ON THE COMPLEXITY OF THE ISOMORPHISM RELATION FOR FIELDS OF FINITE TRANSCENDENCE DEGREE

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ABSTRACT. Confirming a conjecture of Hjorth and Kechris [HK], we prove that the isomorphism relation for fields of finite transcendence degree is a universal essentially countable Borel equivalence relation. We also prove that the theory of fields of finite transcendence degree does not admit canonical models.

1. INTRODUCTION

Given a class $\mathcal{K}$ of structures for a fixed first-order language $\mathcal{L}$, it is natural to try to understand the complexity of the isomorphism relation for $\mathcal{K}$. For those classes consisting of the countable models of some $\mathcal{L}_{\omega_1,\omega}$-sentence, Friedman and Stanley [FS] proposed to use the methods of descriptive set theory to study their isomorphism relations. Hjorth and Kechris [HK] continued this project and situated it within the general theory of Borel equivalence relations. This provided tools for the analysis of the isomorphism relation and generated a host of interesting open problems. The main result of this paper is that the isomorphism relation for fields of finite transcendence degree is as complex as it conceivably could be. (The analogous result for arbitrary countable fields was proved independently in [FK] and [FS], where it was shown that the associated isomorphism relation is complete analytic.)

Before we can give an exact statement of our main result, we first need to describe how to represent the class of countably infinite structures of a given first order language by the elements of a Polish space. For convenience, we shall work only with relational first order languages. If there are function symbols, as is the case with the language of field theory, then we replace them by relation symbols representing the graphs of the associated functions. Given a language $\mathcal{L} = \{R_i \mid i \in I\}$, where

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R_i is an n_i-ary relation symbol, let

$$X_{\mathcal{L}} = \prod_{i \in I} 2^{\mathbb{N}^{n_i}}.$$

Then the elements of $X_{\mathcal{L}}$ code $\mathcal{L}$-structures with universe $\mathbb{N}$ as follows. Given $x = (x_i)_{i \in I} \in X_{\mathcal{L}}$, the structure

$$\mathcal{M}_x = \langle \mathbb{N}, R_i^x \rangle_{i \in I}$$

represented by x is defined by:

$$R_i^x(k_1, \dots, k_{n_i}) \Leftrightarrow x_i(k_1, \dots, x_{n_i}) = 1.$$

The symmetric group S_∞ on $\mathbb{N}$ is a Polish group with the usual product topology inherited from $\mathbb{N}^{\mathbb{N}}$; and $X_{\mathcal{L}}$ together with the natural action of S_∞ is a Polish S_∞-space. Notice that x, $y \in X_{\mathcal{L}}$ lie in the same S_∞-orbit iff $\mathcal{M}_x \cong \mathcal{M}_y$. For this reason, the action is usually called the *logic action* of S_∞ on $X_{\mathcal{L}}$. Given an $\mathcal{L}_{\omega_1,\omega}$ sentence σ,

$$Mod(\sigma) = \{x \in X_{\mathcal{L}} \mid \mathcal{M}_x \models \sigma\}$$

represents the class of all countably infinite models of σ. Note that $Mod(\sigma)$ is an S_∞-invariant Borel subset of $X_{\mathcal{L}}$. We shall denote the restriction of the isomorphism relation $\cong$ to $Mod(\sigma)$ by $\cong_\sigma$. When working with $Mod(\sigma)$, we shall usually identify $\mathcal{M}_x$ with x.

The notion of Borel reducibility will enable us to discuss the complexity of the isomorphism relation $\cong_\sigma$. Suppose that X and Y are Borel sets and that E and F are equivalence relations on X and Y respectively. Then E is *Borel reducible* to F, written $E \leq_B F$ if there exists a Borel function $f : X \to Y$ such that $xEy \Leftrightarrow f(x)Ff(y)$. We say that E and F are *Borel bireducible* and write $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$. We say that E is *smooth* if there exists a Polish space Y such that $E \leq_B \Delta(Y)$, where $\Delta(Y)$ is the equality relation on Y. If $\cong_\sigma$ is smooth, then we say that the models of σ are *concretely classifiable*.

In general, if σ is an arbitrary sentence of $\mathcal{L}_{\omega_1,\omega}$, then the isomorphism relation $\cong_\sigma$ is analytic rather than Borel. (For example, the isomorphism relation for arbitrary countable fields is complete analytic. See either [FK] or [FS].) But there is a large class of theories σ for which $\cong_\sigma$ is a Borel relation. In particular, $\cong_\sigma$ is Borel for those theories σ whose models have "finite rank" in a broad sense, such as: the theory of finitely generated groups; the theory of finite rank torsion-free abelian groups; the theory of fields of finite transcendence degree, etc. In order to study these examples, it is useful to introduce the notion of an essentially countable equivalence relation. An equivalence relation E is said to be *countable* if each equivalence class of E is countable. Given a theory σ, we say that $\cong_\sigma$ is *essentially countable* iff $\cong_\sigma \leq_B E$ for some countable Borel equivalence relation E on a Borel set X. In this case, Kechris [Ke1] has shown that there exists a countable Borel equivalence relation F such that $\cong_\sigma \sim_B F$. There is a natural model-theoretic characterisation of essentially countable theories; namely, Hjorth and Kechris [HK] have shown that the following are equivalent:

(i) $\cong_\sigma$ is essentially countable;

(ii) there is a countable fragment $F \subseteq \mathcal{L}_{\omega_1,\omega}$ containing σ such that if $\mathcal{M} = \langle \mathbb{N}, - \rangle$ is a model of σ, there exists $\bar{a} \in \mathbb{N}^{<\omega}$ such that $\mathrm{Th}_F(\langle \mathcal{M}, \bar{a} \rangle)$ is $\aleph_0$-categorical.

