ASYMPTOTIC CONES OF FINITELY GENERATED GROUPS

SIMON THOMAS AND BOBAN VELICKOVIC

Abstract. Answering a question of Gromov [7], we shall present an example of a finitely generated group Γ and two non-principal ultrafilters $\mathcal{A}$, $\mathcal{B}$ such that the asymptotic cones $\mathrm{Con}_{\mathcal{A}}\,\Gamma$ and $\mathrm{Con}_{\mathcal{B}}\,\Gamma$ are not homeomorphic.

1. Introduction

Let Γ be a finitely generated group equipped with a fixed finite generating set and let d be the corresponding word metric. Consider the sequence of metric spaces $X_n = (\Gamma, d_n)$ for $n \geq 1$, where $d_n(g,h) = d(g,h)/n$. In [5], Gromov proved that if Γ has polynomial growth, then the sequence $(X_n \mid n \geq 1)$ of metric spaces converges in the pointed Gromov-Hausdorff topology to a complete geodesic space $\mathrm{Con}_\infty(\Gamma)$, the asymptotic cone of Γ. (Recall that a *geodesic space* is a metric space (X,d) such that for all points x, $y \in X$, there exists an isometric mapping from the interval $[0, d(x,y)]$ to a path in X joining x and y.) In [4], van den Dries and Wilkie generalised the construction of asymptotic cones to arbitrary finitely generated groups. However, their construction involved the choice of a non-principal ultrafilter $\mathcal{A}$ on the set $\mathbb{N}^+$ of positive natural numbers, and it was not clear whether the resulting asymptotic cone $\mathrm{Con}_{\mathcal{A}}(\Gamma)$ depended on the choice of the ultrafilter $\mathcal{A}$. In this paper, answering a question of Gromov [7], we shall present an example of a finitely generated group Γ and two non-principal ultrafilters $\mathcal{A}$, $\mathcal{B}$ such that the asymptotic cones $\mathrm{Con}_{\mathcal{A}}\,\Gamma$ and $\mathrm{Con}_{\mathcal{B}}\,\Gamma$ are not homeomorphic.

Our group Γ will be chosen from the class of small cancellation groups which was introduced by Bowditch in [1]. For each $n \in \mathbb{N}^+$, let $w_n(a,b)$ be the word $(a^n b^n)^7$ in the two letters a and b; and for each $I \subseteq \mathbb{N}^+$, let Γ_I be the group with presentation

$$\langle a, b \mid (w_n(a,b))_{n \in I} \rangle.$$

Then Γ_I satisfies the $C'(1/6)$ cancellation property; ie. whenever u is a common initial subword of two distinct elements r_1, r_2 of the set $\mathcal{R}_I$ of all cyclic conjugates

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of relators and their inverses, then $6\,\mathrm{length}(u) < \min\{\mathrm{length}(r_1), \mathrm{length}(r_2)\}$. This implies that if w is a reduced word in the letters a, b which represents the identity element in Γ_I, then there exists an element $r \in \mathcal{R}_I$ which has a common subword of length more $\mathrm{length}(r)/2$ with w. (For example, see Section V.4 [8].) Notice that for such an element r, we must have that $\mathrm{length}(r) < 2\,\mathrm{length}(w)$. So an easy induction on $\mathrm{length}(w)$ yields the following result.

Lemma 1.1. *Suppose that w is a (not necessarily reduced) word in the letters a, b which represents the identity element in Γ_I. Then w lies in the normal closure of the set of relators $\{w_\ell(a,b) \mid \ell \in I, \ell < \mathrm{length}(w)/7\}$.*

□

In this section, after giving the definition of an asymptotic cone, we shall prove that if I is any infinite subset of $\mathbb{N}^+$ and $\mathcal{A}$ is any non-principal ultrafilter which contains I, then the asymptotic cone $\mathrm{Con}_{\mathcal{A}}(\Gamma_I)$ is not simply connected. Then in Section 2, we shall prove that if I is a suitably chosen sparse subset of $\mathbb{N}^+$, then there exists a non-principal ultrafilter $\mathcal{B}$ such that the asymptotic cone $\mathrm{Con}_{\mathcal{B}}(\Gamma_I)$ is an $\mathbb{R}$-tree. A geodesic space (X,d) is said to be an $\mathbb{R}$-*tree* iff any two points of X are the endpoints of a unique topological arc (ie. the image of an injective continuous function from a closed interval of $\mathbb{R}$ into X.) In particular, $\mathbb{R}$-trees are simply connected. Hence the asymptotic cones $\mathrm{Con}_{\mathcal{A}}(\Gamma_I)$ and $\mathrm{Con}_{\mathcal{B}}(\Gamma_I)$ are not homeomorphic.

