscr{U}$ be such that $p \in M$. Then there is a weakest condition $p^M \leq p$ with $M \in \mathcal{M}_{p^M}$.*

Proof. Suppose first that M is a Magidor model. Then we let $\mathcal{M}_{p^M} = \mathcal{M}_p$ and $d_{p^M} = d_p$. It is straightforward that $p^M = (\mathcal{M}_{p^M}, d_{p^M})$ is as required.

Now assume that M is countable. We let $\mathcal{M}_{p^M}$ be the closure of $\mathcal{M}_p \cup \{M\}$ under meets. Fix $\delta \in E$. We show that $\mathcal{M}^\delta_p$ is an $\in_\delta$-chain. We may assume that M is active at δ since otherwise $\mathcal{M}^\delta_{p^M} = \emptyset$. By Proposition 2.36 we know that the only models added to $\mathcal{M}^\delta_p$ in order to form $\mathcal{M}^\delta_{p^M}$ are $M \restriction \delta$ and $N \wedge M$ for $N \in \mathcal{M}^\delta_p$ such that $N \wedge M$ is active at δ. Suppose $N \in \mathcal{M}^\delta_p$ is such a model, and let P be the $\in_\delta$-predecessor of N in $\mathcal{M}^\delta_p$, if it exists. First note that $N \cap M \in N$ since N is closed under countable sequence. Therefore, $N \wedge M \in N$. Moreover, if P exists by Proposition 2.35 we have that $P \in_\delta N \wedge M$. This establishes that $\mathcal{M}^\delta_{p^M}$ is a δ-chain.

Let us now define the decoration d_{p^M}. Suppose $N \in \mathrm{dom}(d_p)$ is a δ-model. Then $\delta \in M$. If M is strongly active at δ, then by Proposition 2.32, for every Magidor model

$P \in \mathcal{M}_p$ if $P \wedge M$ is active at δ then it is strongly active at δ. Hence N is $\mathcal{M}_{p^M}$-free. We then keep N in $\mathrm{dom}(p^M)$ and let $d_{p^M}(N) = d_p(N)$. Now, suppose M is not strongly active at δ. This means that δ has uncountable cofinality in M. Let $\bar{\delta} = \sup(M \cap \delta)$ and note that $\bar{\delta}$ is a limit point of E. We claim that $N \restriction \bar{\delta}$ is $\mathcal{M}_{p^M}$-free. Indeed, if there is $P \in \mathcal{M}_{p^M}$ such that $N \in_{\bar{\delta}} P$ and P is not strongly active at $\bar{\delta}$, then $P \in M$, and hence $\eta(P) \geq \delta$. Moreover, P is active but not strongly active at δ as well. Since $N \in_{\bar{\delta}} P$ and $N, P \in M$ it follows that $N \in_\gamma P$, for unboundedly many $\gamma \in E \cap \delta \cap M$. But then by Proposition 2.17 applied in M we conclude that $N \in_\delta P$, and hence N is not $\mathcal{M}_p$-free, a contradiction. Notice also that if $P \in \mathcal{M}_p$ and $N \in_{\bar{\delta}} P$ then by Proposition 2.17 again we must have that $N \in_\delta P$ and thus $d_p(N) \in P$. Therefore, we can replace N by $N \restriction \bar{\delta}$ and let $d_{p^M}(N \restriction \bar{\delta}) = d_p(N)$. It is straightforward to check that p^M is a weakest extension of p such that $M \in \mathcal{M}_{p^M}$. $\boxed{3.11}$

If N, M are virtual models it will be convenient to set $\alpha(N, M) = \max(a(N) \cap a(M))$.

Definition 3.12. *Suppose $p \in \mathbb{P}^\kappa_\lambda$ and $M \in \mathcal{L}(\mathcal{M}_p)$ is a Magidor model. For $N \in \mathcal{M}_p$ we let $N \restriction M = N \restriction \alpha(N, M)$ if $\kappa_N < \kappa_M$, otherwise $N \restriction M$ is undefined. Let*

$$\mathcal{M}_{p \restriction M} = \{N \restriction M : N \in \mathcal{M}_p\}.$$

Let $d_{p\restriction M} = d_p \restriction (\mathrm{dom}(d_p) \cap M)$, and let $p \restriction M = (\mathcal{M}_{p\restriction M}, d_{p \restriction M})$.

Lemma 3.13. *Suppose $p \in \mathbb{P}^\kappa_\lambda$ and $M \in \mathcal{L}(\mathcal{M}_p)$ is a Magidor model. Then $p \restriction M \in \mathbb{P}^\kappa_\lambda \cap M$ and $p \leq p \restriction M$.*

Proof. Since p is a condition, we have that if $N \in \mathcal{M}_p$ and $\kappa_N < \kappa_M$, then $N \in^*_\gamma M$, for all $\gamma \in a(N) \cap a(M)$. By Proposition 2.24 we then conclude that $N \in_\gamma M$, for all such γ. By Proposition 2.28 we have that $\alpha(N, M) \in M$, and hence $N \restriction \alpha(N, M) \in M$. We also have that $d_{p\restriction M} \in M$, thus $p \restriction M \in M$. Let us check that $\mathcal{M}_{p\restriction M} \in \mathbb{M}^\kappa_\lambda$. Suppose $\delta \in E$. If M is not active at δ then $\mathcal{M}^\delta_{p\restriction M}$ is empty, otherwise it is equal to $(\emptyset, M \restriction \delta)^\delta_p$, which is obviously a δ-chain. To check that $\mathcal{M}_{p\restriction M}$ is closed under meets, suppose $N \restriction M, P \restriction M \in \mathcal{M}_{p\restriction M}$ and their meet is defined. Note that then $N \wedge P$ is also defined and, by Proposition 2.34 $(N \wedge P) \restriction M = N \restriction M \wedge P \restriction M$. It is straightforward to check that every $N \in \mathrm{dom}(d_{p\restriction M})$ is $\mathcal{M}_{p\restriction M}$-free, and $(*)$ from Definition 3.5 holds. Finally, the fact that $p \leq p \restriction M$ follows from the definition. $\boxed{3.13}$

Lemma 3.14. *Suppose $p \in \mathbb{P}^\kappa_\lambda$ and $M \in \mathcal{L}(\mathcal{M}_p)$ is a Magidor model. Suppose $q \in M \cap \mathbb{P}^\kappa_\lambda$ extends $p \restriction M$. Then q is compatible with p and the meet $p \wedge q$ exists.*

Proof. We define $r \in \mathbb{P}^\kappa_\lambda$ and check that it is a weakest condition extending p and q. Let $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$. We check that if $\delta \in E$, then $\mathcal{M}^\delta_r$ is a δ-chain closed under meets, meaning if $P, Q \in \mathcal{M}^\delta_r$ and the meet $P \wedge Q$ is defined and active at δ then $P \wedge Q \in \mathcal{M}^\delta_r$. Fix such $\delta \in E$. If M is not active at δ, then $\mathcal{M}^\delta_r = \mathcal{M}^\delta_p$ and thus has the required property since p is a condition. Now, suppose M is active at δ. If $R \in \mathcal{M}^\delta_r$ and $R \in^*_\delta M$, then by Proposition 2.24 we know that $R \in_\delta M$, and by Proposition 2.28 we get that $\delta \in M$. Hence $R \in M$ and therefore $R \in \mathcal{M}^\delta_q$. Therefore, $\mathcal{M}^\delta_r$ is the union of $\mathcal{M}^\delta_q$ and $[M \restriction \delta, V_\lambda)^\delta_p$, and hence is a δ-chain. Now suppose $P, Q \in \mathcal{M}^\delta_r$ and their meet is defined and active at δ. We need to check that $Q \wedge P \in \mathcal{M}^\delta_r$. If both P and Q belong either to $\mathcal{M}^\delta_q$ or $\mathcal{M}^\delta_p$, this follows from the fact that p and q are conditions. Since $Q \in_\delta P$ and

$\mathcal{M}_q^\delta$ is an $\in_\delta^*$-initial segment of $\mathcal{M}_r^\delta$, we may assume $Q \in \mathcal{M}_q^\delta$ and $P \in \mathcal{M}_p^\delta \setminus \mathcal{M}_q^\delta$. The proof goes by induction on the number of Magidor models on the δ-chain $[M \restriction \delta, P)_p^\delta$. If $M \in_\delta P$ then $M \wedge P \in \mathcal{M}_p^\delta$ and is δ-below $M \restriction \delta$, hence belongs to $\mathcal{M}_q^\delta$. On the other hand, by Proposition 2.36 we have $Q \wedge P = Q \wedge (M \wedge P)$, and since $\mathcal{M}_q^\delta$ is closed under meets we get that $Q \wedge P \in \mathcal{M}_q^\delta$. In general, if N is the $\in_\delta^*$-largest Magidor model in $[M \restriction \delta, P)_p^\delta$, by Proposition 2.24, we have that $Q \in_\delta N \in_\delta P$. In particular, $N \wedge P$ is defined and by Proposition 2.36 we have that $Q \wedge P = Q \wedge (N \wedge P)$. Now, we are done if $N \wedge P \in \mathcal{M}_q^\delta$ as q is a condition. Otherwise, it belongs to the interval $[M \restriction \delta, P)_p^\delta$. Then there are fewer Magidor models in $[M \restriction \delta, N \wedge P)_p^\delta$ and thus we can use the induction hypothesis.

