A note on Borel equivalence relations

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Given Borel equivalence relations E and F on Polish spaces X and Y respectively we say that E is *reducible* to F and write $E \leq F$ if there is a Borel function $f : X \to Y$ such that for every x and y in X xEy iff $f(x)Ff(y)$. Let $\mathcal{E}$ be the class of all Borel equivalence relations. The question of which partial orderings can be embedded into the structure $(\mathcal{E}, \leq)$ has been studied by several authors. For a comprehensive survey see ([Ke]). Just ([Ju]) has proved that this structure contains arbitrarily large finite antichains and Woodin ([Wo]) gave an ingenious construction of a perfect antichain as well as an infinite descending chain using metamathematical means. In this note we show by completely elementary means that $(\mathcal{E}, \leq)$ is a very rich structure. We prove the following where $\subseteq_*$ denotes inclusion modulo finite sets.

Theorem 1 *The structure $(\mathcal{P}(\omega), \subseteq_*)$ can be embedded into $(\mathcal{E}, \leq)$.*

Corollary 1 *Every partial ordering of size $\leq \aleph_1$ embeds into $(\mathcal{E}, \leq)$.* $\Box$

Remark It is relatively consistent with ZFC that $2^{\aleph_0}$ is large and $(\omega_2, \leq)$ does not embed into $(\mathcal{E}, \leq)$. To see this start with a model of GCH and add a large number of Cohen reals. The proof is identical to the proof that in this model $\mathcal{P}(\omega)$ does not contain a chain of length ω_2 under $\subseteq_*$ and uses homogeneity of Cohen forcing and the fact that reducibility between Borel equivalence relations is $\mathbf{\Sigma^1_2}$ and hence absolute.

PROOF of **Theorem 1**: Note that if $\mathcal{I}$ is a Borel ideal in $\mathcal{P}(\omega)$ then we can define a Borel equivalence relation E on $\mathcal{P}(\omega)$ as follows

$$XEY \text{ if and only if } X \Delta Y \in \mathcal{I}$$

Our idea is to associate to each subset S of ω a Borel ideal $\mathcal{I}_S$ such that the map which to each S assigns the associated equivalence relation E_S is the required embedding. In fact our embedding will have stronger properties than claimed in **Theorem 1**. When $S \subseteq_* T$ there will be a continuous reduction from E_S to E_T and when $S \not\subseteq_* T$ there will be no reduction from E_S to E_T which is even Baire measurable. Thus, for example, in the context of AD there is such an embedding with respect to arbitrary reductions between Borel equivalence relations. In addition the equivalence relations we construct will be rather simple in complexity, i.e. $\mathbf{\Pi^0_3}$ although it is still possible that there is such an embedding into $\mathbf{\Sigma^0_2}$ equivalence relations.

To commence fix two increasing sequences of natural numbers $(a_n)_{n<\omega}$ and $(b_n)_{n<\omega}$ such that for every $n < \omega$

$$a_{n+1} \geq 2^{n+1}(a_n + 2) \text{ and } b_{n+1} \geq 2^{a_{n+1}+(n+1)b_n}.$$

Define inductively an increasing sequence of integers $(m_n)_{n<\omega}$ by $m_0 = 0$ and $m_{n+1} = m_n + b_n$ and let I_n denote the interval $[m_n, m_{n+1})$. For a subset S of ω let $I_S = \bigcup_{n\in S} I_n$. To each subset X of ω we associate a sequence $(X(n))_{n<\omega}$ of finite sets where $X(n) = X \cap I_n$. Say that a subset G of ω is a *graph* if $\text{card}(G \cap I_k) \leq 1$ for every k. For every n we define an *n-norm* on $\mathcal{P}(\omega)$ as follows

$$||X||_n = \frac{\log(\text{card}X(n) + 1)}{a_n}.$$

The main property of the n-norm which will be used in the proof is the following inequality which is easily verified. For any sets $X_0, \ldots, X_{N-1}$

$$||\bigcup_{i<N} X_i||_n \leq \sup_{i<N} ||X_i||_n + \frac{\log N}{a_n} \tag{1}$$

To each infinite subset S of ω we associate an ideal $\mathcal{I}_S$ defined by

$$\mathcal{I}_S = \{X \subseteq \omega : \lim_{n\in S, n\to\infty} ||X||_n = 0\}.$$

