

Oscillations and their applications in partition calculus

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Abstract. Oscillations are a powerful tool for building examples of colorings witnessing negative partition relations. We survey several results illustrating the general technique and present a number of applications.

1 Introduction

We start by recalling some well known notation. Given three cardinals κ, λ, μ and $n < \omega$, the notation

$$\kappa \rightarrow (\lambda)_\mu^n$$

means that for all function $f : [\kappa]^n \rightarrow \mu$, there exists $H \subseteq \kappa$ with $|H| = \lambda$ and such that $f \upharpoonright [H]^n$ is constant. We say that f is a *coloring of $[\kappa]^n$ in μ colors* and H is an *homogeneous set*. Given $\kappa, \lambda, \mu, \sigma$ and n as before, we write

$$\kappa \rightarrow [\lambda]_\mu^n$$

if for every coloring $f : [\kappa]^n \rightarrow \mu$ there exists $H \subseteq \kappa$ of cardinality λ such that $f''[H]^n \neq \mu$. We write

$$\kappa \rightarrow [\lambda]_{\mu, \sigma}^n$$

if for every coloring $f : [\kappa]^n \rightarrow \mu$ there exists $H \subseteq \kappa$ of cardinality λ such that $|f''[H]^n| \leq \sigma$.

One can extend the above notation to sets with additional structures, such as linear or partial orderings, graphs, trees, topological or vector spaces, etc. For instance, given two topological spaces X and Y , then

$$X \rightarrow (\text{top } Y)_\mu^n$$

means that for all $f : [X]^n \rightarrow \mu$ there exists a subset H of X homeomorphic to Y such that $f \upharpoonright [H]^n$ is constant. Similarly, we can define statements such as $X \rightarrow [\text{top } Y]_\mu^n$, $X \rightarrow [\text{top } Y]_{\mu, \sigma}^n$, etc.

We denote by $[\mathbb{N}]^{<\omega}$ the set of all finite subsets of \mathbb{N} and by $[\mathbb{N}]^\omega$ the set of all infinite subsets of \mathbb{N} . We often identify a set s in $[\mathbb{N}]^{<\omega}$ (or $[\mathbb{N}]^\omega$) with its increasing enumeration. When we do this, we will write $s(i)$ for the i -th element of s , assuming it exists. In this way, we identify $[\mathbb{N}]^\omega$ with $(\omega)^\omega$, the set of strictly increasing sequences from ω to ω , which is a G_δ subset of the Baire space ω^ω , and thus is itself a Polish space. For $s, t \in [\mathbb{N}]^{<\omega}$ we write $s \sqsubseteq t$ to say that s is an initial segment of t . In this way, we can view $([\mathbb{N}]^{<\omega}, \sqsubseteq)$ as a tree. For a

given $s \in [\mathbb{N}]^{<\omega}$ we denote by N_s the set of all infinite increasing sequences of integers which extend s . In general, if T is a subtree of $[\mathbb{N}]^{<\omega}$ then T_s will denote the set of all sequences of T extending s . We will need some basic properties of the Baire space (or rather $[\mathbb{N}]^\omega$) and the Cantor space $\{0, 1\}^\omega$ with the usual product topologies. For these facts and all undefined notions, we refer the reader to [5].

The paper is organized as follows. In §2 we discuss partitions of the rationals as a topological space. The basic tool is oscillations of finite sets of integers. In §3 we consider infinite oscillations of tuples of real numbers and discuss several applications to the study of inner models of set theory. In §4 we discuss finite oscillations of tuples of reals of a slightly different type. Finally, in §5 we present oscillations of pairs of countable ordinals and, in particular, outline Moore's ZFC construction of an L -space. We point out that none of the results of this paper are new and we will give a reference to the original paper for each of the results we mention. Our goal is not to give a comprehensive survey of all applications of oscillations in combinatorial set theory, but rather to present several representative results which illustrate the general method.

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2 Negative Partition Relations on the Rationals

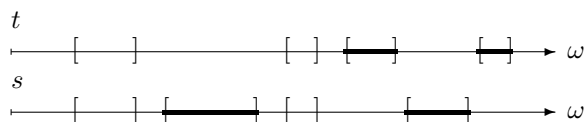
We start with a simple case of oscillations. Given $s, t \in [\mathbb{N}]^{<\omega}$ we define an equivalence relation \sim on $s \Delta t$ by:

$$n \sim m \iff ([n, m] \subseteq (s \setminus t) \vee [n, m] \subseteq (t \setminus s)),$$

for all $n \leq m$ in $s \Delta t$. We now define the function $\text{osc} : ([\mathbb{N}]^{<\omega})^2 \rightarrow \mathbb{N}$ by

$$\text{osc}(s, t) = |(s \Delta t) / \sim|,$$

for all $s, t \in [\mathbb{N}]^{<\omega}$. If, for example, s and t are the two sets represented in the following picture, then $\text{osc}(s, t) = 4$.



The following theorem is due to Baumgartner (see [1]).

Theorem 1 ([1]). $\mathbb{Q} \not\rightarrow [\text{top } \mathbb{Q}]_\omega^2$.

This means, with our notation, that there exists a coloring $c : [\mathbb{Q}]^2 \rightarrow \omega$ such that $c''[A]^2 = \omega$, for all $A \subseteq \mathbb{Q}$ with $A \approx \mathbb{Q}$.

Consider $[\mathbb{N}]^{<\omega}$ with the topology of pointwise convergence. Let $X \subseteq [\mathbb{N}]^{<\omega}$ and $s \in [\mathbb{N}]^{<\omega}$. Then $s \in \overline{X}$ iff for every $n > \sup(s)$ there is $t \in X$ such that $t \cap n = s$. Given $s, t \in [\mathbb{N}]^{<\omega}$ we will write $s < t$ if $\max s < \min t$.

Remark 1. It is well known that $\mathbb{Q} \simeq [\mathbb{N}]^{<\omega}$, so we can view osc as a coloring of $[\mathbb{Q}]^2$.

