COLLAPSING FUNCTIONS

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Abstract. We define what it means for a function on ω_1 to be a *collapsing function for* λ and show that if there exists a collapsing function for $(2^{\omega_1})^+$, then there is no precipitous ideal on ω_1. We show that a collapsing function for ω_2 can be added by forcing. We define what it means to be a *weakly* ω_1 *Erdös cardinal* and show that in $L[E]$, there is a collapsing function for λ iff λ is less than the least weakly ω_1 Erdös cardinal. As a corollary to our results and a theorem of Neeman, the existence of a Woodin limit of Woodin cardinals does not imply the existence of precipitous ideals on ω_1.

We also show that the following statements hold in $L[E]$. The least cardinal λ with the Chang property $(\lambda, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ is equal to the least ω_1-Erdös cardinal. In particular, if j is a generic elementary embedding that arises from non-stationary tower forcing up to a Woodin cardinal, then the minimum possible value of $j(\omega_1)$ is the least ω_1-Erdös cardinal.

One of the striking consequences of large cardinals is that they imply the existence of a generic elementary embedding $j : V \longrightarrow M$ with M transitive and $\mathrm{crit}(j) = \omega_1$. For example, if δ is a Woodin cardinal, then there is a condition in the non-stationary tower $\mathbb{P}_{<\delta}$ that forces the existence of such an embedding; see [6]. The value of $j(\omega_1)$ tends to be rather large. For example, if δ is a Woodin cardinal in an iterable extender model $L[E]$, then forcing with $\mathbb{P}^{L[E]}_{<\delta}$ over $L[E]$ produces an embedding j with $j(\omega_1) \geq$ the least ω_1-Erdös cardinal of $L[E]$. It is natural to ask if large cardinals imply the existence of a precipitous ideal on ω_1 since this would imply the existence of a generic elementary embedding j with $j(\omega_1) < (2^{\omega_1})^+$. One way to disprove this might be to show that there is a set forcing which kills all precipitous ideals on ω_1. In this paper we present some partial results on this question and, in particular, show that Woodin limits of Woodin cardinals do not imply the existence of precipitous ideals on ω_1.

The contents of the paper are as follows. First we define what it means for there to exist a collapsing function for a cardinal λ. Then

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we show that if there exists a collapsing function for $(2^{\omega_1})^+$, then no ideal on ω_1 is precipitous. It is easy, as we show, to add a collapsing function for ω_2 by forcing; whether this can be done for ω_3 is not known.

Next we turn to inner models of the form $L[E]$ where E is a coherent sequence of extenders. We define what it means for a cardinal to be weakly ω_1-Erdös. Then we show that in $L[E]$, there is a collapsing function on λ if and only if λ is less than the least weakly ω_1-Erdös cardinal. In particular, in $L[E]$, there are no precipitous ideas on ω_1. As a corollary to this and an earlier result of Itay Neeman, the existence of a Woodin limit of Woodin cardinals does not imply the existence of a precipitous ideal on ω_1.

We conjecture that in $L[E]$, there is a precipitous ideal on κ if and only if κ is measurable. John Steel [5] has shown this by a method different from ours under hypotheses more restrictive than ours. It is not known if the corollary mentioned above can be proved by Steel's method.

The last part of our paper contains the proof of the fact about nonstationary tower forcing over $L[E]$ mentioned in the first paragraph above. Proposition 14 was inspired by conversations with Doug Burke in 1991 about some of his results that later appeared in [1].

Definition 1. *Define $h : \omega_1 \longrightarrow \omega_1$ to be a* collapsing function for λ *iff for all $\beta < \lambda$, there exists a club $C \subseteq \mathcal{P}_{\omega_1}(\beta)$ such that for all $X \in C$,*

$$h(X \cap \omega_1) > \mathrm{ot}(X).$$

Proposition 2. *Suppose that there exists a collapsing function for $(2^{\omega_1})^+$. Then there is no precipitous ideal on ω_1.*