Note that $\mathcal{I}_S$ is a Polish group under symmetric difference. (Recall that a Borel subgroup of a Polish group is said to be *Polish* if there is a group topology on it which makes it Polish and has the same Borel sets as the original topology. This topology is necessarily unique.) Define as above the equivalence relation E_S on $\mathcal{P}(\omega)$ by saying XE_SY if and only if $X\Delta Y \in \mathcal{I}_S$. Clearly E_S is a $\mathbf{\Pi}^0_3$ equivalence relation. We shall show that the map which to each S associates E_S is the required embedding. It is quite easy to see that if $S \subseteq_* T$ then the function which to each set X associates $X \cap I_S$ is a continuous reduction of E_S to E_T. Thus the theorem will be proved once we establish the following.

Lemma 1 *If S is not almost contained in T then there is no Baire measurable function which reduces E_S to E_T.*

PROOF: Notice that it suffices to prove the lemma for disjoint S and T since otherwise we can replace S by $S \setminus T$. We first show that it suffices to prove that there are no continuous reductions from E_S to E_T.

Lemma 2 *If there is a Baire measurable reduction from E_S to E_T then there is an infinite subset S^* of S and a continuous reduction from E_{S^*} to E_T.*

PROOF: Let f be a Baire measurable reduction from E_S to E_T and fix a dense G_δ subset W of $\mathcal{P}(\omega)$ such that f is continuous on W. Let $W = \bigcap W_n$ where each W_n is dense open in $\mathcal{P}(\omega)$. We build inductively an increasing sequence $(i_n)_{n<\omega}$ of elements of S and sets $(Z_n)_{n<\omega}$ where Z_n is a subset of $[m_{i_n+1}, m_{i_{n+1}})$ such that for any subset X of ω if $X \cap [m_{i_n+1}, m_{i_{n+1}}) = Z_n$ then $X \in W_n$. Then set $S^* = \{i_n : n < \omega\}$ and $Z = \bigcup_{n<\omega} Z_n$. Note that Z is disjoint from I_{S^*}. Finally define a function $f^* : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ by setting

$$f^*(X) = f((X \cap I_{S^*}) \cup Z).$$

Then it is easily seen that f^* is a continuous reduction of E_{S^*} to E_T. $\square$

To simplify notation let us now assume that we have a continuous function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that for every $X, Y \in \mathcal{P}(\omega)$ $X \Delta Y \in \mathcal{I}_S$ if and only if $f(X)\Delta f(Y) \in \mathcal{I}_T$. We shall derive contradiction. Notice that we may assume that the range of f is contained in $\mathcal{P}(I_T)$.

We define inductively increasing sequences $(i_n)_{n<\omega}$ and $(j_n)_{n<\omega}$ of members of S and T respectively, and sequences of finite sets $(A_n)_{n<\omega}$, $(B_n)_{n<\omega}$, $(G_n)_{n<\omega}$ $(A^*_n)_{n<\omega}$, and $(B^*_n)_{n<\omega}$ such that A_n, B_n, and G_n are subsets of m_{i_n}, A^*_n and B^*_n are subsets of m_{j_n} and G_n is a graph. We require that $A_{n+1} \cap m_{i_n} = A_n, B_{n+1} \cap m_{i_n} = B_n$, $G_{n+1} \cap m_{i_n} = G_n$ and that the following conditions be satisfied.

1. For all $X, G \subseteq \omega \setminus m_{i_n}$ with G a graph the following hold
   (a) $f(A_n \cup X) \cap m_{j_n} = A^*_n$ and $f(B_n \cup X) \cap m_{j_n} = B^*_n$
   (b) For all $p \geq j_n$ $||f(A_n \cup X)\Delta f(B_n \cup X)||_p \leq 2^{-(n+1)}$
   (c) For all $p \geq j_n$ $||f(A_n \cup X)\Delta f((A_n \cup X)\Delta(G_n \cup G))||_p \leq 2^{-(n+1)}$
2. (a) $||A_{n+1}\Delta B_{n+1}||_{i_n} \geq 1$
   (b) For $j_n \leq p < j_{n+1}$ $||A^*_{n+1}\Delta B^*_{n+1}||_p \leq 2^{-n}$

In the end define $A = \bigcup_{n<\omega} A_n$ and $B = \bigcup_{n<\omega} B_n$. Then $f(A) = \bigcup_{n<\omega} A^*_n$ and $f(B) = \bigcup_{n<\omega} B^*_n$. It follows from 2(a) that $A \not\!\!E_S B$ and from 2(b) that $f(A) E_T f(B)$. Thus to finish the proof it suffices to do the inductive step of the construction.