Let $d_r = d_q \cup d_p \restriction (\mathrm{dom}(d_p) \setminus M)$. Let us check that every $N \in \mathrm{dom}(d_r)$ is $\mathcal{M}_r$-free. For simplicity, let $\eta = \eta(N)$. If $N \in \mathrm{dom}(d_p) \setminus M$, then there is no $P \in \mathcal{M}_q$ such that $N \in_\eta P$, and hence the conclusion follows from the fact that p is a condition. Suppose now $N \in \mathrm{dom}(d_q)$ and $P \in \mathcal{M}_r$ is such that $N \in_\eta P$. We have to check that P is strongly active at η. We may assume that P is a countable model. If $P \restriction \eta$ is η-below M, then $P \restriction M$ is defined and $P \restriction M \cong_\eta P$, therefore, the conclusion follows from the fact that q is a condition. If $P \restriction \eta$ is η-above M, then $M \wedge P$ is defined and belongs to $\mathcal{M}_p$. Moreover, by Proposition 2.35, $N \in_\eta M \wedge P$. Now $(M \wedge P) \restriction M$ is defined, and belongs to $\mathcal{M}_q^{\alpha(M \wedge P, M)}$, and is strongly active at η since N is $\mathcal{M}_q$-free. Therefore, P is also strongly active at η. The fact that d_r satisfies condition $(*)$ from Definition 3.5 is straightforward. Finally, the fact that r is the weakest common extension of p and q follows readily from the definition. $\boxed{3.14}$

By Lemma 3.11 and Lemma 3.14 we immediately get the following.

Theorem 3.15. *The forcing $\mathbb{P}_\lambda^\kappa$ is $\mathscr{U}$-strongly proper.*

$\boxed{3.15}$

We now proceed to define an analogue of $p \restriction M$ for countable models $M \in \mathcal{L}(\mathcal{M}_p)$. The situation here is more subtle since $p \restriction M$ may not belong to the original forcing, only its version as defined in M. We first analyze the part involving $\mathcal{M}_p$. It will be useful to make the following definition.

Definition 3.16. *Let $\mathcal{M}$ be a subset of $\mathscr{C} \cup \mathscr{U}$ and $M \in \mathscr{C}$. For $\delta \in E$, we let $(\mathcal{M} \restriction M)^\delta = \{N \in \mathcal{M}^\delta : N \in_\delta M\}$.*

Lemma 3.17. *Let $\mathcal{M}_p \in \mathbb{M}_\lambda^\kappa$ and $\delta \in E$. Suppose $M \in \mathcal{M}_p^\delta$ is countable. Then $(\mathcal{M}_p \restriction M)^\delta$ is a δ-chain closed under meets and*

$$(\mathcal{M}_p \restriction M)^\delta = (\emptyset, M \restriction \delta)_p^\delta \setminus \bigcup \{[N \wedge M, N)_p^\delta : N \in (\mathcal{M}_p \restriction M)^\delta \text{ and is a Magidor model}\}.$$

Here, if $N \wedge M$ is defined and not active at δ, by $[N \wedge M, N)_p^\delta$ we mean $(\emptyset, N)_p^\delta$.

Proof. It is clear that $(\mathcal{M}_p \restriction M)^\delta \subseteq (\emptyset, M)_p^\delta$. Suppose $P \in \mathcal{M}_p^\delta$ and $P \in_\delta M$. Then, for any Magidor model $N \in (P, M)_p^\delta$, we have $P \in_\delta N$ by Proposition 2.24. Then by Proposition 2.35 we have that $P \in_\delta N \wedge M$. Conversely, suppose P is in $(\emptyset, M)_p^\delta$, but not in $(\mathcal{M}_p \restriction M)^\delta$. Then, by Proposition 2.24 again, there must be a Magidor model

$N \in \mathcal{M}_p^\delta$ such that $P \in_\delta N \in_\delta M$. Let N be the $\in_\delta^*$-least such model. If $N \wedge M$ is not active at δ, then $P \in (\emptyset, N)_p^\delta$. Suppose $N \wedge M$ is active at δ. We have to show that either $P = N \wedge M$ or $N \wedge M \in_\delta^* P$. Indeed, otherwise we have $P \in_\delta^* N \wedge M$. Note that there cannot be a Magidor model $Q \in \mathcal{M}_p^\delta$ with $P \in_\delta Q \in_\delta N \wedge M$ since then we would have $Q \in_\delta M$ as well, and this contradicts the minimality of N. Since $\mathcal{M}_p^\delta$ is a δ-chain, by Proposition 2.24 we conclude that $P \in_\delta N \wedge M$, but then also $P \in_\delta M$, a contradiction. The fact that $(\mathcal{M}_p \restriction M)^\delta$ is a δ-chain follows from the above analysis. By Proposition 2.37 it is also closed under meets. 3.17

Lemma 3.18. *Let $p \in \mathbb{P}_\lambda^\kappa$ and $M, N \in \mathcal{M}_p$. If there is $\gamma \in a(M) \cap a(N)$ such that $N \in_\gamma M$, then $N \in_\delta M$, for all $\delta \in a(M) \cap a(N)$.*

Proof. Let $\alpha = \alpha(M, N)$. If $N \in_\alpha M$ then $N \in_\gamma M$, for all $\gamma \in a(M) \cap a(N)$, by Proposition 2.15. Now suppose $N \notin_\alpha M$. If M is a Magidor model we have $\kappa_M \leq \kappa_N$, and hence there is no γ such that $N \in_\gamma M$. Assume now that M is countable. Then by Lemma 3.17 there is a Magidor model $P \in \mathcal{M}_p^\delta$ with $P \in_\delta M$ such that $N \restriction \delta$ is in the interval $[P \wedge M, P)_p^\delta$. Now, $P \wedge M$ is active at all $\gamma \in a(N) \cap \alpha$ and $N \restriction \gamma$ is in the interval $[(P \wedge M) \restriction \gamma, P \restriction \gamma)_p^\gamma$, for all such γ. But then, by Lemma 3.17 again, $N \notin_\gamma M$, for all $\gamma \in a(M) \cap a(N)$. 3.18

It would be useful to introduce some notation.

Definition 3.19. *Suppose $M \in \mathscr{V}$ and $\mathcal{M}$ is a finite subset of $\mathscr{V}$. Let $\alpha \in E$. We write $\mathcal{M} \in_\alpha M$ if $N \in_\alpha M$, for all $N \in \mathcal{M}$.*

Lemma 3.20. *Suppose $\mathcal{M}_p \in \mathbb{M}_\lambda^\kappa$ and $\delta \in E$. Suppose $M \in \mathcal{M}_p^\delta$ is a countable model, $\mathcal{M} \in_\delta M$ is a finite δ-chain closed under meets, and $(\mathcal{M}_p \restriction M)^\delta \subseteq \mathcal{M}$. Then the closure of $\mathcal{M}_p^\delta \cup \mathcal{M}$ under meets that are active at δ is a δ-chain.*

Proof. Let us first show that $\mathcal{M}_p^\delta \cup \mathcal{M}$ is a δ-chain. Indeed, by Lemma 3.17 it is obtained by adding to $\mathcal{M}$ the intervals $[N \wedge M, N)_p^\delta$, where $N \in (\mathcal{M}_p \restriction M)^\delta$ is a Magidor model, and the interval $[M, V_\lambda)_p^\delta$. Consider one such interval, say $[N \wedge M, N)_p^\delta$. If P is the last model of $\mathcal{M}$ before N then $P \in_\delta M$ by the assumption that $\mathcal{M} \in_\delta M$, and $P \in_\delta N$ by Proposition 2.24. Hence by Proposition 2.34 we have that $P \in_\delta N \wedge M$. It follows that $\mathcal{M}_p^\delta \cup \mathcal{M}$ is a δ-chain.