(Condition (ii) can be interpreted as saying that the models of σ have "finite rank" in some sense.) For such theories σ, we can use the theory of countable Borel equivalence relations to study the complexity of the isomorphism relation $\cong_\sigma$. (For the general theory of such equivalence relations, see [JKL].)

The least complex countable Borel equivalence relations are those which are smooth. It turns out that there is also a most complex countable Borel equivalence relation. Define a countable Borel equivalence relation E to be *universal* if $F \leq_B E$ for any other countable Borel equivalence relation F. Such an equivalence relation exists, and it is obviously unique up to Borel bireducibility. One particular realization of it is the orbit equivalence relation given by the shift action of the free group F_2 on two generators on 2^{F_2}. Recall that given any set X and any group G, the shift action of G on X^G is given by

$$x(h)g = x(hg).$$

Let $E(F_2, 2^{F_2})$ denote the orbit equivalence relation induced by the shift action of the free group on two generators F_2 on 2^{F_2}. In [DJK], it was shown that $E(F_2, 2^{F_2})$ is a universal countable Borel equivalence relation. If $\cong_\sigma \sim_B E_E(F_2, 2^{F_2})$, then we say that $\cong_\sigma$ is a universal essentially countable Borel equivalence relation. For example, in [JKL], it was shown that the isomorphism relation for locally finite trees is a universal essentially countable Borel equivalence relation; and in [TV], we proved that the isomorphism relation for finitely generated groups is also a universal essentially countable Borel equivalence relation. The main result of this paper is that the isomorphism relation for fields of finite transcendence degree is a universal essentially countable Borel equivalence relation. This confirms Conjecture 5.9 of [HK].

We shall also prove that the theory of fields of finite transcendence degree does not admit canonical models. Here a sentence σ of $\mathcal{L}_{\omega_1,\omega}$ is said to *admit canonical models* if there exists an "effectively definable" function f which to each countable model $\mathcal{M} \in Mod(\sigma)$ assigns a countable model $f(\mathcal{M})$ of σ such that

$$\mathcal{M} \cong \mathcal{N} \text{ implies } f(\mathcal{M}) = f(\mathcal{N}) \cong \mathcal{M}.$$

We usually cannot expect to be able to choose f so that the structure $f(\mathcal{M})$ also has universe $\mathbb{N}$; since Burgess has shown that for any sentence σ of $\mathcal{L}_{\omega_1,\omega}$, the following statements are equivalent:

(i) $Mod(\sigma)$ is concretely classifiable;

(ii) there exists a Borel function $f : Mod(\sigma) \to Mod(\sigma)$ such that $\mathcal{M} \cong \mathcal{N}$ implies $f(\mathcal{M}) = f(\mathcal{N}) \cong \mathcal{M}$.

However, in [HK], Hjorth and Kechris showed that there are many interesting theories which are not concretely classifiable and yet do admit canonical models, including the theories of rank 1 abelian torsion-free groups and of rigid locally finite trees; and they also proved that the theory of locally finite trees does not admit canonical models. In [TV], we proved that the theory of finitely generated groups does not admit canonical models.

It is quite difficult to give a precise formulation of the notion of "effectively definable" in the case when $Mod(\sigma)$ is not concretely classifiable. An exact definition can be found in Section 7 of [HK]; but some set-theoretic sophistication is necessary in order to understand this definition. Fortunately Hjorth and Kechris have found the following algebraic property which is equivalent to admitting canonical models. Suppose that G is a Polish group and that $a : X \times G \to X$ is a Borel action of G on the Borel space X. Let E_a be the associated orbit equivalence relation on X. A Borel function $\alpha : E_a \to G$ is called a *cocycle* if whenever $xE_a yE_a z$, then $\alpha(x,z) = \alpha(x,y)\alpha(y,z)$. We say that the action a has the *cocycle property* if there exists a Borel cocycle α such that $a(x, \alpha(x,y)) = y$. (For the basic facts concerning Polish group actions, we refer the reader to [BK] or [Ke2].) In [HK], Hjorth and Kechris proved that for any sentence σ of $\mathcal{L}_{\omega_1,\omega}$, the following statements are equivalent:

(i) σ admits canonical models;

(ii) the logic action of S_∞ on $Mod(\sigma)$ has the cocycle property.

In Section 2, we shall prove that if σ is the theory of fields of finite transcendence dimension, then the logic action of S_∞ on $Mod(\sigma)$ does not have the cocycle property. Hence the theory of fields of finite transcendence dimension does not admit canonical models.

We shall end this introduction with a few words on our group-theoretic and field-theoretic notation. In this paper, permutation groups always act on the right. If (G, Ω) is a permutation group, $\alpha \in \Omega$ and $g \in G$, then the image of α under g will be denoted by either αg or α^g; and the corresponding orbit equivalence relation will be denoted by $E(G, \Omega)$. Our field-theoretic notation is standard. For example, if $x_1, \dots, x_n$ is a collection of algebraically independent elements over the field F, then $F[x_1, \dots, x_n]$ denotes the corresponding polynomial ring and $F(x_1, \dots, x_n)$ denotes the corresponding field of rational functions. Also if F is a field, then the algebraic closure of F is denoted by $\overline{F}$. Finally if $z \in \mathbb{Z} \smallsetminus \{0\}$, then $\operatorname{sgn}(z) = z/|z|$ denotes the sign of z.