Proof. Suppose the proposition fails. Let $j : V \longrightarrow M$ be a corresponding ultrapower map in a $\mathcal{P}(\omega_1)/I$-generic extension $V[G]$. Work in $V[G]$. Let $\alpha = \omega_1^V$, $\beta = j(h)(\alpha)$ and $\gamma = j(\alpha)$. Then

$$\alpha < \beta < \gamma < ((2^{\omega_1})^+)^V.$$

Say $f : {}^{<\omega}\beta \longrightarrow \beta$ is a function in V such that if

$$X \in (\mathcal{P}_\alpha(\beta))^V$$

and

$$f[{}^{<\omega}X] \subseteq X,$$

then

$$h(X \cap \alpha) > \mathrm{ot}(X).$$

Still in $V[G]$, let

$$S = \{Y \in \mathcal{P}_{\omega_1}(j(\beta)) \mid j(f)[{}^{<\omega}Y] \subseteq Y,\, Y \cap \gamma = \alpha \text{ and } \mathrm{ot}(Y) \geq \beta\}.$$

Clearly $j[\beta] \in S$. It is easy to see that there exists a tree

$$T \subseteq {}^{<\omega} j(\beta)$$

such that $\{b[\omega] \mid \forall n < \omega \quad b \restriction n \in T\} = S$ and $T \in M$. By the absoluteness of the illfoundedness of T, there exists $Y \in S \cap M$. But then

$$\beta = j(h)(\alpha) = j(h)(Y \cap \gamma) > \mathrm{ot}(Y) \geq \beta.$$

Contradiction! □

An interesting fact about a collapsing function for λ is that it is upward absolute to models in which ω_1 and λ are still cardinals. This is basically by the tree argument in the previous proof.

A stronger form of the next result will be given later in Proposition 13. Proposition 3 is just what is needed for the application Corollary 4.

Proposition 3. *Let $L[E]$ be an extender model with the property that if $\mathcal{M}$ is a countable premouse and there exists an elementary embedding from $\mathcal{M}$ to a level of $L[E]$, then $\mathcal{M}$ is $(\omega_1 + 1)$-iterable. Then*

$$L[E] \models \text{there is a collapsing function for } \omega_3.$$

Proof. Let $h(\alpha)$ be the least $\beta < \omega_1$ such that α is countable in $J^E_{\beta+1}$. We claim that h works. Consider a countable $X \prec \mathcal{J}^E_\lambda$ where

$$\lambda = (\omega_3)^{L[E]}.$$

Let $\alpha = X \cap \omega_1$. Let $\pi : \mathcal{M} \longrightarrow \mathcal{J}^E_\lambda$ be the inverse of the Mostowski collapse of X. Then

$$\alpha = \mathrm{crit}(\pi) = (\omega_1)^{\mathcal{M}} = (\omega_1)^{\mathcal{J}^E_{h(\alpha)}}.$$

Let $\beta = (\alpha^+)^{\mathcal{M}} = (\omega_2)^{\mathcal{M}}$ and $\gamma = \mathrm{ot}(X) = \mathrm{OR} \cap |\mathcal{M}|$. It is enough to see that $\gamma < h(\alpha)$ and, for this, it suffices to show that $\mathcal{M}$ is an initial segment of $L[E]$. As an aside, we note that by the Mitchell-Steel Condensation Theorem, we already know that $\mathcal{M}$ and $L[E]$ agree below β. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{M}, \mathcal{J}^E_{h(\alpha)})$. This coiteration is successful by our iterability hypothesis on $L[E]$. That is, either $\mathcal{M}^{\mathcal{S}}_\infty$ is an initial segment of $\mathcal{M}^{\mathcal{T}}_\infty$ or vice-versa.

Lemma 3.1. *$\mathcal{S}$ is trivial.*

Proof. Suppose the lemma fails. Since there are no total measures on the $\mathcal{M}$-sequence, $[0, \infty]_{\mathcal{S}}$ must drop. In the notation of Mitchell and Steel,

$$[0, \infty]_{\mathcal{S}} \cap \mathcal{D}^{\mathcal{S}} \neq \emptyset.$$

Thus $\mathcal{M}^{\mathcal{S}}_\infty$ is not sound. If $\mathcal{T}$ is not trivial, then $\mathcal{M}^{\mathcal{T}}_\infty$ is not sound, so $\mathcal{M}^{\mathcal{S}}_\infty = \mathcal{M}^{\mathcal{T}}_\infty$, which leads to the usual contradiction of the comparison process. Thus $\mathcal{T}$ is trivial. Hence $\mathcal{J}^E_{h(\alpha)}$ is a proper initial segment of $\mathcal{M}^{\mathcal{S}}_\infty$. But this implies that α is countable in $\mathcal{M}^{\mathcal{S}}_\infty$, hence also in $\mathcal{M}$, which is a contradiction. □