Let us now consider what happens when we close $\mathcal{M}_p^\delta \cup \mathcal{M}$ under meets that are active at δ. Suppose $Q, P \in \mathcal{M}_p^\delta \cup \mathcal{M}$, Q is a Magidor model, P is countable, and $Q \in_\delta P$. If $P \in \mathcal{M}$ then $Q \in_\delta P \in_\delta M$, and hence by Proposition 2.24 $Q \in M$, and so $Q \in \mathcal{M}$ as well. Since $\mathcal{M}$ is closed under meets, we have that $Q \wedge P \in \mathcal{M}$. Now suppose $P \in \mathcal{M}_p^\delta \setminus \mathcal{M}$. By Lemma 3.17 we have that $P \in [M, V_\lambda)_p^\delta$ or $P \in [N \wedge M, N)_p^\delta$, for some Magidor model $N \in (\mathcal{M}_p \restriction M)^\delta$. The two cases are only notationally different, so let us assume that there is a Magidor model $N \in (\mathcal{M}_p \restriction M)^\delta$ such that $P \in [N \wedge M, N)_p^\delta$. We may assume that $Q \in \mathcal{M}$. Note that $Q \in_\delta N$ and $Q \in_\delta M$, hence by Proposition 2.35 $Q \in_\delta N \wedge M$. If there is a Magidor model $R \in \mathcal{M}_p$ such that $Q \in_\delta^* R \in_\delta^* P$, let R be the $\in_\delta^*$-largest such model. By Proposition 2.24 we have that $R \in_\delta P$ and hence $R \wedge P$ is defined and is below P on the δ-chain $\mathcal{M}_p^\delta$. Moreover, since $\mathcal{M}_p \cup \mathcal{M}$ is a δ-chain, also by Proposition 2.24, we have $Q \in_\delta R$. Now, by Proposition 2.35 we have that $Q \in_\delta R \wedge P$, and by Proposition 2.36 we have $Q \wedge P = Q \wedge (R \wedge P)$. Therefore, we may assume that there are no Magidor models $R \in \mathcal{M}_p^\delta$ with $Q \in_\delta R \in_\delta P$. Now, let $\{P_i : i < k\}$ list all countable models on the chain $[N \wedge M, N)_p^\delta$ below the first Magidor model, if it exists. Then $P_0 = N \wedge M$ and $P = P_j$, for some j. Note that $Q \in_\delta P_i$, for all $i < k$, again by Proposition 2.24. Now let S be the $\in_\delta^*$-predecessor of N on the δ-chain $\mathcal{M}_p^\delta \cup \mathcal{M}$, if it exists, otherwise let S be $\emptyset$. Note that $S \in_\delta N \wedge M$. Indeed, if $S \in \mathcal{M}_p$ this follows from Proposition 2.24, and the fact that there are no Magidor models in $(S, N \wedge M)_p^\delta$. If $S \in \mathcal{M}$ then $S \in_\delta M$ and thus $S \in_\delta N \wedge M$. Now, by Proposition 2.24 we have that $S \in_\delta P_i$, for all $i < k$. Hence, by Proposition 2.35 we have $S \in_\delta Q \wedge P_i$, for all i. By Proposition 2.37 we have $Q \wedge P_i \in_\delta P_{i+1}$, for all $i < k-1$. Since Q is a Magidor model, we also have that $Q \wedge P_i \in_\delta Q$, for all $i < k$. By Proposition 2.35 again, we have that $Q \wedge P_i \in_\delta Q \wedge P_{i+1}$, for all $i < k-1$. Therefore, $S \in_\delta Q \wedge P_0 \in_\delta \ldots \in_\delta Q \wedge P_{k-1} \in_\delta Q$, and $Q \wedge P$ appears on this chain. If $S = \emptyset$ then an initial segment of this chain may be nonactive at δ, but the remainder is still a δ-chain. $\boxed{3.20}$

Now, suppose $p \in \mathbb{P}_\lambda^\kappa$ and $M \in \mathcal{M}_p$ is a countable β-model, for some $\beta \in E$. Let $A = \operatorname{Hull}(M, V_\beta)$. Then $A \in \mathscr{A}_\beta$. Note that $E_A \cap \beta = E \cap \beta$, and if $\beta \in A$ then $\beta \in E_A$. Also, note that the definitions of $\mathbb{P}_\alpha^\kappa$ and the order relation is Σ_1 with parameter V_α. For $\alpha \in E_A$, let $(\mathbb{P}_\alpha^\kappa)^A$ be the version of $\mathbb{P}_\alpha^\kappa$ as defined in A. Then $(\mathbb{P}_\alpha^\kappa)^A = \mathbb{P}_\alpha^\kappa$ if $\alpha < \beta$, and $(\mathbb{P}_\beta^\kappa)^A \subseteq \mathbb{P}_\beta^\kappa$. We will let $\mathscr{V}_\alpha^M = \mathscr{V}_\alpha^A \cap M$, and $(\mathbb{P}_\alpha^\kappa)^M = (\mathbb{P}_\alpha^\kappa)^A \cap M$, if $\alpha \in E_A \cap M$. Suppose $N \in \mathcal{M}_p$ and $N \in_\delta M$, for some $\delta \in a(M) \cap a(N)$, Then by Lemma 3.18, $N \in_\alpha M$, where $\alpha = \alpha(M, N)$. Note that if M is a standard β-model then $\alpha < \beta$. It may be that $\alpha \notin M$, but then, if we let $\alpha^* = \min(M \cap \mathrm{ORD} \setminus \alpha)$, we have that $\alpha^* \in E_A \cap M$, and α^* is of uncountable cofinality in A. By the previous remarks, if M is a standard β-model or $\beta \in M$ then $\alpha^* \in E \cap (\beta + 1)$, otherwise α^* may be in the nonstandard part of M. Since $N \in_\alpha M$, there is a model $N^* \in M$ with $N^* \in \mathscr{V}^A$ which is α-isomorphic to N. Now, M can compute $N^* \restriction \alpha^*$, hence we may assume $N^* \in \mathscr{V}_{\alpha^*}^A$. Moreover, such N^* is unique. Indeed, if there is another model $N^{**} \in M$ with the same property, since α^* is the least ordinal in M above α and $N^* \cong_\alpha N^{**}$ we would have that $N^* \cong_\delta N^{**}$, for all $\delta \in E_A \cap \alpha^* \cap M$. Hence, by Proposition 2.16 applied in M, we would have that $N^* = N^{**}$. This justifies the following definition.

Definition 3.21. *Suppose $p \in \mathbb{P}_\lambda^\kappa$ and let $M \in \mathcal{L}(\mathcal{M}_p)$ be a countable β-model, for some $\beta \in E$. Let $N \in \mathcal{M}_p$ and let $\alpha = \alpha(M, N)$. If $N \in_\alpha M$ we let $\alpha^* = \min(M \cap \mathrm{ORD} \setminus \alpha)$. We define $N \restriction M$ to be the unique $N^* \in \mathscr{V}_{\alpha^*}^M$ such that $N^* \cong_\alpha N$. Otherwise we leave $N \restriction M$ undefined. Let*

$$\mathcal{M}_{p \restriction M} = \{N \restriction M : N \in \mathcal{M}_p\}, \text{ and}$$

$$\operatorname{dom}(d_{p \restriction M}) = \{N \restriction M : N \in \operatorname{dom}(d_p) \text{ and } N \in_{\eta(N)} M\}.$$

If $N \in \operatorname{dom}(d_p)$ and $N \in_{\eta(N)} M$, let $d_{p \restriction M}(N \restriction M) = d_p(N)$. Let $p \restriction M = (\mathcal{M}_{p \restriction M}, d_{p \restriction M})$.

Remark 3.22. Suppose $N \in \operatorname{dom}(d_p)$ and let $\eta = \eta(N)$. If $N \in_\eta M$ then M is strongly active at η since N is $\mathcal{M}_p$-free. If $\eta \in M$ then we put N in $\operatorname{dom}(d_{p \restriction M})$ and keep the same decoration at N. If $\eta \notin M$ we lift N to the least level η^* of M above η, we put the resulting model N^* in $\operatorname{dom}(d_{p \restriction M})$ and copy the decoration of N to N^*. If $P \in \mathcal{M}_p$ is such that $P \restriction \eta = N$ then $(P \restriction M) \restriction \eta^* = N^*$. Moreover, from N^* we can recover N as $N^* \restriction \sup(\eta^* \cap M)$. Thus, the function $d_{p \restriction M}$ is well defined. Note also that $p \restriction M \in M$.

Proposition 3.23. *Suppose* $p \in \mathbb{P}^\kappa_\lambda$ *and* $M \in \mathcal{L}(\mathcal{M}_p)$ *is a countable* β*-model, for some* $\beta \in E$. *Let*

$$\alpha = \max\{\alpha(N,M) : N \in_{\alpha(N,M)} M \text{ and } N \in \mathcal{M}_p\}.$$

Let $\alpha^* = \min(M \cap \mathrm{ORD} \setminus \alpha)$. *Then* $p \restriction M \in (\mathbb{P}^\kappa_{\alpha^*})^M$.