2. Fields of finite transcendence degree

In this section, we shall first prove our main result which says that the isomorphism relation for fields of finite transcendence degree is a universal essentially countable Borel equivalence relation. Then we shall prove that the theory of fields of finite transcencence degree does not admit canonical models. Our proofs are based closely on those of Fried and Kollár [FK], who showed how to code an arbitrary infinite structure $\mathcal{M}$ in a field of transcendence degree $|\mathcal{M}|$. The following result, which will be proved in Section 3, will enable us to code the orbit equivalence relation $E(F_2, 2^{F_2})$ within the class of fields of finite transcendence degree.

Theorem 2.1. *Let u be a transcendental element over $\mathbb{Q}$. Then there exists a free subgroup $G = \langle a, b \rangle$ of the general linear group $GL(3, \mathbb{Q}(u))$ such that if the matrices*

g_1, g_2, h_1, $h_2 \in G$ satisfy the equality

$$g_1 + 3h_1 = g_2 + 3h_2,$$

then $g_1 = g_2$ and $h_1 = h_2$.

We shall fix such a free group $G = \langle a, b \rangle$ for the rest of this section. The *Cayley graph* of G is the labelled directed graph $\Gamma_G = (G; E_a, E_b)$, where

$$E_a = \{(g, ag) \mid g \in G\} \quad \text{and} \quad E_b = \{(g, bg) \mid g \in G\}.$$

Thus G is the set of vertices of Γ_G; and we regard the elements of E_a and E_b as directed edges labelled by a, b respectively. For each $g \in G$, let g^ρ be the right translation map defined by $h^{g^\rho} = hg$ for all $h \in G$. Then the following result is well-known.

Lemma 2.2. $\operatorname{Aut} \Gamma_G = \{g^\rho \mid g \in G\}$.

□

Clearly the permutation group $\left(G, 2^G\right)$ can be naturally identified with the permutation group $(G, \mathcal{P}(G))$, where G acts on its powerset $\mathcal{P}(G)$ via right translations. Thus $E\left(G, \mathcal{P}(G)\right) \sim_B E\left(G, 2^G\right)$. For technical reasons, we shall work with $\widehat{\mathcal{P}}(G) = \{C \subseteq G \mid |C| \geq 2\}$, rather than with $\mathcal{P}(G)$. Of course, it is easily shown that $E\left(G, \widehat{\mathcal{P}}(G)\right) \sim_B E\left(G, \mathcal{P}(G)\right)$.

For each $C \in \widehat{\mathcal{P}}(G)$, we shall define an associated field F_C of finite transcendence degree such that if C, $D \in \widehat{\mathcal{P}}(G)$, then $F_C \cong F_D$ iff there exists $g \in G$ such that $Cg = D$. (Formally we are required to produce fields with underlying set $\mathbb{N}$. However, this will cause no difficulties, since the fields we define all have *canonical* isomorphisms to such fields.) To begin the construction, let $F_0 = \mathbb{Q}(u)$, so that $G \leqslant GL(3, F_0)$. Now let $\{x_i, y_i, z_i \mid 1 \leq i \leq 3\}$ be a set of algebraically independent elements over F_0. Then the general linear group $GL(3, F_0)$ acts in a natural way as a group of automorphisms of the polynomial ring

$$R = F_0[x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3].$$

Namely, if $\varphi = (\alpha_{ij})$, then

$$x_i^\varphi = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \alpha_{i3}x_3$$
$$y_i^\varphi = \alpha_{i1}y_1 + \alpha_{i2}y_2 + \alpha_{i3}y_3$$
$$z_i^\varphi = \alpha_{i1}z_1 + \alpha_{i2}z_2 + \alpha_{i3}z_3;$$

and if $u = u(x_1, \ldots, z_3) \in R$ is an arbitrary element, then

$$u^\varphi = u(x_1^\varphi, \ldots, z_3^\varphi)$$

is the image of u under the homogeneous linear substitution which replaces x_1 by x_1^φ, etc. Clearly this action on R extends to an action of $GL(3, F_0)$ on the field of fractions

$$L_0 = F_0(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3).$$

So we can regard $GL(3, F_0)$ as a group of automorphisms of L_0. Let $v = x_1+y_2+z_3$; and for each $g \in G$, let $\widetilde{g} = v^g$. Let $\Omega_1 = \{\widetilde{g} \mid g \in G\}$. Note that $\widetilde{g}^h = \widetilde{gh}$ for all $h \in G$; and if $g = (\alpha_{ij})$, then

$$\begin{aligned}\widetilde{g} =&\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3\\ &+\alpha_{21}y_1 + \alpha_{22}y_2 + \alpha_{23}y_3\\ &+\alpha_{31}z_1 + \alpha_{32}z_2 + \alpha_{33}z_3.\end{aligned}$$

Thus the following result is an immediate consequence of Theorem 2.1.