By Lemma 3.1, $\mathcal{M}$ is a proper initial segment of $\mathcal{M}^{\mathcal{T}}_\infty$. The fact that $\mathcal{M}$ has exactly three infinite cardinals easily implies that $\mathcal{T}$ is trivial, which completes the proof of Proposition 3. (This is the part of the proof that will need more work in the proof of Proposition 13.) □

We say that κ is a *Woodin limit of Woodin cardinals* iff κ is a Woodin cardinal and $\{\delta < \kappa \mid \delta \text{ is a Woodin cardinal }\}$ is unbounded in κ.

Corollary 4. *If there is a Woodin limit of Woodin cardinals, then there is a transitive class model with a Woodin limit of Woodin cardinals and no precipitous ideals on ω_1.*

Proof. Neeman [3] proved that if there is a Woodin limit of Woodin cardinals, then there is an extender model $L[E]$ with a Woodin limit of Woodin cardinals such that the iterability hypothesis of Proposition 3 holds for $L[E]$. Corollary 4 is immediate from Neeman's theorem and Propositions 2 and 3. □

Proposition 5. *For all $\delta < \omega_2$, let c_δ be the δ-th canonical function on ω_1. Let h be a function on ω_1. Suppose that for all $\delta < \omega_2$, h dominates c_δ on a club. Then h is a collapsing function for ω_2. In particular, if H is V-generic over the poset of countable partial functions on ω_1, then $\bigcup H$ is a collapsing function in $V[H]$.*

Proof. Let $\mathfrak{A} = \langle H_{\omega_2}, \in, \lhd, h\rangle$ where $\lhd$ is a wellordering of H_{ω_2}. Consider any countable $X \prec \mathfrak{A}$. For $\gamma < \delta$ both ordinal elements of X, there exists $C \in X$ such that C is club in ω_1 and for all $\xi \in C$,

$$c_\gamma(\xi) < c_\delta(\xi) < h(\xi).$$

Thus, if $\alpha = X \cap \omega_1$, then $\delta \mapsto c_\delta(\alpha)$ is a one-to-one function from $X \cap \omega_2$ to $h(\alpha)$. In particular, $\mathrm{ot}(X \cap \omega_2) \leq h(X \cap \omega_1)$ as desired. □

Weakly ω_1-Erdös cardinals, which we define below, fit between $< \omega_1$-Erdös and ω_1-Erdös cardinals. Unless otherwise noted, we use the term "structure" to mean structure in a countable first order language.

Definition 6. *Given a function $h : \omega_1 \longrightarrow \omega_1$ and a cardinal η, we say that η is* h-weakly ω_1-Erdös *iff for all structures $\mathfrak{A}$ with universe η, there exists a set of ordinal indiscernibles I for $\mathfrak{A}$ such that if X is the elementary hull of I in $\mathfrak{A}$, then $\mathrm{ot}(I) \geq h(X \cap \omega_1)$.*

Definition 7. *A cardinal η is* weakly ω_1-Erdös *iff η is h-weakly ω_1-Erdös for all functions h on ω_1.*

Weakly ω_1-Erdös really is a weaker property than ω_1-Erdös by the following result.

Proposition 8. *Suppose that η is a weakly ω_1-Erdös cardinal and M is a transitive inner model of* ZFC *with*

$$(\omega_1)^M = \omega_1.$$

Then

$$M \models \eta \textit{ is a weakly } \omega_1\textit{-Erdös cardinal.}$$

In particular, the existence of a a weakly ω_1-Erdös cardinal does not imply that $0^\#$ exists.