Proof. Let $A = \mathrm{Hull}(M, V_\beta)$ and work in A. It is clear that $\mathcal{M}_{p\restriction M}$ is a finite subset of $\mathscr{C}^A_{\leq\alpha^*} \cup \mathscr{U}^A_{\leq\alpha^*}$. We first show that $\mathcal{M}^\gamma_{p\restriction M}$ is a γ-chain closed under meets, for all $\gamma \in E_A \cap (\alpha^*+1)$. Fix such γ and let $\delta = \min(M \cap \mathrm{ORD} \setminus \gamma)$ and $\bar{\delta} = \sup(M \cap \delta)$. If $\bar{\delta} = \delta$ then $\gamma = \delta$, and hence $\gamma \in M$ and the conclusion follows from the fact that p is a condition and Lemma 3.17. Let us assume now that $\bar{\delta} < \delta$. Note that then $\bar{\delta}, \delta \in E$, δ is of uncountable cofinality in M, and is a limit point of E. Note that if $P \in \mathcal{M}_{p\restriction M}$ is a δ-model that is active at γ then $a(P)$ is cofinal in δ. Moreover, $a(P) \in M$ and since $\bar{\delta} = \sup(M \cap \delta)$ we have that $\bar{\delta} \in a(P)$. This implies that $\mathcal{M}^\gamma_{p\restriction M} \restriction \bar{\delta} = \mathcal{M}^{\bar{\delta}}_p \restriction M$. Therefore, by Lemma 3.17 it is a $\bar{\delta}$-chain closed under active meets. Now, suppose $N, P \in \mathcal{M}^\delta_{p\restriction M}$ and $N \in_{\bar{\delta}} P$. Since $\bar{\delta} = \sup(M \cap \delta)$ we have that $N \in_\xi P$, for unboundedly many $\xi \in E \cap \delta$. We conclude that $N \in_\delta P$. Indeed, if P is countable this follows from Proposition 2.17 applied in A, and if P is a Magidor model this follows from Proposition 2.29, again applied in A. Moreover, assuming N is a Magidor model and P is countable, and $N \restriction \gamma \wedge P \restriction \gamma$ is defined and active at γ then, by Proposition 2.34, $N \wedge P$ is defined and active at unboundedly many $\xi \in E \cap \delta$, and hence it is also active at δ and $\bar{\delta}$. It follows that $\mathcal{M}^\delta_{p\restriction M}$ is a δ-chain closed under meets, and hence $\mathcal{M}^\gamma_{p\restriction M}$ is a γ-chain closed under meets as well.

Let us check that every $P^* \in \mathrm{dom}(d_{p\restriction M})$ is $\mathcal{M}_{p\restriction M}$-free. If $P^* \in \mathrm{dom}(d_p)$ this is immediate. Otherwise, P^* is of the form $P \restriction M$, for some $P \in \mathrm{dom}(d_p)$ such that $\eta(P) \notin M$. Let $\eta = \eta(P)$ and $\eta^* = \eta(P^*)$. Note that M is strongly active at η and η^* is the least ordinal of M above η. Suppose $N \in \mathcal{M}_{p\restriction M}$ is such that $P^* \in_{\eta^*} N$. Then $N \restriction \eta \in \mathcal{L}(\mathcal{M}_p)$ and $P \in_\eta N$. Since P is $\mathcal{M}_p$-free, N must be strongly active at η. Since $\eta = \sup(M \cap \eta^*)$ and $N \in M$ we must have that N is strongly active at η^* as well. This also establishes $(*)$ from Definition 3.5. Indeed, if $P^* \in_{\eta^*} N$ then $P \in_\eta N \restriction \eta$, and hence $d_p(P) \subseteq N$, since $N \restriction \eta \in \mathcal{L}(\mathcal{M}_p)$, and p is a condition. This completes the proof that $p \restriction M = (\mathbb{P}^\kappa_{\alpha^*})^A$. 3.23

Remark 3.24. Note that if $p, q \in \mathbb{P}^\kappa_\lambda$ are such that $q \leq p$ and $M \in \mathcal{M}_p$ then $q \restriction M \leq p \restriction M$.

We are planning to show that if p is a condition and $M \in \mathcal{M}_p$ is a countable β-model then, for any $q \leq p \restriction M$ with $q \in M$, p and $q \restriction \beta$ are compatible, and in fact the meet $p \wedge q \restriction \beta$ exists. Before that we show the following special case of this statement.

Lemma 3.25. *Suppose* $p \in \mathbb{P}^\kappa_\lambda$ *and* $\delta \in E$. *Suppose* $M \in \mathcal{M}^\delta_p$ *is a countable model,* $\mathcal{M} \in_\delta M, \mathcal{M} \in \mathbb{M}^\kappa_\delta$, *and* $(\mathcal{M}_p \restriction M)^\gamma \subseteq \mathcal{M}^\gamma$, *for all* $\gamma \in E \cap (\delta+1)$. *Suppose further that* $P \notin_{\eta(P)} M$, *for all* $P \in \mathrm{dom}(d_p)$. *Let* $\mathcal{M}_q$ *be the closure of* $\mathcal{M}_p \cup \mathcal{M}$ *under meets and let* $d_q = d_p$. *Finally, let* $q = (\mathcal{M}_q, d_q)$. *Then* $q \in \mathbb{P}^\kappa_\lambda$.

Proof. Let us first check that $\mathcal{M}^\gamma_q$ is a γ-chain closed under active meets, for all $\gamma \in E$. Fix $\gamma \in E$. If M is not active at γ then $\mathcal{M}^\gamma_q = \mathcal{M}^\gamma_p$, so this follows from the fact that p is a condition. If M is active at γ then this follows from Lemma 3.20.

Thus, it remains to check that every $P \in \mathrm{dom}(d_p)$ is $\mathcal{M}_q$-free and $d_p(P) \in Q$, for all $Q \in \mathcal{M}_q$ such that $P \in_{\eta(P)} Q$. Now, fix one such $P \in \mathrm{dom}(d_p)$ and let $\eta = \eta(P)$. If M is not active at η, then no model of $\mathcal{M}$ is active at η, and hence $Q \in \mathcal{M}_p$, for all $Q \in \mathcal{M}_q$ such that $P \in_\eta Q$. The conclusion then follows from the fact that p is a condition and d_p is its decoration. Suppose now that M is active at η, but P is either equal to $M \restriction \eta$ or is above $M \restriction \eta$ on the η-chain $\mathcal{M}_p^\eta$. Then, again any $Q \in \mathcal{M}_q$ such that $P \in_\eta Q$ is in $\mathcal{M}_p$, and the conclusion follows as above. Suppose now that M is active at η and $P \in_\eta^* M$. Note that $\mathcal{M}_q^\eta$ is obtained by closing $\mathcal{M}_p^\eta \cup \mathcal{M}^\eta$ under meets that are active at η. Suppose P is below $M \restriction \eta$ on $\mathcal{M}_p^\eta$. By the assumption, $P \notin_\eta M$, hence by Lemma 3.17, there must be a Magidor model $N \in (\mathcal{M}_p \restriction M)^\eta$ such that P is in the interval $[N \wedge M \restriction \eta, N)_p^\eta$. By Proposition 2.24, we have that $P \in_\eta N$ and thus $d_q(P) \in N$. Note that P also belongs to the interval $[N \wedge M \restriction \eta, N)_q^\eta$. Suppose $Q \in \mathcal{M}_q$ and $P \in_\eta Q$. By replacing Q with $Q \restriction \eta$, we may assume that $Q \in \mathcal{M}^\eta$. Note that Q cannot be a countable model since then we would have $P \in_\eta M$. If Q is a Magidor model in $\mathcal{M}^\eta$ then Q cannot be below N since then it would be below $N \wedge M \restriction \eta$ on $\mathcal{M}_q^\eta$. Therefore, Q must be either equal to N or above N on the η-chain $\mathcal{M}^\eta$. Then we would have $N \cap V_\kappa \subseteq Q \cap V_\kappa$, and hence $d_p(P) \in Q$. If $Q \in \mathcal{M}_p$ then Q is strongly active at η and $d_p(P) \in Q$, since p is a condition. It remains to consider the case when Q is of the form $R \wedge S$, for some Magidor model $R \in \mathcal{M}^\eta$ and countable $S \in \mathcal{M}_p^\eta \setminus \mathcal{M}^\eta$. Now, we must have $R = N$ or $N \in_\eta R$ since otherwise R, and hence also $R \wedge S$, would be below $N \wedge M \restriction \eta$. Since $d_p(P) \in N$, we must have $d_p(P) \in R$. Moreover, since $S \in \mathcal{M}_p^\eta$, and d_p is the decoration of p, S must be strongly active at η and $d_p(P) \in S$. By Proposition 2.33, $R \wedge S$ is strongly active at η. By Proposition 2.32, we have $R \wedge S \cap V_\eta = R \cap S \cap V_\eta$, and hence $d_p(P) \in R \wedge S = Q$. $\boxed{3.25}$