Suppose now the construction has been carried out through stage n. We do it at stage $n+1$. Pick a family $\mathcal{F}$ of disjoint subsets of I_{i_n} each of cardinality $2^{a_{i_n}-1}$ such that card$\mathcal{F} = 2^{m_{i_n}} + 1$. Fix some $X_0 \subseteq \omega \setminus m_{i_n+1}$. By the pigeon-hole principle there are distinct A and B in $\mathcal{F}$ such that

$$f(A_n \cup A \cup X_0) \upharpoonright m_{i_n} = f(A_n \cup B \cup X_0) \upharpoonright m_{i_n}.$$

By continuity there is some $i' > i_n + 1$ such that for any $X \subseteq \omega \setminus m_{i_n+1}$ if $X \cap [m_{i_n+1}, m_{i'}) = X_0 \cap [m_{i_n+1}, m_{i'})$ then the above equality is true with X instead of X_0 and the value is fixed. Let $X^*_0 = X_0 \cap [m_{i_n+1}, m_{i'})$.

Claim 1 *For every $X \subseteq \omega \setminus m_{i'}$ and $p \geq j_n$*

$$||f(A_n \cup A \cup X^*_0 \cup X)\Delta f(B_n \cup B \cup X^*_0 \cup X)||_p \leq 2^{-n}.$$

PROOF: The main point is that $i_n \notin T$. This is so since we have that the range of f is contained in $\mathcal{P}(I_T)$. Thus we have two cases to consider.

Case 1: $p < i_n$. First of all from the choice of A and B it follows that $f(A_n \cup A \cup X_0^* \cup X) \restriction m_{i_n} = f(A_n \cup B \cup X_0^* \cup X) \restriction m_{i_n}$. Also by 1(b)

$$||f(A_n \cup B \cup X_0^* \cup X) \Delta f(B_n \cup B \cup X_0^* \cup X)||_p \leq 2^{-(n+1)}$$

Combining these two we obtain the claim in this case.

Case 2: $p > i_n$. Enumerate the set $A \cup B$ as $\{t_0, \ldots, t_{2^{a_{i_n}}-1}\}$ and define $A^{(k)} = A \Delta \{t_i : i < k\}$. Note that $A^{(0)} = A$ and $A^{(2^{a_{i_n}})} = B$. Fix any $X \subseteq \omega \setminus m_{i'}$ and a graph $G \subseteq \omega \setminus m_{i_n+1}$. For each k consider the set

$$U_k = f(A_n \cup A^{(k)} \cup X_0^* \cup X) \Delta f((A_n \cup A^{(k)} \cup X_0^* \cup X) \Delta (G_n \cup \{t_k\} \cup G)).$$

Then by 1(c) $||U_k||_p \leq 2^{-(n+1)}$. On the other hand 1(b) implies that

$$||f(A_n \cup B \cup X_0^* \cup X) \Delta f(B_n \cup B \cup X_0^* \cup X)||_p \leq 2^{-(n+1)}.$$

Now since $f(A_n \cup A \cup X_0^* \cup X) \Delta f(B_n \cup B \cup X_0^* \cup X)$ is contained in the union of $f(A_n \cup B \cup X_0^* \cup X) \Delta f(B_n \cup B \cup X_0^* \cup X)$ and $\bigcup_{k \leq 2^{a_{i_n}}} U_k$, using inequality (1) we obtain that

$$||f(A_n \cup A \cup X_0^* \cup X) \Delta f(B_n \cup B \cup X_0^* \cup X)||_p \leq 2^{-(n+1)} + \frac{a_{i_n} + 2}{a_p} \leq 2^{-n}.$$

This proves **Claim 1**. □

Let now $A'_n = A_n \cup A \cup X_0^*$ and $B'_n = B_n \cup B \cup X_0^*$. To define A_{n+1} and B_{n+1} we shall end-extend A'_n and B'_n by the same set. Since $||A'_n \Delta B'_n||_{i_n} = 1$ and by **Claim 1** we have guaranteed conditions 2(b) no matter which common extension we pick, we only have to worry about satisfying conditions 1(a)-(c).