Definition 1.2. A non-principal ultrafilter on $\mathbb{N}^+$ is a collection $\mathcal{A}$ of subsets of $\mathbb{N}^+$ satisfying the following properties.

(i) If A, $B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.
(ii) If $A \in \mathcal{A}$ and $A \subseteq B \subseteq \mathbb{N}^+$, then $B \in \mathcal{A}$.
(iii) For all $A \subseteq \mathbb{N}^+$, either $A \in \mathcal{A}$ or $\mathbb{N}^+ \smallsetminus A \in \mathcal{A}$.
(iv) If F is a finite subset of $\mathbb{N}^+$, then $F \notin \mathcal{A}$.

Equivalently, if $\omega : \mathcal{P}(\mathbb{N}^+) \to \{0,1\}$ is the function such that $\omega(A) = 1$ iff $A \in \mathcal{A}$, then ω is a finitely additive probability measure on $\mathbb{N}^+$ such that $\omega(F) = 0$ for all finite subsets F of $\mathbb{N}^+$. It is easily checked that if (r_n) is a bounded sequence of real numbers, then there exists a unique real number ℓ such that

$$\{n \in \mathbb{N}^+ \mid |r_n - \ell| < \varepsilon\} \in \mathcal{A}$$

for all $\varepsilon > 0$. We write $\ell = \lim_{\mathcal{A}} r_n$.

Definition 1.3. Let Γ be a finitely generated group equipped with a fixed finite generating set and let d be the corresponding word metric. For each $n \geq 1$, let $X_n = (\Gamma, d_n)$ be the metric space defined by $d_n(g,h) = d(g,h)/n$. Let e be the identity element of Γ. Then X_∞ is the set of all sequences (g_n) of elements of Γ such that there exists a constant c with $d_n(g_n, e) \leq c$ for all $n \in \mathbb{N}^+$.

Let $\mathcal{A}$ be a non-principal ultrafilter on $\mathbb{N}^+$. Define an equivalence relation $\sim$ on X_∞ by $(g_n) \sim (h_n)$ iff $\lim_{\mathcal{A}} d_n(g_n, h_n) = 0$; and for each $(g_n) \in X_\infty$, let $(g_n)_{\mathcal{A}}$ denote the corresponding equivalence class. Then the asymptotic cone is defined to be

$$\mathrm{Con}_{\mathcal{A}}(\Gamma) = \{(g_n)_{\mathcal{A}} \mid (g_n) \in X_\infty\}$$

endowed with the metric $d_{\mathcal{A}}((g_n)_{\mathcal{A}}, (h_n)_{\mathcal{A}}) = \lim_{\mathcal{A}} d_n(g_n, h_n)$.

Proposition 1.4 (van den Dries and Wilkie [4]). *If Γ is a finitely generated group equipped with a fixed finite generating set and $\mathcal{A}$ is a non-principal ultrafilter on $\mathbb{N}^+$, then $\mathrm{Con}_{\mathcal{A}}(\Gamma)$ is a complete geodesic space.*

□

The asymptotic cone $\mathrm{Con}_{\mathcal{A}}(\Gamma)$ can be understood intuitively as follows. (Our account borrows heavily from those in [7] and [2].) Imagine an observer who moves away from the Cayley graph of Γ. Then X_n is what he sees if he pauses to observe the Cayley graph when the points of Γ that are actually at a distance n apart appear to be only at a distance 1 apart. As the observer continues to move away, any finite configuration will eventually become indistinguishable from a single point; but he may observe certain finite configurations which resemble earlier configurations. The asymptotic cone $\mathrm{Con}_{\mathcal{A}}(\Gamma)$ is a space which encodes all of these recurring finite configurations. For example, let I be an infinite subset of $\mathbb{N}^+$ and let $\mathcal{A}$ be a non-principal ultrafilter which contains I. Then for each $n \in I$, the relator $w_n(a,b) = (a^n b^n)^7$ can be regarded as a loop of length $14n$ in the Cayley graph of Γ_I, and hence as a loop of length 14 in X_n. The sequence of loops $(w_n(a,b) \mid n \in I)$ gives rise to a closed path of length 14 in $\mathrm{Con}_{\mathcal{A}}(\Gamma_I)$ as follows. For each $n \in I$, write $w_n(a,b) = x_1 \cdots x_{14n}$, where each $x_i \in \{a,b\}$; and define $f_n : [0,14] \to \Gamma_I$ by $f_n(t) = x_1 \cdots x_{\lceil tn \rceil}$. If $n \notin I$, let $f_n : [0,14] \to \Gamma_I$ be the function such that

$f_n(t) = e$ for all $t \in [0, 14]$. Now define $f : [0, 14] \to \mathrm{Con}_{\mathcal{A}}(\Gamma_I)$ by $f(t) = (f_n(t))_{\mathcal{A}}$. Then it is easily checked that f is a closed path of length 14 in $\mathrm{Con}_{\mathcal{A}}(\Gamma_I)$. We shall shortly prove that f is not null-homotopic, and hence that $\mathrm{Con}_{\mathcal{A}}(\Gamma_I)$ is not simply connected.