In order to prove Theorem 1, we recall the definition of the Cantor-Bendixson derivative:

$$\begin{aligned}\delta(X) &= \{x \in X : x \in \overline{X \setminus \{x\}}\}, \\ \delta^0(X) &= X, \\ \delta^{k+1}(X) &= \delta(\delta^k(X)).\end{aligned}$$

We need the following lemma.

Lemma 1. *Suppose $X \subseteq [\mathbb{N}]^{<\omega}$ and $k > 0$ is an integer such that $\delta^{(k)} \neq \emptyset$. Then $\text{osc}''[X]^2 \supseteq \{1, 2, \dots, 2k - 1\}$.*

Proof. The proof is by induction on k . First assume $k = 1$, then let $s \in \delta(X)$. This means that we can find $t, u \in X$ such that $s \sqsubset t, u$ and $t \setminus s < u \setminus s$. It follows that $\text{osc}(t, u) = 1$. Assume that the property holds for all $l < k$, we show that $\text{osc}''[X]^2$ takes values $2k - 2, 2k - 1$. Fix $s \in \delta^k(X)$. Recursively pick $u_i, t_i \in \delta^{k-i}(X)$, for all $i \leq k$, such that the following hold:

1. $t_0 = u_0 = s$;
2. $s \sqsubset t_1 \sqsubset t_2 \cdots \sqsubset t_k$;
3. $s \sqsubset u_1 \sqsubset u_2 \cdots \sqsubset u_k$;
4. $t_i \setminus t_{i-1} < u_i \setminus u_{i-1} < t_{i+1} \setminus t_i$, for all $i \in \{1, 2, \dots, k\}$.

Then $\text{osc}(t_{k-1}, u_{k-1}) = 2k - 2$ and $\text{osc}(t_k, u_{k-1}) = 2k - 1$. □

Proof. (of **Theorem 1**). By Remark 1, it is sufficient to check that for all $A \subseteq [\mathbb{N}]^{<\omega}$ homeomorphic to \mathbb{Q} , $\text{osc}''[A]^2 = \omega$. Since $A \approx \mathbb{Q}$, we have $\delta^k(A) \neq \emptyset$, for all integers k . Hence we can apply Lemma 1 and this completes the proof. □

An unpublished result of Galvin states that

$$\eta \rightarrow [\eta]_{n,2}^2$$

when η is the order type of the rational numbers, and n is any integer. Therefore, the order theoretic version of Theorem 1 does not hold. Also, the coloring we build to prove Baumgartner's theorem is not continuous. In fact, if we only consider continuous colorings, then we have

$$\mathbb{Q} \rightarrow_{\text{cont}} [\text{top } \mathbb{Q}]_2^2.$$

If we want a continuous coloring, we need to work in $[\mathbb{Q}]^3$. The following result is due to Todorćević ([10]).

Theorem 2 ([10]). *There is a continuous coloring $c : [\mathbb{Q}]^3 \rightarrow \omega$ such that $c''[A]^3 = \omega$, for all $A \subseteq \mathbb{Q}$ with $A \approx \mathbb{Q}$.*

Proof. Given $s, t, u \in [\mathbb{N}]^{<\omega}$, we define

$$\Delta(s, t) = \min(s \triangle t)$$

$$\Delta(s, t, u) = \max\{\Delta(s, t), \Delta(t, u), \Delta(s, u)\}.$$

The value of $\Delta(s, t, u)$ is equal to the least $n \in \mathbb{N}$ such that $|\{s \cap (n+1), t \cap (n+1), u \cap (n+1)\}| = 3$. So, in particular, for such an integer n , we have $|\{s \cap n, t \cap n, u \cap n\}| = 2$. Let $\{v, w\} = \{s \cap n, t \cap n, u \cap n\}$, then we define

$$\text{osc}_3(s, t, u) = \text{osc}(v, w).$$

The coloring osc_3 is obviously continuous. The proof that this coloring works is similar to the one given for Theorem 1. We can prove, analogously, that if $X \subseteq [\mathbb{N}]^{<\omega}$ and $\delta^k(X) \neq \emptyset$, for some integer $k > 0$, then $\text{osc}_3''[X]^2 \supseteq \{1, 2, \dots, 2k-1\}$. Let us just see the case $k = 1$. Fixing $s \in \delta(X)$ we can find $t, u \in X$ such that $s \sqsubset t, u$ and $t \setminus s < u \setminus s$. Then $\text{osc}_3(s, t, u) = 1$. Finally one can apply this result to all subsets of $[\mathbb{N}]^{<\omega}$ that are homeomorphic to \mathbb{Q} , and this completes the proof. \square

3 Oscillations of Real Numbers - Part 1

We now discuss infinite oscillations and their applications.

Let $x \subseteq \mathbb{N}$, we define an equivalence relation \sim_x on $\mathbb{N} \setminus x$:

$$n \sim_x m \iff [n, m] \cap x = \emptyset,$$

for all $n \leq m$ in $\mathbb{N} \setminus x$. Thus, the equivalence classes of \sim_x are the intervals between consecutive elements of x . Given $x, y, z \subseteq \mathbb{N}$, suppose that $(I_k)_{k \leq t}$ for $t \leq \omega$ is the natural enumeration of those equivalence classes of x which meet both y and z . We define a function $o(x, y, z) : t \rightarrow \{0, 1\}$ as follows:

$$o(x, y, z)(k) = 0 \iff \min(I_k \cap y) \leq \min(I_k \cap z).$$

Notice that o is a continuous function from

$$\{(x, y, z) \in [[\mathbb{N}]^\omega]^3 : |(\mathbb{N} \setminus x) / \sim_x| = \aleph_0\}$$

to $2^{\leq \omega}$.

Note that $[\mathbb{N}]^{<\omega}$ ordered by \sqsubseteq is a tree. A subset T of $[\mathbb{N}]^{<\omega}$ is a *subtree* if it is closed under initial segments.

