POSITIONAL STRATEGIES IN LONG EHRENFEUCHT-FRAÏSSÉ GAMES

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ABSTRACT. We prove that it is relatively consistent with ZF + CH that there exist two models of cardinality $\aleph_2$ such that the second player has a winning strategy in the Ehrenfeucht-Fraïssé-game of length ω_1 but there is no σ-closed back-and-forth set for the two models. If CH fails, no such pairs of models exist.

1. INTRODUCTION

Suppose $\mathcal{A} = (A, \ldots)$ and $\mathcal{B} = (B, \ldots)$ are structures for the same vocabulary $\mathcal{L}$ of cardinality $< \kappa$. We say that a set $\mathcal{I}$ of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ has the *κ-back-and-forth property* if for every $p \in \mathcal{I}$, and every $A_0 \subseteq A$ and $B_0 \subseteq B$ of size $< \kappa$ there is $q \in \mathcal{I}$ extending p such that $A_0 \subseteq \mathrm{dom}(q)$ and $B_0 \subseteq \mathrm{ran}(q)$. We say that $\mathcal{A}$ and $\mathcal{B}$ *κ-partially isomorphic* and write $\mathcal{A} \simeq^p_\kappa \mathcal{B}$ if there is a κ-back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. The relation $\mathcal{A} \simeq^p_\kappa \mathcal{B}$ has a metamathematical interpretation. Namely, for regular κ it coincides with elementary equivalence relative to the infinitary language $L_{\infty\kappa}$. In particular, $\simeq^p_\kappa$ is an equivalence relation on the class of all $\mathcal{L}$-structures. If κ is uncountable then even for models of cardinality κ the relation $\simeq^p_\kappa$ is strictly weaker than isomorphism. This was first proved by Morley (1968, unpublished, see [8]). For instance, for $\kappa = \aleph_1$, one can take a pair of $\aleph_1$-like dense linear orders one of which contains a closed copy of ω_1 while the other doesn't.

In this paper we investigate a strengthening of the relation $\simeq^p_\kappa$. Namely, given two cardinals κ and λ and two structures $\mathcal{A}$ and $\mathcal{B}$ in a vocabulary of size $< \kappa$, we say that $\mathcal{A}$ and $\mathcal{B}$ are (κ, λ)-*partially isomorphic* and write $\mathcal{A} \simeq^p_{\kappa,\lambda} \mathcal{B}$ if there is a κ-back-and-forth set $\mathcal{I}$ between $\mathcal{A}$ and $\mathcal{B}$ such that any increasing chain of length $< \lambda$ in $\mathcal{I}$ has an upper bound in $\mathcal{I}$. The point is that the relation $\simeq^p_{\kappa,\kappa}$, unlike the weaker version $\simeq^p_\kappa$, implies isomorphism in the case that the models are of cardinality at most κ, and many classical isomorphism-proofs can be interpreted as results about the relation $\simeq^p_{\kappa,\lambda}$. Indeed, suppose κ is regular. Then any two η_κ-sets are in the relation $\simeq^p_{\kappa,\kappa}$. If they are of cardinality κ, they are isomorphic.

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Also, it is well known that any two real closed fields whose underlying orders are of type η_{ω_1} and are of cardinality ω_1 are isomorphic, see [3]. In fact, if κ is regular then any two real closed fields whose underlying orders are of type η_κ are in the relation $\simeq^p_{\kappa,\kappa}$, see [2]. Another example concerns saturated models. Any two κ-saturated elementary equivalent structures of cardinality κ are isomorphic, and the proof shows that any two κ-saturated elementary equivalent structures are in the relation $\simeq^p_{\kappa,\kappa}$. Finally, consider two κ-homogeneous structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \simeq^p_\kappa \mathcal{B}$. If they happen to be of cardinality κ they are isomorphic and the proof goes by showing that $\mathcal{A} \simeq^p_{\kappa,\kappa} \mathcal{B}$.

Thus, the relation $\simeq^p_{\kappa,\kappa}$ seems like an attractive weaker version of isomorphism. However, there are some simple questions concerning it that are still open. The most important one was raised by Dickmann [1] and Kueker [7], and asks if $\simeq^p_{\kappa,\kappa}$ is equivalent to elementary equivalence in some logic. In fact, it is not clear if $\simeq^p_{\kappa,\kappa}$ is even transitive. This was a serious obstacle to generalizing first order logic. In order to overcome this Karttunen [6] defined *tree-like partial isomorphisms*. This leads to a transitive relation which coincides with elementary equivalence in a certain logic called $\mathcal{N}_{\infty\kappa}$ and implies isomorphism for models of size κ. One can translate Karttunen's concept in terms of the existence of a winning strategy in a certain Ehrenfeucht-Fraïssé game which we now describe. To begin, we fix two regular cardinals κ and λ and two structures $\mathcal{A}$ and $\mathcal{B}$ in the same vocabulary $\mathcal{L}$ of size $< \kappa$.

Definition 1.1 ($\mathrm{EF}^\lambda_\kappa(\mathcal{A},\mathcal{B})$)**.** *There are two players $\forall$ and $\exists$. The game runs in λ rounds and proceeds as follows.*

$$\begin{array}{c|cccccc} \forall & A_0, B_0 & & \dots & A_\alpha, B_\alpha & & \dots \\ \hline \exists & & p_0 & \dots & & p_\alpha & \dots \end{array} \qquad (\alpha < \lambda)$$

At stage $\alpha < \lambda$, player $\forall$ picks $A_\alpha \subseteq A$ and $B_\alpha \subseteq B$, both of size $< \kappa$. Player $\exists$ responds by a partial isomorphism p_α between a substructure of $\mathcal{A}$ of size $< \kappa$ containing A_α and a substructure of $\mathcal{B}$ containing B_α. We require that p_α extends the p_ξ, for $\xi < \alpha$. Player $\exists$ wins the game if she plays λ rounds while obeying the rules. Otherwise player $\forall$ wins.

We write $\mathcal{A} \equiv_{\kappa,\lambda} \mathcal{B}$ if $\exists$ has a winning strategy in $\mathrm{EF}^\lambda_\kappa(\mathcal{A},\mathcal{B})$. This is clearly transitive. This concept has allowed the study of infinitary languages to take off and has been very fruitful (see e.g. [10]). One of the first new results was obtained by Hyttinen [5] who proved the Craig Interpolation Theorem and other classical results for this new logic. Still the following question remains.

Question 1. *What is the relation between $\simeq^p_{\kappa,\lambda}$ and $\equiv_{\kappa,\lambda}$?*

Clearly, if $\mathcal{A} \simeq^p_{\kappa,\lambda} \mathcal{B}$ then $\mathcal{A} \equiv_{\kappa,\lambda} \mathcal{B}$. Indeed, if $\mathcal{A} \simeq^p_{\kappa,\lambda} \mathcal{B}$ then there is a *positional* winning strategy for $\exists$ in $\mathrm{EF}^\lambda_\kappa(\mathcal{A},\mathcal{B})$, in the sense that $\exists$ only needs to know the current position in order to know how to play and win. Thus, Question 1 simply asks if the converse is true. Note that the positive answer implies that $\simeq^p_{\kappa,\lambda}$ is transitive. We concentrate on the first nontrivial case, namely the relation between $\simeq^p_{\aleph_1,\aleph_1}$ and $\equiv_{\aleph_1,\aleph_1}$. Let us first note the well known fact that $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$ can be expressed as the existence of *potential isomorphism*[1] an isomorphism in a forcing extension obtained by σ-closed forcing.

Proposition 1.2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures in the same vocabulary $\mathcal{L}$. Then $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$ if and only if there is a σ-closed forcing notion $\mathcal{P}$ such that $\Vdash_{\mathcal{P}} \mathcal{A} \cong \mathcal{B}$.* □

We recall the following results from [9] where the equivalence of $\simeq^p_{\aleph_1,\aleph_1}$ and $\equiv_{\aleph_1,\aleph_1}$ has been established in some special cases.

Theorem 1.3. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two structures in the same vocabulary $\mathcal{L}$. Then $\mathcal{A} \simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$ and $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$ are equivalent in any of the following cases:*

(1) $|\mathcal{A}|, |\mathcal{B}| \leq 2^{\aleph_0}$.
(2) *$\mathcal{A}$ and $\mathcal{B}$ have different cardinality.*
(3) *$\mathcal{A}$ and $\mathcal{B}$ are trees of height $\aleph_1$.* □

On the basis of these results it seems interesting to investigate the case when $\mathcal{A}$ and $\mathcal{B}$ are of cardinality $\aleph_2$ and CH holds. Even in this case we can have a positive result if we look at partial isomorphisms of size $\aleph_1$ rather than of size $\aleph_0$. The following result was proved in [9].

Theorem 1.4. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two structures of cardinality $\aleph_2$ in the same vocabulary $\mathcal{L}$. Then $\mathcal{A} \simeq^p_{\aleph_2,\aleph_1} \mathcal{B}$ if and only if $\mathcal{A} \equiv_{\aleph_2,\aleph_1} \mathcal{B}$.* □

The main result of this paper is that the relations $\simeq^p_{\aleph_1,\aleph_1}$ and $\equiv_{\aleph_1,\aleph_1}$ may not be equivalent for structures of size $\aleph_2$.

