Hereditary normality of $\gamma\mathbf{N}$

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One of the classical separation axioms of topology is complete normality. A topological space X is *completely normal* if for every pair of subsets A and B of X which are separated (i.e. $\bar{A} \cap B = \emptyset = A \cap \bar{B}$) there are disjoint open sets containing A and B respectively. A standard exercise is to show that this is equivalent to hereditary normality; that is, the property that all subspaces of X are normal. Hausdorff spaces satisfying this property are commonly designated as T_5 spaces.

Until now it has been somewhat of a mystery how well-behaved countably compact T_5 spaces can be. Under Gödel's axiom of constructibility ($V = L$) they can be quite pathological: in [**Fe**] there is a $V = L$ construction of a compact T_5 space X of cardinality $2^{\mathbf{c}}$ with a countable dense subspace, yet X has no nontrivial convergent sequences. Until the results of this paper, some of which were announced in [**NSSV**], it was not known whether such a space can be constructed just using the usual axioms of set theory [ZFC].

In this paper we show how this pathology disappears under the Proper Forcing Axiom (PFA), introduced in [**Bau**]. We will show:

Theorem 1 [PFA] *Every countably compact T_5 space is sequentially compact.*

In fact, we will show that the following, much stronger result follows from the PFA: every countable subset of a countably compact T_5 space has compact, Fréchet-Urysohn closure.

Definition 1 A space X is *Fréchet-Urysohn* if whenever a point x is in the closure of a subset A of X, there is a sequence from A converging to x.

A quick corollary is that, assuming PFA, every countably compact T_5 space with a countable dense subspace is compact and of cardinality at most $\mathbf{c}$. Other corollaries, whose proofs are liberally outlined in [**NSSV**], include:

Corollary 1 [PFA] *If X is a product of countably compact T_5 spaces, then X is countably compact.*

Indeed, one need only take a countably infinite set S in the product, project it to each factor space, take the respective compact closures, and find an accumulation point of S in the compact product of these subspaces.

Corollary 2 [PFA] *In every compact T_5 space X, the closure of a subset A can be obtained by adding, to the set $\hat{A}$ of all limits of convergent sequences in A, the set of all points x such that x is the limit of a well-ordered net in $\hat{A}$.*

Corollary 3 [PFA] *Every locally compact, T_5, separable, first countable space of cardinality $\aleph_1$ is a normal Moore space.*

Corollary 1 shows that an affirmative answer to the Scarborough-Stone problem is consistent in the T_5 case. Scarborough and Stone showed [**SS**] that the product of $\aleph_1$ sequentially compact spaces is countably compact and asked whether this continued to hold for any number of factors. The first author has solved this problem [**NV**] by producing a family of T_2 sequentially compact spaces whose product is not countably compact, but it is still important to know what happens if higher separation axioms are imposed on the factor spaces. Corollary 1, coupled with the results of [**Va**], give an independence result for the Scarborough-Stone problem for T_5 and T_6 spaces. For the case of T_3, Tychonoff, and T_4 spaces we only know that the negative answer is consistent.

The key to these results is a seemingly specialized class of spaces generically designated $\gamma\mathbf{N}$. This notation is used for any locally compact Hausdorff space X with a countable dense set of isolated points, identified with the set $\mathbf{N}$ of positive integers, such that $X \setminus \mathbf{N}$ is homeomorphic to ω_1. We will also identify $X \setminus \mathbf{N}$ with ω_1 using a definition of $\mathbf{N}$ that makes it disjoint from ω_1.