Definition 1. *Let T be a subtree of $[\mathbb{N}]^{<\omega}$. We say that $t \in T$ is ∞ -splitting if for all k there exists $u \in T$ such that $t \sqsubseteq u$ and $u(|t|) > k$.*

Definition 2. A subtree T of $[\mathbb{N}]^{<\omega}$ is superperfect if for all $s \in T$ there exists $t \in T$ such that $s \sqsubseteq t$ and t is ∞ -splitting in T .

Definition 3. $X \subseteq [\mathbb{N}]^\omega$ is superperfect if there is a superperfect tree $T \subseteq [\mathbb{N}]^{<\omega}$ such that $X = [T] = \{A \in [\mathbb{N}]^\omega : A \cap k \in T, \text{ for all } k\}$.

The following theorem is due to Veličković and Woodin ([11]).

Theorem 3 ([11]). Let $X, Y, Z \subseteq [\mathbb{N}]^\omega$ be superperfect sets. Then $o''[X \times Y \times Z] \supseteq 2^\omega$.

Proof. Let $T_1, T_2, T_3 \subseteq [\mathbb{N}]^{<\omega}$ be superperfect trees such that $X = [T_1]$, $Y = [T_2]$ and $Z = [T_3]$. Given an $\alpha \in 2^\omega$, we build sequences $\langle s_k \rangle_k, \langle t_k \rangle_k, \langle u_k \rangle_k$, of nodes of T_1, T_2 and T_3 , respectively, such that the following properties hold:

1. s_0, t_0, u_0 are the least ∞ -splitting node of T_1, T_2 and T_3 , respectively;
2. $s_0 \sqsubset s_1 \sqsubset s_2 \sqsubset \dots \sqsubset s_k \sqsubset \dots$;
3. $t_0 \sqsubset t_1 \sqsubset t_2 \sqsubset \dots \sqsubset t_k \sqsubset \dots$;
4. $u_0 \sqsubset u_1 \sqsubset u_2 \sqsubset \dots \sqsubset u_k \sqsubset \dots$;
5. $t_i \setminus t_{i-1}, u_i \setminus u_{i-1} < s_i \setminus s_{i-1}$;
6. $t_i \setminus t_{i-1} < u_i \setminus u_{i-1}$, if $\alpha(i) = 0$ and
 $u_i \setminus u_{i-1} < t_i \setminus t_{i-1}$, if $\alpha(i) = 1$.

If $x = \bigcup_{k < \omega} s_k, y = \bigcup_{k < \omega} t_k$ and $z = \bigcup_{k < \omega} u_k$, then $o(x, y, z) = \alpha$ and this completes the proof. \square

Corollary 1 ([11]). If $X \subseteq [\mathbb{N}]^\omega$ is a superperfect set, then $o''[X]^3 \supseteq 2^\omega$. \square

We now apply the previous theorem to prove some results about reals of inner models of set theory.

Theorem 4 ([11]). Let V, W be models of set theory such that $W \subseteq V$. If there is a superperfect set X in V such that $X \subseteq W$ then $\mathbb{R}^W = \mathbb{R}^V$.

Proof. This is trivial by applying Corollary 1. \square

Question 1. Can we replace superperfect by perfect in the previous theorem?

Surprisingly, the answer depends on whether CH holds in the model W , as it is asserted in the following theorem due to Groszek and Slaman (see [4]).

Theorem 5 ([4]). Suppose that W and V are two models of set theory such that $W \subseteq V$. Assume that there is a perfect set P in V such that $P \subseteq W$. If CH holds in W , then $\mathbb{R}^W = \mathbb{R}^V$.

In order to prove this theorem, let us introduce the following notion.

Definition 4. Given two models of set theory W and V such that $W \subseteq V$ we say that (W, V) satisfies the countable covering property for the reals if, for all X in V such that $X \subseteq \mathbb{R}^W$ and X is countable in V , there is an Y in W such that $X \subseteq Y$ and Y is countable in W .

We prove first the following theorem.

Theorem 6. Given two models of set theory W and V such that $W \subseteq V$, suppose that there is a perfect set P in V such that $P \subseteq W$. If (W, V) satisfies the countable covering property for the reals, then $\mathbb{R}^W = \mathbb{R}^V$.

Proof. Work in V and fix a perfect subset P of $(2^\omega)^W$. Let X be a countable dense subset of P . By the countable covering property for the reals we can cover X by some set $D \in W$ such that D is countable in W , is a dense subset of 2^ω and $D \cap P$ is dense in P . In W , fix an enumeration $\{d_n; n < \omega\}$ of D . For $x, y \in 2^\omega$ with $x \neq y$ let

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Given $x \in 2^\omega \setminus D$ first define a sequence $\langle k_x(n); n < \omega \rangle$ by induction as follows

$$k_x(n) = \min\{k : \Delta(x, d_k) > \Delta(x, d_{k_x(i)}), \text{ for all } i < n\}.$$

Note that $k_x(0) = 0$. Since D is dense in P and $x \in P \setminus D$ then $k_x(n)$ is defined, for all n . Now define $f : P \setminus D \rightarrow [\mathbb{N}]^\omega$ by setting

$$f(x)(n) = \Delta(x, d_{k_x(n)}).$$

Clearly, f is continuous and $f(x)$ is a strictly increasing function, for all $x \in 2^\omega \setminus D$. Since $D \in W$ then f is coded in W . We can now prove that $f''[P \setminus D]$ is superperfect. Let $T = \{f(x) \upharpoonright n : x \in P \setminus D \wedge n \in \omega\}$. First note that $f''[P \setminus D]$ is closed, i.e. it is equal to $[T]$. To see this, note that if $b \in [T]$, then for every i there is $x_i \in P \setminus D$ such that $b \upharpoonright i = f(x_i) \upharpoonright i$. Since P is compact, it follows that the sequence $(x_i)_i$ converges to some $x \in P$. Note then $k_x(n) = k_{x_m}(n)$, for all $m > n$, in particular, $k_x(n)$ is defined, for all n . It follows that $x \notin D$. Since $f(x) = b$ it follows that $b \in f''[P \setminus D]$, as desired.