Next we shall address the question of where the ultrafilter $\mathcal{A}$ comes into the intuitive picture of $\mathrm{Con}_{\mathcal{A}}(\Gamma)$. As the definition of the closed path f indicates, if $C \notin \mathcal{A}$, then the observations of the metric spaces $\{X_n \mid n \in C\}$ can safely be ignored. So we can imagine that our observer walks backwards away from the Cayley graph of Γ with his eyes tightly shut, only opening them at rare intervals. It will then hardly be surprising if two observers, who open their eyes at very different times, should get very different impressions of the asymptotic geometry of the Cayley graph of Γ. For example, suppose that $I = \{i_m \mid m \in \mathbb{N}\}$ is an extremely sparse subset of $\mathbb{N}^+$ and that a second observer only opens his eyes at various stages $\{t_m \mid m \in \mathbb{N}\}$ such that $i_m \ll t_m \ll i_{m+1}$ for some $m \in \mathbb{N}$. Suppose that N is a large natural number and that our second observer examines the ball of radius N around e in X_{t_m}, which corresponds to the ball of radius Nt_m around e in the Cayley graph of Γ_I. By choosing m sufficiently large, we can suppose that $Nt_m \ll i_{m+1}$. This implies that the ball of radius Nt_m around e in the Cayley graph of Γ_I will be the same as that in the Cayley graph of the group Γ_m, which has the presentation

$$\langle a, b \mid w_{i_0}(a, b), \ldots, w_{i_m}(a, b)\rangle.$$

Since Γ_m is a finitely presented group which satisfies the $C'(1/6)$ cancellation property, it follows that Γ_m is a hyperbolic group. Hence by Gromov [6], every asymptotic cone of Γ_m is an $\mathbb{R}$-tree. Since $i_m \ll t_m$, this implies that the ball of radius N around e in X_{t_m} will already look very much like a tree to our second observer. Since N could be chosen to be arbitrarily large, our second observer will believe that the asymptotic cone of Γ_I is an $\mathbb{R}$-tree.

Now that we have described the intuition which motivates our construction, we are ready to present the details.

Theorem 1.5. *If I is any infinite subset of $\mathbb{N}^+$ and $\mathcal{A}$ is any non-principal ultrafilter which contains I, then the asymptotic cone $\mathrm{Con}_{\mathcal{A}}(\Gamma_I)$ is not simply connected.*

Proof. (This argument is essentially just a proper subset of subsections 2.3 and 2.4 of Bridson [2].) Let $S = [0,1] \times [0,1]$ be the unit square and let ∂S be the boundary of S. Let $f : [0,14] \to \mathrm{Con}_{\mathcal{A}}(\Gamma_I)$ be the closed path which was defined earlier. Then it is easy to convert this path into a corresponding continuous map $h : \partial S \to \mathrm{Con}_{\mathcal{A}}(\Gamma_I)$ which parametrizes the path proportionately to the length of ∂S. Suppose that h is null-homotopic. Then h can be extended to a continuous map $H : S \to \mathrm{Con}_{\mathcal{A}}(\Gamma_I)$. Let M be an integer such that if we subdivide S into M^2 equally small squares with vertices $\{p_{i,j} \mid 0 \leq i,j \leq M\}$, then $d_{\mathcal{A}}(H(p_{i,j}), H(p_{i',j'})) < 1/2$ for every pair of adjacent vertices $p_{i,j}$ and $p_{i',j'}$. For each $0 \leq i,j \leq M$, let $(x^n_{i,j} \mid n \in \mathbb{N}^+)$ be a sequence of elements of Γ_I such that $H(p_{i,j}) = (x^n_{i,j})_{\mathcal{A}}$, chosen so that if $p_{i,j} \in \partial S$ and $h(p_{i,j}) = f(t)$, then $(x^n_{i,j}) = (f_n(t))$. Let $n \in I$ be an integer such that whenever $p_{i,j}$ and $p_{i',j'}$ are adjacent vertices, then $d_n(x^n_{i,j}, x^n_{i',j'}) < 1$ and hence $d(x^n_{i,j}, x^n_{i',j'}) < n$. Notice that if $p_{i,j}$, $p_{i',j'} \in \partial S$ are adjacent vertices, then the edge between $p_{i,j}$ and $p_{i',j'}$ parametrizes an arc of length $14/4M$ of the closed path $f : [0,14] \to \mathrm{Con}_{\mathcal{A}}(\Gamma_I)$. Consequently, if we choose $M > 14$ and $n > 4$, then the corresponding arc between $x^n_{i,j}$and $x^n_{i',j'}$ in the loop $w_n(a,b)$ has length at most $\lceil 14n/4M \rceil + 1 < n$. If the adjacent vertices $p_{i,j}$ and $p_{i',j'}$ do not both lie in ∂S, then choose a geodesic segment between $x^n_{i,j}$ and $x^n_{i',j'}$ in the Cayley graph of Γ_I. Thus to the boundary of each of the M^2 small squares in S, we have now associated a loop of length less than $4n$ in the Cayley graph of Γ_I. For each such small square S', the labels on the edges of the associated loop yield a word w' of length less than $4n$ which represents the identity element in Γ_I. Applying Lemma 1.1, we see that each such word w' lies in the normal closure of the set of relators $\{w_\ell(a,b) \mid \ell \in I, \ell < n\}$. But this implies that $w_n(a,b)$ also lies in the normal closure of the set $\{w_\ell(a,b) \mid \ell \in I, \ell < n\}$, which is a contradiction. □