Proof. The downward absoluteness is by a familiar tree argument. If $0^\#$ exists, then there is an L-generic G over the Levy collapse of the real ω_1. Take $M = L[G]$ to see why the second claim holds.

□

It is relatively clear that if η is a weakly ω_1 Erdös cardinal, then there is no collapsing function for η^+. In fact, slightly more is true.

Proposition 9. *Suppose that η is a weakly ω_1-Erdös cardinal. Then there is no collapsing function for η.*

Proof. Clearly, it is enough to show that for all functions $h : \omega_1 \longrightarrow \omega_1$, there exist arbitrarily large $\lambda < \eta$ such that for all structures $\mathfrak{A}$ with universe λ, there exists a set of indiscernibles I for $\mathfrak{A}$ with

$$\mathrm{ot}(I) > h(\mathrm{Hull}^{\mathfrak{A}}(I) \cap \omega_1).$$

We prove this by contradiction. So suppose that for some function $h : \omega_1 \longrightarrow \omega_1$ and ordinal $\lambda_0 < \eta$, if $\lambda_0 \leq \lambda < \eta$, then there exists a structure $\mathfrak{A}_\lambda$ with universe λ for which there is no set of indiscernibles I with

$$\mathrm{ot}(I) > h(\mathrm{Hull}^{\mathfrak{A}_\lambda} \cap \omega_1).$$

We may assume that h is non-decreasing. Let $\mathfrak{A}$ be a structure with universe η that codes $\lambda \mapsto \mathfrak{A}_\lambda$. Because η is weakly ω_1 Erdös, there exists a set I of indiscernibles for $\mathfrak{A}$ with

$$\mathrm{ot}(I) > h(X \cap \omega_1) + 1$$

where $X = \mathrm{Hull}^{\mathfrak{A}}(I)$. Since λ_0 is definable in $\mathfrak{A}$, by indiscernibility, either $I \subseteq \lambda_0$ or $I \cap (\lambda_0 + 1) = \emptyset$. First suppose that $I \subseteq \lambda_0$. Then I is a set of indiscernibles for $\mathfrak{A}_{\lambda_0}$. This is a contradiction since

$$X \cap \omega_1 \supseteq \mathrm{Hull}^{\mathfrak{A}_{\lambda_0}}(I) \cap \omega_1$$

and h is non-decreasing. Therefore $\lambda > \lambda_0$ for all $\lambda \in I$. Let μ be the $h(\alpha)$-th element of I where $\alpha = X \cap \omega_1$. Then $I \cap \mu$ is a set of $h(\alpha)$ many indiscernibles for $\mathfrak{A}_\mu$, which is again a contradiction. □

Proposition 10. *For each function $h : \omega_1 \longrightarrow \omega_1$, let η_h be the least h-weakly ω_1-Erdös cardinal. Suppose that there is a club C such that $h(\alpha) < k(\alpha)$ for all $\alpha \in C$. Then $\eta_h < \eta_k$.*

Proof. The proof of Proposition 9 almost literally shows that $\eta_h < \eta_{h+1}$. An easy modification proves the claim made here. For each $\lambda < \eta_h$, pick a witnessing structure $\mathfrak{A}_\lambda$. Then form $\mathfrak{A}$ as before. The new wrinkle is that C should be a predicate of $\mathfrak{A}$. For contradiction, suppose $\eta_h = \eta_k$. Say I is a set of indiscernibles for $\mathfrak{A}$ and $\text{type}(I) > k(\alpha)$ where

$$\alpha = \text{Hull}^{\mathfrak{A}} \cap \omega_1.$$

Then $\alpha \in C$, which justifies defining μ as before and gives the same contradiction. □