Now, we show that every node of T is ∞ -splitting. Let $s \in T$ and suppose $n = |s|$. Then there is some $x \in P \setminus D$ such that $s \sqsubseteq f(x)$. Therefore, $s(i) = \Delta(x, d_{k_x(i)})$, for all $i < n$. Let $l = k_x(n)$. Since P is perfect we can find, for every $j \geq \Delta(x, d_l)$, some $x_j \in P \setminus D$ such that $\Delta(x_j, d_l) \geq j$. It follows that $f(x_j) \upharpoonright n = s$ and $f(x_j)(n) \geq j$. This shows that s is ∞ -splitting.

Since $P \subseteq W$ and f is coded in W we have $f''[P \setminus D] \subseteq W$, that is W contains a superperfect set. By Corollary 1, we have $\mathbb{R}^W = \mathbb{R}^V$ and this completes the proof. \square

Proof. (of **Theorem 5**). By the previous theorem, it is enough to prove that (W, V) satisfies the countable covering property for the reals. By assumption, W satisfies CH, so we can fix in W a well-ordering on $(\mathbb{R})^W$ of height $(\omega_1)^W$. Since every perfect set is uncountable and $P \subseteq W$, then $\omega_1^W = \omega_1^V$. Therefore, any

$X \subseteq (\mathbb{R})^W$ which is countable in V , is contained in a proper initial segment Y of the well-ordering. Then $Y \in W$ and Y is countable in W . This completes the proof. \square

In particular we can state the following corollary.

Corollary 2 ([4]). *If there is a perfect set of constructible reals, then $\mathbb{R} \subseteq L$.* \square

Is the countable covering condition necessary to obtain this result? Theorem 7 below (see [11]) gives a partial answer to this question.

Theorem 7 ([11]). *There is a pair (W, V) of generic extensions of L with $W \subseteq V$, such that $\aleph_1^W = \aleph_1^V$ and V contains a perfect set of W -reals, but $\mathbb{R}^W \neq \mathbb{R}^V$.*

On the other hand, in [11] we have also the following theorem.

Theorem 8 ([11]). *Suppose that M is an inner model of set theory and \mathbb{R}^M is analytic, then either \aleph_1^M is countable, or all reals are in M .*

In order to prove Theorem 8, let us introduce a generalization of the notion of a superperfect set.

Definition 5. *Suppose λ is a limit ordinal and T is a subtree of $[\lambda]^{<\omega}$. We say that $t \in T$ is λ -splitting if for all $\xi < \lambda$ there exists $u \in T$ such that $t \sqsubseteq u$ and $u(|t|) > \xi$.*

Definition 6. *Suppose λ is a limit ordinal and let T be a subtree of $[\lambda]^{<\omega}$. We say T is λ -superperfect if for all $s \in T$ there exists $t \in T$ such that $s \sqsubseteq t$ and t is λ -splitting.*

Definition 7. *A set $P \subseteq [\lambda]^\omega$ is λ -superperfect if there is a λ -superperfect tree $T \subseteq [\lambda]^{<\omega}$ such that $P = \{x \in [\lambda]^\omega : \forall n < \omega (x \upharpoonright n \in T)\}$. Here $x \upharpoonright n$ denotes the set of the first n elements of x in the natural order.*

The definition of $o : ([\mathbb{N}]^\omega)^3 \rightarrow \{0, 1\}^\omega$ can be trivially generalized to a coloring

$$o_\lambda : ([\lambda]^\omega)^3 \rightarrow \{0, 1\}^\omega.$$

As for o one can easily check that for all λ -superperfect P , we have $o''_\lambda[P^3] \supseteq \{0, 1\}^\omega$ (the proof is the same as for Theorem 3). Moreover, we have $o_\lambda(x, y, z) \in L[x, y, z]$, for all $x, y, z \in [\lambda]^\omega$. To complete the proof of Theorem 8 it suffices to prove the following lemma.

Lemma 2. *Suppose that A is an analytic set such that $\sup\{\omega_1^{\text{CK},x} : x \in A\} = \omega_1$. Then every real is hyperarithmetical in a quadruple of elements of A .*

Proof. Let $T \subset (\omega \times \omega)^{<\omega}$ be a tree such that $A = p[T]$. Note that the statement $\sup\{\omega_1^{\text{CK},x} : x \in p[T]\} = \omega_1$ is $\Pi_2^1(T)$ and thus absolute.

For an ordinal α let $\text{Coll}(\aleph_0, \alpha)$ be the usual collapse of α to \aleph_0 using finite conditions. Let \mathcal{P} denote $\text{Coll}(\aleph_0, \aleph_1)$. If G is V -generic over \mathcal{P} , by Shoenfield's absoluteness theorem, in $V[G]$ there is $x \in p[T]$ such that $\omega_1^{\text{CK},x} > \omega_1^V$. In V fix a name \dot{x} for x and a name σ for a cofinal ω -sequence in ω_1^V such that the maximal condition in \mathcal{P} forces that $\dot{x} \in p[T]$ and $\sigma \in L[\dot{x}]$.

Claim 1 *For every $p \in \mathcal{P}$ there is $k < \omega$ such that for every $\alpha < \omega_1$ there is $q \leq p$ such that $q \Vdash \sigma(k) > \alpha$.*

Proof. Assume otherwise and fix p for which the claim is false. Then for every k there is $\alpha_k < \omega_1$ such that $p \Vdash \sigma(k) < \alpha_k$. Let $\alpha = \sup\{\alpha_k : k < \omega\}$. Then $p \Vdash \text{ran}(\sigma) \subset \alpha$, contradicting the fact that σ is forced to be cofinal in ω_1^V . \square

Let \mathcal{Q} denote $\text{Coll}(\aleph_0, \aleph_2)$ as defined in V . Suppose H is V -generic over \mathcal{Q} . Work for a moment in $V[H]$. If G is a V -generic filter over \mathcal{P} let σ_G denote the interpretation of σ in $V[G]$. Let B be the set of all σ_G where G ranges over all V -generic filters over \mathcal{P} .