Lemma 2.3. *If $\widetilde{g}_1$, $\widetilde{g}_2$, $\widetilde{h}_1$, $\widetilde{h}_2 \in \Omega_1$ satisfy the equality*

$$\widetilde{g}_1 + 3\widetilde{h}_1 = \widetilde{g}_2 + 3\widetilde{h}_2,$$

then $\widetilde{g}_1 = \widetilde{g}_2$ and $\widetilde{h}_1 = \widetilde{h}_2$.

□

Next let F_1 be the splitting field over L_0 of the set of polynomials

$$P_0 = \{y^{3^m} - d \mid d \in L_0, m \geq 1\}.$$

Definition 2.4. Let K be a field of characteristic 0 and let p be a prime. Let t be a transcendental element over K, and let $K(t)$ be the corresponding purely transcendental extension of K. Then $K(t, p)$ denotes the field which is obtained by adjoining elements $\{t(\ell) \mid \ell \in \mathbb{N}\}$ to $K(t)$ such that

(a) $t(0) = t$, and

(b) $t(\ell+1)^p = t(\ell)$.

Note that the field $K(t,p)$ is uniquely defined up to isomorphism. We shall usually denote the element $t(\ell)$ by t^{1/p^ℓ}.

For the rest of this section, $p_0 = 3, \dots, p_4 = 13$ will denote the first 5 odd primes. For each $C \in \widehat{\mathcal{P}}(G)$, let $\Omega_1^C = \Omega_1$, $\Omega_2^C = \Omega_2 = \{\widetilde{g} + 3\widetilde{ag} \mid g \in G\}$ and $\Omega_3^C = \Omega_3 = \{\widetilde{g}+3\widetilde{bg} \mid g \in G\}$. Suppose inductively that F_n has been defined for some $1 \leq n \leq 3$. Let t_n be a transcendental element over F_n and let $L_n = F_n(t_n, p_n)$. Then F_{n+1} is the splitting field over L_n of the set of polynomials

$$P_n = \{y^2 - (t_n - d) \mid d \in \Omega_n\}.$$

Finally let $\Omega_4^C = \{\widetilde{c} \mid c \in C\}$. Let t_4 be a transcendental element over F_4 and $L_4 = F_4(t_4, p_4)$. Then F_C is the splitting field over L_4 of the set of polynomials

$$P_4^C = \{y^2 - (t_4 - d) \mid d \in \Omega_4^C\}.$$

Lemma 2.5. *If C, $D \in \widehat{\mathcal{P}}(G)$ and there exists $g \in G$ such that $Cg = D$, then $F_C \cong F_D$.*

Proof. Regard g as an element of $\operatorname{Aut} L_0$. Then it is easily checked that there exists an automorphism $\pi \in \operatorname{Aut} L_4$ such that $\pi \restriction L_0 = g$ and $\pi(t_n) = t_n$ for $1 \leq n \leq 4$. Since $\widetilde{c}^g = \widetilde{cg}$ for all $c \in C$, it follows that π maps Ω_4^C onto Ω_4^D. Hence $F_C \cong F_D$. □

The proof of the converse of Lemma 2.5 is much more involved. Our arguments will closely follow those of Fried and Kollár [FK]. We shall only give the statements of their main two technical lemmas; but in order to make this paper relatively self-contained, we shall sketch some of their other arguments.

Lemma 2.6 (Fried and Kollár [FK]). *Let K be a field of characteristic 0, and let t be a transcendental element over K. Suppose that $T_1, \dots, T_n$ are mutually prime nonconstant polynomials in $K[t]$, each having no multiple factors; and for each $1 \leq i \leq n$, let $\vartheta_i \in \overline{K[t]}$ be an element such that $\vartheta_i^2 = T_i$. Let $K_0 = K(t)$ and for each $1 \leq i \leq n$, let $K_i = K(t, \vartheta_1, \dots, \vartheta_i)$. Then the following statements hold for each $1 \leq i \leq n$.*

(a) $\vartheta_i \notin K_{i-1}$.

(b) *If* $\eta \in K_i$ *satisfies* $\eta^2 \in K_0$*, then there exist an element* $c \in K_0$ *and a subset* Z *of* $\{1, \ldots, i\}$ *such that* $\eta = c \prod_{\ell \in Z} \vartheta_\ell$.

(c) *If* $\eta \in K_i$ *and* η *is algebraic over* K*, then* $\eta \in K$.

□

Corollary 2.7. *Let* $C \in \widehat{\mathcal{P}}(G)$ *and* $\eta \in F_C$*. If* $1 \leq n \leq 4$ *and* η *is algebraic over* F_n*, then* $\eta \in F_n$.

Proof. We shall prove that for each $1 \leq n \leq 3$, if $\eta \in F_{n+1}$ is algebraic over F_n, then $\eta \in F_n$. The same argument will show that if $\eta \in F_C$ is algebraic over F_4, then $\eta \in F_4$. The result then follows easily.

Fix some $1 \leq n \leq 3$. Let $p = p_n$ and $K = F_n$. Suppose that $\eta \in F_{n+1}$ is algebraic over F_n. Then there exists a finite subset $\{d_i \mid 1 \leq i \leq s\}$ of Ω_n such that η is in the splitting field over L_n of the set of polynomials

$$\{y^2 - (t_n - d_i) \mid 1 \leq i \leq s\}.$$

For each $1 \leq i \leq s$, let $\vartheta_i \in F_{n+1}$ be an element such that $\vartheta_i^2 = t_n - d_i$. Choose an integer $k > \ell$ and an element $t \in L_n$ such that

(a) $t^{p^k} = t_n$, and

(b) $\eta \in K(t, \vartheta_1, \ldots, \vartheta_s)$.

