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The Basis Problem For CCC Posets

Boban Velickovic

Abstract. Given a family Σ of forcing notions a subfamily Σ_0 of Σ is called a *basis* provided for every $\mathcal{P} \in \Sigma$ there is $\mathcal{Q} \in \Sigma_0$ such that forcing with $\mathcal{P}$ adds a generic for $\mathcal{Q}$. We investigate the problem of finding a small basis for the class of nonatomic ccc partial orderings. Prikry conjectured that it is consistent that $\{\mathcal{C}, \mathcal{R}\}$ forms such a basis, where $\mathcal{C}$ is Cohen forcing and $\mathcal{R}$ is random real forcing. We survey what is known about this problem and present some new results. Finally we list some open questions.

Introduction

Given two partially ordered sets $\mathcal{P}$ and $\mathcal{Q}$ we write $\mathcal{P} \leq \mathcal{Q}$ iff forcing with $\mathcal{Q}$ introduces a $\mathcal{P}$-generic over V. This is equivalent to saying that there is a function $f : \mathcal{P} \to RO(\mathcal{Q})$ such that:

(a) if $p \leq q$ then $f(p) \leq f(q)$
(b) if $A \subseteq \mathcal{P}$ is a maximal antichain then $f"[A]$ is a maximal antichain in $RO(\mathcal{Q})$.

To see this note that if $\dot{H}$ is a $\mathcal{Q}$-name for a $\mathcal{P}$-generic filter we can define f by letting $f(p) = \|p \in \dot{H}\|$. Conversely given f, in the forcing extension by $\mathcal{Q}$ we can define a $\mathcal{P}$-generic filter H by letting

$$p \in H \text{ iff } f(p) \in G.$$

One can also show that these conditions are equivalent to saying there is $p \in \mathcal{P}$ and an embedding of the complete Boolean algebra $RO(\mathcal{P} \restriction p)$ into $RO(\mathcal{Q})$.

We will say that a poset $\mathcal{Q}$ is *basic* if for any nontrivial poset $\mathcal{P} \leq \mathcal{Q}$ we have $\mathcal{Q} \leq \mathcal{P}$. Posets such as Cohen forcing, Random real forcing, Sacks forcing, etc are basic in this sense. Let Σ be a given class of posets. We say that $\Sigma_0 \subseteq \Sigma$ is a *basis* for Σ if for every $\mathcal{Q} \in \Sigma$ there is $\mathcal{P} \in \Sigma_0$ such that $\mathcal{P} \leq \mathcal{Q}$. The basis problem for the class of ccc posets asks if it is consistent that there be a small (preferably finite) basis for this class of forcing notions. Another version of this problem is the identify a basis for the class of appropriately definable ccc posets. Under ZFC definable is interpreted to mean the class of Souslin posets, i.e. posets $\mathcal{P}$ such that the domain of $\mathcal{P}$ is an analytic set of reals and both the order and the incompatibility relation

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of $\mathcal{P}$ are analytic. Under suitable large cardinal or determinacy assumptions we can replace Souslin by some higher order definability. Clearly, both Cohen forcing $\mathcal{C}$ and random real forcing $\mathcal{R}$ should be members of any such basis. Prikry asked if it is consistent that $\{\mathcal{C}, \mathcal{R}\}$ form a basis for all ccc posets. This is equivalent to saying that every nontrivial ccc posets adds a Cohen or a random real. There is an obvious ZFC analog of this question of Souslin ccc posets. One related problem is a well-known question of von Neumann [**Ma**] who asked if there is an (ω, ω)-weakly distributive complete Boolean algebra which is not a measure algebra. Recall that a poset $\mathcal{P}$ is *weakly distributive* iff every real (i.e. element of ω^ω) in $V^{\mathcal{P}}$ is dominated by a ground model real, i.e. $\mathcal{P}$ is ω^ω-bounding.

The paper is organized as follows. In Section 1 we deal with the Sacks property. Under addditional set theoretic assumptions it is possible to extract ccc posets of perfect trees which have the Sacks property, add a minimal real, etc. On the other hand, some such assumptions are necessary since, for instance, the Open Coloring Axiom implies that no nontrivial ccc forcing notion has the Sacks property. We also present a new proof of a previous result of Shelah saying that there are no such Souslin forcings. Although this follows from the above result we think that the proof is interesting in its own right. In Section 2 we deal with adding Cohen reals and present another result of Shelah saying that every ccc Souslin forcing which adds an unbounded real actually adds a Cohen real. Finally, Section 3 contains some open problems and directions for further research. Our forcing terminology is standard and can be found in [**Jech**].

1. The Sacks Property

Most Souslin forcing notions for adding a real can be represented by a class of subtrees of $2^{<\omega}$ or $\omega^{<\omega}$. Perhaps, the best known of them is Sacks forcing $\mathcal{S}$ which consists of all perfect subtrees of $2^{<\omega}$ ordered under inclusion. Sacks forcing has some strong properties which prevent it from adding either a Cohen or a random real. We recall the following two definitions.