Claim 2 *B contains an ω_1^V -superperfect set in $(\omega_1^V)^\omega$.*

Proof. Let $\{D_n : n < \omega\}$ be an enumeration of all dense subsets of \mathcal{P} which belong to the ground model. For each $t \in (\omega_1^V)^{<\omega}$ we define a condition p_t in the regular open algebra of \mathcal{P} as computed in V and $s_t \in (\omega_1^V)^{<\omega}$ inductively on the length of t such that

1. $p_t \in D_{lh(t)}$
2. $p_t \Vdash s_t \subset \sigma$
3. if $t \subseteq r$ then $p_r \leq p_t$ and $s_t \subset s_r$
4. if t and r are incomparable then s_t and s_r are incomparable
5. for every t the set $\{\alpha < \omega_1^V : \text{there is } q \leq p \text{ } q \Vdash s_t \hat{\wedge} \alpha \subset \sigma\}$ is unbounded in ω_1^V .

Suppose p_t and s_t have been defined. Using 5. choose in V a 1-1 order preserving function $f_t : \omega_1^V \rightarrow \omega_1^V$ and for every α $q_{t,\alpha} \leq p_t$ such that $q_{t,\alpha} \Vdash s_t \hat{\wedge} f_t(\alpha) \subset \sigma$. By extending $q_{t,\alpha}$ if necessary, we may assume that it belongs to $D_{lh(t)+1}$. Now, by applying Claim 1, we can find a condition $p \leq q_{t,\alpha}$ and $k > lh(s_t) + 1$ such that for some $s \in (\omega_1^V)^k$ $p \Vdash s \subset \sigma$ and for every $\gamma < \omega_1^V$ there is $q \leq p$ such that $q \Vdash \sigma(k) > \gamma$. Let then $s_t \hat{\wedge} \alpha = s$ and $p_t \hat{\wedge} \alpha = p$. This completes the inductive construction.

Now if $b \in (\omega_1^V)^\omega$ then $\{p_{b \upharpoonright n} : n < \omega\}$ generates a filter G_b which is V -generic over \mathcal{P} . The interpretation of σ under G_b is $s_b = \bigcup_{n < \omega} s_{b \upharpoonright n}$. Since the set $R = \{s_b : b \in (\omega_1^V)^\omega\}$ is ω_1^V -superperfect, this proves Claim 2. \square

Now, using the remark following Definition 7, for any real $r \in \{0, 1\}^\omega$ we can find $b_1, b_2, b_3 \in (\omega_1^V)^\omega$ such that $r \in L[s_{b_1}, s_{b_2}, s_{b_3}]$. Let x_i be the interpretation

of \dot{x} under G_{b_i} . Then it follows that $x_i \in p[T]$ and $s_{b_i} \in L[x_i]$, for $i = 1, 2, 3$. Thus $r \in L[x_1, x_2, x_3]$. Pick a countable ordinal δ such that $r \in L_\delta[x_1, x_2, x_3]$. Using the fact that in $V[H]$ $\sup\{\omega_1^{\text{CK},x} : x \in p[T]\} = \omega_1$, we can find $y \in p[T]$ such that $\omega_1^{\text{CK},y} > \delta$. Then we have that r is $\Delta_1^1(x_1, x_2, x_3, y)$. Note that the statement that there are $x_1, x_2, x_3, y \in p[T]$ such that $r \in \Delta_1^1(x_1, x_2, x_3, y)$ is $\Sigma_2^1(r, T)$. Thus for any real $r \in V$, by Shoenfield absoluteness again, it must be true in V . This proves Lemma 2. \square

We complete this section by stating some related results.

Theorem 9 ([11]). *There is a pair of generic extensions of L , $W \subseteq V$ such that \mathbb{R}^W is an uncountable F_δ set in V , and $\mathbb{R}^W \neq \mathbb{R}^V$.*

Theorem 10 ([3]). *Suppose that $W \subseteq V$ are two models of set theory, $\kappa > \omega_1^V$ and there exists $C \subseteq [\kappa]^\omega$ which is a club in V such that $C \subseteq W$. Then $\mathbb{R}^W = \mathbb{R}^V$.*

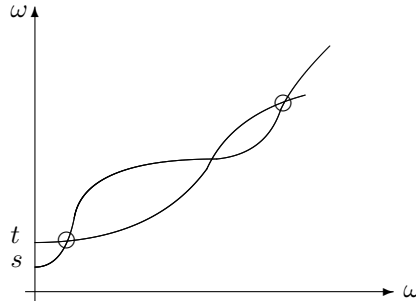
Theorem 11 ([2]). *Suppose that $W \subseteq V$ are two models of set theory such that $V, W \models \text{PFA}$ and $\aleph_2^W = \aleph_2^V$. Then $\mathbb{R}^W = \mathbb{R}^V$, in fact $\mathcal{P}(\omega_1)^W = \mathcal{P}(\omega_1)^V$.*

4 Oscillations of Real Numbers - part 2

The results of this section are taken from [9]. We look increasing sequences of integers and slightly change the definition of oscillation. Given $s, t \in (\omega)^{\leq \omega}$ we define

$$\text{osc}(s, t) = |\{n < \omega : s(n) \leq t(n) \wedge s(n+1) > t(n+1)\}|.$$

In the next picture, s and t are two functions in $(\omega)^{< \omega}$ with $\text{osc}(s, t) = 2$.