Baumgartner and the first author have independently observed that using only the usual axioms of set theory one can construct versions of $\gamma\mathbf{N}$ in which the union of $\mathbf{N}$ with the successor ordinals is not normal. On the other hand, it is also consistent that there are other versions which are T_5. It is shown in [**N**] that there can be a $\gamma\mathbf{N}$ in a model where MA+$2^{\aleph_0} \geq \aleph_3$, but the construction does not work if $2^{\aleph_0} = \aleph_2$. In this paper we will show:

Theorem 2 *Let $\kappa > \omega_1$ be a cardinal such that $\kappa^{<\kappa} = \kappa$. Then there is a ccc forcing notion $\mathcal{P}$ such that $V^{\mathcal{P}}$ satisfies* MA $+ 2^{\aleph_0} = \kappa +$ *"there is a T_5 $\gamma\mathbf{N}$-space".*

On the other hand, we will also show that the PFA, which implies both MA and $2^{\aleph_0} = \aleph_2$ (see [**Ve3**]), implies that no $\gamma\mathbf{N}$ can be T_5. This turns out to be the key to the PFA results given above. The link is provided by Theorem 5 below. It uses the following concepts.

Definition 3 A space X is *countably tight* if whenever $x \in X$ and x is in the closure of $A \subset X$, then there is a countable $B \subset A$ such that x is in the closure of B.

Definition 4 A *free sequence* in a space X is a sequence $\langle x_\xi : \xi < \alpha \rangle$ such that, for each $\beta < \alpha$, the closures of $\{x_\xi : \xi < \beta\}$ and $\{x_\xi : \xi \geq \beta\}$ are disjoint.

Definition 5 A space X is *separable* if it has a countable dense subspace. The following theorem is proved in [**Ny**].

Theorem 3 *The following are equivalent.*

- *(a) Every separable, T_5, compact space is countably tight.*
- *(b) Every free sequence in a separable, T_5, countably compact space is countable.*
- *(c) A separable, T_5, countably compact space cannot contain ω_1.*
- *(d) No version of $\gamma\mathbf{N}$ is T_5.*

1 Consequences of PFA

In this section, we will show how to derive the main PFA results mentioned in the introduction. We begin by showing that the fourth equivalent condition in Theorem 4 is implied by the following version of the Open Coloring Axiom (OCA):

If X is a separable metric space and

$$[X]^2 = K_0 \cup K_1$$

is a partition with K_0 open in the product topology then either there exists an uncountable 0-homogeneous subset of X, or else X can be covered by countably many 1-homogeneous sets.

As usual, $[A]^2$ stands for the collection of two-element subsets of A. A subset H of X is called *i-homogeneous* if $[H]^2 \subseteq K_i$. In saying K_0 is open in the product topology, what we really mean is that $\{\langle x, y\rangle : \{x, y\} \in K_0\}$ is open.

This version of OCA was introduced and proved relatively consistent with ZFC + MA $+2^{\aleph_0} = \aleph_2$ and also deduced from PFA by Todorčević ([**To1**]), who extended and refined the previous work of Abraham, Rubin, and Shelah ([**ARS**]). (For many other applications of OCA see [**To1, Ve1, Ve2**].)

Theorem 4 *Under* OCA *no version of $\gamma\mathbf{N}$ can be hereditarily normal.*

PROOF: For each $\alpha < \omega_1$ let $a_\alpha \subseteq \omega$ be such that $a_\alpha \cup [0, \alpha]$ is a compact neighborhood of $[0, \alpha]$. Since every subset of $\mathbf{N}$ must have a limit point, it follows that $a_\alpha \subset^* a_\beta$ and $a_\beta \setminus a_\alpha \subset^* U$, for every neighborhood U of $(\alpha, \beta]$ whenever $\alpha < \beta$. [As usual, $a \subset^* b$ means $a \setminus b$ is finite.] We identify $\{a_\alpha : \alpha < \omega_1\}$ with a subset of the Cantor set.