Lemma 3.26. *Suppose $p \in \mathbb{P}_\lambda^\kappa$ and $M \in \mathcal{L}(M_p)$ is a countable β-model, for some $\beta \in E$. Let $\alpha^* \in M$ be such that $p \restriction M \in (\mathbb{P}_{\alpha^*}^\kappa)^M$. The for any $q \in (\mathbb{P}_{\alpha^*}^\kappa)^M$ with $q \leq p \restriction M$, p and $q \restriction \beta$ are compatible, and the meet $p \wedge q \restriction \beta$ exists.*

Proof. Let $\mathcal{M}_r$ be the closure of $\mathcal{M}_p \cup \mathcal{M}_{q \restriction \beta}$ under meets. By Lemma 3.20 we already know that $\mathcal{M}_r^\delta$ is a δ-chain, for all $\delta \in E$. Hence $\mathcal{M}_r \in \mathbb{M}_\lambda^\kappa$. It remains to define the decoration d_r, and check that it satisfies $(*)$ from Definition 3.5. Let

$$D_p = \{P \in \mathrm{dom}(d_p) : P \notin_{\eta(P)} M\}.$$

Now, suppose $P \in \mathrm{dom}(d_q)$. Let $\delta(P)$ be the largest ordinal $\gamma \in E \cap (\eta(P)+1)$ such that M is strongly active at γ. Let

$$D_q = \{P \restriction \delta(P) : P \in \mathrm{dom}(d_q)\}.$$

Note that, for every $P \in \mathrm{dom}(d_q)$, we have $(P \restriction \delta(P)) \restriction M = P$, and P is active at $\delta(P)$. Observe that D_p and D_q are disjoint. Let $\mathrm{dom}(d_r) = D_p \cup D_q$ and define d_r by:

$$d_r(P) = \begin{cases} d_p(P) & \text{if } P \in D_p \\ d_q(P \restriction M) & \text{if } P \in D_q \text{ and } \eta(P) \leq \beta \end{cases}$$

We have to check that every $P \in \mathrm{dom}(d_r)$ is $\mathcal{M}_r$-free and condition $(*)$ holds. By Lemma 3.25 we have that $(\mathcal{M}_r, d_p \restriction D_p)$ is already a condition, so we may assume $P \in D_q$. Fix one such $P \in D_q$, and let $\eta = \eta(P)$. Note that it suffices to show that the least model, say R, on the η-chain $\mathcal{M}_r^\eta$ above P is strongly active at η, and $d_r(P) \in R$. By Lemma 3.17 either $R \in_\eta M$ or $R = N \wedge M \restriction \eta$, for some Magidor model $N \in (\mathcal{M}_p \restriction M)^\eta$. Now, if R is of the form $N \wedge M$, then, since N and M are strongly active at η, by Proposition 2.33, so is R. Moreover, $N \restriction M \in \mathcal{L}(\mathcal{M}_q)$ and $d_q(P \restriction M) \in M \cap N$. It follows that $d_r(P) \in R$. Suppose now that $R \in_\eta M$. Let $\rho = \min(E \cap M \setminus \eta)$. Then $P \restriction M$ and $R \restriction M$ are ρ-models, $R \restriction M \in \mathcal{L}(\mathcal{M}_q)$, and $P \restriction M \in_\rho R \restriction M$. Therefore, $R \restriction M$ is strongly active at ρ, and $d_q(P \restriction M) \in R \restriction M$. Since $(R \restriction M) \cap V_\kappa = R \cap V_\kappa$, we get that $d_q(P \restriction M) \in R$, and hence $d_r(P) \in R$. Moreover, since $R \restriction M$ is strongly active at ρ, it follows that R is strongly active at η. This shows that all the models in $\mathrm{dom}(d_r)$ are $\mathcal{M}_r$-free and condition $(*)$ holds for r. The fact that $r \leq p, q \restriction \beta$ and is in fact the weakest such condition follows from the definition. $\boxed{3.26}$

Remark 3.27. Suppose $p \in \mathbb{P}^\kappa_\lambda$ and $M \in \mathcal{M}_p$ is a countable β-model, for some $\beta \in E$. If either M is standard or $\beta \in M$ we have that $p \restriction M \in \mathbb{P}^\kappa_\lambda$. In particular, Lemma 3.26 shows that if $p \in \mathbb{P}^\kappa_\lambda$ then p and $p \restriction M$ are compatible. Now, we have already observed that, if $q \in \mathbb{P}^\kappa_\lambda$ and $q \leq p$, then $q \restriction M \leq p \restriction M$. Therefore, even though it may not be the case that $p \leq p \restriction M$, every p forces $p \restriction M$ to belong to the generic filter, and hence p is stronger than $p \restriction M$.

Now, by Lemma 3.11 and Lemma 3.26 we immediately get the following.

Theorem 3.28. *$\mathbb{P}^\kappa_\lambda$ is $\mathscr{C}_{\mathrm{st}}$-strongly proper.*

Proof. Suppose $M \in \mathscr{C}$ and $p \in M \cap \mathbb{P}^\kappa_\lambda$. Let p^M be the condition defined in Lemma 3.11. If $q \leq p^M$ then $M \in \mathcal{L}(\mathcal{M}_q)$ and $q \restriction M \in M$, and by Remark 3.27, $q \in \mathbb{P}^\kappa_\lambda$. Then, by Lemma 3.26 any extension r of $q \restriction M$ with $r \in M$ is compatible with q, and moreover $q \wedge r$ exists. Thus, p^M is a $(M, \mathbb{P}^\kappa_\lambda)$-strongly generic condition extending p. $\boxed{3.28}$

Remark 3.29. A similar proof shows that the forcing $\mathbb{P}^\kappa_\alpha$ is strongly proper for the collection of all $M \in \mathscr{C}$ such that $\alpha \in M$.

Notation 3.30. *Let F be a filter in $\mathbb{P}^\kappa_\lambda$. Then we let $\mathcal{M}_F$ denotes $\bigcup\{\mathcal{M}_p : p \in F\}$.*

Let G be a $\mathbb{P}^\kappa_\lambda$-generic filter over V. We let $G_\alpha = G \cap \mathbb{P}^\kappa_\alpha$, for all $\alpha \in E$. The following is straightforward.

Proposition 3.31. *Let G be V-generic over $\mathbb{P}^\kappa_\lambda$. Then $\mathcal{M}^\delta_G$ is a δ-chain, for all $\delta \in E$.*

$\boxed{3.31}$

Theorem 3.32. *Assume κ is supercompact. Then $\mathbb{P}^\kappa_\lambda$ preserves ω_1 and κ, and collapses all cardinals between ω_1 and κ to ω_1.*

Proof. By Proposition 2.19, $\mathscr{C}_{\mathrm{st}}$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$, and by Lemma 3.26, $\mathbb{P}^\kappa_\lambda$ is $\mathscr{C}_{\mathrm{st}}$-strongly proper. Hence ω_1 is preserved. By Proposition 2.23, $\mathscr{U}$ is stationary in $\mathcal{P}_\kappa(V_\lambda)$, and by Theorem 3.15, $\mathbb{P}^\kappa_\lambda$ is $\mathscr{U}$-strongly proper. Hence κ is preserved. Now, fix a cardinal $\mu < \kappa$. Let G be a $\mathbb{P}^\kappa_\lambda$-generic filter over V. Fix $\alpha \in E$ of cofinality less than κ. A standard density argument shows that there exists a Magidor model $N \in \mathcal{M}^\alpha_G$ such that $\mu \in N$. By Proposition 3.31 $\mathcal{M}^\alpha_G$ is an $\in_\alpha$-chain. Let N^* be the least Magidor model above N in $\mathcal{M}^\alpha_G$, and let $I = (N, N^*)^\alpha_{\mathcal{M}_G}$. Note that every model in I is countable and $\in_\alpha$ is transitive on I. Hence if $P, Q \in I$ and $P \in_\alpha Q$ then $P \cap V_\alpha \subseteq Q \cap V_\alpha$. Another standard density argument shows that, for every $x \in N \cap V_\alpha$, there is $P \in I$ such that $x \in P$. Thus, $\{P \cap V_\alpha : P \in I\}$ is an increasing chain of countable sets whose union covers $N \cap V_\alpha$. It follows that $N \cap V_\alpha$ is of cardinality at most ω_1. Since μ belongs to the transitive part of N, we also get that $|\mu| \leq \omega_1$. $\boxed{3.32}$