Proposition 11. *Let η be the least weakly ω_1-Erdös cardinal. Then η is a strong limit cardinal and*

$$\omega_2 \leq \text{cf}(\eta) \leq 2^{\omega_1}.$$

Proof. Consider an arbitrary $\lambda < \eta$. Since η is a weakly ω_1-Erdös cardinal, for all $h : \omega_1 \longrightarrow \omega_1$ and structures $\mathfrak{A}$ with universe λ, if we let $\mathfrak{B} = \langle H_\eta, \in, \lhd, \lambda, \mathfrak{A} \rangle$, then there is a set $I_{h,\mathfrak{A}}$ of indiscernibles for $\mathfrak{B}$ such that $\text{ot}(I_{h,\mathfrak{A}}) > \omega \cdot h(X \cap \omega_1)$ where $X = \text{Hull}^{\mathfrak{B}}(I_{h,\mathfrak{A}})$. Moreover, either $\max(I_{h,\mathfrak{A}}) < \lambda$ or $\min(I_{h,\mathfrak{A}}) > \lambda$. Since λ is not a weakly ω_1-Erdös cardinal, there exists a pair $(h, \mathfrak{A})$ such that $\min(I_{h,\mathfrak{A}}) > \lambda$. But just from the existence of ω many indiscernibles above λ for H_η, we may conclude by standard arguments that $\eta \neq \lambda^+$ and $2^\lambda < \eta$. In other words, η is a strong limit cardinal.

It is easy to see that, in the terminology of Proposition 10,

$$\eta = \sup\left(\{\eta_h \mid h \in {}^{\omega_1}\omega_1\}\right).$$

The statement on the cofinality of η follows from Proposition 10. □

The following result will be used in the proof of Proposition 13.

Proposition 12. *Let $\mathcal{P}$ be an active premouse and $\alpha = (\omega_1)^{\mathcal{P}}$. Suppose that $\rho_1(\mathcal{P}) \geq \alpha$. Let F be the top extender of $\mathcal{P}$ and $\mu = \text{crit}(F)$. Then μ is weakly ω_1 Erdös in $\mathcal{P}$.*

Proof. We may assume that $\mathcal{P} = \mathcal{H}_1^{\mathcal{P}}(\alpha)$. In particular, that $\rho_1(\mathcal{P}) = \alpha$. We may also assume that the only generator of F is μ since otherwise μ would be a measurable cardinal in $\mathcal{P}$.

Consider a structure $\mathfrak{A} \in |\mathcal{P}|$ such that μ is the universe of $\mathfrak{A}$, and, also, an increasing function $h : \alpha \longrightarrow \alpha$ with $h \in |\mathcal{P}|$. Let $\nu = (\mu^+)^{\mathcal{P}}$. So actually $\mathfrak{A}, h \in J^{\mathcal{P}}_{\nu}$.

Because

$$\rho_1(\mathcal{P}) = \alpha = (\omega_1)^{\mathcal{P}},$$

there exists a Σ_1 elementary embedding $\tau : \mathcal{N} \longrightarrow \mathcal{P}$ with $\mathcal{N} \in J^{\mathcal{P}}_{\alpha}$ and $\mathfrak{A}, h \in \operatorname{ran}(\tau)$. Namely,

$$\mathcal{N} = \mathcal{H}^{\mathcal{P}}_1 \left(\{\mathfrak{A}, h\}\right).$$

Because $\rho_1(\mathcal{P})$ has uncountable Σ_1 cofinality over $\mathcal{P}$, there exists $\sigma < \mathrm{OR} \cap |\mathcal{P}|$ such that

$$\mathcal{N} = \mathcal{H}^{\mathcal{P}\restriction\sigma}_1 \left(\{\mathfrak{A}, h\}\right).$$

Hence also $\tau \in |\mathcal{P}|$.

Say $\tau(\mu_0) = \mu$. For $\xi < \alpha$, let $\mathcal{N}_\xi$ be the ξ-th internal iterate of $\mathcal{N}$ always by the top extender and $j_\xi : \mathcal{N} \longrightarrow \mathcal{N}_\xi$ be the corresponding cofinal Σ_1 elementary iteration map. Let

$$\mu_\xi = j_\xi(\mu_0).$$

Since

$$\mathcal{P} \models F \text{ is a countably complete extender,}$$

for all $\xi < \alpha$, there exists a Σ_1 elementary embedding $\tau_\xi : \mathcal{N}_\xi \longrightarrow \mathcal{P}$ such that

$$\tau_\xi \circ j_\xi = \tau$$

and $\tau_\xi \in |\mathcal{P}|$. It follows that

$$I_\xi = \{\tau_\xi(\mu_\zeta) \mid \zeta < \xi\}$$

is a set of indiscernibles for $\mathfrak{A}$. Moreover if

$$X_\xi = \mathrm{Hull}^{\mathfrak{A}}(I_\xi),$$

then

$$X_\xi \cap \alpha \leq (\omega_1)^{\mathcal{N}}.$$

To finish, take

$$\xi = h((\omega_1)^{\mathcal{N}}).$$

Then I_ξ witnesses that μ is h-weakly ω_1-Erdös in $\mathcal{P}$ with respect to $\mathfrak{A}$. □