Theorem 1.5. *It is relatively consistent with* ZFC + CH *that there exist two relational structures $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$ in a countable vocabulary such that $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$ and $\mathcal{A} \not\simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$.* □

The remainder of the paper is organized as follows. In §2 we introduce the persistency game played on a given family of countable partial functions from ω_2 to ω_1. Given an $(\omega_1, 1)$-simplified morass $\mathfrak{M}$ we define a family $\mathcal{F} = \mathcal{F}(\mathfrak{M})$ which is strategically persistent. If $\mathfrak{M}$ is a generic morass we show that $\mathcal{F}$ does not have a σ-closed persistent subfamily. In §3 we use the family $\mathcal{F}$ from the previous section to define two structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$. If $\mathcal{F}$ is derived from a generic morass we show that $\mathcal{A} \not\simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$. Finally, in §4 we state some open questions and directions for further research.

[1]Recall that for purely relational structures $\mathcal{A} \equiv_{\omega,\omega} \mathcal{B}$ is equivalent to the existence of an isomorphism of $\mathcal{A}$ and $\mathcal{B}$ in *some* forcing extension.

2. Persistent families of functions

In this section we change the original problem and instead of considering the Ehrenfeucht-Fraïssé game on a pair of structures, we consider a certain game on a given family of countable partial functions from ω_2 to ω_1.

Let $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ be the collection of all countable partial functions from ω_2 to ω_1. We say that a subfamily $\mathcal{F}$ of $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ is *persistent* if for every $p \in \mathcal{F}$ and $\alpha < \omega_2$ there is $q \in \mathcal{F}$ extending p such that $\alpha \in \mathrm{dom}(q)$. We will also consider the following *persistency game* on $\mathcal{F}$.

Definition 2.1 ($\mathcal{G}_{\omega_1}(\mathcal{F})$)**.** *Suppose $\mathcal{F}$ is a subfamily of $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$. The game $\mathcal{G}_{\omega_1}(\mathcal{F})$ is played by players $\forall$ and $\exists$ and runs as follows:*

$$\begin{array}{c|cccccccc} \forall & \alpha_0 & & \alpha_1 & & \ldots\ \alpha_\xi & & \ldots \\ \hline \exists & & p_0 & & p_1\ \ldots & & p_\xi & \ldots \end{array} \quad (\xi < \omega_1)$$

At stage ξ player $\forall$ plays an ordinal $\alpha_\xi < \omega_2$ and $\exists$ plays $p_\xi \in \mathcal{F}$ extending p_η, for $\eta < \xi$, such that $\alpha_\xi \in \mathrm{dom}(p_\xi)$. $\exists$ wins the game if she is able to play ω_1 moves. Otherwise, $\forall$ wins.

We say that $\mathcal{F}$ is *strategically persistent* if $\exists$ has a winning strategy in $\mathcal{G}_{\omega_1}(\mathcal{F})$. One way to guarantee the existence of a winning strategy for $\exists$ is that there exist a persistent subfamily $\mathcal{D}$ of $\mathcal{F}$ which is *σ-closed*, i.e. for every sequence $(p_n)_n$ which is increasing under inclusion there is $q \in \mathcal{D}$ such that $p_n \subseteq q$, for all n. Indeed, given such a family $\mathcal{D}$, $\exists$ has a trivial winning strategy in $\mathcal{G}_{\omega_1}(\mathcal{F})$: at stage ξ she plays any $p_\xi \in \mathcal{D}$ which extends $\bigcup_{\eta<\xi} p_\eta$ and such that $\alpha_\xi \in \mathrm{dom}(p_\xi)$. The main goal of this section is to show that it is relatively consistent with ZFC that there exist a downward closed family $\mathcal{F}$ which is strategically persistent but does not have a σ-closed persistent subfamily. Indeed, given a simplified $(\omega_1, 1)$-morass $\mathfrak{M}$ we can read off a certain family $\mathcal{F} = \mathcal{F}(\mathfrak{M})$ which is strategically persistent. If $\mathfrak{M}$ is obtained by the standard forcing construction we show that $\mathcal{F}$ does not have a σ-closed persistent subfamily.

We start by recalling the relevant definitions from Velleman [11].

Definition 2.2 ([11])**.** *A simplified $(\omega_1, 1)$-morass is a pair*

$$\mathfrak{M} = \langle \langle \theta_\alpha : \alpha \leq \omega_1 \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \omega_1 \rangle \rangle,$$

where $\langle \theta_\alpha : \alpha \leq \omega_1 \rangle$ is a sequence of countable ordinals, $\mathcal{F}_{\alpha,\beta}$ is a family of order preserving embeddings from θ_α to θ_β, for $\alpha < \beta \leq \omega_1$, and the following conditions are satisfied:

(1) **(Successor)** *For every α there are $\gamma_\alpha, \eta_\alpha \leq \theta_\alpha$ such that $\theta_\alpha = \gamma_\alpha + \eta_\alpha$, $\theta_{\alpha+1} = \theta_\alpha + \eta_\alpha$ and $\mathcal{F}_{\alpha,\alpha+1} = \{\mathrm{id}_{\theta_\alpha}, s_\alpha\}$, where $\mathrm{id}_{\theta_\alpha}$ is the identity on θ_α and $s_\alpha \restriction \gamma_\alpha = \mathrm{id}_{\gamma_\alpha}$ and $s_\alpha(\gamma_\alpha + \xi) = \theta_\alpha + \xi$, for all $\xi < \eta_\alpha$. We call s_α the* shift *at α. (Figure 2).*

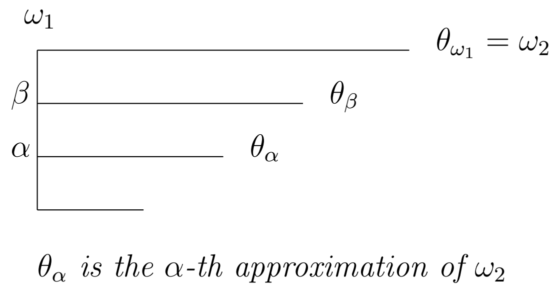

FIGURE 1. A simplified morass.

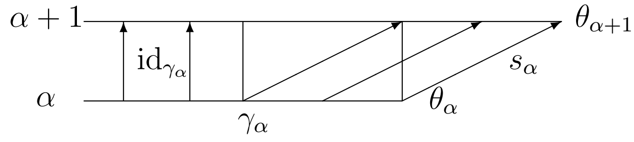

FIGURE 2. A shift

(2) **(Composition)** *If* $\alpha < \beta < \gamma$ *then* $\mathcal{F}_{\alpha\gamma} = \{g \circ f : f \in \mathcal{F}_{\alpha\beta}, g \in \mathcal{F}_{\beta\gamma}\}$.

(3) **(Factoring)** *Suppose* γ *is limit,* $\alpha < \gamma$ *and* $f, g \in \mathcal{F}_{\alpha\gamma}$. *Then there exists* β *such that* $\alpha < \beta < \gamma$, *and* $f', g' \in \mathcal{F}_{\alpha,\beta}$ *and* $h \in \mathcal{F}_{\beta\gamma}$ *such that* $f = h \circ f'$ *and* $g = h \circ g'$. *(Figure 3).*

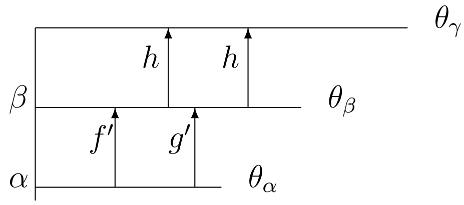

FIGURE 3. Factoring

(4) **(Fullness)** *If* $\alpha < \beta$ *then* $\theta_\beta = \bigcup\{f[\theta_\alpha] : f \in \mathcal{F}_{\alpha\beta}\}$. *Moreover,* $\theta_{\omega_1} = \omega_2$.

We then have (see [11]) that if $\alpha < \beta$ and $\xi < \theta_\beta$, then there is a *unique* predecessor of ξ on level α, i.e. there is a unique $\eta < \theta_\alpha$ such that $f(\eta) = \xi$, for some $f \in \mathcal{F}_{\alpha\beta}$. Moreover, any such f is uniquely determined on $\eta + 1$. We call η the α-th *predecessor* of ξ and write

$$\pi^\beta_\alpha(\xi) = \eta.$$

Definition 2.3. *Given a simplified* $(\omega_1, 1)$*-morass* $\mathfrak{M}$ *we define the ordering* $\preceq^{\mathfrak{M}}$ *on* ω_2 *as follows:*

$$\xi \preceq^{\mathfrak{M}} \eta \quad \textit{iff} \quad \pi^{\omega_1}_\alpha(\xi) \leq \pi^{\omega_1}_\alpha(\eta),\ \textit{for all}\ \alpha < \omega_1.$$

We also define the ordering $\preceq_\alpha^{\mathfrak{M}}$ *by:*

$$\xi \preceq_\alpha^{\mathfrak{M}} \eta \quad \textit{iff} \quad \xi \preceq^{\mathfrak{M}} \eta \quad \& \quad \pi_\alpha^{\omega_1}(\xi) = \pi_\alpha^{\omega_1}(\eta).$$

If $\mathfrak{M}$ *is clear from the context we write* $\preceq$ *for* $\preceq^{\mathfrak{M}}$ *and* $\preceq_\alpha$ *for* $\preceq_\alpha^{\mathfrak{M}}$.