Definition 1.1. A forcing notion $\mathcal{P}$ has the *Laver property* iff for every function $g \in \omega^\omega$, a $\mathcal{P}$-name τ for an element of $\Pi_n g(n)$ and for every condition $p \in \mathcal{P}$ there is a condition $q \leq p$ and a sequence $\langle I_n : n < \omega \rangle$ such that $I_n \in [\omega]^{2^n}$, for all n, and $q \Vdash \tau \in \Pi_n I_n$. We say that $\mathcal{P}$ has the *Sacks property* if the above holds for any $\mathcal{P}$-name τ for an element of ω^ω, i.e. if $\mathcal{P}$ has the Laver property and is ω^ω-bounding at the same time.

Definition 1.2. A real r is *minimal* over a model of set theory V if for every real $s \in V[r]$, $V[s]$ is equal either to V or $V[r]$.

One can show that $\mathcal{S}$ has the Sacks property and the generic real it adds is minimal in the above sense. Either of these properties guarantees that no Cohen or random real is added when forcing with $\mathcal{S}$. However, $\mathcal{S}$ does not have the ccc and it is natural to ask if it is possible to extract a ccc suborder of $\mathcal{S}$ which has one or both of these properties. In the late 1960s Jensen [**Jen**] constructed a ccc suborder of $\mathcal{S}$ in order to 'construct' a nonconstructible Π^1_2-singleton. He used $\diamond$ and a fusion argument. It is possible to isolate abstractly the properties of his construction. Let $\mathcal{P}$ be a poset, CCC($\mathcal{P}$) denotes the following statement.

> For every family $\mathcal{D}$ of $2^{\aleph_0}$ dense open subsets of $\mathcal{P}$ there is a ccc perfect suborder $\mathcal{Q}$ of $\mathcal{P}$ such that $D \cap \mathcal{Q}$ is dense in $\mathcal{Q}$, for every $D \in \mathcal{D}$.

Here *perfect suborder* mean that the incompatibility relation of $\mathcal{Q}$ is the restriction of the incompatibility relation of $\mathcal{P}$. In [**Ve2**] by using forcing instead of $\diamond$ we showed the following.

THEOREM 1.3. *Let $\kappa > \aleph_1$ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Then there is a generic extension in which Martin's Axiom holds, $2^{\aleph_0} = \kappa$, and CCC($\mathcal{S}$) holds.*

PROPOSITION 1.4. *Assume* CCC($\mathcal{S}$). *Then there is a ccc perfect suborder of $\mathcal{S}$ which has the Sacks property and adds a minimal real.*

PROOF. In order to do this we have to define a family of $2^{\aleph_0}$ dense open subsets of $\mathcal{S}$. Suppose $\bar{A} = \langle A_n : n < \omega \rangle$ is a sequence of countable antichains in $\mathcal{S}$. Let $D_{\bar{A}}$ be the set of all $T \in \mathcal{S}$ such that either there is n such that T is incompatible with all members of A_n or for all n there is $X_n \subseteq A_n$ with $|X_n| \leq 2^n$ such that $T \leq \bigvee X_n$. By a standard fusion argument one shows that each $D_{\bar{A}}$ is dense and open in $\mathcal{S}$. Let

$$\mathcal{D} = \{D_{\bar{A}} : \bar{A} \text{ a sequence of countable antichains }\}.$$

A *partial name* for a real is a sequence $\tau = \{\langle A_n^0, A_n^1 \rangle : n < \omega\}$, where $A_n = A_n^0 \cup A_n^1$ is a countable antichain in $\mathcal{S}$, for each n. We say that a perfect tree T is *good* for τ if $T \leq \bigvee A_n$, for each n. Finally, we say that T *interprets* τ if if for every n there is k such that for every $s \in T \cap 2^k$ either $T_s \leq \bigvee A_n^0$ or $T_s \leq \bigvee A_n^1$. Thus if T interprets τ we can define a function $val_\tau : [T] \to 2^\omega$ by

$$val_\tau(b)(n) = 0 \text{ iff there is } k \text{ such that } T_{b\restriction k} \leq \bigvee A_n^0.$$

Now for each partial name for a real τ let E_τ be the set of all $T \in \mathcal{S}$ such that either T is not good for τ or else T interprets τ and the function val_τ is either constant or 1-1 on $[T]$. By a fusion argument one can show that E_τ is a dense and open subset of $\mathcal{S}$. Let

$$\mathcal{E} = \{E_\tau : \tau \text{ a partial name for a real }\}.$$

By applying CCC($\mathcal{S}$) to $\mathcal{D} \cup \mathcal{E}$ we obtain a ccc perfect suborder $\mathcal{Q}$ of $\mathcal{S}$ in which the sets $D_{\bar{A}}$ and E_τ are all dense. One can then show that $\mathcal{Q}$ has the Sacks property and adds a minimal real. □

If one is looking for a counterexampe to Prikry's question it is therefore natural to ask if it is possible to find a ccc partial order which has the Sacks property. Shelah in [**Sh1**] proved that there is no such Souslin partial ordering.

THEOREM 1.5. *Let $\mathcal{P}$ be a Souslin ccc poset. Then $\mathcal{P}$ does not have the Laver property.*

PROOF. We present a descriptive set theoretic proof of this result which may have some interest in its own. For a subset a of ω let e_a denote the increasing enumeration of a. Recall that $<^*$ denotes the order of eventual dominance in ω^ω. We shall need the following.