For each $1 \leq i \leq s$, let $T_i = t^{p^k} - d_i \in K[t]$. Then the polynomials $T_1, \ldots, T_s$ satisfy the hypotheses of Lemma 2.6. So applying Lemma 2.6(c), we obtain that $\eta \in F = F_n$. □

Definition 2.8. Let K be a field and let p be a prime. Then $u \in K$ is said to be a *p-high* element of K if the equation $y^{p^m} = u$ is solvable in K for all $m \in \mathbb{N}$.

Lemma 2.9 (Fried and Kollár [FK]). *Let* K *be a field of characteristic 0, and let* t *be a transcendental element over* K*. Let* p *be an odd prime, and let* $\{t(\ell) \mid \ell \in \mathbb{N}\}$ *be a set of elements of* $\overline{K[t]}$ *such that* $t(0) = t$ *and* $t(\ell+1)^p = t(\ell)$*. Let*

$$L = K(t(0), t(1), \ldots, t(\ell), \ldots) = K(t, p).$$

Let $\{T_i \mid i \in I\}$ be a set of mutually prime polynomials in $K[t]$, none of which is divisible by t or has a multiple factor; and for each $i \in I$, let $\vartheta_i \in \overline{K[t]}$ be an element such that $\vartheta_i^2 = T_i$. Let $M = L(\ldots, \vartheta_i, \ldots)$.

(a) *If $u \in M$ is a p-high element of M, then $u = ct(\ell)^m$, for some $\ell \in \mathbb{N}$, $m \in \mathbb{Z}$ and some p-high element c of K.*
(b) *If p' is an odd prime such that $p' \neq p$ and $u \in M$ is a p'-high element of M, then $u \in K$ and u is a p'-high element of K.*

□

Lemma 2.10. *Suppose that $p \in \{p_n \mid 0 \leq n \leq 4\}$ and that d is a p-high element of F_C for some $C \in \widehat{\mathcal{P}}(G)$.*

(a) *If $p = 3$, then $d \in F_1$.*
(b) *If $p \neq 3$, then $d = et_n^{m/p^\ell}$ for some $\ell \in \mathbb{N}$, $m \in \mathbb{Z}$ and some p-high element e of F_1.*

Proof. This follows easily from Lemma 2.9. □

The following result is a slight variant of Lemma 4.5 of [FK].

Lemma 2.11. *Let $C \in \widehat{\mathcal{P}}(G)$ and let $1 \leq n \leq 4$. Suppose that $e \in F_1 \setminus \{0\}$, $d \in F_n \setminus \{0\}$ and $r = m/p_n^\ell$ for some $m \in \mathbb{Z} \setminus \{0\}$ and $\ell \in \mathbb{N}$. If the equation $y^2 = e(t_n^r - d)$ has a solution in F_C, then $r = 1$, $d \in \Omega_n^C$ and e is a square in F_C.*

Proof. To simplify notation, we shall just consider the case when $1 \leq n \leq 3$. Let $p = p_n$ and $K = F_n$. Let $\eta \in F_C$ be a solution of the equation $y^2 = e(t_n^r - d)$. By Corollary 2.7, $\eta \in F_{n+1}$. Thus there exists a finite subset $\{d_i \mid 1 \leq i \leq s\}$ of Ω_n such that η is in the splitting field over L_n of the set of polynomials

$$\{y^2 - (t_n - d_i) \mid 1 \leq i \leq s\}.$$

For each $1 \leq i \leq s$, let $\vartheta_i \in F_{n+1}$ be an element such that $\vartheta_i^2 = t_n - d_i$. Choose an integer $k > \ell$ and an element $t \in L_n$ such that

(a) $t^{p^k} = t_n$, and
(b) $\eta \in K(t, \vartheta_1, \ldots, \vartheta_s)$.

For each $1 \leq i \leq s$, let $T_i = t^{p^k} - d_i \in K[t]$. Then the polynomials $T_1, \ldots, T_s$ satisfy the hypotheses of Lemma 2.6. Note that

$$\eta^2 = e(t_n^r - d) = e(t^{mp^{k-\ell}} - d) \in K(t).$$

By Lemma 2.6, there exist polynomials $f(t)$, $g(t) \in K[t]$ and a subset Z of $\{1, \ldots, s\}$ such that

$$\eta = \frac{f(t)}{g(t)} \prod_{i \in Z} \vartheta_i.$$

It follows that

$$g^2(t)e(t^{mp^{k-\ell}} - d) = f^2(t) \prod_{i \in Z} (t^{p^k} - d_i). \tag{2.11}$$

First suppose that $m > 0$. Then we must have that $eg^2(t) = f^2(t)$, since the other factors do not have multiple roots. Hence

$$(t^{mp^{k-\ell}} - d) = \prod_{i \in Z} (t^{p^k} - d_i).$$

This implies that $|Z| = 1$, and so $t^{mp^{k-\ell}} - d = t^{p^k} - d_i$ for some $1 \leq i \leq s$. Consequently, $r = m/p^\ell = 1$ and $d = d_i \in \Omega_n$. Since $e = f^2(t)/g^2(t)$, we also see that e is a square in F_C.