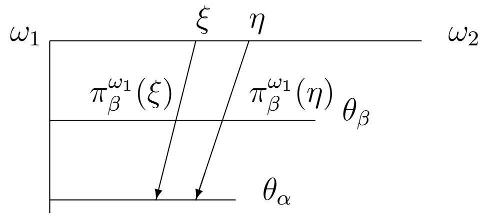

FIGURE 4. The ordering $\preceq$.

Given a simplified $(\omega_1, 1)$-morass $\mathfrak{M}$, we define a certain subfamily $\mathcal{F}(\mathfrak{M})$ of $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ and show that it is strategically persistent.

Definition 2.4. *Suppose* $\mathfrak{M}$ *is a simplified* $(\omega_1, 1)$*-morass. Let* $\mathcal{F}(\mathfrak{M})$ *be the set of all* $f \in \mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ *such that:*

(1) *if* $\xi, \eta \in \mathrm{dom}(f)$, $f(\eta) = \alpha$ *and* $\xi \preceq_\alpha \eta$, *then* $f(\xi) = \alpha$.
(2) $f^{-1}\{\alpha\}$ *is* $\preceq$*-bounded, for all* $\alpha \in \mathrm{ran}(f)$.

Note that the family $\mathcal{F}(\mathfrak{M})$ is closed under subfunctions. If $\mathfrak{M}$ is clear from the context, we will write $\mathcal{F}$ for $\mathcal{F}(\mathfrak{M})$. We first show the following.

Lemma 2.5. *Suppose* $\mathfrak{M}$ *is a simplified* $(\omega_1, 1)$*-morass. Then* $\mathcal{F}(\mathfrak{M})$ *is strategically persistent.*

Proof. Given $\xi, \eta < \omega_2$, by (4) and (3) of Definition 2.2 there exists $\alpha < \omega_1$ and $f \in \mathcal{F}_{\alpha,\omega_1}$ such that $\xi, \eta \in \mathrm{ran}(f)$. Let $\mu(\xi, \eta)$ be the least such α. If $\xi < \eta$ it follows that $\pi_\beta^{\omega_1}(\xi) < \pi_\beta^{\omega_1}(\eta)$, for every β such that $\mu(\xi, \eta) \leq \beta < \omega_1$. We now describe a strategy for $\exists$ in the persistency game on $\mathcal{F}(\mathfrak{M})$. At every stage j if player $\forall$ plays some $\xi_j < \omega_2$ then player $\exists$ picks an ordinal $\alpha_j < \omega_1$ and plays $p_j = \{\langle \xi_i, \alpha_i \rangle : i \leq j\}$. Thus, we only need to describe how to choose the ordinals α_j and check that the corresponding function p_j belongs to $\mathcal{F}(\mathfrak{M})$. Suppose we are at stage j and player $\forall$ plays ξ_j. Player $\exists$ first asks if there is an ordinal $i < j$ such that $\xi_j \preceq_{\alpha_i} \xi_i$. If so, then $\exists$ picks the least such i and sets $\alpha_j = \alpha_i$. Otherwise, $\exists$ picks any ordinal α_j strictly bigger than the α_i, for $i < j$, and $\mu(\alpha_i, \alpha_j)$, for $i < j$. In order to check that the corresponding functions p_j are in $\mathcal{F}(\mathfrak{M})$ we need the following.

Claim. *At every stage* j *there is at most one* α *for which there is* $i < j$ *such that* $\xi_j \preceq_\alpha \xi_i$ *and* $\alpha_i = \alpha$.

Proof. Suppose there were two distinct such ordinals, say α and β. Let k be the least such that $\alpha_k = \alpha$ and $\xi_j \preceq_\alpha \xi_k$ and, similarly, let l be the least such that $\alpha_l = \beta$ and $\xi_j \preceq_\beta \xi_l$. Suppose that $k < l$. Notice that, by the minimality of l, there is no $i < l$ such that $\alpha_i = \beta$ and $\xi_l \preceq_\beta \xi_i$. Therefore, by the definition of α_l, it follows that β is bigger than α and $\mu(\xi_k, \xi_l)$. We consider two cases.

Case 1. $\xi_k < \xi_l$. Since $\beta > \mu(\xi_k, \xi_l)$ we have that $\pi^{\omega_1}_\beta(\xi_k) < \pi^{\omega_1}_\beta(\xi_l)$. Since $\xi_j \preceq_\alpha \xi_k$ and $\alpha < \beta$ we have that $\pi^{\omega_1}_\beta(\xi_j) \leq \pi^{\omega_1}_\beta(\xi_k)$. Therefore, we have that $\pi^{\omega_1}_\beta(\xi_j) < \pi^{\omega_1}_\beta(\xi_l)$. On the other hand, we have that $\xi_j \preceq_\beta \xi_l$, which means that, in particular, $\pi^{\omega_1}_\beta(\xi_j) = \pi^{\omega_1}_\beta(\xi_l)$, a contradiction.

Case 2. $\xi_l < \xi_k$. Since $\beta > \mu(\xi_k, \xi_l)$ we have that $\pi^{\omega_1}_\gamma(\xi_l) < \pi^{\omega_1}_\gamma(\xi_k)$, for all $\gamma \geq \beta$. We also have that $\pi^{\omega_1}_\beta(\xi_j) = \pi^{\omega_1}_\beta(\xi_l)$. Since $\xi_j \preceq_\alpha \xi_k$ and $\alpha < \beta$ it follows that $\xi_l \preceq_\alpha \xi_k$. Therefore, at stage l we should have let $\alpha_l = \alpha$, a contradiction.
□

Now, we check that the functions p_j belong to $\mathcal{F}(\mathfrak{M})$, for all j. Condition (1) of Definition 2.4 is satisfied by the construction. To verify (2), suppose $\alpha \in \operatorname{ran}(p_j)$ and notice that if i is the least such that $\alpha_i = \alpha$ then ξ_i is the $\preceq_\alpha$-largest element of $p_j^{-1}\{\alpha\}$. Therefore, $p_j^{-1}\{\alpha\}$ is $\preceq$-bounded. This completes the proof of Lemma 2.5. □

In order to show that $\mathcal{F}(\mathfrak{M})$ does not have a σ-closed persistent family we will need to assume certain properties of $\mathfrak{M}$.

Definition 2.6. *Let $\mathfrak{M}$ be a simplified $(\omega_1, 1)$-morass.*

1. *We say that $\mathfrak{M}$ is* stationary *if $\mathcal{S}(\mathfrak{M}) = \{f[\theta_\alpha] : \alpha < \omega_1$ and $f \in \mathcal{F}_{\alpha,\omega_1}\}$ is a stationary subset of $[\omega_2]^\omega$.*
2. *We say that $\mathfrak{M}$ satisfies the* $\aleph_2$-antichain condition *if for every $X \subseteq (\omega_2)^\omega$ of size ω_2 there are distinct $s, t \in X$ such that $s(n) \preceq t(n)$, for all n, i.e. there is no antichain of size $\aleph_2$ in $(\omega_2, \preceq)^\omega$ under the product ordering.*

We first show that if $\mathfrak{M}$ has the above properties then $\mathcal{F}(\mathfrak{M})$ does not have a σ-closed persistency subfamily. Then we show that if $\mathfrak{M}$ is obtained by the standard forcing for adding a simplified $(\omega_1, 1)$-morass then $\mathfrak{M}$ has the above properties.

Lemma 2.7. *Suppose* CH *holds and $\mathfrak{M}$ is a simplified $(\omega_1, 1)$-morass which satisfies the $\aleph_2$-antichain condition. Let $\mathcal{A}$ be a subset of $\mathcal{F}(\mathfrak{M})^\omega$ of size $\aleph_2$. Then there is $\vec{g} \in \mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{A}$ of size $\aleph_2$ such that for every $\vec{h} \in \mathcal{B}$, every n, and every $f \in \mathcal{F}(\mathfrak{M})$, if f extends h_n and* $\operatorname{dom}(g_n) \subseteq \operatorname{dom}(f)$ *then f extends g_n.*

Proof. First, observe that if X is a subset of $(\omega_2)^\omega$ of size $\aleph_2$ then there is $s \in X$ and $Y \subseteq X$ of size $\aleph_2$ such that $s(n) \preceq t(n)$, for all $t \in Y$ and all n. To see this, let Z be a maximal antichain in X. Then every element of X is comparable with an element of Z. Since $\preceq$ refines the usual ordering on ω_2, by CH, for every $s \in Z$ the set of $t \in X$ such that $t(n) \preceq s(n)$, for all n, has size at most $\aleph_1$. Therefore, there is $s \in Z$ such that the set

$$Y = \{t \in X : s(n) \preceq t(n), \text{ for all } n\}$$

is of size $\aleph_2$. Then s and Y are as required.