LEMMA 1.6. *Let $A \subseteq \mathcal{P}(\omega)$ be analytic. Then there is an increasing $f : \omega \to \omega$ such that $e_a <^* f$, for all $a \in A$, or there is a perfect subset P of A consisting of pairwise almost disjoint sets.*

Proof. Given a subtree T of $\omega^{<\omega}$ a node $s \in T$ is called *infinitely splitting* if the set $\{n : s^\frown n \in T\}$ is infinite. Recall that T is called *superperfect* iff for every $s \in T$ there is $t \in T$ extending s which is infinitely splitting. A *superperfect set* is the set of all branches of a superperfect tree. We shall use a result of Kechris and St. Raymond (see [**Ke**] Cor.21.23, page 163) which says that any analytic subset of ω^ω which is unbounded under $<^*$ contains a superperfect subset. We may assume that A consists of infinite sets and hence $e_a \in \omega^\omega$, for all $a \in A$. Since the set $A^* = \{e_a : a \in A\}$ is also analytic we can apply the above result to it and conclude that it contains a superperfect subset, say P. Let T be a superperfect tree such that the set $P = [T]$ is contained in A^*. Note that T consists of finite strictly increasing sequences. Now we build a perfect subset of P consisting of increasing enumerations of pairwise almost disjoint sets. In order to do this we construct an embedding $\varphi : 2^{<\omega} \to T$ such that

(a) $\varphi(s)$ is infinitely splitting, for all $s \in 2^{<\omega}$
(b) if $s \subseteq t$ then $\varphi(s) \subseteq \varphi(t)$
(c) $\mathrm{ran}(\varphi(s)) \cap \mathrm{ran}(\varphi(t)) = \mathrm{ran}(\varphi(s \wedge t))$ for all $s, t \in 2^{<\omega}$.

First enumerate the nodes of $2^{<\omega}$ as $\{s_n : n < \omega\}$ such that the initial segments of any node appear before it in the enumeration. Now we define $\varphi(s_n)$ by induction on n. To begin let $\varphi(s_0)$ be any infinitely splitting node of T. Suppose $\varphi \restriction \{s_0, \ldots s_{n-1}\}$ has been defined satisfying the above conditions. We need to define $\varphi(s_n)$. Let $l < n$ be such that s_l is the immediate predecessor of s_n. Since $\varphi(s_l)$ is an infinitely splitting node of T we can find an integer k bigger than $\sup\{\bigcup_{i<n} \mathrm{ran}(\varphi(s_i))\}$, such that $\varphi(s_l)^\frown k \in T$. Then find an infinitely splitting node t of T extending $\varphi(s_0)^\frown k$ and set $\varphi(s_n) = t$. This completes the inductive construction of φ. Now let $f : 2^\omega \to \omega^\omega$ be defined by $f(b) = \bigcup_n \varphi(b \restriction n)$. Then f is a continuous function and $\mathrm{ran}(f) \subseteq P$. Finally, let

$$Q = \{\mathrm{ran}(f(b)) : b \in 2^\omega\}.$$

Then Q is a perfect subset of A consisting of pairwise almost disjoint infinite sets, as desired. $\square$

Let now $\mathcal{P}$ be a nontrivial ccc Souslin poset. First note that $\mathcal{P}$ must add a real. Let τ be a $\mathcal{P}$-name for a new element of 2^ω. Given any condition $p \in \mathcal{P}$ we define "the tree of possibilities" $T_p[\tau]$ for τ below p by letting:

$$T_p[\tau] = \{s \in 2^{<\omega} : \text{ there is } q \leq p \text{ such that } q \Vdash s \subseteq \tau\}.$$

Since τ is forced to be a new real, $T_p[\tau]$ is a perfect tree. Let also a_p be the set of splitting levels of $T_p[\tau]$, i.e.

$$a_p = \{n : \text{ there is } s \in 2^n \text{ such that } s^\frown 0, s^\frown 1 \in T_p\}.$$

Note that if a_p and a_q are almost disjoint then $p \perp q$. Now, the set $A = \{a_p : p \in \mathcal{P}\}$ is analytic. By Lemma 1 and the fact that $\mathcal{P}$ is ccc the set of increasing enumerations of elements of A must be $<^*$-bounded in ω^ω by some increasing function, say f. Now define a new name σ by letting:

$$\sigma(k) = \tau \restriction f(2^k) + 1.$$

Suppose we could find a sequence $\langle I_k : k < \omega \rangle \in V$, with $|I_k| \leq 2^k$, for all k, and a condition $p \in \mathcal{P}$ such that $p \Vdash \sigma(k) \in I_k$, for all k. Since $e_{a_p} <^* f$ we could find k

such that $|a_p \cap f(2^k)| \geq 2^k$. But this implies that the set $T_p[\tau] \cap 2^{f(2^k)+1}$ has size at least $2^k + 1$ and therefore cannot be contained in I_k. This contradiction shows that $\mathcal{P}$ does not have the Sacks property.

□

Recall that the Open Coloring Axiom (OCA) is the following statement (see for instance [**To**] or [**Ve1**]).

> Let X be a set of reals and
> $$[X]^2 = K_0 \cup K_1$$
> a given coloring with K_0 open in the product topology. Then either X can be written as $X = \bigcup_n X_n$ with $[X_n]^2 \subseteq K_1$, for each n, or else there is an uncountable subset H of X such that $[H]^2 \subseteq K_0$.