Now suppose that $m < 0$. Then multiplying equation (2.11) by $t^{-mp^{k-\ell}}$, we obtain that

$$g^2(t)e(1 - dt^{-mp^{k-\ell}}) = f^2(t) \prod_{i \in Z} (t^{p^k} - d_i) t^{-mp^{k-\ell}},$$

and so

$$-edg^2(t)(t^{-mp^{k-\ell}} - d^{-1}) = f^2(t) \prod_{i \in Z} (t^{p^k} - d_i) t^{-mp^{k-\ell}}.$$

Arguing as above, we must have that

$$-edg^2(t) = f^2(t)t^{2a}$$

for some integer a such that $0 \leq -mp^{k-\ell} - 2a \leq 1$; and

$$t^{-mp^{k-\ell}} - d^{-1} = t^b \prod_{i \in Z} (t^{p^k} - d_i),$$

where $b = -mp^{k-\ell} - 2a$. It follows that $b = 0$ and that $-mp^{k-\ell} = p^k$. But then $p^k = 2a$, which is impossible since p is odd. $\square$

We are now ready to prove the converse of Lemma 2.5.

Theorem 2.12. *If C, $D \in \widehat{\mathcal{P}}(G)$ and $F_C \cong F_D$, then there exists $g \in G$ such that $Cg = D$.*

Proof. During this proof, isomorphisms will act on the left. (This will not cause any confusion, since we will never need to compose two isomorphisms.) Suppose that $\pi : F_C \to F_D$ is an isomorphism. If $s \in F_1$, then $\pi(s)$ is a 3-high element of F_D; and so Lemma 2.10(a) yields that $\pi(s) \in F_1$. Similarly if $1 \leq n \leq 3$, then $\pi(t_n)$ is a p_n-high element of F_D, and so $\pi(t_n) \in F_{n+1}$. Applying Corollary 2.7, we obtain inductively that $\pi[F_n] \subseteq F_n$ for all $1 \leq n \leq 4$. By also considering the isomorphism π^{-1}, we find that $\pi[F_n] = F_n$ for all $1 \leq n \leq 4$.

By Lemma 2.10(b), we have that $\pi(t_1) = et_1^r$, where e is a 5-high element of F_1 and $r = m/5^\ell$ for some $m \in \mathbb{Z} \smallsetminus \{0\}$ and $\ell \in \mathbb{N}$. Let $\widetilde{g} \in \Omega_1$. Then $\pi(\widetilde{g}) \in F_1$ and the equation

$$y^2 = et_1^r - \pi(\widetilde{g}) = e(t_1^r - e^{-1}\pi(\widetilde{g}))$$

has a solution in F_D. So Lemma 2.11 yields that $r = 1$, $e^{-1}\pi(\widetilde{g}) \in \Omega_1$ and e is a square in F_D. In particular, there exists an injection $\varphi : \Omega_1 \to \Omega_1$ such that $\pi(\widetilde{g}) = e\varphi(\widetilde{g})$ for all $\widetilde{g} \in \Omega_1$.

Claim 2.13. *φ is a bijection.*

Proof of Claim 2.13. Let $\widetilde{g} \in \Omega_1$. Since e is a square in F_D, the equation

$$y^2 = e(t_1 - \widetilde{g}) = et_1 - e\widetilde{g}$$

has a solution in F_D. Applying π^{-1}, we find that the equation $y^2 = t_1 - \pi^{-1}(e\widetilde{g})$ has a solution in F_C. By Lemma 2.11, there exists $\widetilde{h} \in \Omega_1$ such that $\pi^{-1}(e\widetilde{g}) = \widetilde{h}$. It follows that $\varphi(\widetilde{h}) = \widetilde{g}$. □

We shall prove that there exists an element $g \in G$ such that $\varphi(\widetilde{h}) = \widetilde{hg}$ for all $\widetilde{h} \in \Omega_1$. To see this, we shall next consider $\pi[\Omega_2]$. By Lemma 2.10(b), $\pi(t_2) = ft_2^r$, where e is a 7-high element of F_1 and $r = m/7^\ell$ for some $m \in \mathbb{Z} \smallsetminus \{0\}$ and $\ell \in \mathbb{N}$. Arguing as above, we find that $r = 1$ and that $f^{-1}\pi(w) \in \Omega_2$ for every $w \in \Omega_2$.

Claim 2.14. *$f = e$.*

Proof of Claim 2.14. Let $w = \widetilde{c} + 3\widetilde{ac} \in \Omega_2$. Then

$$f^{-1}\pi(w) = f^{-1}e\left(\varphi(\widetilde{c}) + 3\varphi(\widetilde{ac})\right) \in \Omega_2.$$

Thus there exists $d \in G$ such that

$$f^{-1}e = \frac{\varphi(\widetilde{c}) + 3\varphi(\widetilde{ac})}{\widetilde{d} + 3\widetilde{ad}};$$

and for each $h \in G$,

$$\left(\frac{\varphi(\widetilde{c}) + 3\varphi(\widetilde{ac})}{\widetilde{d} + 3\widetilde{ad}}\right)\left(\varphi(\widetilde{h}) + 3\varphi(\widetilde{ah})\right) \in \Omega_2 \subseteq F_0[x_1, \dots, z_3].$$

This is possible only if the linear polynomials $\varphi(\widetilde{c}) + 3\varphi(\widetilde{ac})$ and $\widetilde{d} + 3\widetilde{ad}$ are equal. Hence $f^{-1}e = 1$. □

In particular, for all $c \in G$, there exists $d \in G$ such that

$$\varphi(\widetilde{c}) + 3\varphi(\widetilde{ac}) = \widetilde{d} + 3\widetilde{ad}.$$