We now turn to the proof of the lemma. First of all, we may assume that there is a fixed ordinal $\alpha < \omega_1$ such that $\alpha = \sup(\bigcup_n \operatorname{ran}(g_n))$, for all $\vec{g} \in \mathcal{A}$. By CH, we may assume that there is a fixed ordinal $\mu > \alpha$ and, for each n a subset E_n of θ_μ such that, for every $\vec{g} \in A$, there is $f_{\vec{g}} \in \mathcal{F}_{\mu,\omega_1}$ such that $f_{\vec{g}}[E_n] = \operatorname{dom}(g_n)$. Consider now the functions $e_{n,\vec{g}} = g_n \circ f_{\vec{g}}$, for $\vec{g} \in \mathcal{A}$ and $n < \omega$. By CH again, we may assume that there are fixed functions e_n, such that $e_{n,\vec{g}} = e_n$, for all $\vec{g} \in \mathcal{A}$ and n. By the first paragraph of this proof, there is $\vec{g} \in \mathcal{A}$ and a subset $\mathcal{B}$ of $\mathcal{A}$ of size $\aleph_2$ such that $f_{\vec{g}}(\xi) \preceq f_{\vec{h}}(\xi)$, for all $\vec{h} \in \mathcal{B}$ and $\xi < \theta_\mu$. We claim that $\vec{g}$ and $\mathcal{B}$ are as required. To see this, consider some $\vec{h} \in \mathcal{B}$ and some integer n. Let u be any extension of h_n which belongs to $\mathcal{F}(\mathfrak{M})$ and is defined on $\operatorname{dom}(g_n)$. We check that u extends g_n. Let $\rho \in \operatorname{dom}(g_n)$. Then there is $\xi \in E_n$ such that $f_{\vec{g}}(\xi) = \rho$. Let $\rho' = f_{\vec{h}}(\xi)$. Then $\rho \preceq_\mu \rho'$. Since u extends h_n and $h_n(\rho') \leq \mu$, by (1) of Definition 2.4 it follows that $u(\rho) = h_n(\rho')$. On the other hand, $g_n(\rho) = h_n(\rho') = e_n(\xi)$. Therefore, $u(\rho) = g_n(\rho)$. Since ρ was arbitrary it follows that u extends g_n. $\square$

Lemma 2.8. *Assume* CH *and let $\mathfrak{M}$ be a simplified $(\omega_1, 1)$-morass which is stationary and satisfies the $\aleph_2$-antichain condition. Then there is no σ-closed persistent subfamily of $\mathcal{F}(\mathfrak{M})$.*

Proof. Fix a persistent subfamily $\mathcal{G}$ of $\mathcal{F}(\mathfrak{M})$. We need to show that $\mathcal{G}$ is not σ-closed. Let τ be a sufficiently large regular cardinal. Since $\mathcal{S}(\mathfrak{M})$ is stationary in $[\omega_2]^\omega$, we can find a countable elementary submodel M of H_τ containing all the relevant objects such that $M \cap \omega_2 \in \mathcal{S}(\mathfrak{M})$. Let $\zeta = \sup(M \cap \omega_2)$ and fix an increasing sequence $\{\zeta_n\}_n$ of ordinals in M which is cofinal in ζ.

We now work in M. For each $\delta < \omega_2$ fix $g^0_\delta \in \mathcal{G}$ such that $\delta \in \operatorname{dom}(g^0_\delta)$. We can find $\alpha < \omega_1$ and $X_0 \subseteq \omega_2 \setminus \zeta_0$ of size $\aleph_2$ such that $g^0_\delta(\delta) = \alpha$, for all $\delta \in X_0$. Since $\mathfrak{M}$ satisfies the $\aleph_2$-antichain condition, by Lemma 2.7 we can fix $\delta_0 \in X_0$ and $X_1 \subseteq X_0 \setminus \zeta_1$ of size $\aleph_2$ such that, for all $\delta \in X_1$, any extension of g^0_δ to a function in $\mathcal{F}(\mathfrak{M})$ which is defined on $\operatorname{dom}(g^0_{\delta_0})$ must extend $g^0_{\delta_0}$. For each $\delta \in X_1$ fix some $g^1_\delta \in \mathcal{G}$ which extends g^0_δ and is defined on $\operatorname{dom}(g^0_{\delta_0})$. It follows that $g^0_{\delta_0} \cup g^0_\delta \subseteq g^1_\delta$. By Lemma 2.7 again, we can fix $\delta_1 \in X_1$ and $X_2 \subseteq X_1 \setminus \zeta_2$ of size $\aleph_2$ such that, for all $\delta \in X_2$ and all $h \in \mathcal{F}(\mathfrak{M})$, if h extends g^1_δ and is defined on $\operatorname{dom}(g^1_{\delta_1})$ then h extends $g^1_{\delta_1}$. We continue like this and get an increasing sequence $(\delta_n)_n$ of ordinals from M, a decreasing sequence $(X_n)_n$ of subsets of ω_2 of size $\aleph_2$, and, for each n and $\delta \in X_n$, a function $g^n_\delta \in \mathcal{G}$ such that:

(1) $\delta_n \geq \zeta_n$, for all n,
(2) $g^n_{\delta_n} \cup g^n_\delta \subseteq g^{n+1}_\delta$, for all $\delta \in X_{n+1}$.

While the sequence $(\zeta_n)_n$ does not belong to M, at each stage we need to know only finitely many of the ζ_n. Therefore, we can perform each step of the construction inside M. It follows that $(g^n_{\delta_n})_n$ is an increasing sequence of functions from $\mathcal{G}$ and $g^n_{\delta_n}(\delta_n) = \alpha$, for all n. The sequence $(\delta_n)_n$ is cofinal in ζ and, since $M \cap \omega_2 \in \mathcal{S}(\mathfrak{M})$, it follows that it is unbounded in the sense of $\preceq$. Therefore, any functions which extends $\bigcup_n g^n_{\delta_n}$ violates (2) of Definition 2.4 and cannot be in $\mathcal{F}(\mathfrak{M})$. It follows that $\mathcal{G}$ is not σ-closed. □

We now consider the standard forcing notion for adding a simplified $(\omega_1, 1)$-morass and show that the generic morass is stationary and satisfies the $\aleph_2$-antichain condition. Before we start, it will be convenient to make the following definition.

Definition 2.9. *For $\beta < \omega_2$ let I_β be the interval $[\omega_1 \cdot \beta, \omega_1 \cdot (\beta+1))$. We say that a subset A of ω_2 is ω_1-*full *if $A \cap I_\beta$ is an initial segment of I_β, for all $\beta < \omega_2$.*

We now state a slight variation of the standard forcing for adding a simplified $(\omega_1, 1)$-morass from [11].

Definition 2.10 ([11])**.** *The forcing notion $\mathcal{P}$ consists of tuples*

$$p = \langle \langle \theta^p_\alpha : \alpha \leq \delta_p \rangle, \langle \mathcal{F}^p_{\alpha,\beta} : \alpha < \beta \leq \delta_p \rangle, A_p, i_p \rangle,$$

where $\delta_p < \omega_1$, $\langle \theta^p_\alpha : \alpha \leq \delta_p \rangle$ is a sequence of limit ordinals $< \omega_1$, $\mathcal{F}^p_{\alpha,\beta}$ is a collection of order-preserving mappings from θ^p_α to θ^p_β, A_p is an ω_1-full subset of ω_2, i_p is an order preserving bijection between $\theta^p_{\delta_p}$ and A_p, and the following conditions hold:

(1) *$\mathcal{F}^p_{\alpha,\alpha+1} = \{\mathrm{id}_{\theta_\alpha}, s_\alpha\}$, where s_α is a shift as in Definition 2.2 (1).*
(2) *If $\alpha < \beta < \gamma \leq \delta_p$ then $\mathcal{F}^p_{\alpha,\gamma} = \{g \circ f : f \in \mathcal{F}^p_{\alpha,\beta}, g \in \mathcal{F}^p_{\beta,\gamma}\}$.*
(3) *Suppose $\alpha < \gamma \leq \delta_p$, γ limit and $f, g \in \mathcal{F}^p_{\alpha,\gamma}$. Then there is β such that $\alpha < \beta < \gamma$ and there are $f', g' \in \mathcal{F}^p_{\alpha,\beta}$ and $h \in \mathcal{F}^p_{\beta,\gamma}$ such that $f = h \circ f'$ and $g = h \circ g'$.*
(4) *If $\alpha < \beta \leq \delta_p$ then $\theta^p_\beta = \bigcup \{f[\theta^p_\alpha] : f \in \mathcal{F}^p_{\alpha\beta}\}$.*

The ordering of $\mathcal{P}$ is defined as follows. We say that $q \leq p$ if $\delta_p \leq \delta_q$, $\theta^p_\alpha = \theta^q_\alpha$ for $\alpha \leq \delta_p$, $\mathcal{F}^p_{\alpha,\beta} = \mathcal{F}^q_{\alpha,\beta}$ if $\alpha < \beta \leq \delta_p$, and $i_p = i_q \circ h$, for some $h \in \mathcal{F}^q_{\delta_p,\delta_q}$. Note that, in particular, this means that $A_p \subseteq A_q$.