Let S be the set of all $\langle a_\xi, a_\eta, a_\mu\rangle$ such that $\xi < \eta < \mu$. Then S is a separable metric space. Define the partition

$$[S]^2 = K_0 \cup K_1$$

by $\{\langle a, b, c\rangle, \langle \bar{a}, \bar{b}, \bar{c}\rangle\} \in K_0$ iff

$$(*) \qquad a \neq \bar{a} \text{ and } [(a \setminus b) \cap (\bar{c} \setminus \bar{b}) \neq \emptyset \text{ or } (c \setminus b) \cap (\bar{b} \setminus \bar{a}) \neq \emptyset].$$

Then K_0 is open in the product topology. Indeed, one base for S^2 consists of closed-and-open subsets formed by fixing n and taking all points $\langle\, \langle a, b, c\rangle, \langle \bar{a}, \bar{b}, \bar{c}\rangle\,\rangle$ with a fixed $\langle\, \langle a \cap n, b \cap n, c \cap n\rangle, \langle \bar{a} \cap n, \bar{b} \cap n, \bar{c} \cap n\rangle\,\rangle$. But if (*) holds, it will also hold if n is large enough and each of the six sets involved is replaced by its intersection with n.

Next we show in ZFC that S can not be the union of a sequence $\{S_n : n < \omega\}$ of 1-homogeneous sets. Let T_n be the set of all ξ for which there are uncountably many η such that $\langle a_\xi, a_\eta, a_\mu\rangle \in S_n$, for some μ. Clearly some T_n must be uncountable. Fix such n and some $\xi \in T_n$. Let $\langle a_{\bar{\xi}}, a_{\bar{\eta}}, a_{\bar{\mu}}\rangle \in S_n$ be such that $\xi < \bar{\xi}$ and find $\mu > \eta > \bar{\mu}$ such that $\langle a_\xi, a_\eta, a_\mu\rangle \in S_n$. Since $\xi <$

$\bar\eta < \bar\mu < \eta$ we have $a_{\bar\mu} \setminus a_{\bar\eta} \subset^* a_\eta \setminus a_\xi$. Thus, $\{\langle a_\xi, a_\eta, a_\mu\rangle, \langle a_{\bar\xi}, a_{\bar\eta}, a_{\bar\mu}\rangle\} \in K_0$, which contradicts the fact that S_n is 1-homogeneous.

So, by OCA, there is an uncountable 0-homogeneous subset H of S. By cutting H down if necessary we may assume $\mu < \bar\xi$ whenever $\langle a_\xi, a_\eta, a_\mu\rangle$ and $\langle a_{\bar\xi}, a_{\bar\eta}, a_{\bar\mu}\rangle$ are two distinct members of H such that $\xi < \bar\xi$. Then

$$A = \bigcup\{(\xi, \eta] : \langle a_\xi, a_\eta, a_\mu\rangle \in H\}$$

and

$$B = \bigcup\{(\eta, \mu] : \langle a_\xi, a_\eta, a_\mu\rangle \in H\}$$

are separated in $\gamma\mathbf{N}$. If there were an open subset U of $\gamma\mathbf{N}$ such that $A \subset U$ and $\mathrm{cl}U \cap B = \emptyset$, we could let $c = U \cap \mathbf{N}$ and have $a_\eta \setminus a_\xi$ almost contained in c and $a_\mu \setminus a_\eta$ almost disjoint from c whenever $\langle a_\xi, a_\eta, a_\mu\rangle \in H$. Now, for every ξ there are at most one η and μ such that $\langle a_\xi, a_\eta, a_\mu\rangle \in H$. If this happens choose $n(\xi) \in \mathbf{N}$ such that

$$[(a_\eta \setminus a_\xi) \setminus c] \cup [(a_\mu \setminus a_\eta) \cap c] \subseteq [0, n(\xi)].$$

Then there is an uncountable subset I of H, $n \in \mathbf{N}$, and $a \subseteq [0, n]$ such that whenever $\langle a_\xi, a_\eta, a_\mu\rangle \in I$ then $n(\xi) = n$ and $a_\eta \cap [0, n] = a$. But then any pair of distinct elements of I is in K_1, a contradiction. □