THEOREM 1.7. *Assume OCA and Souslin's Hypothesis. Then no nontrivial ccc poset has the Sacks property.*

PROOF. Let $\mathcal{P}$ be a nontrivial ccc poset and assume towards contradiction that $\mathcal{P}$ has the Sacks property. The assumption of Souslin's Hypothesis is there only to ensure that $\mathcal{P}$ adds a new real. Let now τ be a $\mathcal{P}$-name for a new element of 2^ω and define, for every condition $p \in \mathcal{P}$, the tree of possibilities $T_p[\tau]$ for τ below p, as in the previous proof. As before $T_p[\tau]$ is a perfect subtree of $2^{<\omega}$. For each perfect tree T define a function $f_T : \omega \to \omega$ by:

$$f_T(n) = \min\{l : |T \cap 2^l| \geq n\}.$$

CLAIM 1.8. *For every $p \in \mathcal{P}$ the set $\{f_{T_q} : q \leq p\}$ is cofinal in ω^ω under $<^*$.*

PROOF. Suppose for some p $\{f_{T_q} : q \leq p\}$ does not $<^*$-dominate some function, say g. Then define a $\mathcal{P}$-name σ by $\sigma(k) = \tau \restriction g(2^k + 1)$, for all k. Then as in the proof of the previous theorem we obtain that $\mathcal{P} \restriction p$ does not have the Sacks property. Contradiction. □

CLAIM 1.9. *For every countable subset A of 2^ω and for every $p \in \mathcal{P}$ there is $q \leq p$ such that $A \cap [T_q[\tau]] = \emptyset$.*

PROOF. Assume otherwise and fix such an A. Let us assume for simplicity that p is the maximal condition in $\mathcal{P}$. Enumerate A as $\{a_n : n < \omega\}$ and define a $\mathcal{P}$-name σ for a function in ω^ω by

$$\sigma(n) = \min\{l : \tau(l) \neq a_n(l)\}.$$

Since τ is forced not to belong to V this function is well-defined. Now, since $\mathcal{P}$ has the Sacks property it is also ω^ω-bounding and therefore we can find a function $g \in \omega^\omega$ and a condition $q \leq p$ such that $q \Vdash \sigma(n) \leq g(n)$, for all n. It now follows that $A \cap [T_q[\tau]] = \emptyset$, as desired. □

Now let $\mathcal{T} = \{T_p[\tau] : p \in \mathcal{P}\}$. Then $\mathcal{T}$ is a subset of $\mathcal{P}(2^{<\omega})$ which with the usual product topology is homeomorphic to 2^ω. Thus we can identify $\mathcal{T}$ with a set of reals. We now define an open coloring $[\mathcal{T}]^2 = K_0 \cup K_1$ as follows.

$$\{T, R\} \in K_0 \text{ iff there is } l \text{ such that } T \cap R \cap 2^l = \emptyset.$$

It is easily seen that K_0 is open as a subset of $[\mathcal{T}]^2$. We can therefore apply OCA to this coloring. We cannot have an uncountable K_0-homogeneous set since this would yield an uncountable antichain in $\mathcal{P}$. Therefore we may assume that $\mathcal{T} = \bigcup_n \mathcal{T}_n$, where each $\mathcal{T}_n$ is K_1-homogeneous. For each n let

$$\mathcal{F}_n = \{f_T : T \in \mathcal{T}_n\}.$$

Let I be the set of n such that $\mathcal{F}_n$ is unbounded in ω^ω under $<^*$.

Claim 1.10. *Suppose $n \in I$. Then there is a finite subset A_n of 2^ω such that $[T] \cap A_n \neq \emptyset$, for every $T \in \mathcal{T}_n$.*

Proof. To see this fix the least integer l such that the set $\{f_T(l) : T \in \mathcal{T}_n\}$ is unbounded. We can therefore find a sequence $\{T_k\}_k$ of elements of $\mathcal{T}_n$ such that $\{f_{T_k}(l)\}_k$ converges to infinity. We may moreover assume that $\{T_k\}_k$ converges in $\mathcal{P}(2^{<\omega})$ to a tree say T. Then we have that every level of T has at most $l-1$ elements and therefore so does $[T]$. We claim that setting $A_n = [T]$ works. To see this suppose $R \in \mathcal{T}_n$, but $[R] \cap A_n = \emptyset$. Let i be such that $R \cap T \cap 2^i = \emptyset$. Then find k such that $T \cap 2^i = T_k \cap 2^i$. It follows that $\{R, T_k\} \in K_0$, which contradicts the fact that $\mathcal{T}_n$ is K_1-homogeneous. □

Now fix sets A_n, for $n \in I$, as in the above claim and let $A = \bigcup_n A_n$. For $n \in \omega \setminus I$ let g_n be a function in ω^ω which $<^*$-dominates all members of $\mathcal{F}_n$ and let g be a function which $<^*$-dominates $\{g_n : n \in \omega \setminus I\}$. Now using Claims 1.8 and 1.9 we can find a condition $p \in \mathcal{P}$ such that $[T_p[\tau]] \cap A = \emptyset$ and f_{T_p} $<^*$-dominates g. However, then it follows that $T_p[\tau]$ does not belong to any $\mathcal{T}_n$, a contradiction. This completes the proof of Theorem 1.7.