We now define two orders \leq_m and \leq_* on $(\omega)^\omega$:

$$f \leq_m g \iff \forall n \geq m (f(n) \leq g(n));$$

$$f \leq_* g \iff \exists m (f \leq_m g).$$

Given $X \subseteq (\omega)^\omega$ and $s \in (\omega)^{< \omega}$ we let $X_s = \{f \in X : s \sqsubseteq f\}$ and

$$T_X = \{s \in (\omega)^{< \omega} : X_s \text{ is unbounded under } \leq_*\}.$$

Lemma 3. *Suppose that $X \subseteq (\omega)^\omega$ is unbounded under \leq_* and $X = \bigcup_{n < \omega} A_n$. Then there exists n such that A_n is unbounded.*

Proof. Suppose that every A_n is bounded and, for all n , let g_n be such that $f \leq_* g_n$, for all $f \in A_n$. If we define $g(n) = \sup\{g_k(n) : k \leq n\}$, then X is bounded by g with respect to \leq_* . This leads to a contradiction. \square

Lemma 4. *Suppose that $X \subseteq (\omega)^\omega$ is unbounded under \leq_* . Then T_X is superperfect.*

Proof. Suppose, by way of contradiction, that there is a node $s \in T_X$ with no ∞ -splitting extensions in T_X . We define a function $g_s : \omega \rightarrow \omega$ as follows:

$$g_s(n) = \sup\{t(n) : t \in (T_X)_s \wedge n \in \text{dom}(t)\},$$

where $(T_X)_s = \{f \in T_X : s \sqsubseteq f\}$. First note that $g_s(n) < \omega$, for all $n < \omega$. Let $Q = \{t \in (\omega)^{<\omega} : X_t \text{ is bounded under } \leq_*\}$. By Lemma 3, $\bigcup\{X_t : t \in Q\}$ is bounded under \leq_* by some function g . Now let $h = \max(g, g_s)$. It follows that X_s is \leq_* -bounded by h , a contradiction. \square

We first consider oscillations of elements of $(\omega)^{<\omega}$. Our first goal is to prove that if T is a superperfect subtree of $(\omega)^{<\omega}$ then $\text{osc}''[T]^2 = \omega$. In fact, we prove a slightly stronger lemma.

Lemma 5. *Let S and T be two superperfect subtrees of $(\omega)^{<\omega}$ and let s and t be ∞ -splitting nodes of S and T respectively. Then for all n there are ∞ -splitting nodes s' in S and t' in T such that $s \sqsubseteq s'$, $t \sqsubseteq t'$ and*

$$\text{osc}(s', t') = \text{osc}(s, t) + n.$$

Proof. We may assume without loss of generality that $|s| < |t|$ and $s(|s| - 1) \leq t(|s| - 1)$. We can recursively pick some ∞ -splitting extensions $s_i \in S$ and $t_i \in T$, for $i \leq n$ such that:

- $s_0 = s$ and $t_0 = t$;
- $s_0 \sqsubset s_1 \sqsubset \dots \sqsubset s_n$;
- $t_0 \sqsubset t_1 \sqsubset \dots \sqsubset t_n$;
- $\text{osc}(s_i, t_i) = \text{osc}(s, t) + i$, for all i ;
- $|s_i| < |t_i|$ and $s_i(|s_i| - 1) \leq t_i(|s_i| - 1)$.

Given s_i and t_i , since S is superperfect and s_i is ∞ -splitting in S , we can find some ∞ -splitting extension u of s_i in S such that $u(|s_i|) > t_i(|t_i| - 1)$ and such that $|u| > |t_i| + 1$. In the same way, we can take an ∞ -splitting extension v of t_i in T such that $v(|t_i|) > u(|u| - 1)$ and $|v| > |u| + 1$. Since u and v are strictly increasing, we have $\text{osc}(u, v) = \text{osc}(s_i, t_i) + 1$, so we can define $s_{i+1} = u$ and $t_{i+1} = v$.

Finally, $\text{osc}(s_n, t_n) = \text{osc}(s, t) + n$ and this completes the proof. \square

Corollary 3. *If T is a superperfect subtree of $(\omega)^{<\omega}$ then $\text{osc}''[T]^2 = \omega$. \square*

We now turn to oscillations of elements of $(\omega)^\omega$. We will need the following definition.

Definition 8. *A subset X of $(\omega)^\omega$ is σ -directed under \leq_* if, and only if, for all countable $D \subseteq X$ there is $f \in X$ such that $d \leq_* f$, for all $d \in D$.*

Lemma 6. *Suppose $X \subseteq (\omega)^\omega$ and σ -directed and unbounded under \leq_* and $Y \subseteq (\omega)^\omega$ is such that for every $a \in X$ there is $b \in Y$ such that $a \leq_* b$. There is an integer n_0 such that for all $k < \omega$ there is $f \in X$ and $g \in Y$ such that $\text{osc}(f, g) = n_0 + k$.*

Proof. Fix a countable dense subset D of X . Since X is σ -directed, there is a function $a \in X$ such that $d \leq_* a$, for all $d \in D$. The set $Y' = \{g \in Y : a \leq_* g\}$ is unbounded under \leq_* . We define $Y_m = \{g \in Y' : a \leq_m g\}$, for all $m < \omega$. By Lemma 3 and the fact that $Y' = \bigcup\{Y_m : m < \omega\}$, there exists $m_0 < \omega$ such that Y_{m_0} is also \leq_* -unbounded. Let $s_0 \in T_X$ and $t_0 \in T_{Y_{m_0}}$ be the two least ∞ -splitting nodes of T_X and $T_{Y_{m_0}}$ respectively. Let $n_0 = \text{osc}(s_0, t_0)$. Now, fix $k < \omega$. By Lemma 5, there are two ∞ -splitting $s \in T_X$ and $t \in T_{Y_{m_0}}$ such that $\text{osc}(s, t) = n_0 + k$. We may assume without loss of generality that $|t| \leq |s|$ and $t(|t| - 1) > s(|t| - 1)$. Since D is dense, there is $f \in D$ such that $s \sqsubseteq f \leq_* a$. Fix $m \geq m_0$ such that $f \leq_m a$. Since t is ∞ -splitting in $T_{Y_{m_0}}$, we can pick $i > f(m)$ and $g \in Y_{m_0}$, such that $t \hat{\ } i \sqsubseteq g$. We know that for all $k \geq m_0$, $a(k) \leq g(k)$, so for all $k \geq m$, $f(k) \leq g(k)$. Moreover, f and g are increasing and $t \hat{\ } i \sqsubseteq g$, so for all k between $|t|$ and m we have $g(k) > f(k)$. It follows that $\text{osc}(f, g) = \text{osc}(s, t) = n_0 + k$ and this completes the proof. \square

The following theorem is due to S. Todorćević (see [9]).