2. The construction

If (X,d) is a metric space and $w \in X$ is a base point, then we define the *Gromov inner product* on X by

$$(x.y)_w = \frac{1}{2}(d(x,w) + d(w,y) - d(x,y)).$$

Let $\delta \geq 0$. Then the metric space (X, d) is said to be *δ-hyperbolic* if for every w, x, y, $z \in X$, we have that

$$(x.y)_w \geq \min\{(x.z)_w, (z.y)_w\} - \delta.$$

If Γ is a finitely generated group equipped with a fixed finite generating set, then Γ is said to be *hyperbolic* iff its Cayley graph is δ-hyperbolic for some $\delta \geq 0$. (It is well-known that the hyperbolicity of Γ does not depend on the choice of the finite generating set.) We shall make use of the following characterization of δ-hyperbolicity, a proof of which can be found in Section 1.1 [3].

Lemma 2.1. *The metric space (X, d) is δ-hyperbolic iff*

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta$$

for all points w, x, y, $z \in X$.

□

We shall also make use of the following well-known characterization of $\mathbb{R}$-trees, a proof of which can be found in Section 3.4 [3].

Theorem 2.2. *Let (X, d) be a geodesic space. Then (X, d) is an $\mathbb{R}$-tree iff (X, d) is 0-hyperbolic.*

□

We are now ready to begin our construction. We shall define inductively two strictly increasing sequences of natural numbers $\langle i_m \mid m \in \mathbb{N}\rangle$ and $\langle t_m \mid m \in \mathbb{N}\rangle$. First we set $i_0 = 1$. Now suppose that $m \geq 0$ and that we have already defined $\langle i_n \mid n \leq m\rangle$ and $\langle t_n \mid n < m\rangle$. Let Γ_m be the group with the presentation

$$\langle a, b \mid w_{i_0}(a, b), \ldots, w_{i_m}(a, b)\rangle.$$

Since Γ_m is a finitely presented group which satisfies the $C'(1/6)$ cancellation property, it follows that Γ_m is a hyperbolic group. So there exists a constant $\delta_m \geq 0$ such that the Cayley graph of Γ_m is δ_m-hyperbolic. Let t_m be any integer such that $t_{m-1} < t_m$ and $2\delta_m / t_m < 1/m$; and let i_{m+1} be any integer such that $i_m < i_{m+1}$ and $i_{m+1} > 8mt_m$. Finally we let $I = \{i_m \mid m \in \mathbb{N}\}$ and let $\mathcal{B}$ be any non-principal ultrafilter which contains the set $T = \{t_m \mid m \in \mathbb{N}\}$.