□

Remark OCA does not imply that no ccc poset has the Laver property. To see this note that MA + OCA is consistent (it is for example a consequence of the Proper Forcing Axiom) and thus in the context of OCA one could have a selective ultrafilter $\mathcal{U}$. Now let $\mathcal{P}_\mathcal{U}$ be Prikry forcing relative to $\mathcal{U}$, i.e. elements of $\mathcal{P}_\mathcal{U}$ are pairs $\langle s, A\rangle$, where s is a finite subset of ω and A is a member of $\mathcal{U}$. The ordering is $\langle s, A\rangle \leq \langle t, B\rangle$ iff s end extends t, $A \subseteq B$, and $s \setminus t \subseteq B$. It is well-known that if $\mathcal{U}$ is selective then $\mathcal{P}_\mathcal{U}$ has the Laver property. It is possible to obtain a model of ZFC in which no ccc poset has the Laver property. This is achieved by iterating Mathias forcing $\aleph_2$ times over a model of CH. This was proved independantly by Shelah [**Sh4**] and the author.

2. Adding Cohen reals

In the positive direction of Prikry's conjecture there is a natural dividing line into ω^ω-bounding posets and those which add an unbounded real. Thus it is natural to conjecture that consistently all ccc posets which add an unbounded real add a Cohen real and all ω^ω-bounding ccc posets add a random real. The first part of the conjecture is much more likely. Indeed Shelah [**Sh1**] has shown that all ccc Souslin posets which add an unbounded real add a Cohen real and Shelah and Blaszyk [**BłSh**] have shown that it is consistent that all nontrivial σ-centered posets add a Cohen real (such posets necessarily add an unbounded real). However the full consistency result is still lacking. The second part of the conjecture is much more dubious even for Souslin ccc ω^ω-bounding posets. It is quite difficult to embed a measure algebra into a given ccc poset which does not already explicitly contain it. Moreover, if there is a counterexample to the Control Measure Problem it is likely to yield a ccc Souslin ω^ω-bounding poset which does not add a random real, i.e. a counterexample to Prikry's conjecture. We now present a descriptive set theoretic proof of the above mentioned result of Shelah [**Sh1**]. The referee informed us that a similar proof but using the von Neumann uniformization theorem instead of Solecki's thereom appread in [**Ka**].

THEOREM 2.1. *let $\mathcal{P}$ be a Souslin ccc poset which adds an unbounded real. Then $\mathcal{P}$ adds a Cohen real.*

PROOF. This will be an application of a combinatorical result concerning analytic subsets of $\mathcal{P}(\omega\times\omega)$ which is a version of Lemma 1 for the ideal $\mathcal{I}$ of subsets of $\omega\times\omega$ which have finite vertical sections. For $a\subseteq\omega\times\omega$ we call $a(n)=\{j|(n,j)\in a\}$ the n-th section of a. If $a(0)\subseteq a(1)\subseteq a(2)\subseteq\dots$ we call a *increasing*.

LEMMA 2.2. *Let $A\subseteq\mathcal{P}(\omega\times\omega)$ be analytic whose members are all increasing. Then either there is some increasing $f\in\omega^\omega$ such that for every a in A there is some n such that $e_{a(n)}<^* f$ or there is a perfect set $P\subseteq A$ such that the intersection of any two distinct elements of P is in $\mathcal{I}$.*

PROOF. In order to prove this we shall make use of the following beautiful result of Solecki [**So**] generalizing a previous result of Petruska [**Pe**].

THEOREM 2.3. *Let $A\subseteq 2^\omega$ be analytic, and let $\mathcal{F}$ be a collection of closed sets. If A is not covered by countably many elements of $\mathcal{F}$, then there is a G_δ-set $G\subseteq A$ which has the same property. Moreover no nonempty relatively open subset of G can be covered by countably many members of $\mathcal{F}$.* □

For $f\in\omega^\omega$ increasing let

$$F_{f,n}=\{a\subseteq\omega\times\omega:|a(n)\cap f(k)|\leq k,\text{ for all }k<\omega\}.$$

Note that each $a\in A$ belongs to some $F_{f,n}$. Therefore,

$$\mathcal{F}=\{F_{f,n}:f\in\omega^\omega\text{ increasing and }n\in\omega\}$$

is a family of closed sets which covers A, and Lemma 2.3 can be applied. Again, we have to consider two cases:

CASE 1: A can be covered by a countable subfamily $\mathcal{F}_0$ of $\mathcal{F}$. Enumerate $\mathcal{F}_0$ as $\{F_{f_i,n_i}:i<\omega\}$. Let $f\in\omega^\omega$ be defined by $f(k)=\max\{f_j(k)|j\leq k\}+1$, i.e. f is the diagonal function of the f_i's. Then for each $a\in A$ there is n such that $e_{a(n)}<^* f$, as desired.