Lemma 2.11. *Let $(p_n)_n$ be a decreasing sequence of conditions in $\mathcal{P}$. Then there is $q \in \mathcal{P}$ such that $A_q = \bigcup_n A_{p_n}$ and $q \leq p_n$, for all n. In particular, $\mathcal{P}$ is σ-closed.*

Proof. Suppose $(p_n)_n$ is a decreasing sequence of conditions in $\mathcal{P}$. We define the required condition q. We let $A_q = \bigcup_n A_{p_n}$ and $\delta_q = \sup_n \delta_{p_n}$. Note that, since the sequence of the A_{p_n} is increasing and each of them is ω_1-full, then so is A_q. Let $\theta^q_{\delta_q}$ be the order type of A_q and i_q the order preserving bijection between $\theta^q_{\delta_q}$ and A_q. For $\alpha < \delta_q$ we let θ^q_α be equal to $\theta^{p_n}_\alpha$, for any sufficiently large n. Also, for $\alpha < \beta < \delta^q$ we let $\mathcal{F}^q_{\alpha,\beta}$ be equal to $\mathcal{F}^{p_n}_{\alpha,\beta}$, for any sufficiently large n. It remains to define the collections $\mathcal{F}^q_{\alpha,\delta_q}$, for $\alpha < \delta_q$. Fix some $\alpha < \delta_q$ and let n be sufficiently large such that $\alpha < \delta_{p_n}$. We let

$$\mathcal{F}^q_{\alpha,\delta_q} = \{i_q^{-1} \circ i_{p_n} \circ f : f \in \mathcal{F}^{p_n}_{\alpha,\delta_{p_n}}\}.$$

It is straightforward to check that $q = \langle\langle\theta^q_\alpha : \alpha \le \delta_q\rangle, \langle\mathcal{F}^q_{\alpha,\beta} : \alpha < \beta \le \delta_q\rangle, A_q, i_q\rangle$ is a condition and $q \le p_n$, for all n. □

It follows that $\mathcal{P}$ preserves ω_1. We now need a lemma on the compatibility of conditions in $\mathcal{P}$. First, let us say that two condtions p and q are *isomorphic* if $\delta_p = \delta_q$, $\theta^p_\alpha = \theta^q_\alpha$, for all $\alpha \le \delta_p$, and $\mathcal{F}^p_{\alpha,\beta} = \mathcal{F}^q_{\alpha,\beta}$, for all $\alpha < \beta \le \theta^p_{\delta_p}$. If p and q are isomorphic, we say that they are *directly compatible* if there is $r \le p, q$ such that $\delta_r = \delta_p + 1$. We call such r the *amalgamation* of p and q.

Lemma 2.12. *Suppose p and q are two isomorphic conditions in $\mathcal{P}$ such that $A_p \cap A_q$ is an initial segment of both A_p and A_q, and* $\sup(A_p \setminus A_q) < \inf(A_q \setminus A_p)$. *Then p and q are directly compatible.*

Proof. We define a condition r which is the amalgamation of p and q. For simplicity, set $\delta = \delta_p = \delta_p$ and $\theta_\alpha = \theta^p_\alpha = \theta^q_\alpha$, for all $\alpha \le \delta$. Let $\delta_r = \delta + 1$ and $A_r = A_p \cup A_q$. Note that, since A_p and A_q are ω_1-full, then so is A_r. Let $\theta^r_{\delta_r}$ be the order type of A_r and i_r the order preserving bijection between $\theta^r_{\delta_r}$ and A_r. For $\alpha < \beta \le \delta$ let $\mathcal{F}^r_{\alpha,\beta} = \mathcal{F}^p_{\alpha,\beta}$. Let $R = A_p \cap A_q$, let γ be the order type of R and η the order type of $A_p \setminus A_q$ and $A_q \setminus A_p$. Since $\sup(A_p \setminus A_q) < \inf(A_q \setminus A_p)$ it follows that $\theta^r_{\delta_r} = \theta_\delta + \eta$. Let $s : \theta_\delta \to \theta^r_{\delta_r}$ be the shift of θ_δ at γ, i.e. it is the identity on γ and $s(\gamma + \xi) = \theta_\delta + \xi$, for all $\xi < \eta$. We let $\mathcal{F}^r_{\delta,\delta_r} = \{\mathrm{id}_{\theta_\delta}, s\}$. Finally, for $\alpha < \delta$ let

$$\mathcal{F}^r_{\alpha,\delta_r} = \{g \circ f : f \in \mathcal{F}^p_{\alpha,\delta}, g \in \mathcal{F}^r_{\delta,\delta_r}\}.$$

Then r is as required. □

Remark Let p and q be as in Lemma 2.12 and let r be the amalgamation of p and q. Let i be the order preserving bijection between A_p and A_q. What is important for our purposes is that r forces that $\xi \preceq_{\delta_p} i(\xi)$, for all $\xi \in A_p$.

Lemma 2.13. *Let $\alpha < \omega_2$. Then, for every $p \in \mathcal{P}$ there is $r \le p$ such that $\alpha \in A_r$.*

Proof. Let β be such that $\alpha \in I_\beta$. We show that every condition p has an extension r such that $A_r \cap I_\beta$ is a proper extension of $A_p \cap I_\beta$. Since $A_r \cap I_\beta$ is an initial segment of I_β, for every r, the order type of I_β is ω_1 and $\mathcal{P}$ is σ-closed, by iterating this operation countably many times we can find a condition $s \leq p$ such that $\alpha \in A_s$. So, fix some $p \in \mathcal{P}$. Assume first that $A_p \setminus \omega_1 \cdot (\beta+1)$ is non empty and let η be its order type. Note that η is a countable ordinal. Let $\mu = \min(I_\beta \setminus A_p)$. Since A_p is ω_1-full we have that $A_p \cap [\mu, \omega_1 \cdot (\beta+1)) = \emptyset$. Let $\nu = \mu + \eta$ and let $A_q = (A_p \cap \omega_1 \cdot \beta) \cup [\omega_1 \cdot \beta, \nu)$. Then A_p and A_q have the same order type, $A_p \cap A_q$ is an initial segment of both of them, and $\sup(A_q \setminus A_p) < \inf(A_p \setminus A_q)$. Also note that $A_p \cup A_q$ is ω_1-full. Let i_q be the isomorphism between $\theta^p_{\delta_p}$ and A_q. Let $\delta_p = \delta_q$, $\theta^p_\alpha = \theta^q_\alpha$, for all $\alpha \leq \delta_p$, $\mathcal{F}^p_{\alpha,\beta} = \mathcal{F}^q_{\alpha,\beta}$, for all $\alpha < \beta \leq \theta^p_{\delta_p}$. Then p and q satisfy the assumptions of Lemma 2.12. Let r be their amalgamation. Then $r \leq p$ and $A_r \cap I_\beta$ is a proper extension of $A_p \cap I_\beta$, as required.

Assume now that $A_p \subseteq \omega_1 \cdot (\beta+1)$. For simplicity, let $\delta = \delta_p$ and $\theta_\alpha = \theta^p_\alpha$, for $\alpha \leq \delta$. Recall that this implies that θ_δ is the order type of A_p. Let $\mu = \min(I_\beta \setminus A_p)$. We are going to define the condition r directly. We let $A_r = A_p \cup [\mu, \mu + \theta_\delta)$. We let $\delta_r = \delta + 1$. We let $\theta^r_\alpha = \theta_\alpha$, for all $\alpha \leq \delta$ and $\theta^r_{\delta+1} = \theta_\delta \cdot 2$. We let $\mathcal{F}^r_{\alpha,\beta} = \mathcal{F}^p_{\alpha,\beta}$, for all $\alpha < \beta \leq \delta$. We let $\mathcal{F}^r_{\delta,\delta+1} = \{\mathrm{id}_{\theta_\delta}, s\}$, where $\mathrm{id}_{\theta_\delta}$ is the identity on θ_δ and s is the shift of θ_δ at 0, i.e. $s(\rho) = \theta_\delta + \rho$, for all $\rho < \theta_\delta$. For $\alpha < \delta$ we let $\mathcal{F}^r_{\alpha,\delta+1}$ consist of all functions of the form $g \circ f$, where $f \in \mathcal{F}^p_{\alpha,\delta}$ and $g \in \mathcal{F}^r_{\delta,\delta+1}$. Finally, let i_r be the order preserving bijection between $\theta_\delta \cdot 2$ and A_r. Then r is an extension of p and $A_r \cap I_\beta$ is a proper extension of $A_p \cap I_\beta$. □

Lemma 2.14. *Assume* CH. *Then $\mathcal{P}$ satisfies the $\aleph_2$-chain condition.*

Proof. Let $\mathcal{A}$ be a subset of $\mathcal{P}$ of size $\aleph_2$. By CH we may assume that all the conditions in $\mathcal{A}$ are compatible. Therefore, we can fix an ordinal δ, a sequence $\langle \theta_\alpha : \alpha \leq \delta \rangle$ and a sequence $\langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \delta$ such that every condition p in A is of the form $p = \langle \langle \theta_\alpha : \alpha \leq \delta \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha \leq \beta \leq \delta \rangle, A_p, i_p \rangle$, for some A_p of order type θ_δ where i_p is the order preserving bijection between θ_δ and A_p. By CH again and the Δ-system lemma, we may find distinct $p, q \in \mathcal{A}$ such that $A_p \cap A_q$ is an initial segment of both A_p and A_q and such that $\sup(A_p \setminus A_q) < \inf(A_q \setminus A_p)$. By Lemma 2.12 p and q are compatible, as required. □

Assume CH. By Lemmas 2.11 and 2.14 $\mathcal{P}$ preserves cardinals. Let G be a $\mathcal{P}$-generic filter over V. For $\alpha < \omega_1$, we let θ^G_α be equal to θ^p_α, for any $p \in G$ such that $\alpha \leq \delta_p$. We also let $\theta^G_{\omega_1} = \omega_2$. For $\alpha < \beta < \omega_1$ we let $\mathcal{F}^G_{\alpha,\beta}$ be equal to $F^p_{\alpha,\beta}$, for any $p \in G$ such that $\beta \leq \delta_p$. For $\alpha < \omega_1$ we define:

$$\mathcal{F}^G_{\alpha,\omega_1} = \{ i_p \circ f : f \in \mathcal{F}^p_{\alpha,\delta_p}, p \in G \text{ and } \alpha \leq \delta_p \}.$$

It follows that

$$\mathfrak{M}_G = \langle \langle \theta^G_\alpha : \alpha \leq \omega_1 \rangle, \langle \mathcal{F}^G_{\alpha,\beta} : \alpha < \beta \leq \omega_1 \rangle \rangle$$

is a simplified $(\omega_1, 1)$-morass. Let $\dot{\mathfrak{M}}$ be the canonical $\mathcal{P}$-name for $\mathfrak{M}_G$.