Theorem 12 ([9]). *Suppose $X \subseteq (\omega)^\omega$ is unbounded under \leq_* and σ -directed, then $\text{osc}''[X]^2 = \omega$.*

Proof. The proof is the same as for Lemma 6, by assuming $Y = X$. Thus $s_0 = t_0$ and, consequently, $n_0 = 0$ in the previous proof. Hence, for all $k < \omega$ there are $f, g \in X$ such that $\text{osc}(f, g) = k$. This completes the proof. \square

We recall that \mathfrak{b} is the least cardinal of an \leq_* -unbounded subset of $(\omega)^\omega$. Fix an unbounded $\mathcal{F} \subseteq (\omega)^\omega$ of cardinality \mathfrak{b} . We may assume that \mathcal{F} is well ordered under \leq_* and (\mathcal{F}, \leq_*) has order type \mathfrak{b} .

Remark 2. Every unbounded subset of \mathcal{F} is σ -directed and cofinal in \mathcal{F} under \leq_* .

Corollary 4. *Let $X, Y \subseteq \mathcal{F}$ be unbounded under \leq_* . There exists $n_0 < \omega$ such that for all $k < \omega$ there exist $f \in X$ and $g \in Y$ such that $\text{osc}(f, g) = n_0 + k$.*

Proof. Trivial by Remark 2 and Lemma 6. \square

In [9], Todorćević proved a more general result:

Theorem 13 ([9]). *Suppose \mathcal{F} is \leq_* -unbounded and well ordered by \leq_* in order type \mathfrak{b} . Suppose $\mathfrak{A} \subseteq [\mathcal{F}]^n$, $|\mathfrak{A}| = \mathfrak{b}$ and \mathfrak{A} consists of pairwise disjoint n -tuples. Then there exists $h : n \times n \rightarrow \omega$ such that for all $k < \omega$ there exist $A, B \in \mathfrak{A}$ such that $A \neq B$ and $\text{osc}(A(i), B(j)) = h(i, j) + k$, for all $i, j < n$. Here $A(i)$ denotes the i -th element of A in increasing order, and similarly $B(j)$ denotes the j -th element of B .*

Proof. For any $A, B \in [\mathcal{F}]^n$, we will write $A <_m B$ if, and only if, $a <_m b$ for all $a \in A$ and $b \in B$. Similarly, with $A \leq_* B$ we mean that $a \leq_* b$, for all $a \in A$ and $b \in B$. Finally, if $A \in \mathfrak{A}$ and $m < \omega$, we denote by $A \upharpoonright m$ the sequence $\langle A(i) \upharpoonright m \rangle_{i < n}$.

We may assume that \mathfrak{A} is *everywhere unbounded*, that is for all $m < \omega$ and $A \in \mathfrak{A}$, the set $\{B \in \mathfrak{A} : B \upharpoonright m = A \upharpoonright m\}$ is also unbounded in $(\omega)^\omega$ under \leq_* . Take a countable dense $\mathfrak{D} \subseteq \mathfrak{A}$. There is $A \in \mathfrak{A}$ such that $D \leq_* A$, for all $D \in \mathfrak{D}$. For all $m < \omega$, let $\mathfrak{A}_m = \{B \in \mathfrak{A} : A <_m B\}$. As before, there is $m_0 < \omega$ such that \mathfrak{A}_{m_0} is everywhere unbounded.

Given any $\mathbf{t} \in (\omega^{<\omega})^n$, we denote by t_i the i -th element of \mathbf{t} in increasing order. If $B \in (\omega^\omega)^n$, then $\mathbf{t} \sqsubseteq B$ means $t_i \sqsubseteq B(i)$, for all $i < n$. Now, we define

$$T_{\mathfrak{A}_{m_0}} = \{\mathbf{t} \in (\omega^{<\omega})^n : \forall i < n (|t_i| < |t_{i+1}|) \wedge \exists B \in \mathfrak{A}_{m_0} (\mathbf{t} \sqsubseteq B)\}.$$

For any sequence $\mathbf{s} \in T_{\mathfrak{A}_{m_0}}$, we say that \mathbf{s} is ∞ -splitting if for all $l < \omega$, there is $\mathbf{t} \in T_{\mathfrak{A}_{m_0}}$ such that $\mathbf{s} \sqsubseteq \mathbf{t}$ and $t_i(|s_i|) > l$, for all $i < n$.

Claim 3 $T_{\mathfrak{A}_{m_0}}$ is superperfect, that is for all $\mathbf{s} \in T_{\mathfrak{A}_{m_0}}$, there is an ∞ -splitting sequence $\mathbf{t} \in T_{\mathfrak{A}_{m_0}}$ such that $\mathbf{s} \sqsubseteq \mathbf{t}$.

Proof. Given $\mathbf{s} \in T_{\mathfrak{A}_{m_0}}$, define t_0 as the least ∞ -splitting extension of s_0 in $T_{Z(0)}$, where $Z(0) := \{B(0) : B \in \mathfrak{A}_{m_0}\}$. Assume that $\mathbf{t} \upharpoonright i$ is defined, the set $Z(i) := \{B(i) : B \in \mathfrak{A}_{m_0} \text{ and } B \upharpoonright i = \mathbf{t} \upharpoonright i\}$ is unbounded (because \mathfrak{A}_{m_0} is everywhere unbounded). Put t_i any ∞ -splitting extension of s_i in $T_{Z(i)}$, such that $|t_i| > |t_{i-1}|$. The sequence \mathbf{t} , so defined, is ∞ -splitting in $T_{\mathfrak{A}_{m_0}}$. This completes the proof of the claim. \square

Now, let $\mathbf{r} \in T_{\mathfrak{A}_{m_0}}$ be the least ∞ -splitting sequence, we define for all $i, j < n$,

$$h(i, j) = \text{osc}(r_i, r_j).$$

Claim 4 For all $k < \omega$, there are two ∞ -splitting sequences $\mathbf{s}, \mathbf{t} \in T_{\mathfrak{A}_{m_0}}$ such that $\mathbf{r} \sqsubseteq \mathbf{s}, \mathbf{t}$ and $\text{osc}(s_i, t_j) = \text{osc}(r_i, r_j) + k$, for all $i, j < n$.