By Lemma 2.3, it follows that $\varphi(\widetilde{c}) = \widetilde{d}$ and $\varphi(\widetilde{ac}) = \widetilde{ad}$. Hence if we define $\widetilde{E}_a = \{(\widetilde{h}, \widetilde{ah}) \mid h \in G\}$, then $\varphi[\widetilde{E}_a] \subseteq \widetilde{E}_a$. Arguing as in the proof of Claim 2.13, we obtain that $\varphi[\widetilde{E}_a] = \widetilde{E}_a$. A similar argument shows that φ also preserves the relation $\widetilde{E}_b = \{(\widetilde{h}, \widetilde{bh}) \mid h \in G\}$. By Lemma 2.2, there exists an element $g \in G$ such that $\varphi(\widetilde{h}) = \widetilde{hg}$ for all $\widetilde{h} \in \Omega_1$.

Finally, arguing as above, we find that $\varphi[\Omega_4^C] = \Omega_4^D$. (The analogue of Claim 2.14 is the only point in the proof where we make use of the assumption that $|C|$, $|D| \geq 2$.) This implies that $Cg = D$. □

This completes the proof of our main result.

Theorem 2.15. *Let σ be the theory of fields of finite transcendence degree. Then $(Mod(\sigma), \cong)$ is a universal essentially countable Borel equivalence relation.*

□

In the remainder of this section, we shall prove that the theory of fields of finite transcendence degree does not admit canonical models. We shall make use of the following results of Hjorth and Kechris [HK].

Theorem 2.16. *If $\sigma \in \mathcal{L}_{\omega_1,\omega}$, then the following are equivalent.*

(i) *σ admits canonical models.*

(ii) *The logic action of S_∞ on $Mod(\sigma)$ has the cocycle property.*

□

Theorem 2.17. *The shift action of the free group $G = \langle a, b\rangle$ on $\mathcal{P}(F)$ does not have the cocycle property.*

□

It follows easily that the shift action of G on $\widehat{\mathcal{P}}(G)$ also does not have the cocycle property.

Theorem 2.18. *Let σ be the theory of fields of finite transcendence degree. Then the logic action of S_∞ on $Mod(\sigma)$ does not have the cocycle property.*

Proof. We shall use the notation which was introduced in the proof of Theorem 2.12, except that we shall once again let all isomorphisms act on the right. Suppose that the Borel cocycle α witnesses that the logic action of S_∞ on $Mod(\sigma)$ has the cocycle property. For each $C \in \widehat{\mathcal{P}}(G)$, there exists an element $\mathcal{M}_C \in Mod(\sigma)$ and a canonical isomorphism $\pi_C : F_C \to \mathcal{M}_C$. Thus whenever $F_C \cong F_D$, then the map

$$\beta(F_C, F_D) = \pi_C \alpha(\mathcal{M}_C, \mathcal{M}_D)\pi_D^{-1}$$

is an isomorphism from F_C onto F_D. Furthermore, if $F_C \cong F_D \cong F_E$, then

$$\beta(F_C, F_E) = \beta(F_C, F_D)\beta(F_D, F_E).$$

Now define γ as follows. If C, $D \in \widehat{\mathcal{P}}(G)$, let $e \in F_1$ be such that $t_1^{\beta(F_C,F_D)} = et_1$. Then $\gamma(C, D) = g \in G$ is the element such that

$$\widetilde{h}^{\beta(F_C,F_D)} = e\widetilde{hg}$$

for all $\widetilde{h} \in \Omega_1$. The proof of Theorem 2.12 shows that $Cg = D$.

Now suppose that $F_C \cong F_D \cong F_E$. Let $f \in F_1$ be such that $t_1^{\beta(F_D,F_E)} = ft_1$ and let $\gamma(D, E) = k$. Note that

$$\begin{aligned} t_1^{\beta(F_C,F_E)} &= t_1^{\beta(F_C,F_D)\beta(F_D,F_E)} \\ &= (et_1)^{\beta(F_D,F_E)} \\ &= e^{\beta(F_D,F_E)} ft_1. \end{aligned}$$

Hence for every $\widetilde{h} \in \Omega_1$,

$$\begin{aligned}\widetilde{h}^{\beta(F_D,F_E)} &= \widetilde{h}^{\beta(F_C,F_D)\beta(F_D,F_E)} \\ &= (e\widetilde{hg})^{\beta(F_D,F_E)} \\ &= e^{\beta(F_D,F_E)} f\widetilde{hgk}.\end{aligned}$$

It follows that $\gamma(C,E)\gamma(C,D) = \gamma(D,E)$. But then γ witnesses that the action of G on $\widehat{\mathcal{P}}(G)$ has the cocycle property, which is a contradiction. □

Corollary 2.19. *The theory of fields of finite transcendence degree does not admit canonical models.*

□

3. A linear representation of the free group

In this section, we shall prove Theorem 2.1. Let u be a real number which is transcendental over $\mathbb{Q}$; and let $0 < \theta < \pi$ be the real number such that $\theta = 2\arctan u$. Then

$$\sin\theta = \frac{2u}{1+u^2} \quad \text{and} \quad \cos\theta = \frac{1-u^2}{1+u^2}.$$

Let A and B be the real orthogonal matrices defined by

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

In [dG], de Groot proved that $\langle A, B\rangle$ is a free subgroup of $GL(3,\mathbb{Q}(u))$. We shall prove that G satisfies the requirements of Theorem 2.1.