To show Theorem 1, we combine Theorem 4 (b) and (a) with the following results from [**BDFN**]:

Theorem 5 [PFA] *Every regular space is either α-realcompact or has an uncountable free sequence.*

Theorem 6 [PFA] *Every compact space of countable tightness is sequential.*

PROOF OF THEOREM 1: Let σ be a sequence in a countably compact T_5 space X. By Theorem 5 and Theorem 4 (b), every free sequence in the closure of the range of σ is countable; hence, by Theorem 6, this closure is α-realcompact; but every α-realcompact, countably compact space is compact. [Recall that an α-realcompact space is one in which every filterbase of closed sets with the countable intersection property has nonempty intersection, whereas a space is (1) compact if, and only if every filterbase of closed sets has nonempty intersection and (2) countably compact if, and only if, every filterbase of closed sets has the countable intersection property.] Thus the range R of the sequence has compact closure. But an old result of Arhangel'skii is that a compact Hausdorff space is countably tight if, and only if, every free sequence is countable. Another application of Theorem 4 (b) shows that R has countably tight, hence (by Theorem 7) sequential closure. $\square$

Corollary 4 [PFA] *In a countably compact T_5 space, every countable subset has compact, Fréchet-Urysohn closure.*

PROOF: We need only put together the following ZFC results, and add them to the preceding proof. (1) A compact sequential space has the property that every countably compact subset is compact [**IN**]. (2) Every pseudocompact subset of a T_5 space is countably compact [**En §3.10.21**]. And (3) if a countably compact space has the property that every pseudocompact subset is compact, then the space is Fréchet-Urysohn [**Zh**]. $\square$

2 MA and a hereditarily normal $\gamma\mathbf{N}$

In this section, we prove Theorem 2 of the introduction. That is we produce models of MA in which there is a T_5 $\gamma\mathbf{N}$ and the continuum is arbitrary uncountable regular cardinal κ. To begin suppose $\mathcal{A} = \langle a_\alpha : \alpha < \omega_1 \rangle$ is an ω_1-tower in $[\mathbf{N}]^\omega$. Given $\xi < \alpha < \omega_1$ and $k \in \mathbf{N}$ let

$$u^{\mathcal{A}}(\xi, \alpha, k) = (a_\alpha \setminus a_\xi) \setminus [0, k] \text{ and } U^{\mathcal{A}}(\xi, \alpha, k) = u^{\mathcal{A}}(\xi, \alpha, k) \cup (\xi, \alpha].$$

If $\mathcal{A}$ is clear from the context we usually omit the superscript $\mathcal{A}$. Let $X(\mathcal{A})$ denote the topological space $\langle \mathbf{N} \cup \omega_1, \tau \rangle$ where the points of $\mathbf{N}$ are isolated and the neighbourhood base at $\alpha < \omega_1$ is given by $\{U(\xi, \alpha, k) : \xi < \alpha \text{ and } k \in \mathbf{N}\}$. Thus $X(\mathcal{A})$ is a typical $\gamma\mathbf{N}$ space. Given a club $C \subseteq \omega_1$ let

$$\mathcal{B}(\mathcal{A}, C) = \{\langle \xi, \alpha, k \rangle : \xi < \alpha, k \in \mathbf{N} \text{ and } (\xi, \alpha] \cap C = \emptyset\}.$$

If $p \in \mathcal{B}(\mathcal{A}, C)$ we write $p = \langle \xi_p, \alpha_p, k_p \rangle$. Let $u(p) = u(\xi_p, \alpha_p, k_p)$ and similarly $U(p) = U(\xi_p, \alpha_p, k_p)$. Given a finite subset F of $\mathcal{B}(\mathcal{A}, C)$ let

$$u(F) = \bigcup \{u(p) : p \in F\} \text{ and } U(F) = \bigcup \{U(p) : p \in F\}.$$