Lemma 2.15. $\Vdash_{\mathcal{P}} \dot{\mathfrak{M}}$ *is stationary.*

Proof. Suppose $p \Vdash_{\mathcal{P}} \dot{C}$ is a club in $[\omega_2]^\omega$. Set $p_0 = p$. By using Lemmma 2.13 and 2.11 repeatedly and the fact that p forces $\dot{C}$ to be unbounded in $[\omega_2]^\omega$, we can build a decreasing sequence $(p_n)_n$ of conditions in $\mathcal{P}$ and an increasing sequence $(B_n)_n$ of countable subsets of ω_2 such that $A_{p_n} \subseteq B_n \subseteq A_{p_{n+1}}$ and $p_{n+1} \Vdash_{\mathcal{P}} B_n \in \dot{C}$, for all n. Let q be the limit of the sequence $(p_n)_n$ as in Lemma 2.11. Then $A_q = \bigcup_n A_{p_n} = \bigcup_n B_n$. Since $\dot{C}$ is forced by p to be closed and $q \leq p$ it follows that $q \Vdash_{\mathcal{P}} A_q \in \dot{C}$. Since $q \Vdash_{\mathcal{P}} A_q \in \mathcal{S}(\dot{\mathfrak{M}})$ and $\dot{C}$ was arbitrary, it follows that $\dot{\mathfrak{M}}$ is forced to be stationary. □

Lemma 2.16. *Assume* CH *holds in* V*. Then* $\Vdash_{\mathcal{P}} \dot{\mathfrak{M}}$ *satisfies the* $\aleph_2$*-antichain condition.*

Proof. Suppose $p \in \mathcal{P}$ forces that $\dot{X}$ is a subset of $(\omega_2)^\omega$ of size $\aleph_2$. We can find a subset S of $(\omega_2)^\omega$ of size $\aleph_2$ and, for each $s \in S$, a condition $p_s \leq p$ such that $p_s \Vdash_{\mathcal{P}} s \in \dot{X}$. By Lemma 2.13 we may assume that $\mathrm{ran}(s) \subseteq A_{p_s}$, for all s. By CH we may assume that the conditions p_s, for $s \in S$, are all isomorphic. Let us fix an ordinal δ, a sequence $\langle \theta_\alpha : \alpha \leq \delta \rangle$ and a sequence $\langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \delta \rangle$ such that every condition p_s, for $s \in S$, is of the form $p_s = \langle \langle \theta_\alpha : \alpha \leq \delta \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha \leq \beta \leq \delta \rangle, A_{p_s}, i_{p_s} \rangle$, for some A_{p_s} of order type θ_δ, where i_{p_s} is the order preserving bijection between θ_δ and A_{p_s}. Further, again by CH, we may assume that there are fixed ordinals $\xi_n < \theta_\delta$, for $n < \omega$, such that $s(n) = i_{p_s}(\xi_n)$, for all $s \in S$ and all n. By the Δ-system lemma, we may find distinct $s, t \in S$ such that $A_{p_s} \cap A_{p_t}$ is an initial segment of both A_{p_s} and A_{p_t} and such that $\sup(A_{p_s} \setminus A_{p_t}) < \inf(A_{p_t} \setminus A_{p_s})$. Let r be the amalgamation of p_s and p_t. Then $r \Vdash_{\mathcal{P}} s, t \in \dot{X}$. By the remark following Lemma 2.12 it follows that that $r \Vdash_{\mathcal{P}} s(n) \preceq_\delta t(n)$, for all n. Therefore r forces that $\dot{X}$ is not an antichain in $(\omega_2, \preceq)^\omega$, as required. □

By putting together the results of this section we obtain the following.

Theorem 2.17. *It is relatively consistent with* $\mathrm{ZFC} + \mathrm{CH}$ *that there exist a downward closed subfamily* $\mathcal{F}$ *of* $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ *which is strategically persistent but does not have a* σ*-closed persistent subfamily.* □

3. The main theorem

The goal of this section is to prove Theorem 1.5. Before we do that we show that if $\mathcal{A} \simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$ then we can find an ω_1-back and forth family $\mathcal{I}$ of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ with additional closure properties. Recall that we defined $\mathcal{I}$ to be σ-closed if every increasing sequence $(p_n)_n$ of members of $\mathcal{I}$ has an upper bound in $\mathcal{I}$. We will say that $\mathcal{I}$ is *strongly σ-closed* if $\bigcup_n p_n \in \mathcal{I}$, for every such sequence $(p_n)_n$. We will need the following.

Lemma 3.1. *Assume* CH *and let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two structures of size* $\aleph_2$ *in the same vocabulary such that* $\mathcal{A} \simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$*. Then there is an* ω_1*-back and forth set* $\mathcal{J}$ *for* $\mathcal{A}$ *and* $\mathcal{B}$ *which is strongly* σ*-closed.*

Proof. Let $\mathcal{I}$ be a σ-closed ω_1-back and forth set of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$. We build another ω_1-back and forth set $\mathcal{J}$ which is strongly σ-closed. We may assume that the base set of both $\mathcal{A}$ and $\mathcal{B}$ is ω_2. Since $\mathcal{I}$ consists of countable partial functions from ω_2 to ω_2, by CH it follows that it is of cardinality ω_2. Let us fix an enumeration $\{p_\alpha : \alpha < \omega_2\}$ of $\mathcal{I}$. We may assume that the empty function belongs to $\mathcal{I}$ and is enumerated as p_0. We let $q \in \mathcal{J}$ if q is a permutation of a countable subset D_q of ω_2 containing 0 and the following hold:

(1) if $\alpha \in D_q$ then $\mathrm{dom}(p_\alpha) \cup \mathrm{ran}(p_\alpha) \subseteq D_q$,
(2) if $\alpha \in D_q$ and $p_\alpha \subseteq q$ then for every $\xi \in D_q$ there is $\beta \in D_q$ such that $p_\alpha \subseteq p_\beta \subseteq q$, and $\xi \in \mathrm{dom}(p_\beta) \cap \mathrm{ran}(p_\beta)$.

Note that if $q \in \mathcal{J}$ then, by (2) and the fact that $0 \in D_q$, we can find a sequence $(\alpha_n)_n$ of elements of D_q such that $p_{\alpha_0} \subseteq p_{\alpha_1} \subseteq \ldots \subseteq p_{\alpha_n} \subseteq \ldots$, and $q = \bigcup_n p_{\alpha_n}$. Since each p_{α_n} is a countable partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$, then so is q. Moreover, since $\mathcal{I}$ is σ-closed there is $p \in \mathcal{I}$ such that $q \subseteq p$. Note also that $\mathcal{J}$ is strongly σ-closed. In order to show that $\mathcal{J}$ has the ω_1-back and forth property it suffices to show the following.

Claim. *For every* $p \in \mathcal{I}$ *there is* $q \in \mathcal{J}$ *such that* $p \subseteq q$*.*

Proof. Fix a sufficiently large regular cardinal τ and a countable elementary submodel M of H_τ containing p and the enumeration of $\mathcal{I}$. Fix an enumeration $\{\alpha_n : n < \omega\}$ of $M \cap \omega_2$. We define an increasing sequence $(r_n)_n$ of elements of $\mathcal{I} \cap M$ as follows. Let $r_0 = p$. Suppose we have defined r_n. By the fact that $\mathcal{I}$ is an ω_1-back and forth set and M is elementary, we can find $r_{n+1} \in M \cap \mathcal{I}$ extending r_n such that:

(a) r_{n+1} either extends p_{α_n} or is incompatible with it,
(b) $\alpha_n \in \mathrm{dom}(r_{n+1}) \cap \mathrm{ran}(r_{n+1})$.