Theorem 3.33. *$\mathbb{P}^\kappa_\lambda$ collapses cardinals of the interval between κ and λ to κ.*

Proof. Let $\alpha \in E$ be of cofinality less than κ, and let G be a V-generic filter over $\mathbb{P}^\kappa_\lambda$.. Let U_α be the set of Magidor models in $\mathcal{M}^\alpha_G$. By Proposition 2.24, we have that $\in_\alpha$ is transitive on U_α. Note that if $P, Q \in U_\alpha$ then $P \cap V_\alpha \subseteq Q \cap V_\alpha$. Now, a standard density argument using the stationarity of $\mathscr{U}$ shows that, for every $x \in V_\alpha$, there is $P \in U_\alpha$ such that $x \in P$. It follows that $\{P \cap V_\alpha : P \in U_\alpha\}$ is an increasing family of sets of size $< \kappa$ whose union is V_α. Therefore, V_α has cardinality $\leq \kappa$ in $V[G]$. $\boxed{3.33}$

Theorem 3.34. *Suppose λ is an inaccessible cardinal. Then $\mathbb{P}^\kappa_\lambda$ is λ-c.c.*

Proof. For each $p \in \mathbb{P}^\kappa_\lambda$, let $a(p) = \bigcup\{a(M) : M \in \mathcal{M}_p\}$. Note that $a(p)$ is a closed subset of E of size $< \kappa$, for all p. Suppose A is a subset of P^κ_λ of cardinality λ. Since λ is inaccessible, by a standard Δ-system argument, we can find a subset B of A of size λ and a subset a of E such that $a(p) \cap a(q) = a$, for all distinct $p, q \in B$. Note that a is closed, and if we let $\gamma = \max(a)$ then $\gamma \in E$. Since B has size λ, by a simple counting argument, we may assume there is $\mathcal{M} \in \mathbb{M}^\kappa_\gamma$ such that $\mathcal{M}_p \restriction \gamma = \mathcal{M}$, for all $p \in B$. Now, pick distinct $p, q \in B$, and define $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$ and $d_r = d_p \cup d_q$. Let $r = (\mathcal{M}_r, d_r)$. It is straightforward to check that $r \in \mathbb{P}^\kappa_\lambda$ and $r \leq p, q$. $\boxed{3.34}$

Definition 3.35. *Suppose G is V-generic over $\mathbb{P}^\kappa_\lambda$ and $\alpha \in E$ is of cofinality less than κ. Let $C_\alpha(G) = \{\kappa_M : M \in \mathcal{M}^\alpha_G\}$.*

Lemma 3.36. *Let G be a V-generic filter over $\mathbb{P}^\kappa_\lambda$. Then $C_\alpha(G)$ is a club in κ, for all $\alpha \in E$ of cofinality $< \kappa$. Moreover, if $\alpha < \beta$ then $C_\beta(G) \setminus C_\alpha(G)$ is bounded in κ.*

Proof. Let us check the second statement first. Fix $\alpha, \beta \in E$ such that $\mathrm{cof}(\alpha), \mathrm{cof}(\beta) < \kappa$, and $\alpha < \beta$. By a standard density argument using the stationarity of $\mathscr{U}$ there is $p \in G$ and a Magidor model $M \in \mathcal{M}_G$ which is active at both α and β. Therefore, any model N above $M \restriction \beta$ on the β-chain $\mathcal{M}^\beta_G$ is also active at α. It follows that $C_\beta(G) \setminus C_\alpha(G) \subsetneq \kappa_M$.

We work in V and prove the first statement by induction on α. Let $\dot{\mathcal{M}}^\alpha$ and $\dot{C}_\alpha$ be canonical $\mathbb{P}^\kappa_\lambda$-names for $\mathcal{M}^\alpha_G$ and $C_\alpha(G)$, for $\alpha \in E$. Now, fix $\alpha \in E$ of cofinality less than κ and suppose the statement has been proved for all $\bar{\alpha} \in E \cap \alpha$ of cofinality $< \kappa$. Suppose $\gamma < \kappa$ and $p \in \mathbb{P}^\kappa_\lambda$ forces that γ is a limit point but not a member of $\dot{C}_\alpha$ We may assume that there is a model $M \in \mathcal{M}^\alpha_p$ such that p forces that M is the least model on the α-chain $\dot{\mathcal{M}}^\alpha$ such that $\gamma \leq \kappa_M$. Then we must have $\gamma < \kappa_M$. Let P be the previous model on $\mathcal{M}^\alpha_p$ before M. We may assume that such a model exists since p forces that γ is a limit point of $\dot{C}_\alpha$. Note that $\kappa_P < \gamma$ since p forces that $\gamma \notin \dot{C}_\alpha$.

Case 1. Suppose M is strongly active at α. Since P is $\mathcal{M}_p$-free and we may assume that $P \in \mathrm{dom}(d_p)$, by defining $d_p(P) = \emptyset$ if necessary. Since $\gamma < \kappa_M$, we can find $\delta \in M$ such that $\gamma \leq \delta < \kappa_M$. Define a condition q as follows. Let $\mathcal{M}_q = \mathcal{M}_p$, and let $\mathrm{dom}(d_q) = \mathrm{dom}(d_p)$. Let $d_q(P) = d_p(P) \cup \{\delta\}$, and $d_q(Q) = d_p(Q)$, for any other $Q \in \mathrm{dom}(d_p)$. Let $q = (\mathcal{M}_q, d_q)$. Then q is a condition and forces that the next model

of $\dot{\mathcal{M}}^\alpha$ above P contains δ. Hence, it forces that there is no element of $\dot{C}_\alpha$ between κ_P and γ, and so it forces that γ is not a limit point of $\dot{C}_\alpha$, a contradiction.
Case 2. Suppose now that M is not strongly active at α. Then M is countable. Let $A = \mathrm{Hull}(M, V_\alpha)$, let α^* be the least ordinal of M above α, and let $\bar{\alpha} = \sup(M \cap \alpha)$. Note that $\alpha^* \in E_A$, $\bar{\alpha}$ is a limit point of E of cofinality ω, and that P is also active at $\bar{\alpha}$. Now, by the proof of the second part of the lemma, p forces that $\dot{C}_\alpha \setminus \dot{C}_{\bar{\alpha}} \subseteq \kappa_P$, and so it also forces that γ is a limit point of $\dot{C}_{\bar{\alpha}}$. By the inductive assumption $\dot{C}_{\bar{\alpha}}$ is forced to be a club, so there is $q \leq p$ and some $N \in \mathcal{M}_q^{\bar{\alpha}}$ such that $\kappa_N = \gamma$. Now, for each $Q \in (\mathcal{M}_q \restriction M)^{\bar{\alpha}}$, we can find a unique model $Q^* \in M$ with $Q^* \in \mathscr{V}_{\alpha^*}^A$ such that $Q^* \restriction \bar{\alpha} = Q$. Let $\mathcal{M}^* = \{Q^* : Q \in (\mathcal{M}_p \restriction M)^{\bar{\alpha}}\}$. Working in A, $\mathcal{M}^*$ is an α^*-chain closed under meets that are active at α^*. Let $\mathcal{M} = \{Q^* \restriction \alpha : Q^* \in \mathcal{M}^*\}$. Then $\mathcal{M} \in_\alpha M$, is an α-chain closed under meets that are active at α, and $(\mathcal{M}_q \restriction M)^\alpha \subseteq \mathcal{M}$. We now define a condition r. Let $\mathcal{M}_r$ be the closure of $\mathcal{M}_p$ and $\mathcal{M}$ under meets. By applying Lemma 3.20, for all levels $\delta \in E \cap (\bar{\alpha}, \alpha]$, we have that $\mathcal{M}_r \in \mathbb{M}_\lambda^\kappa$. Let $d_r = d_q$ and $r = (\mathcal{M}_r, d_r)$. Observe that $\mathcal{M}_r^\eta = \mathcal{M}_q^\eta$, for all $\eta \in E \setminus (\bar{\alpha}, \alpha]$. Also, if $R \in \mathrm{dom}(d_q)$ and $\eta(R) \in (\bar{\alpha}, \alpha]$ then $R \notin_{\eta(R)} M$, since M is not strongly active at $\eta(R)$. By Lemma 3.25, we conclude that r is a condition. Also, we have that $r \leq q$. Recall that $N \in \mathcal{M}_q^{\bar{\alpha}}$ and $\kappa_N = \gamma$. Let Q be the model on the $\bar{\alpha}$-chain $\mathcal{M}_r^{\bar{\alpha}}$ immediately before $M \restriction \bar{\alpha}$. Then $Q^* \in \mathcal{M}^*$, and hence $Q^* \restriction \alpha \in \mathcal{M}_r$. Let $R = Q^* \restriction \alpha$. In other words, we lifted the model Q to level α and called this model R. Note that $\kappa_R = \kappa_Q$. Then r forces that $R \in \dot{\mathcal{M}}_\alpha$ and $\gamma \leq \kappa_R < \kappa_M$, which contradicts the fact that p forces that $\gamma \notin \dot{C}_\alpha$ and M is the least model on $\dot{\mathcal{M}}_\alpha$ with $\gamma \leq \kappa_M$. This completes the proof of the lemma. 3.36