In light of Proposition 9, the next result gives a characterization in $L[E]$ of the cardinals that carry collapsing functions: they are exactly the cardinals less than the first weakly ω_1-Erdös cardinal.

Proposition 13. *Let $L[E]$ be an extender model with the property that if $\mathcal{M}$ is a countable premouse and there exists an elementary embedding from $\mathcal{M}$ to a level of $L[E]$, then $\mathcal{M}$ is (ω_1+1)-iterable. Let λ be strictly less than any weakly ω_1-Erdös cardinal of $L[E]$. Then*

$$L[E] \models \text{there is a collapsing function for } \lambda.$$

Proof. We may assume that λ is a successor cardinal in $L[E]$. Say

$$\lambda = (\kappa^+)^{L[E]}.$$

Define $h(\alpha)$ as in the proof of Proposition 3 even though it may turn out that h is not fast enough. Ultimately, we will define the required collapsing function to be g, where $g(\alpha)$ is the ordinal height of a certain non-dropping linear iterate of $\mathcal{J}^E_{h(\alpha)}$. Consider an arbitrary countable $X \prec \mathcal{J}^E_\lambda$. Let π, $\mathcal{M}$, α, $\mathcal{S}$, and $\mathcal{T}$ be as in the proof of Proposition 3.

Lemma 13.1. *$\mathcal{S}$ is trivial.*

The proof of Lemma 13.1 is identical to that of Lemma 3.1.

Lemma 13.2. *$\mathcal{T}$ is thorough, hence linear.*

Proof. By definition, $\mathcal{T}$ being thorough means that whenever $\xi+1 < \mathrm{lh}(\mathcal{T})$ if $\mathcal{P}$ is the level of $\mathcal{M}^{\mathcal{T}}_\xi$ whose top extender is $E^{\mathcal{T}}_\xi$, then $E^{\mathcal{T}}_\xi$ is the only total-on-$\mathcal{P}$ extender on the $\mathcal{P}$-sequence. If $\mathcal{T}$ is not thorough, then we can argue that there is a measurable cardinal in $\mathcal{M}$, which easily leads to a contradiction. Thorough iterations are obviously linear. □

Lemma 13.3. *Let $\xi+1 < \mathrm{lh}(\mathcal{T})$ and $\mathcal{P}$ be the initial segment of $\mathcal{M}^{\mathcal{T}}_\xi$ whose top extender is E_ξ. Then*

$$\rho_1(\mathcal{P}) = 1.$$

Moreover,

$$\mathcal{P} = \mathcal{M}^{\mathcal{T}}_\xi.$$

and

$$\deg^{\mathcal{T}}(\xi+1) = 0.$$

Proof. Let $\mu = \mathrm{crit}(E_\xi)$. Suppose that $\rho_1(\mathcal{P}) \neq 1$. Then

$$\rho_1(\mathcal{P}) \geq (\omega_1)^{\mathcal{P}} = \alpha,$$

so μ is weakly ω_1 Erdös in $\mathcal{P}$ by Proposition 12. It follows that μ is weakly ω_1 Erdös in $\mathcal{M}^{\mathcal{T}}_\infty$. But then $\pi(\mu)$ is weakly ω_1-Erdös in $\mathcal{J}^E_\lambda$ and hence in $L[E]$. This is a contradiction.

If $\mathcal{P}$ is a proper initial segment of $\mathcal{M}^{\mathcal{T}}_\xi$, then α is not a cardinal in $\mathcal{M}^{\mathcal{T}}_\xi$ because $\rho_1(\mathcal{P}) = 1$. It follows that α is not a cardinal in

$$\mathcal{M}^{\mathcal{T}}_0 = \mathcal{J}^E_{h(\alpha)}.$$

This is in direct contradiction with the definition of $\mathcal{J}^E_{h(\alpha)}$.