This completes the definition of the sequence $(r_n)_n$. Let $q = \bigcup_n r_n$. Clearly, q is a permutation of $M \cap \omega_2$, i.e. $D_q = M \cap \omega_2$. We check that $q \in \mathcal{J}$. Condition (1) is satisfied by elementary of M. To see that condition (2) is satisfied consider some $\alpha, \xi \in D_q$ such that $p_\alpha \subseteq q$. Let k and l be such that $\alpha = \alpha_k$ and $\xi = \alpha_l$. Choose some $n > k, l$. Then $p_\alpha \subseteq r_n$ and $\xi \in \mathrm{dom}(r_n) \cap \mathrm{ran}(r_n)$. By elementary of M there is $\beta \in D_q$ such that $r_n = p_\beta$. Then β witnesses condition (2) for α and ξ. □

This completes the proof that $\mathcal{J}$ is a strongly σ-closed ω_1-back and forth set of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$. □

We now turn to the proof of Theorem 1.5. We work in a model of ZFC + CH in which there is a simplified $(\omega_1, 1)$-morass $\mathfrak{M}$ which is stationary and satisfies the $\aleph_2$-antichain condition. Let $\mathcal{F} = \mathcal{F}(\mathfrak{M})$ be the family defined in Definition 2.4. Our plan is to define one structure $\mathcal{C}$ and two distinct elements a and b of $\mathcal{C}$ and let $\mathcal{A} = (\mathcal{C}, a)$ and $\mathcal{B} = (\mathcal{C}, b)$. $\mathcal{C}$ will consist of two parts, one is ω_2 with the usual ordering. Its only role is to ensure certain amount of rigidity of $\mathcal{C}$. The second part of $\mathcal{C}$ consists of layers indexed by countable subsets of ω_2. Given $u \in [\omega_2]^\omega$ let

$$\mathcal{F}_u = \{f \in \mathcal{F} \ : \ \mathrm{dom}(f) = u\}.$$

We let $\mathcal{G}_u$ be $[\mathcal{F}_u]^{<\omega}$. Since we wish these structures to be disjoint and $\emptyset$ belongs to all them, we will replace $\emptyset$ in $\mathcal{G}_u$ by another object, which we denote by $\emptyset_u$, such that the $\emptyset_u$ are all distinct. We still denote the modified structure by $\mathcal{G}_u$. Let $\mathcal{G} = \bigcup\{\mathcal{G}_u : u \in [\omega_2]^\omega\}$. For $a \in \mathcal{G}$ we let $u(a)$ be the unique u such that $a \in \mathcal{G}_u$. The base set of $\mathcal{C}$ will be

$$C = \omega_2 \cup \mathcal{G}.$$

We now describe the language of $\mathcal{C}$. First, we will have two binary relation symbols, $\leq$ and E. The interpretation $\leq^{\mathcal{C}}$ of $\leq$ will be the usual ordering on ω_2. The interpretation of E is as follows:

$$(\alpha, a) \in E^{\mathcal{C}} \text{ iff } \alpha < \omega_2, a \in \mathcal{G} \text{ and } \alpha \in u(a).$$

This guarantees that any isomorphism of $\mathcal{C}$ is the identity on ω_2 and maps each $\mathcal{G}_u$ to itself. We now put some structure on the $\mathcal{G}_u$. Note that $(\mathcal{G}_u, \Delta)$ is a Boolean group, where Δ denotes the symmetric difference. We will keep only the affine structure of this group, i.e. we want the automorphisms of $\mathcal{G}_u$ to be precisely the shifts by some member of $\mathcal{G}_u$, i.e. maps of the form:

$$x \mapsto x \Delta a,$$

for some fixed element a of $\mathcal{G}_u$. In order to achieve this, we will add countably many binary relation symbols $R_{n,i}$, for $i = 0, 1$ and $n < \omega$. In each $\mathcal{G}_u$ we will interpret these relation symbols as follows. First, we index the members of $\mathcal{F}_u$ by elements of 2^ω, say $\mathcal{F}_u = \{f^u_x : x \in 2^\omega\}$. If $a, b \in \mathcal{G}_u$ and $a \Delta b$ is a singleton, say $\{f^u_x\}$, for each n and i, we let

$$R^{\mathcal{C}}_{n,i}(a, b) \text{ if and only if } x(n) = i.$$

Otherwise, no relation between a and b holds. Also, if $u \neq v$ then no relation $R^{\mathcal{C}}_{n,i}$ holds between elements of $\mathcal{G}_u$ and $\mathcal{G}_v$. We also need to connect the different layers of our structure. Suppose $u, v \in [\omega_2]^\omega$ and $u \subseteq v$. We define a homomorphism $\pi_{u,v} : \mathcal{G}_v \to \mathcal{G}_u$ as follows. First, for $f \in \mathcal{F}_u$ we let $\pi_{u,v}(\{f\}) = \{f \restriction u\}$. Then we extend $\pi_{u,v}$ to a homomorphism of $\mathcal{G}_v$ to $\mathcal{G}_u$. Note that, in general, $\pi_{u,v}(a)$ may be different from $\{f \restriction u \ : \ f \in a\}$, since there may be cancelation, i.e.

there could exist $f, f' \in a$ with $f \neq f'$ but $f \restriction u = f' \restriction u$. Now we add a binary relation symbol S and we let:

$$S^{\mathcal{C}}(a, b) \text{ iff } [a, b \in \mathcal{G}, u(a) \subseteq u(b) \text{ and } \pi_{u(a),u(b)}(b) = a].$$

This guarantees the following: if τ is an automorphism of our structure $\mathcal{C}$ then, for each layer u, τ is the shift by some $a_u \in \mathcal{G}_u$ and if $u \subseteq v$ then $\pi_{u,v}(a_v) = a_u$. This completes the definition of the structure $\mathcal{C}$.

Now, we turn to the definition of $\mathcal{A}$ and $\mathcal{B}$. Recall that $\exists$ has a winning strategy, say σ, in the persistency game on $\mathcal{F}$. Consider the play of length ω in which, at stage n, player $\forall$ plays n and player $\exists$ responds by following σ. Let p^* be the resulting position after ω moves and let f^* be the corresponding function. So, $f^* \in \mathcal{F}_\omega$. Now, we introduce a new constant symbol, c. Then we let $\mathcal{A}$ be the expansion of $\mathcal{C}$ obtained by interpreting c as $\emptyset_\omega$ and $\mathcal{B}$ the expansion of $\mathcal{C}$ in which we interpret c as $\{f^*\}$.

Lemma 3.2. *$\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$.*

Proof. We describe informally a winning strategy for player $\exists$ in $\mathrm{EF}^{\aleph_1}_{\aleph_1}(\mathcal{A}, \mathcal{B})$. Suppose player $\forall$ starts by playing A_0 and B_0, where A_0 is a countable subset of $\mathcal{A}$ and B_0 is a countable subset of $\mathcal{B}$. Since the base sets of $\mathcal{A}$ and $\mathcal{B}$ are the same, we may assume $A_0 = B_0$. Let's call this set C_0. Let $C_0' = C_0 \cap \omega_2$ and $C_0'' = C_0 \cap \mathcal{G}$. Now, let $U_0 = \{u(a) : a \in C_0''\}$. Then, U_0 is a countable collection of countable subsets of ω_2. Let $u_0 = \bigcup U_0$. Then player $\exists$ simulates a play in the persistency game on $\mathcal{F}$ continuing the play p^* in which player $\forall$ enumerates the elements of $u_0 \setminus \omega$ in some order after the ω-th move and $\exists$ uses her winning strategy σ. Let p_0 be the resulting position and f_0 the corresponding function. Then $f_0 \in \mathcal{F}_{u_0}$. Let φ_0 be the function on C_0'' defined by:

$$\varphi_0(a) = a \Delta \{f_0 \restriction u(a)\},$$

and let $\psi_0 = \varphi_0 \cup \mathrm{id}_{C_0'}$. Note that ψ_0 is an involution and $\psi_0(\emptyset_\omega) = \{f^*\}$, since f_0 extends f^*. Thus, we can consider ψ_0 as a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ such that $A_0 \subseteq \mathrm{dom}(\psi_0)$ and $B_0 \subseteq \mathrm{ran}(\psi_0)$. Player $\exists$ then plays ψ_0 as her first move in $\mathrm{EF}^{\aleph_1}_{\aleph_1}(\mathcal{A}, \mathcal{B})$.

In general, in the ξ-th move of $\mathrm{EF}^{\aleph_1}_{\aleph_1}(\mathcal{A}, \mathcal{B})$ player $\forall$ plays a countable subset A_ξ of $\mathcal{A}$ and a countable subset B_ξ of $\mathcal{B}$. We may assume that $A_\xi = B_\xi$ and we call this set C_ξ. We let $C_\xi' = C_\xi \cap \omega_2$ and $C_\xi'' = C_\xi \cap \mathcal{G}$. We let $U_\xi = \{u(a) : a \in C_\xi''\}$ and

$$u_\xi = \bigcup \{u_\eta : \eta < \xi\} \cup \bigcup U_\xi.$$

Player $\exists$ simulates a play p_ξ in the persistency game on $\mathcal{F}$ which extends the p_η, for $\eta < \xi$, such that after $\bigcup_{\eta<\xi} p_\eta$ player $\forall$ continues by enumerating in some order the elements of $u_\xi \setminus \bigcup_{\eta<\xi} u_\eta$ and player $\exists$ plays by following her strategy

σ. Let f_ξ be the function corresponding to p_ξ. Notice that f_ξ extends f_η, for $\eta < \xi$. Now, let φ_ξ be the function defined on C''_ξ by

$$\varphi_\xi(a) = a\Delta\{f_\xi \restriction u(a)\}.$$

Finally, let

$$\psi_\xi = \bigcup_{\eta<\xi} \psi_\eta \cup \varphi_\xi \cup \mathrm{id}_{C'_\xi}.$$

It is easy to see that ψ_ξ extends ψ_η, for $\eta < \xi$. Since σ is a winning strategy for player $\exists$ in the persistency game on $\mathcal{F}$, player $\exists$ can continue playing like this for ω_1 moves. Therefore, she has a winning strategy in $\mathrm{EF}^{\aleph_1}_{\aleph_1}(\mathcal{A}, \mathcal{B})$, as required. $\square$

Lemma 3.3. $\mathcal{A} \not\simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$.