Finally let $\mathcal{P}(\mathcal{A}, C)$ denote the following poset. Members of $\mathcal{P}(\mathcal{A}, C)$ are pairs $\langle F, G\rangle$ of finite subsets of $\mathcal{B}(\mathcal{A}, C)$ such that $U(F) \cap U(G) = \emptyset$. Say that $\langle F^*, G^*\rangle \leq \langle F, G\rangle$ iff $F \subseteq F^*$ and $G \subseteq G^*$. Note that if $C \subseteq D$ then $\mathcal{P}(\mathcal{A}, C)$ is a subposet of $\mathcal{P}(\mathcal{A}, D)$ and two conditions are compatible in $\mathcal{P}(\mathcal{A}, C)$ iff they are compatible in $\mathcal{P}(\mathcal{A}, D)$. Thus if $\mathcal{P}(\mathcal{A}, D)$ is ccc then so is $\mathcal{P}(\mathcal{A}, C)$.

Lemma 1 *Assume* $\mathrm{MA}_{\aleph_1}$. *Then for every* ω_1*-tower* $\mathcal{A}$ *the following are equivalent.*

> *(a)* $X(\mathcal{A})$ *is* T_5
>
> *(b)* $\mathcal{P}(\mathcal{A}, C)$ *satisfies the ccc for every club* $C \subseteq \omega_1$

PROOF: (a) implies (b). Suppose for some club $\subseteq \omega_1$ there is an uncountable antichain $\{\langle F_\eta, G_\eta\rangle : \eta < \omega_1\}$ in $\mathcal{P}(\mathcal{A}, C)$. Let

$$I_\eta = \bigcup\{(\xi_p, \alpha_p] : p \in F_\eta\} \text{ and } J_\eta = \bigcup\{(\xi_q, \alpha_q] : q \in G_\eta\}.$$

Then $I_\eta \cap J_\eta = \emptyset$, for all $\eta < \omega_1$. By a standard Δ-system and counting argument we may assume without loss of generality that if $\eta < \nu$ there is $\gamma \in C$ such that $\sup I_\eta < \inf I_\nu$ and $\sup J_\eta < \inf J_\nu$. This implies that $I_\eta \cap J_\nu = \emptyset$ for every $\eta, \nu < \omega_1$. Let

$$A = \bigcup\{I_\eta : \eta < \omega_1\} \text{ and } B = \bigcup\{J_\eta : \eta < \omega_1\}.$$

It is now easy to see that A and B are separated, i.e. $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Suppose now U and W are disjoint open sets containing A and B respectively. Let $u = U \cap \mathbf{N}$ and $w = W \cap \mathbf{N}$. Note that for every $\alpha < \omega_1$ there is $k_\alpha \in \mathbf{N}$ such that $u(F_\alpha) \setminus [0, k_\alpha] \subseteq u$ and $u(G_\alpha) \setminus [0, k_\alpha] \subseteq w$. Then there is an uncountable set $X \subseteq \omega_1$, $k \in \mathbf{N}$, and $t, s \subseteq [0, k]$ such that $k_\alpha = k$, $u(F_\alpha) \cap [0, k] = s$, and $u(G_\alpha) \cap [0, k] = s$, for every $\alpha \in X$. It now follows that for every $\alpha, \beta \in X$ $\langle F_\alpha, G_\alpha\rangle$ and $\langle F_\beta, G_\beta\rangle$ are compatible. Contradiction.