Claim 2 *There is $j > i'$ and $Z \subseteq [m_{i'}, m_j)$ such that for every $X \subseteq \omega \setminus m_j$ and for every $p \geq j$*

$$||f(A'_n \cup Z \cup X) \Delta f(B'_n \cup Z \cup X)||_p \leq 2^{-(n+2)}$$

PROOF: Otherwise we can build strictly increasing sequences of integers $(s_k)_{k<\omega}$ and $(t_k)_{k<\omega}$, and a sequence of sets $(Z_k)_{k<\omega}$ such that $s_0 = i'$, $Z_0 = \emptyset$, and for every k $Z_k \subseteq [m_{i'}, m_{s_k})$, $Z_{k+1} \cap m_{s_k} = Z_k$, and for every $X \subseteq \omega \setminus m_{s_k}$

$$||f(A'_n \cup Z_k \cup X) \Delta f(B'_n \cup Z_k \cup X)||_{t_k} \geq 2^{-(n+2)}.$$

Note that since the range of f is contained in $\mathcal{P}(I_T)$ each t_k is in T. In the end define $X = A'_n \cup \bigcup_{k<\omega} Z_k$, $Y = B'_n \cup \bigcup_{k<\omega} Z_k$. It follows that $X \Delta Y = A'_n \Delta B'_n \in \mathcal{I}_S$ but $f(X) \Delta f(Y) \notin \mathcal{I}_T$. Hence f is not a reduction of E_S to E_T. □

Fix now Z and j is in **Claim 2** and define $A''_n = A'_n \cup Z$ and $B''_n = B'_n \cup Z$. A_{n+1} and B_{n+1} will be built as end-extensions of A''_n and B''_n respectively and thus by **Claim 2** condition 1(b) is guaranteed provided that we choose $j_{n+1} \geq j$. We now take care of condition 1(c).

Claim 3 *There is $j' > j$ and $W, H \subseteq [m_j, m_{j'})$ such that H is a graph and such that for every $X, G \subseteq \omega \setminus m_{j'}$ with G a graph and every $p \geq j'$*

$$||f(A''_n \cup W \cup X)\Delta f((A''_n \cup W \cup X)\Delta(G_n \cup H \cup G))||_p \leq 2^{-(n+2)}.$$

PROOF: Otherwise as in **Claim 2** we can build strictly increasing sequences of integers $(s_k)_{k<\omega}$ and $(t_k)_{k<\omega}$ and sequences of sets $(W_k)_{k<\omega}$ and $(H_k)_{k<\omega}$ such that $s_0 = j'$, $W_0 = H_0 = \emptyset$, and for every k $s_k < t_k < s_{k+1}$, $W_k, H_k \subseteq [m_j, m_{s_k})$ with H_k a graph, $W_{k+1} \cap m_{s_k} = W_k$, $H_{k+1} \cap m_{s_k} = H_k$, and for every $X, G \subseteq \omega \setminus m_{s_k}$ with G a graph

$$||f(A''_n \cup W_k \cup X)\Delta f((A''_n \cup W_k \cup X)\Delta(G_n \cup H_k \cup G))||_{t_k} \geq 2^{-(n+2)}.$$

As before it follows that each t_k is in T. Finally let $X = A''_n \cup \bigcup_{k<\omega} W_k$, $G = G_n \cup \bigcup_{k<\omega} H_k$, and $Y = X\Delta G$. Then $X\Delta Y = G$ is a graph and hence in $\mathcal{I}_S$ but $f(X)\Delta f(Y) \notin \mathcal{I}_T$. Contradiction. □

Fix W, H and j' as in **Claim 3**. Set $j_{n+1} = j'$ and define $A'''_n = A''_n \cup W$ and $B'''_n = B''_n \cup W$. Now by continuity of f find $i^* \geq j_{n+1}$ and $U \subseteq [m_{j_{n+1}}, m_{i^*})$ such that for some $A^*_{n+1}, B^*_{n+1} \subseteq m_{j_{n+1}}$ for every $X \subseteq \omega \setminus m_{i^*}$

$$f(A'''_n \cup U \cup X) \cap m_{j_{n+1}} = A^*_{n+1} \text{ and } f(B'''_n \cup U \cup X) \cap m_{j_{n+1}} = B^*_{n+1}.$$

Then set $i_{n+1} = i^*$ and define $A_{n+1} = A'''_n \cup U$, $B_{n+1} = B'''_n \cup U$, and $G_{n+1} = G_n \cup H$. It follows that conditions 1(a) and 1(c) are satisfied. This finishes the proof of **Lemma 1** and **Theorem 1**. □

References

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