CASE 2: A cannot be covered by countably many elements of $\mathcal{F}$. Now let G be a G_δ subset of A as in Lemma 2.3. Suppose $G=\bigcap_n G_n$, where each G_n is open. We have that no nonempty open subset of G can be covered by countably many members of $\mathcal{F}$. In what follows we shall identify $\mathcal{P}(\omega\times\omega)$ and $2^{\omega\times\omega}$ via characteristic functions. For $i\in\omega$ and $t\in 2^{i\times i}$ let $N_t=\{a\in 2^{\omega\times\omega}:t\subseteq a\}$. Let

$$S=\{t\in\bigcup_i 2^{i\times i}:N_t\cap G\neq\emptyset\}.$$

Let us call t (l,k)-*good* iff for every $m \geq k$ there is $a \in N_t \cap G$ such that $a \restriction l \times [k,m) \equiv 0$.

Claim 2.4. *For any $s \in S$ and $l < \omega$ there is $t \in S$ and $k < \omega$ such that $t \supseteq s$ and t is (l,k)-good.*

Proof. Suppose not and fix such s and l. Suppose $\mathrm{dom}(s) = k_0 \times k_0$. By extending s if necessary we may assume $k_0 \geq l$. Since s itself is not (l,k_0)-good there is k_1 such that $a \restriction l \times [k_0,k_1) \not\equiv 0$, for all $a \in N_s \cap G$. Let $\{t_0,\dots,t_n\}$ enumerate all $t \in 2^{k_1 \times k_1}$ such that $t \supseteq s$ and $t \in S$. Since none of the t_i is (l,k_1)-good and $N_s \cap G = \bigcup_{i=0}^{n}(N_{t_i} \cap G)$ we can find $k_2 > k_1$ such that for every $a \in N_s \cap G$ we have $a \restriction l \times [k_1,k_2) \not\equiv 0$. Continuing in this fashion we construct a sequence $\{k_n\}_n$ such that $a \restriction l \times [k_n,k_{n+1}) \not\equiv 0$, for all $a \in N_s \cap G$ and all n. Now using the fact that the members of G are increasing we have that for every $a \in N_s \cap G$ $a(l)$ intersects the interval $[k_n,k_{n+1})$, for every n. Therefore if we define the function $f \in \omega^\omega$ by $f(n) = k_{n+1}$, we have that $N_s \cap G \subseteq F_{f,l}$, contradicting the fact that no nonempty open subset of G is contained in the union of a countable subfamily of $\mathcal{F}$. □

Now we construct an embedding $\varphi : 2^{<\omega} \to S$ and for every $s \in 2^{<\omega}$ an integer k_s such that:

(a) if $s \subseteq t$ then $\varphi(s) \subseteq \varphi(t)$
(b) $\varphi(s)$ is $(|s|,k_s)$-good, for every $s \in 2^{<\omega}$
(c) $N_{\varphi(s)} \subseteq G_{|s|}$, for every $s \in 2^{<\omega}$
(d) for all $s,t \in 2^{<\omega}$ and all $i < \min\{|s|,|t|\}$ if $j = \max\{i+1, |s \wedge t|\}$ then

$$\{l : \varphi(s)(i,l) = \varphi(t)(i,l) = 1\} \subseteq \min\{k_{s\restriction j}, k_{t\restriction j}\}.$$

To begin let $\varphi(s_0)$ be any node of S such that $N_{\varphi(s_0)} \subseteq G_0$. Note that $\varphi(s_0)$ is vacuously $(0,0)$-good, so we can set $k_{s_0} = 0$. To do the inductive step suppose $n > 0$ and $\varphi(s_i)$ and k_{s_i} have been defined for every $i < n$. Now let $l = |s_n|$ and suppose the immediate predecessor of s_n is s_j. By the inductive assumption $\varphi(s_j)$ is $(l-1,k_{s_j})$-good. First choose an integer h such that $\mathrm{dom}(\varphi(s_i) \subseteq h \times h$, for every $i < n$. Then using the fact that $\varphi(s_j)$ is $(l-1,k_{s_j})$-good find $t \in S$ extending $\varphi(s_j)$ such that $t \restriction (l-1) \times [k_{s_j},h) \equiv 0$. By further extending t, if necessary, we may assume that $N_t \subseteq G_l$. Now by applying Claim 2.4 we can find an integer $k > h$ and a node $t^* \in S$ extending t which is (l,k)-good. Finally we set $\varphi(s_n) = t^*$ and $k_{s_n} = k$. This completes the inductive construction of $\varphi(s)$ and the k_s, for $s \in 2^{<\omega}$.

Now define a function $f : 2^\omega \to 2^{\omega\times\omega}$ by $f(b) = \bigcup_n \varphi(b \restriction n)$. Then f is continuous and (c) guarantees that $\mathrm{ran}(f) = P \subseteq G$. Moreover (d) guarantees that the intersection of any two distinct members of P is in $\mathcal{I}$. This completes the proof of Lemma 2.2. □

Let now $\mathcal{P}$ be a Souslin ccc poset and τ a $\mathcal{P}$-name for an unbounded real. For any condition p let us define "the tree of possibilities" $T_p[\tau]$ for τ below p as before, i.e.

$$T_p[\tau] = \{s \in \omega^{<\omega} : \text{ there is } q \leq p \text{ such that } q \Vdash s \subseteq \tau\}.$$