Suppose that $C \in G$ is any element. Then each entry c of C has a unique expression of the form

$$c = \alpha + \sum_{i=1}^{\infty} (\beta_i \sin\theta + \gamma_i \cos\theta)\cos^{i-1}\theta,$$

where α, β_i, $\gamma_i \in \mathbb{Z}$ and $\beta_i = \gamma_i = 0$ for all but finitely many integers i. This expression will be called the *normal form* of c; and throughout this section, we shall suppose that all of the entries of each $C \in G$ have been written in normal form. For example, suppose that $n > 0$ and that $C = A^n \in G$. Then

$$C = \begin{pmatrix} \cos n\theta & -\sin n\theta & 0 \\ \sin n\theta & \cos n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the identities,

$$\cos n\theta = 2^{n-1} \cos^n \theta + \text{ terms of lower degree in } \cos\theta$$

and

$$\sin n\theta = \sin\theta \left(2^{n-1} \cos^{n-1} \theta + \text{ terms of lower degree in } \cos\theta\right)$$

we easily obtain the following result.

Lemma 3.1. *Suppose that $C = A^n \in G$, where $n \in \mathbb{Z} \smallsetminus \{0\}$. Then*

$$C = \begin{pmatrix} Q\cos\theta & -\operatorname{sgn}(n) P \sin\theta & 0 \\ \operatorname{sgn}(n) P \sin\theta & Q\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where P and Q are polynomials in $\cos\theta$ with integer coefficients of degree $d = |n| - 1$ and leading coefficient $2^{|n|-1}$.

□

Similary we obtain the following result.

Lemma 3.2. *Suppose that $C = B^m \in G$, where $m \in \mathbb{Z} \smallsetminus \{0\}$. Then*

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q\cos\theta & -\operatorname{sgn}(m) P \sin\theta \\ 0 & \operatorname{sgn}(m) P \sin\theta & Q\cos\theta \end{pmatrix}$$

where P and Q are polynomials in $\cos\theta$ with integer coefficients of degree $d = |m| - 1$ and leading coefficient $2^{|m|-1}$.

□

Finally we shall require the following the result from page 59 of Wagon [W].

Lemma 3.3. *Suppose that* $C = A^{n_1}B^{m_1}\dots A^{n_s}B^{m_s} \in G$, *where* $s \geq 1$ *and* n_i, $m_i \in \mathbb{Z} \smallsetminus \{0\}$ *for* $1 \leq i \leq s$. *Let* $\Sigma = |n_1| + |m_1| + \cdots + |n_s| + |m_s|$. *Then*

$$C = \begin{pmatrix} P_1 \cos\theta & -\operatorname{sgn}(n_1)Q_1 \sin\theta & -\operatorname{sgn}(n_1 m_s)Q_2 \cos\theta \\ P_2 \sin\theta & Q_3 \cos\theta & -\operatorname{sgn}(m_s)Q_4 \sin\theta \\ P_3 \cos\theta & P_4 \sin\theta & P_5 \cos\theta \end{pmatrix}$$

where

(a) *each* Q_ℓ *is a polynomial in* $\cos\theta$ *with integer coefficients of degree* $d = \Sigma - 1$ *and leading coefficient* $2^{\Sigma - 2s}$; *and*

(b) *each* P_ℓ *is a polynomial in* $\cos\theta$ *with integer coefficients of degree strictly less than* d.

□

Suppose now that the matrices C_1, C_2, D_1, $D_2 \in G$ satisfy the equality

$$C_1 + 3D_1 = C_2 + 3D_2.$$

If $C_1 = C_2$, then $D_1 = D_2$. So suppose that $C_1 \neq C_2$. After multiplying both sides by C_2^{-1} if necessary, we can suppose that C_2 is the identity matrix I. So we have the equality

$$C_1 - I = 3\,(D_2 - D_1)\,.$$

Also, after conjugating both sides by a suitable element of G if necessary, we can suppose that C_1 has either the form A^n for some $n \in \mathbb{Z} \smallsetminus \{0\}$, or B^m for some $m \in \mathbb{Z} \smallsetminus \{0\}$, or $A^{n_1}B^{m_1}\dots A^{n_s}B^{m_s}$ for some $s \geq 1$ and n_i, $m_i \in \mathbb{Z} \smallsetminus \{0\}$. Let $C_1 - I = (c_{ij})$. Then applying Lemmas 3.1, 3.2 and 3.3, we see that c_{22} is a polynomial in $\cos\theta$ with integer coefficients of degree $d \geq 1$ and leading coefficient 2^ℓ for some $\ell \geq 0$. On the other hand, since $(c_{ij}) = 3\,(D_2 - D_1)$, it follows that each coefficient of c_{22} is divisible by 3, which is a contradiction. This completes the proof of Theorem 2.1.

4. Concluding remarks

In this paper, we have made no attempt to keep the transcendence degree of F_C as low as possible. (For example, the proof of Theorem 2.1 shows that we could have taken $R = F_0[y_1, y_2, y_3]$; which would have lowered the transcendence degree from 14 to 8.) However, we believe that the following question is of some interest.

Question 4.1. Suppose that φ is the theory of fields of transcendence degree 1. Is $(Mod(\varphi), \cong)$ a universal essentially countable Borel equivalence relation?

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