Proof. This is similar to the proof of Lemma 2.8. Suppose Ω is a σ-closed family of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$ with the back-and-forth property. By Lemma 3.1, we may assume that Ω is strongly σ-closed. Let ψ be a member of Ω. Then, the domain of ψ is a countable subset A_ψ of $\mathcal{A}$ and the range is a countable subset B_ψ of $\mathcal{B}$. Let $A'_\psi = A_\psi \cap \omega_2$ and let $A''_\psi = A_\psi \cap \mathcal{G}$. Since Ω has the back and forth property, it is easy to see that ψ has to be the identity on A'_ψ and preserve the layers of $\mathcal{G}$. Let $U_\psi = \{u(a) : a \in A''_\psi\}$. Since Ω is also strongly σ-closed, the set of $\psi \in \Omega$ such that U_ψ is directed under inclusion is dense in Ω. By replacing Ω by this set we may assume that U_ψ is directed, for all $\psi \in \Omega$. Let $u(\psi) = \bigcup U_\psi$, for $\psi \in \Omega$. For $u \in U_\psi$ let $A_{\psi,u} = A''_\psi \cap \mathcal{G}_u$. It follows that $\psi \restriction A_{\psi,u}$ has to be the shift by some element of $\mathcal{G}_u$, say $a_{\psi,u}$. Moreover, if $u, v \in U_\psi$ and $u \subseteq v$ then $\pi_{u,v}(a_{\psi,v}) = a_{\psi,u}$. Each $a_{\psi,u}$ is finite and since U_ψ is directed under inclusion and ψ can be extended to a function ρ in Ω which is defined on some point of $\mathcal{G}_{u(\psi)}$, it follows that there exists $a_\psi \in \mathcal{G}_{u(\psi)}$ such that $\psi \restriction A_{\psi,u}$ is the shift by $\pi_{u,u(\psi)}(a_\psi)$, for every $u \in U_\psi$. Let n_ψ be the cardinality of a_ψ. Note that $n_\psi > 0$, since $\psi(\emptyset_\omega) = \{f^*\}$, so ψ cannot be the identity on its domain. Moreover, since Ω is σ-closed and $n_\psi \leq n_\rho$, for every $\psi, \rho \in \Omega$ such that $\psi \subseteq \rho$, there is $\psi_0 \in \Omega$ and an integer n such that $n_\psi = n$, for all $\psi \in \Omega$ such that $\psi_0 \subseteq \psi$. We can replace Ω by $\{\psi \in \Omega : \psi_0 \subseteq \psi\}$, so without loss of generality we may assume that $n_\psi = n$, for all $\psi \in \Omega$.

Now, we proceed as in the proof of Lemma 2.8. We fix a sufficiently large regular cardinal τ. Since $\mathcal{S}(\mathfrak{M})$ is stationary in $[\omega_2]^\omega$, we can find a countable elementary submodel M of H_τ containing all the relevant objects such that $M \cap \omega_2 \in \mathcal{S}(\mathfrak{M})$. Let $\zeta = \sup(M \cap \omega_2)$ and fix an increasing sequence $\{\zeta_n\}_n$ of ordinals in M which is cofinal in ζ. We now work in M. For each $\delta < \omega_2$, fix $\psi_{\delta,0} \in \Omega$ such that $\delta \in u(\psi_{\delta,0})$. Let us enumerate $a_{\psi_{\delta,0}}$ as, say $\{f^0_{\delta,0}, \dots f^{n-1}_{\delta,0}\}$. We can find $\alpha < \omega_1$ and $X_0 \subseteq \omega_2 \setminus \zeta_0$ of size $\aleph_2$ such that $f^0_{\delta,0}(\delta) = \alpha$, for all $\delta \in X_0$. Since $\mathfrak{M}$ satisfies the $\aleph_2$-antichain condition, by Lemma 2.7 we can fix

$\delta(0) \in X_0$ and $X_1 \subseteq X_0 \setminus \zeta_1$ of size $\aleph_2$ such that, for all $\delta \in X_1$, and all $i < n$, any extension of $f^i_{\delta,0}$ to a function in $\mathcal{F}$ which is defined on $\mathrm{dom}(f^i_{\delta(0),0})$ must extend $f^i_{\delta(0),0}$. For each $\delta \in X_1$ fix some $\psi_{\delta,1} \in \Omega$ which extends $\psi_{\delta,0}$ and is defined on $A_{\psi_{\delta(0),0}}$. Then $\psi_{\delta,1}$ must be the identity on $A'_{\psi_{\delta(0),0}}$ and

$$\pi_{u(\psi_{\delta,0}),u(\psi_{\delta,1})}(a_{\psi_{\delta,1}}) = a_{\psi_{\delta,0}}.$$

Since $a_{\psi_{\delta,1}}$ has the same size as $a_{\psi_{\delta,0}}$, we can enumerate it as $\{f^0_{\delta,1}, \dots f^{n-1}_{\delta,1}\}$ such that $f^i_{\delta,1}$ extends $f^i_{\delta,0}$, for all $i < n$. Moreover, $f^i_{\delta,1}$ is defined on $\mathrm{dom}(f^i_{\delta(0),0})$ and so it must extend $f^i_{\delta(0),0}$. In other words, $f^i_{\delta(0),0} \cup f^i_{\delta,0} \subseteq f^i_{\delta,1}$, for all $i < n$. It follows that $\psi_{\delta,1}$ extends $\psi_{\delta(0),0}$, for all $\delta \in X_1$. By Lemma 2.7 again, we can fix $\delta(1) \in X_1$ and $X_2 \subseteq X_1 \setminus \zeta_2$ of size $\aleph_2$ such that, for all $\delta \in X_2$ and all $i < n$, any extension of $f^i_{\delta,1}$ to a function in $\mathcal{F}$ which is defined on $\mathrm{dom}(f^i_{\delta(1),1})$ must extend $f^i_{\delta(1),1}$. For each $\delta \in X_2$ fix some $\psi_{\delta,2} \in \Omega$ which extends $\psi_{\delta,1}$ and is defined on $A_{\psi_{\delta(1),1}}$. As before, $\psi_{\delta,2}$ must be the identity on $A'_{\psi_{\delta,2}}$ so it must agree with $\psi_{\delta(1),1}$ on $A'_{\psi_{\delta(1),1}}$ Also, we can enumerate $a_{\psi_{\delta,2}}$ as $\{f^0_{\delta,2}, \dots, f^{n-1}_{\delta,2}\}$ such that $f^i_{\delta(1),1} \cup f^i_{\delta,1} \subseteq f^i_{\delta,2}$. We conclude that $\psi_{\delta,2}$ extends $\psi_{\delta(1),1} \cup \psi_{\delta,1}$, for all $\delta \in X_2$. Continuing in this way we get an increasing sequence $(\delta(k))_k$ of ordinals from M, a decreasing sequence $(X_k)_k$ of subsets of ω_2 of size $\aleph_2$, and, for each k and $\delta \in X_k$, $\psi_{\delta,k} \in \Omega$ and an enumeration $\{f^0_{\delta,k}, \dots, f^{n-1}_{\delta,k}\}$ of $a_{\psi_{\delta,k}}$ such that:

(1) $\delta(k) \geq \zeta_k$, for all k,
(2) $\psi_{\delta(k),k} \cup \psi_{\delta,k} \subseteq \psi_{\delta,k+1}$, for all $\delta \in X_{k+1}$.
(3) $f^i_{\delta(k),k} \cup f^i_{\delta,k} \subseteq f^i_{\delta,k+1}$, for all $i < n$ and all $\delta \in X_{k+1}$.

Now, $(\psi_{\delta(k),k})_k$ is an increasing sequence of members of Ω and since Ω is σ-closed there is $\rho \in \Omega$ extending all the $\psi_{\delta(k),k}$. It follows that there is an enumeration $\{f^0, \dots, f^{n-1}\}$ of a_ρ such that $f^i_{\delta(k),k} \subseteq f^i$, for each $i < n$ and $k < \omega$. Recall that $f^0_{\delta,0}(\delta) = \alpha$, for all $\delta \in X_0$. Moreover, $f^i_{\delta,0} \subseteq \ldots \subseteq f^i_{\delta,k}$, for all $i < n$ and $\delta \in X_k$. It follows that $f^0(\delta(k)) = \alpha$, for all k. However, all the $\delta(k)$ belong to $M \cap \omega_2$ and the sequence $(\delta(k))_k$ is cofinal in ζ. Since $M \cap \omega_2$ belongs $\mathcal{S}(\mathfrak{M})$ it follows that this sequence is $\preceq$-unbounded. Therefore, f^0 violates condition (2) of Definition 2.4 and so it cannot belong to $\mathcal{F}$, a contradiction. □

This completes the proof of Theorem 1.5. □

4. Open questions

We mention a couple of questions which remain open.

Question 2. *Is it consistent that* $\equiv_{\aleph_1,\aleph_1}$ *and* $\simeq^p_{\aleph_1,\aleph_1}$ *are equivalent for structures of size* $\aleph_2$ *in the context of* CH *?*

Question 3. *Is it consistent that* $\simeq^p_{\aleph_1,\aleph_1}$ *is not transitive? This would show that* $\simeq^p_{\aleph_1,\aleph_1}$ *is not the right concept, i.e. it does not represent equivalence in some logic.*

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