Since τ is forced to be unbounded each $T_p[\tau]$ is a superperfect tree. Fix an enumeration $\{s_n : n < \omega\}$ of $\omega^{<\omega}$. For each $p \in \mathcal{P}$ we define the set $a_p \subseteq \omega \times \omega$ by:

$$a_p = \{(n,k) : \text{ there is } i < n \text{ such that } s_i{}^\frown k \in T_p[\tau]\}.$$

Then clearly we have $a_p(0) \subseteq a_p(1) \subseteq \ldots$. The family $\mathcal{A} = \{a_p : p \in \mathcal{P}\}$ is analytic. Now if $p, q \in \mathcal{P}$ and $a_p \cap a_q \in \mathcal{I}$ then $p \perp q$. To see this note that $T_p[\tau] \cap T_q[\tau]$ is finitely splitting and thus if p and q were compatible any $r \leq p, q$ would force τ to be bounded by a ground model real, a contradiction. Now by Lemma 2.2 and the fact that $\mathcal{P}$ is ccc there is an increasing function $f \in \omega^\omega$ such that for every $p \in \mathcal{P}$ there is n such that $e_{a_p(n)} <^* f$. Define recursively a sequence $\{n_k\}_k$, by $n_0 = f(0)$ and $n_k = f(n_{k-1})$, for all $k > 0$. Let $I_0 = [0, n_0)$ and $I_k = [n_{k-1}, n_k)$, for all $k > 0$. Then $\{I_k\}_k$ is a partition of ω into finite blocks. Moreover we have that for every $p \in \mathcal{P}$ there is $n < \omega$ such that for all but finitely many k we have $a_p(n) \cap I_k \neq \emptyset$. Now fix a bijection $e : 2^{<\omega} \to \omega$ and define a function $\varphi : \omega \to 2^{<\omega}$ by setting $\varphi(i) = s$, for s such that $i \in I_{e(s)}$. Now let σ be a $\mathcal{P}$-name for the real:

$$\varphi(\tau(0))^\frown\varphi(\tau(1))^\frown\varphi(\tau(2))^\frown\ldots$$

Claim 2.5. *σ is Cohen generic over V.*

Proof. Take $p \in \mathcal{P}$ and a dense open subset D of $2^{<\omega}$. We know that there is n such that $a_p(n)$ hits all but finitely many of the blocks I_k. Look at $s_0, \ldots, s_{n-1}$. For $i < n$ define

$$u_i = \varphi(s_i(0)^\frown\ldots^\frown\varphi(s_i(|s_i| - 1)).$$

Now there are infinitely many $t \in 2^{<\omega}$ such that $u_i{}^\frown t \in D$, for each $i < n$. So, since $a_p(n)$ hits all but finitely many blocks there is such t such that $a_p(n) \cap I_{e(t)} \neq \emptyset$. Pick an element $l \in I_{e(t)}$ such that for some $i < n$ $u_i{}^\frown l \in T_p[\tau]$. It follows now that there is $q \leq p$ such that $q \Vdash s_i{}^\frown l \subset \tau$. But then $q \Vdash u_i{}^\frown t \subseteq \sigma$, and therefore q forces that σ meets D. □

This completes the proof of Theorem 2.1. □

We can use some of these ideas to show the following.

Theorem 2.6. *MA(σ-centered) implies that every poset $\mathcal{P}$ of size $< 2^{\aleph_0}$ which adds a new real must also add a Cohen real.*

Proof. We first show that $\mathcal{P}$ adds an unbounded real. Let τ be a name for a new element of 2^ω and define for every condition $p \in \mathcal{P}$ the tree of possibilities $T_p[\tau]$ for τ below p as before and let $F_p = [T_p[\tau]]$. Then F_p is a perfect set for every p. Since $\mathcal{P}$ is of size $< 2^{\aleph_0}$ we can pick $x_p, y_p \in F_p$ such that they are all distinct. Let $A = \{x_p : p \in \mathcal{P}\}$, $B = \{y_p : p \in \mathcal{P}\}$ and $C = A \cup B$. Now by MA(σ-centered) A is a relative F_σ subset of C. Pick closed sets K_n, for $n < \omega$, such that

$$A = (\bigcup_n K_n) \cap C.$$

So each F_p intersects some K_n, but does not contain any of them. Let R_n be the tree of intial segments of K_n. Now define the name σ for an element of ω^ω by:

$$\sigma(n) = \min\{k : \tau \restriction k \notin R_n\}.$$

It is easy to see now that σ is forced to be an unbounded real. It follows that $T_p[\sigma]$ is a superperfect tree, for each $p \in \mathcal{P}$. Now enumerate $\omega^{<\omega}$ as $\{s_n : n < \omega\}$ and define the sets $a_p \subseteq \omega \times \omega$ as before:

$$a_p = \{(n,k) : \text{ there is } i < n \text{ such that } s_i{}^\frown k \in T_p[\sigma]\}.$$

Since there are $< 2^{\aleph_0}$ of these sets by MA(σ-centered) there is an increasing function $f : \omega \to \omega$ such that for every p there is n such that $e_{a_p(n)} <^* f$, in fact this happens for the least n such that $a_p(n)$ is infinite. Now we can repeat the last part of the proof of Theorem 2.1 to obtain a $\mathcal{P}$-name for a Cohen real. □

Much more is known on forcing notions adding Cohen reals. For instance, Shelah and Zapletal [**ShZa**] have shown that under PFA every forcing notion of uniform density $\aleph_1$ adds a generic for $\mathcal{C}_{\aleph_1}$ the usual poset for adding $\aleph_1$ Cohen reals.