The final claim, which is that $\mathcal{M}^{\mathcal{T}}_{\xi+1}$ is the internal ultrapower of $\mathcal{M}^{\mathcal{T}}_{\xi}$ by $E^{\mathcal{T}}_{\xi}$, is now clear. □

Shortly, we will use the fact that iteration trees with the property given in Lemma 13.3 are completely determined by their starting models and their lengths.

Lemma 13.4. *Let $\beta < \lambda$ and $\mathfrak{A}$ be a structure with universe β. Suppose that $\mathfrak{A}$ is definable in $\mathcal{J}^E_{\lambda}$. Then there exists a set D of ordinal indiscernibles for $\mathfrak{A}$ with $\operatorname{ot}(D) = \operatorname{lh}(\mathcal{T})$.*

Proof. Let C be the set of critical points of extenders used on $\mathcal{T}$ and $D = \pi[C]$. □

Now because λ is not a weakly ω_1-Erdös cardinal, there exists a structures $\mathfrak{A}$ with universe some $\beta < \lambda$ and a non-decreasing function $g : \omega_1 \longrightarrow \omega_1$ such that for all sets I of indiscernibles for $\mathfrak{A}$,

$$\operatorname{ot}(I) < g\left(\operatorname{Hull}^{\mathfrak{A}}(I) \cap \omega_1\right).$$

Let $(\mathfrak{A}, g)$ be the pair with this property that is least in the order of construction of $\mathcal{J}^E_{\lambda}$.

Lemma 13.5. $\operatorname{lh}(\mathcal{T}) < g(\alpha)$.

Proof. By the properties of g just mentioned,

$$\operatorname{lh}(\mathcal{T}) = \operatorname{ot}(D) < g\left(\operatorname{Hull}^{\mathfrak{A}}(D) \cap \omega_1\right) \leq g(\alpha)$$

where D comes from the proof of Lemma 13.4. □

Define $f : \omega_1 \longrightarrow \omega_1$ as follows. Given $\widehat{\alpha} < \omega_1$, let $\widehat{\mathcal{T}}$ be the internal iteration of $\mathcal{J}^E_{h(\widehat{\alpha})}$ in which the top extender of $\mathcal{M}^{\widehat{\mathcal{T}}}_{\xi}$ is used at all $\xi < g(\widehat{\alpha})$ and $\operatorname{lh}(\widehat{\mathcal{T}}) = g(\widehat{\alpha})$. If $\mathcal{J}^E_{h(\widehat{\alpha})}$ does not have a top extender, then $\widehat{\mathcal{T}}$ is trivial. It is clear from Lemmas 13.3 and 13.5 that f is a collapsing function for λ. That completes the proof of Proposition 13. □

Finally, we turn to the facts about non-stationary tower forcing mentioned in the introduction, beginning with a review of some well-known results. A cardinal κ is an ω_1-Erdös cardinal iff

$$\kappa \rightarrow (\omega_1)^{<\omega}.$$

Let $\mathcal{E}(\omega_1)$ be the least ω_1-Erdös cardinal. Let $\mathcal{C}(\omega_1)$ be the least λ such that the Chang property

$$(\lambda, \omega_1) \twoheadrightarrow (\omega_1, \omega)$$

holds. In general, $\mathcal{C}(\omega_1) \leq \mathcal{E}(\omega_1)$. Silver [4] proved that the consistency of

$$\mathsf{ZFC} + \mathcal{E}(\omega_1) \text{ exists}$$

implies that of

$$\mathsf{ZFC} + \mathcal{C}(\omega_1) = \omega_2,$$

and Donder [2] proved the reverse relative consistency.