(b) implies (a). Let A and B be two subsets of $X(\mathcal{A})$ such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. We have to find disjoint open sets U and W containing A and B respectively. We may assume without loss of generality that $A, B \subseteq \omega_1$. Let $C = \bar{A} \cap \bar{B}$. Clearly C is a closed subset of ω_1 in the order topology. If it is countable then either A or B is countable and then the open sets U and W

are easy to find. Thus we may assume that C is club in ω_1. By assumption $\mathcal{P}(\mathcal{A}, C)$ satisfies the ccc. Let us define a suborder $\mathcal{Q}$ of $\mathcal{P}(\mathcal{A}, C)$. Members of $\mathcal{Q}$ are pairs $\langle F, G\rangle \in \mathcal{P}(\mathcal{A}, C)$ such that $\bar{A} \cap U(G) = \bar{B} \cap U(F) = \emptyset$. Note that two conditions in $\mathcal{Q}$ are compatible in $\mathcal{Q}$ iff they are compatible in $\mathcal{P}(\mathcal{A}, C)$. Consequently $\mathcal{Q}$ is a ccc poset. For each $\alpha \in A$ and $\beta \in B$ the set

$$D_{\alpha,\beta} = \{\langle F, G\rangle \in \mathcal{Q} : \alpha \in U(F) \text{ and } \beta \in U(G)\}$$

is dense in $\mathcal{Q}$. Applying $\mathrm{MA}_{\aleph_1}$ find a filter K in $\mathcal{Q}$ which meets all of these dense sets. Let U be the union of all $U(F)$ such that there is G such that $\langle F, G\rangle \in K$ and let W be the union of al $U(G)$ such that there is F such that $\langle F, G\rangle \in K$. Then U and W are disjoint open sets containing A and B respectively. □

Given a collection $\mathcal{C}$ of partial orders let us say that $\mathcal{C}$ is *productively ccc* if the product of any finite subset of $\mathcal{C}$ possibly with repetitions satisfies the ccc. Thus $\mathcal{C}$ is productively ccc iff the poset $\mathcal{Q}$ which is the finite support product of ω copies of $\mathcal{P}$ for every $\mathcal{P} \in \mathcal{C}$ is ccc. For an ω_1-tower $\mathcal{A}$ let $\mathcal{C}(\mathcal{A})$ be the collection of all $\mathcal{P}(\mathcal{A}, C)$, where C is a club in ω_1. Our goal is to produce an ω_1-tower $\mathcal{A}$ such that $\mathcal{C}(\mathcal{A})$ is productively ccc. Let $\mathcal{P}$ denote the standard Cohen forcing for adding a subset of $\mathbf{N}$ with finite conditions.

Lemma 2 *In $V^{\mathcal{P}}$ there is an ω_1-tower $\mathcal{A}$ such that $\mathcal{C}(\mathcal{A})$ is productively ccc.*

PROOF: Let $\mathcal{B} = \langle b_\alpha : \alpha < \omega_1\rangle$ be any ω_1-tower in V. Let c be the Cohen subset of $\mathbf{N}$ introduced by $\mathcal{P}$. In $V^{\mathcal{P}}$ define $a_\alpha = b_\alpha \cap c$ and let $\mathcal{A} = \langle a_\alpha : \alpha < \omega_1\rangle$. We claim that $\mathcal{C}(\mathcal{A})$ is productively ccc. By the remark following the definition of $\mathcal{P}(\mathcal{A}, C)$ and the fact that every club in $V^{\mathcal{P}}$ contains a club in V it suffices to show that for any club $C \subseteq \omega_1$ which is in V the poset $\mathcal{P}(\mathcal{A}, C)$ is ccc. Fix such a club C. It suffices to show that for any n, any uncountable subset X of $\mathcal{P}^n(\mathcal{A}, C)$ which is in V, and any $s \in \mathcal{P}$ there is an extension t of s which forces that some two members of X are compatible. Fix such X and s. Let $X = \{\langle\langle F^i_\eta, G^i_\eta\rangle : i < n\rangle : \eta < \omega_1\}$ and suppose $s : [0, k] \to \{0, 1\}$. Let

$$I^i_\eta = \bigcup\{(\xi_p, \alpha_p] : p \in F^i_\eta\} \text{ and } J^i_\eta = \bigcup\{(\xi_q, \alpha_q] : q \in G^i_\eta\}.$$