Now one may conjecture from Theorem 2.1 that some strong forcing axiom such as PFA would imply that any ccc forcing which adds an unbounded real actually adds a Cohen real. However this is not the case. To see this fix a nonprincipal ultrafilter $\mathcal{U}$ on ω. Consider the following poset $\mathcal{P}_\mathcal{U}$: elements of $\mathcal{P}_\mathcal{U}$ are subtrees T of $\omega^{<\omega}$ such that T has a stem s and for every $t \in T$ extending s the set $\{n : t^\frown n \in T\}$ belongs to $\mathcal{U}$. $\mathcal{P}_\mathcal{U}$ is always σ-centered and adds a dominating real. If $\mathcal{U}$ is a Ramsey ultrafilter or even a P-point then $\mathcal{P}_\mathcal{U}$ does not add a Cohen real. In fact, it is possible to completely characterize ultrafilters $\mathcal{U}$ for which $\mathcal{P}_\mathcal{U}$ adds a Cohen real. The following definition is due to Baumgartner.

Definition 2.7. $\mathcal{U}$ is called a *nowhere dense ultrafilter* iff for every function $f : \omega \to \mathbb{R}$ there is $a \in \mathcal{U}$ such that $f[a]$ is nowhere dense.

For instance, it is easily seen that if $\mathcal{U}$ is a P-point then $\mathcal{U}$ is nowhere dense. In [**BłSh**] Shelah and Błaszczyk showed that $\mathcal{P}_\mathcal{U}$ adds a Cohen real iff $\mathcal{U}$ is not a nowhere dense ultrafilter. They also showed that there is a nontrivial σ-centered poset which does not add a Cohen real iff there is a nowhere dense ultrafilter. On the other hand in [**Sh2**] Shelah showed that it is relatively consistent with ZFC that there are no nowhere dense ultrafilter. Thus, we obtain that it is relatively consistent that every nontrivial σ-centered poset adds a Cohen real. However it is not known if it is consistent that every ccc poset which adds an unbounded real adds a Cohen real.

3. Open Questions

In this section we list some open problems in this area. First looking at the other side of the dividing line we may ask how close are ccc ω^ω-bounding posets to adding a random real. We mention some test questions in this direction.

Question 3.1. *Does every ccc Souslin forcing add a splitting real?*

It is possible to give a descriptive set theoretic translation of this problem. For this we will need the following.

Definition 3.2. Let $\mathcal{A}$ be a collection of perfect subtrees of $2^{<\omega}$. We say that $\mathcal{A}$ has *partition property* if for every $T \in \mathcal{A}$ and $A \in [\omega]^\omega$ and every partition

$$T \restriction A = K_0 \cup K_1$$

there is $B \in [A]^\omega$ and $R \subseteq T$ and $\epsilon \in \{0,1\}$ such that $R \in \mathcal{A}$ and $R \restriction B \in K_\epsilon$.

Then Question 3.1 reduces to the statement that no analytic family $\mathcal{A}$ of perfect trees which is ccc under inclusion has the partition property.

It is well-known that if $\mathcal{R}$ is the random real forcing that $\mathcal{R} \times \mathcal{R}$ adds a Cohen real. Thus we may ask the following.

QUESTION 3.3. *If $\mathcal{P}$ is a nontrivial ccc Souslin forcing does $\mathcal{P} \times \mathcal{P}$ necessarily add a Cohen real?*

In order to answer this question positively it suffices of course to show that $\mathcal{P} \times \mathcal{P}$ adds an unbounded real. Again it is possible to reformulate this as follows. Suppose $\mathcal{A}$ is an analytic ccc family of perfect sets under inclusion. Let

$$\mathcal{A}^{(2)} = \{P \times Q; P, Q \in \mathcal{A}\}.$$

Does there always exist an F_σ subset of $2^\omega \times 2^\omega$ which splits every element of $\mathcal{A}^{(2)}$?

The following question is well-known.

QUESTION 3.4. *Is there a ccc Souslin forcing which adds a minimal real?*

One can ask for a natural basis for various subclasses of ccc forcings. In this regard we mention only the following question from [**Sh3**]

QUESTION 3.5. *Does every ccc Souslin forcing which adds a dominating real add a Hechler real, i.e. is Hechler forcing a basis for the class of ccc Souslin posets which add a dominating real?*

Concerning consistency results we will mention two open problems.

QUESTION 3.6. *Is it relatively consistent with CH that no nontrivial ccc forcing has the Sacks property?*

Groszek [**Gr**] has shown that CH implies that there is a ccc forcing notion which adds a minimal real.

QUESTION 3.7. *Is it relatively consistent that every ccc forcing which adds an unbounded real adds a Cohen real?*

Added in proof. Question 3.6 was recently answered positively by S. Quickert [**Qu**].

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Equipe de Logique UPRESA 7056, Université de Paris 7, 2 Place Jussieu, 75251 Paris, France

E-mail address: boban@logique.jussieu.fr