Suppose for the moment that δ is a Woodin cardinal. Let $\mathbb{P}_{<\delta}$ be the non-stationary tower. Hugh Woodin [6] proved that forcing with $\mathbb{P}_{<\delta}$ adds a generic elementary embedding

$$j : V \longrightarrow M$$

with M transitive and ${}^{<\delta}M \subseteq M$. It is easy to see that the Chang property

$$(\lambda, \omega_1) \twoheadrightarrow (\omega_1, \omega)$$

is equivalent to the existence of a condition p of $\mathbb{P}_{<\delta}$ that forces

$$\omega_1 < j(\omega_1) \leq \lambda.$$

It is also easy to see that $\mathcal{C}(\omega_1)$ is the least $\lambda > \omega_1$ such that some some condition p of $\mathbb{P}_{<\delta}$ forces $j(\omega_1) = \lambda$.

Proposition 14. *Let $L[E]$ be an extender model with the property that if $\mathcal{M}$ is a premouse of cardinality ω_1 and there exists an elementary embedding from $\mathcal{M}$ into a level of $L[E]$, then $\mathcal{M}$ is $(\omega_2 + 1)$-iterable. Then*

$$L[E] \models \mathcal{C}(\omega_1) = \mathcal{E}(\omega_1).$$

Proof. Work in $L[E]$. Let $\lambda = \mathcal{C}(\omega_1)$. It suffices to see that λ is ω_1-Erdös. Consider an arbitrary $A \subseteq \lambda$. Pick $\eta < \lambda^+$ so that $A \in J^E_\eta$ and

$$J^E_\eta \models \lambda \text{ is the largest cardinal.}$$

It suffices to see that there exists a set $I \subseteq \lambda$ of indiscernibles for $(\mathcal{J}^E_\eta, A)$ such that I has order type ω_1.

Now apply definition of $\mathcal{C}(\omega_1)$ to find a premouse $\mathcal{J}^{\overline{E}}_{\overline{\eta}}$ whose cardinality is ω_1, an ordinal $\alpha < \omega_1$, an elementary embedding

$$\pi : \mathcal{J}^{\overline{E}}_{\overline{\eta}} \longrightarrow \mathcal{J}^E_\eta$$

with

$$\pi(\alpha) = \omega_1$$

and

$$\pi(\omega_1) = \lambda,$$

and $\overline{A} \subseteq \omega_1$ with

$$\pi(\overline{A}) = A.$$

Let $(\overline{\mathcal{T}}, \mathcal{T})$ be the coiteration of

$$\left(\mathcal{J}^{\overline{E}}_{\overline{\eta}}, \mathcal{J}^{E}_{\omega_1}\right).$$

By our iterability hypothesis, the coiteration is successful, i.e., either

$$\overline{\mathcal{M}}_\infty = \mathcal{M}^{\overline{\mathcal{T}}}_\infty$$

is an initial segment of

$$\mathcal{M}_\infty = \mathcal{M}^{\mathcal{T}}_\infty$$

or vice-versa. It is easy to see that $\overline{\mathcal{T}}$ is trivial and $\mathcal{T}$ is a thorough iteration of length exactly ω_1. Moreover, $\mathcal{T}$ has drops, in particular $1 \in \mathcal{D}^{\mathcal{T}}$. Hence $\mathcal{J}^{\overline{E}}_{\overline{\eta}}$ is an initial segment of

$$\mathcal{M}_\infty = \mathcal{M}^{\mathcal{T}}_{\omega_1}.$$

It is also possible to use the fact that $\mathcal{J}^{\overline{E}}_{\overline{\eta}}$ has no Ramsey cardinals to see that

$$\deg^{\mathcal{T}}(\xi + 1) = 0$$

for all $\xi \in \omega_1 - \mathcal{D}^{\mathcal{T}}$. Let $\xi_0 < \omega_1$ such that

$$\xi \notin \mathcal{D}^{\mathcal{T}}$$

and

$$\overline{A} \in \operatorname{ran}\left(i^{\mathcal{T}}_{\xi,\omega_1}\right)$$

for all countable $\xi > \xi_0$. Let

$$I = \left\{\operatorname{crit}(E^{\mathcal{T}}_\xi) \mid \xi \in \omega_1 - \xi_0\right\}.$$

Then $\pi[I]$ is a set of indiscernibles for $(\mathcal{J}^{E}_{\eta}, A)$ of order type ω_1. □

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