4. Guessing Models in $V[G]$

We assume κ is supercompact and λ is inaccessible and analyze ω_1-guessing models in the the generic extension by $\mathbb{P}_\lambda^\kappa$. Suppose $\alpha \in E$. We have already established in Corollary 3.10 that $\mathbb{P}_\alpha^\kappa$ is a complete suborder of $\mathbb{P}_\lambda^\kappa$. Let us fix a V-generic filter G_α over $\mathbb{P}_\alpha^\kappa$, and let $\mathbb{P}_\lambda^\kappa/G_\alpha$ denote the quotient forcing. Recall that $\mathbb{P}_\lambda^\kappa/G_\alpha$ consists of all $p \in \mathbb{P}_\lambda^\kappa$ such that $p \restriction \alpha \in G_\alpha$, with the induced ordering. Forcing with this poset over $V[G_\alpha]$ produces a V-generic filter G for $\mathbb{P}_\lambda^\kappa$ such that $G \cap \mathbb{P}_\alpha^\kappa = G_\alpha$. We first show that the pair $(V[G_\alpha], V)$ has the ω_1-approximation property. We will need the following definition.

Definition 4.1. *Let $\mathscr{C}_{\mathrm{st}}^\alpha$ denote the set of all $M \in \mathscr{C}_{\mathrm{st}}$ such that $\eta(M) > \alpha$, $\alpha \in M$, and $M \restriction \max(a(M) \cap (\alpha+1)) \in \mathcal{M}_{G_\alpha}$.*

Lemma 4.2. *$\mathscr{C}_{\mathrm{st}}^\alpha$ is stationary subset of $\mathcal{P}_{\omega_1}(V_\lambda)$ in the model $V[G_\alpha]$.*

Proof. We work in V. Let $\dot{\mathscr{C}}_{\mathrm{st}}^\alpha$ and $\dot{M}^\alpha$ be the canonical $\mathbb{P}_\alpha^\kappa$-names for $\mathscr{C}_{\mathrm{st}}^\alpha$ and $\mathcal{M}_{G_\alpha}$. Suppose $p \in \mathbb{P}_\alpha^\kappa$ forces that and $\dot{F} : [V_\lambda]^{<\omega} \to V_\lambda$ is a function. Let θ be a sufficiently large regular cardinal. By Proposition 2.19, $\mathscr{C}_{\mathrm{st}}$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$, hence we can find a countable $M^* \prec H(\theta)$ containing all the relevant objects. Let $M = M^* \cap V_\lambda$ and note that $M \in \mathscr{C}_{\mathrm{st}}$. Let $\bar{\alpha} = \max(a(M) \cap (\alpha+1))$ and $M' = M \restriction \bar{\alpha}$. Note that $p \in M'$, so we can form the condition $p^{M'}$. Then $p^{M'}$ is $(M', \mathbb{P}_\alpha^\kappa)$-strongly generi and $p^{M'} \leq p$. Let σ be the α-isomorphism between M and M'. Note that $\sigma(q) = q$, for all $q \in M \cap \mathbb{P}_\alpha^\kappa$. Hence, $M \cap \mathbb{P}_\alpha^\kappa = M' \cap \mathbb{P}_\alpha^\kappa$. Therefore, $p^{M'}$ is also $(M, \mathbb{P}_\alpha^\kappa)$-strongly generic, and thus it is $(M^*, \mathbb{P}^\kappa_\alpha)$-generic. Since $\dot{F} \in M^*$, it follows that $p^{M'}$ forces that M is closed under $\dot{F}$. It also forces that M' belongs to $\dot{\mathcal{M}}_{G_\alpha}$, hence it forces that M belongs to $\dot{\mathscr{C}}^\alpha_{\rm st}$. $\boxed{4.2}$

Lemma 4.3. *Suppose $\alpha \in E$ and let G_α be V-generic over $\mathbb{P}^\kappa_\alpha$. Then $\mathbb{P}^\kappa_\lambda/G_\alpha$ is $\mathscr{C}^\alpha_{\rm st}$-strongly proper.*

Proof. Work in $V[G_\alpha]$. Let $p \in \mathbb{P}^\kappa_\lambda/G_\alpha$, and $M \in \mathscr{C}^\alpha_{\rm st}$ be such that $p \in M$. Let p^M be the condition defined in Lemma 3.11. Since $\alpha \in M$ we have $p \restriction \alpha \in M$, and also $p \restriction \alpha \in M \restriction \alpha$. Note that $p^M \restriction \alpha = (p \restriction \alpha)^{M \restriction \alpha}$. Since $p \restriction \alpha \in G_\alpha$ and $M \restriction \alpha \in \mathcal{M}_{G_\alpha}$, we have that $p^M \restriction \alpha \in G_\alpha$, thus $p^M \in \mathbb{P}^\kappa_\lambda/G_\alpha$. Let us show that p^M is $(M, \mathbb{P}^\kappa_\lambda/G_\alpha)$-strongly generic. Suppose $q \leq p^M$ and $q \restriction \alpha \in G_\alpha$. Since $\alpha \in M \restriction \alpha$ we have $(q \restriction M) \restriction \alpha = (q \restriction \alpha) \restriction (M \restriction \alpha)$, and hence $q \restriction M \in M \cap \mathbb{P}^\kappa_\lambda/G_\alpha$. Let $r \leq q \restriction M$ be such that $r \in M \cap \mathbb{P}^\kappa_\lambda/G_\alpha$. By Lemma 3.26, r and q are compatible in $\mathbb{P}^\kappa_\lambda$ and the meet $r \wedge q$ exists. Now, observe that the meet of $r \restriction \alpha$ and $q \restriction \alpha$ exists, and $r \restriction \alpha \wedge q \restriction \alpha = (r \wedge q) \restriction \alpha$. Since $r \restriction \alpha, q \restriction \alpha \in G_\alpha$, we conclude that $r \restriction \alpha \wedge q \restriction \alpha \in \mathbb{P}^\kappa_\lambda/G_\alpha$. It follows that q and r are compatible in $\mathbb{P}^\kappa_\lambda/G_\alpha$. $\boxed{4.3}$

Now, by Lemma 4.2, Lemma 4.3, and Proposition 1.14, we get the following.

Corollary 4.4. *The pair $(V[G_\alpha], V)$ has the ω_1-approximation property.*

$\boxed{4.4}$

Suppose now $N \in \mathscr{U}$. Let $\mathbf{1}^N = (\{N\}, \emptyset)$. By Lemma 3.14, $\mathbf{1}^N$ is $(N, \mathbb{P}^\kappa_\lambda)$-strongly generic. Moreover, for every $q \leq \mathbf{1}^N$ and $r \leq q \restriction N$, q and r are compatible, and the meet $q \wedge r$ exists. Let $\mathbb{P}_N = \mathbb{P}^\kappa_\lambda \cap N$ and let

$$\mathbb{P}^\kappa_\lambda \restriction N = \{q \in \mathbb{P}^\kappa_\lambda : N \in \mathcal{M}_q\}.$$

Then the map $p \mapsto p^N$ is a complete embedding from $\mathbb{P}_N$ to $\mathbb{P}^\kappa_\lambda \restriction N$. Now, fix a V-generic filter G_N over $\mathbb{P}_N$.

Definition 4.5. *Let G_N be a V-generic filter over $\mathbb{P}_N$. Let $\mathscr{C}^N_{\rm st}$ denote the set of all $M \in \mathscr{C}_{\rm st}$ such that $N \in M$ and $N \wedge M \in \mathcal{M}_{G_N}$.*

Lemma 4.6. *The collection $\mathscr{C}^N_{\rm st}$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$ in the model $V[G_N]$.*

Proof. This is very similar to the proof of Lemma 4.2. We work in V. Let $\dot{\mathscr{C}}^N_{\rm st}$ be the canonical $\mathbb{P}_N$-name for $\mathscr{C}^N_{\rm st}$. Suppose $p \in \mathbb{P}_N$ forces that $\dot{F} : [V_\lambda]^{<\omega} \to V_\lambda$ is a function. Let θ be a sufficiently large regular cardinal. By Proposition 2.19, $\mathscr{C}_{\rm st}$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$, hence we can find a countable $M^* \prec H(\theta)$ containing all the relevant objects. Let $M = M^* \cap V_\lambda$, and note that $M \in \mathscr{C}_{\rm st}$. Since $N \in_{\eta(N)} M$, the meet $N \wedge M$ is defined. Let $\eta = \eta(N \wedge M)$ and let σ be the η-isomorphism between $N \cap M$ and $N \wedge M$. Note that $\sigma(q) = q$, for all $q \in \mathbb{P}_N$. Now, $p^{N \wedge M}$ is $(N \wedge M, \mathbb{P}_N)$-strongly generic, hence also $(N \cap M, \mathbb{P}_N)$-strongly generic, and therefore it is $(M^*, \mathbb{P}_N)$-generic. It follows that $p^{N \wedge M}$ forces that $M \in \dot{\mathscr{C}}^N_{\rm st}$ and is closed under $\dot{F}$. $\boxed{4.6}$

Let $\mathbb{Q}_N$ denotes the quotient forcing $(\mathbb{P}^\kappa_\lambda \restriction N)/G_N$.