Theorem 2.3. $\mathrm{Con}_{\mathcal{B}}(\Gamma_I)$ *is an* $\mathbb{R}$*-tree.*

Proof. Applying Lemma 2.1 and Theorem 2.2, we see that it is enough to show that

$$d_{\mathcal{B}}(P_1,P_2)+d_{\mathcal{B}}(P_3,P_4) \\ \leq \max\{d_{\mathcal{B}}(P_1,P_3)+d_{\mathcal{B}}(P_2,P_4), d_{\mathcal{B}}(P_1,P_4)+d_{\mathcal{B}}(P_2,P_3)\}$$

for all points $P_1,\dots,P_4 \in \mathrm{Con}_{\mathcal{B}}(\Gamma_I)$. So let $P_1,\dots,P_4$ be an arbitrary 4-tuple of elements of $\mathrm{Con}_{\mathcal{B}}(\Gamma_I)$; and for each $1 \leq k \leq 4$, let $(P_k^n \mid n \in \mathbb{N}^+)$ be a sequence of elements of Γ_I such that $P_k = (P_k^n)_{\mathcal{B}}$. Let N be a natural number such that $d_{\mathcal{B}}(P_k,(e)_{\mathcal{B}}) < N$ for each $1 \leq k \leq 4$. Now fix some $\varepsilon > 0$ and let n_0 be an integer such that $n_0 > N$ and $5/n_0 < \varepsilon$. Then there exists a set $A \in \mathcal{B}$ such that if $n \in A$, then the following conditions are satisfied.

(a) $n > t_{n_0}$.
(b) If $1 \leq k,\ell \leq 4$, then $|d_n(P_k^n,P_\ell^n) - d_{\mathcal{B}}(P_k,P_\ell)| < 1/n_0$.
(c) If $1 \leq k \leq 4$, then $d_n(P_k^n,e) < 2N$.

Let $B = A \cap T$. Then $B \in \mathcal{B}$ and so there exists an integer $t_m \in B$. Let $1 \leq k,\ell \leq 4$. Then $d_{t_m}(P_k^{t_m},e)$, $d_{t_m}(P_\ell^{t_m},e) < 2N$ and so $d(P_k^{t_m},e)$, $d(P_\ell^{t_m},e) < 2Nt_m$. Let p_k, p_ℓ be geodesic paths from e to $P_k^{t_m}$, $P_\ell^{t_m}$ respectively in the Cayley graph of Γ_I and let $p_{k,\ell}$ be a geodesic path from $P_k^{t_m}$ to $P_\ell^{t_m}$. Then the labels on the edges of the loop $p_k * p_{k,\ell} * p_\ell^{-1}$ yield a word w of length less than $8Nt_m$ which represents the identity element in Γ_I. Since $8Nt_m < 8n_0t_m < 8mt_m < i_{m+1}$, Lemma 1.1 implies that w lies in the normal closure of the set of relators $\{w_{i_0}(a,b),\dots,w_{i_m}(a,b)\}$. It follows that $p_{k,\ell}$ is also a geodesic path from $P_k^{t_m}$ to $P_\ell^{t_m}$ in the Cayley graph of the hyperbolic group Γ_m. Thus each pair of points $P_k^{t_m}$, $P_\ell^{t_m}$ is the same distance apart in the Cayley graphs of Γ_I and Γ_m. Consequently we must have that

$$d(P_1^{t_m},P_2^{t_m})+d(P_3^{t_m},P_4^{t_m}) \\ \leq \max\{d(P_1^{t_m},P_3^{t_m})+d(P_2^{t_m},P_4^{t_m}), d(P_1^{t_m},P_4^{t_m})+d(P_2^{t_m},P_3^{t_m})\}+2\delta_m.$$

Using the fact that $2\delta_m/t_m < 1/m < 1/n_0$, we now obtain that

$$
\begin{aligned}
&d_{\mathcal{B}}(P_1,P_2)+d_{\mathcal{B}}(P_3,P_4)\\
&< d_{t_m}(P_1^{t_m},P_2^{t_m})+d_{t_m}(P_3^{t_m},P_4^{t_m})+2/n_0\\
&< \max\{d_{t_m}(P_1^{t_m},P_3^{t_m})+d_{t_m}(P_2^{t_m},P_4^{t_m}),d_{t_m}(P_1^{t_m},P_4^{t_m})+d_{t_m}(P_2^{t_m},P_3^{t_m})\}+3/n_0\\
&< \max\{d_{\mathcal{B}}(P_1,P_3)+d_{\mathcal{B}}(P_2,P_4),d_{\mathcal{B}}(P_1,P_4)+d_{\mathcal{B}}(P_2,P_3)\}+2/n_0+3/n_0\\
&< \max\{d_{\mathcal{B}}(P_1,P_3)+d_{\mathcal{B}}(P_2,P_4),d_{\mathcal{B}}(P_1,P_4)+d_{\mathcal{B}}(P_2,P_3)\}+\varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

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Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019, USA

Equipe de Logique, URA 753, Universite de Paris 7, 2 Place Jussieu, France