As in the proof of the previous lemma, by applying a standard Δ-system and counting argument we may assume that for every $\eta < \nu$ there is $\gamma \in C$ such that $\sup I^i_\eta < \gamma < \inf I^i_\nu$ and $\sup J^i_\eta < \gamma < \inf J^i_\nu$, for every $i < n$. Note that for evey $F \in \mathcal{P}(\mathcal{B}, C)$ $u^{\mathcal{A}}(F) = u^{\mathcal{B}}(F) \cap c$, where c is the Cohen subset of $\mathbf{N}$ introduced by $\mathcal{P}$. Let $x^i_\eta = u^{\mathcal{B}}(F^i_\eta) \cap s^{-1}\{1\}$ and Let $y^i_\eta = u^{\mathcal{B}}(G^i_\eta) \cap s^{-1}\{1\}$. Then $x^i_\eta \cap y^i_\eta = \emptyset$, for all $\eta < \omega_1$ and $i < n$. Find distinct $\eta, \nu < \omega_1$ such that $x^i_\eta = x^i_\nu$ and $y^i_\eta = y^i_\nu$, for all $i < n$. We claim that s can be extended to force that $\langle F^i_\eta, G^i_\eta \rangle$ and $\langle F^i_\nu, G^i_\nu \rangle$ are compatible in $\mathcal{P}(\mathcal{A})$, for all $i < n$. To see this note that for every $i < n$ $u^{\mathcal{B}}(F^i_\eta) \cap u^{\mathcal{B}}(G^i_\nu)$ is finite and disjoint from $[0, k]$. Find $m \in \mathbf{N}$ sufficiently large such that $u^{\mathcal{B}}(F^i_\eta) \cap u^{\mathcal{B}}(G^i_\nu) \subseteq (k, m]$, for every $i < n$. Define $t : [0, m] \to \{0, 1\}$ by $t \restriction [0, k] = s$ and let $t \restriction (k, m]$ be constantly equal to 0. Then t is an extension of s which forces that $\langle F^i_\eta, G^i_\eta \rangle$ and $\langle F^i_\nu, G^i_\nu \rangle$ are compatible in $\mathcal{P}(\mathcal{A}, C)$ for all $i < n$, as desired. $\Box$

We are now ready to complete the proof of Theorem 2. Let in V $\kappa > \omega_1$ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Fix an ω_1-tower $\mathcal{B} = \langle b_\xi : \xi < \omega_1 \rangle$. Let $\mathcal{P}$ be the Cohen forcing for adding a subset of $\mathbf{N}$ with finite conditions. In $V^{\mathcal{P}}$ define $\mathcal{A} = \langle a_\xi : \xi < \omega_1 \rangle$ as in Lemma 2 by letting $a_\xi = b_\xi \cap c$, where c is the Cohen subset of $\mathbf{N}$ introduced by $\mathcal{P}$. Then by Lemma 2 $\mathcal{C}(\mathcal{A})$ is productively ccc. In $V^{\mathcal{P}}$ define $\mathcal{Q}$ to be the finite support product of ω copies of $\mathcal{P}(\mathcal{A}, C)$, for every club $C \subseteq \omega_1$ which is in V. Then $\mathcal{Q}$ makes each such $\mathcal{P}(\mathcal{A}, C)$ σ-centered. Now in $V^{\mathcal{P} * \mathcal{Q}}$ let $\mathcal{R}$ be the standard finite support iteration of ccc posets forcing MA and $2^{\aleph_0} = \kappa$. Since $\mathcal{P} * \mathcal{Q} * \mathcal{R}$ is ccc every club $C \subseteq \omega_1$ in $V^{\mathcal{P} * \mathcal{Q} * \mathcal{R}}$ contains a club D from V. Thus $\mathcal{P}(\mathcal{A}, C)$ is σ-centered as a subposet of $\mathcal{P}(\mathcal{A}, D)$. Now by Lemma 1 it follows that $X(\mathcal{A})$ is hereditarily normal, as desired. $\Box$.

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