Lemma 4.7. *$\mathbb{Q}_N$ is $\mathscr{C}^N_{\rm st}$-strongly proper.*

Proof. Work in $V[G_N]$. Let $p \in \mathbb{Q}_N$ and $M \in \mathscr{C}^N_{\rm st}$ be such that $p \in M$. Let p^M be the condition defined in Lemma 3.11. Since $p, N \in M$, we have $p \restriction N \in M$. Thus,

$p \restriction N \in N \cap M$. Observe that $p^M \restriction N = (p \restriction N)^{N \wedge M}$. Since $p \restriction N \in G_N$ and $N \wedge M \in \mathcal{M}_{G_N}$, we have that $p^M \restriction N \in G_N$, thus $p^M \in \mathbb{Q}_N$. Let us show that p^M is $(M, \mathbb{Q}_N)$-strongly generic. Suppose $q \leq p^M$ and $q \in \mathbb{Q}_N$. Observe that $(q \restriction M) \restriction N = (q \restriction N) \restriction (N \wedge M)$, and hence $q \restriction M \in \mathbb{Q}_N$. Let $r \leq q \restriction M$ be such that $r \in M \cap \mathbb{Q}_N$. By Lemma 3.26, r and q are compatible in $\mathbb{P}^\kappa_\lambda$ and the meet $r \wedge q$ exists. Note that $r \restriction N \in N \cap M \subseteq N \wedge M$, and $r \restriction N$ extends $(q \restriction N) \restriction (N \wedge M)$. Hence, again by Lemma 3.26, the meet of $r \restriction N$ and $q \restriction N$ exists, and $r \restriction N \wedge q \restriction N = (r \wedge q) \restriction N$. Since $r \restriction N, q \restriction N \in G_N$, we have that $r \restriction N \wedge q \restriction N \in G_N$. It follows that r and q are compatible in $\mathbb{Q}_N$. $\boxed{4.7}$

Suppose G is a V-generic filter over $\mathbb{P}^\kappa_\lambda$. For $\alpha \in E$, let $G_\alpha = G \cap \mathbb{P}^\kappa_\alpha$.

Lemma 4.8. *Let $\alpha \in E$. Suppose $N \in \mathcal{M}_G$ is a Magidor model. Then $N[G_\alpha]$ is an ω_1-guessing model in $V[G]$.*

Proof. Let $\overline{N}$ be the transitive collapse of N, and let π be the collapsing map. For convenience, let us write $\bar\kappa$ for κ_N. Then $\overline{N} = V_{\bar\gamma}$, for some $\bar\gamma$ with $\mathrm{cof}(\bar\gamma) \geq \bar\kappa$ and $\pi(\kappa) = \bar\kappa$. Let $\bar\alpha = \pi(\alpha)$. Note that $\pi(\mathbb{P}_N) = \mathbb{P}^{\bar\kappa}_{\bar\alpha}$. Let $G_N = G_\alpha \cap N$ and $G^{\bar\kappa}_{\bar\alpha} = \pi[G_N]$. Then the transitive collapse $\overline{N[G_N]}$ of $N[G_N]$ is equal to $V_{\bar\gamma}[G^{\bar\kappa}_{\bar\alpha}] = V_{\bar\gamma}^{V[G_N]}$. Hence $N[G_N]$ is an ω_1-guessing model in $V[G_N]$. On the other hand, by Lemma 4.7, the quotient forcing $\mathbb{Q}_N$ is $\mathscr{C}^N_{\mathrm{st}}$-strongly proper, and $\mathscr{C}^N_{\mathrm{st}}$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$. It follows by Proposition 1.14 that the pair $(V[G_N], V[G])$ has the ω_1-approximation property. Thus, $N[G_N]$ remains an ω_1-guessing model in $V[G]$. $\boxed{4.8}$

A similar argument shows the following.

Lemma 4.9. *Suppose $\mu > \lambda$ and $N \prec V_\mu$ is a κ-Magidor model containing all the relevant parameters. Then $N[G]$ is an ω_1-guessing model in $V[G]$.*

$\boxed{4.9}$

Corollary 4.10. *The principle* $\mathrm{ISP}(\omega_2)$ *holds in $V[G]$.*

$\boxed{4.10}$

Theorem 4.11. *The principle* $\mathrm{FS}(\omega_2)$ *holds in $V[G]$.*

Proof. Fix $X \in H(\omega_3)^{V[G]}$. We have to find a collection $\mathcal{G}$ of ω_1-guessing models containing X such that $\{M \cap \omega_2 : M \in \mathcal{G}\}$ is an ω_1-closed unbounded subset of ω_2. Back in V we can find $\alpha \in E$, and a canonical $\mathbb{P}^\kappa_\lambda$-name $\dot X$, such that $\dot X[G_\alpha] = X$. Fix some $\beta \in E \setminus (\alpha+1)$ with $\mathrm{cof}(\beta) < \kappa$. By a standard density argument, we can find a Magidor model $M \in \mathcal{M}_{G_\beta}$ such that $\alpha, \dot X \in M$. Suppose $P \in \mathcal{M}_{G_\beta}$ is also a Magidor model and $M \in_\beta P$. Notice that $M \cap V_\beta \subseteq P \cap V_\beta$, so $\dot X \in P$, and hence $X \in P[G_\alpha]$. By Lemma 4.8, $P[G_\alpha]$ is an ω_1-guessing model, for all such P. Now, by Lemma 3.36, the set $C_\beta(G)$ is club in ω_2, and hence the family $\mathcal{G} = \{P \in \mathcal{M}_{G_\beta} \cap \mathscr{U} : M \in_\beta P\}$ is as required. $\boxed{4.11}$

Finally, we observe that if λ is also supercompact, then $\mathrm{ISP}(\omega_3)$ holds in $V[G]$ as well. In fact, we show that for all $\mu > \lambda$ the set of ω_1-guessing models is stationary in $\mathcal{P}_{\omega_3}(V_\mu[G])$.

Lemma 4.12. *Suppose $\mu > \lambda$ and $N \prec V_\mu$ is a λ-Magidor model containing all the relevant parameters. Then $N[G]$ is an ω_1-guessing model.*

Proof. Suppose the transitive collapse $\overline{N}$ equals $V_{\bar{\gamma}}$. Let $\bar{\lambda} = N \cap \lambda$. Note that $\mathrm{cof}(\bar{\lambda}) \geq \kappa$, and hence $N[G] = N[G_{\bar{\lambda}}]$. Observe that the transitive collapse $\overline{N[G]}$ equals $V_{\bar{\gamma}}[G_{\bar{\lambda}}]$, and is therefore an ω_1-guessing model in $V[G_{\bar{\lambda}}]$. On the other hand, by Corollary 4.4, the pair $(V[G_{\bar{\lambda}}], V)$ has the ω_1-approximation property. Therefore, $N[G]$ remains an ω_1-guessing model in $V[G]$. 4.12

Corollary 4.13. *Suppose λ is also supercompact. Then* $\mathrm{ISP}(\omega_3)$ *holds in* $V[G]$.

4.13

Corollary 4.14. *Suppose κ and λ are supercompact cardinals. Let G be V-generic over $\mathbb{P}^{\kappa}_{\lambda}$. Then in $V[G]$ we have* $\mathrm{ISP}(\omega_2)$, $\mathrm{ISP}(\omega_3)$, *and* $I[\omega_2] \restriction S^{\omega_1}_{\omega_2} = \mathrm{NS} \restriction S^{\omega_1}_{\omega_2}$.

4.14

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Institut de Mathématiques de Jussieu - Paris Rive Gauche, Université Paris Diderot, 8 Place Aurélie Nemours, 75205 Paris Cedex 13, France

E-mail address: rahmanmohammadpour@gmail.com

URL: https://sites.google.com/site/rahmanmohammadpour

E-mail address: boban@math.univ-paris-diderot.fr

URL: http://www.logique.jussieu.fr/~boban