Proof. We prove this by induction on $k < \omega$. The case $k = 0$ is trivial. Let $\mathbf{s}, \mathbf{t} \in T_{\mathfrak{A}_{m_0}}$ be ∞ -splitting, such that $\mathbf{r} \sqsubseteq \mathbf{s}, \mathbf{t}$ and $\text{osc}(s_i, t_j) = \text{osc}(r_i, r_j) + k$, for all i, j . Assume without loss of generality that $|s_i| < |t_j|$ and $s_i(|s_i| - 1) \leq t_j(|s_i| - 1)$, for all i, j . Since \mathbf{s} is ∞ -splitting, there is an ∞ -splitting sequence $\mathbf{u} \in T_{\mathfrak{A}_{m_0}}$ such that $\mathbf{s} \sqsubseteq \mathbf{u}$ and $u_i(|s_i|) > t_j(|t_j| - 1)$, for all i, j . We ask, also, for $|u_i| > |t_j| + 1$, for all i, j . Similarly, we can find an ∞ -splitting sequence $\mathbf{v} \in T_{\mathfrak{A}_{m_0}}$ such that $\mathbf{t} \sqsubseteq \mathbf{v}$ and $v_i(|t_i|) > u_j(|u_j| - 1)$, for all i, j . It follows that $\text{osc}(u_i, v_j) = \text{osc}(s_i, t_j) + 1$ for all i, j . This completes the proof of the claim. \square

Fix \mathbf{s} and \mathbf{t} as in Claim 4, assume without loss of generality that $|s_i| \leq |t_j|$ and $s_i(|s_i| - 1) > t_j(|s_j| - 1)$, for all i, j . Consider, now, the families $X = \{B \in \mathfrak{A} : \mathbf{t} \sqsubseteq B\}$ and $\mathfrak{D}' = \mathfrak{D} \cap X$. We have that X is everywhere unbounded and \mathfrak{D}' is dense in X . Take any $D \in \mathfrak{D}'$, then $\mathbf{t} \sqsubseteq D <_m A$ for some $m > m_0$. Since \mathbf{s} is ∞ -splitting there is $l \geq D(n-1)(m)$ and $B \in \mathfrak{A}_{m_0}$ such that $\mathbf{s} \hat{\wedge} l := \langle s_i \hat{\wedge} l \rangle_{i < n} \sqsubseteq B$. By construction, $\text{osc}(D(i), B(j)) = \text{osc}(s_i, t_j) = h(i, j) + k$, for all $i, j < n$. This completes the proof. \square

Sometimes we need to improve osc to get an even better coloring. First we want to get rid of the function h of Theorem 13. We fix a bijection $\omega \xrightarrow{e} \omega \times \omega$. We define a new partial function o on pairs of elements of $(\omega)^{<\omega}$ or $(\omega)^\omega$ as follows. Suppose $\text{osc}(f, g) = 2^{i_0} + 2^{i_1} + \dots + 2^{i_k}$ for $i_0 > i_1 > \dots > i_k$, be the binary expansion of $\text{osc}(f, g)$. We define $o(f, g) = \pi_0 \circ e(i_0)$ where π_0 is the projection on the first component.

Lemma 7. *Suppose \mathcal{F} and $\mathfrak{A} \subseteq [\mathcal{F}]^n$ are as in Theorem 13. Then for all $k < \omega$ there exists $A, B \in \mathfrak{A}$ such that $A \neq B$ and $o(A(i), B(j)) = k$, for all $i, j < n$.*

Proof. Given k , consider the function $h : n \times n \rightarrow \omega$ of Theorem 13. For all $i, j < n$, let $l_{i,j}$ be the largest integer such that $2^{l_{i,j}} \leq h(i, j)$ and let $l = \max\{l_{i,j}; i, j < n\}$. The set $\{m : \exists p(e(m) = (k, p))\}$ is infinite so we can find $m > l$ such that $\pi_0 \circ e(m) = k$. By definition of h there exist two different $A, B \in \mathfrak{A}$ such that $\text{osc}(A(i), B(j)) = h(i, j) + 2^m$, for all $i, j < n$. It follows that $o(A(i), B(j)) = \pi_0 \circ e(m) = k$, for all $i, j < n$. This completes the proof. \square

Finally we want to be able to choose the color of $\{A(i), B(j)\}$ independently for all i, j . First we need the following lemma.

Lemma 8. *Given $\mathfrak{A} \subseteq [\mathcal{F}]^n$ an unbounded family of pairwise disjoint sets. There are $k < \omega$ and $\mathfrak{A}^* \subseteq \mathfrak{A}$ unbounded such that, for all $i < n$, there exists $a_i \in (\omega)^k$ such that $A(i) \upharpoonright k = a_i$, for all $A \in \mathfrak{A}^*$ and $a_i \neq a_j$, for all $i \neq j < n$.*

Proof. We prove it by induction on $n < \omega$. It is trivial for $n = 1$. Assume that the statement is true for n , we prove it for $n + 1$. Given $\mathfrak{A} \subseteq [\mathcal{F}]^{n+1}$, let be $k < \omega$, $\mathfrak{A}' \subseteq \mathfrak{A} \upharpoonright n$, and $\{a_i\}_{i < n}$ as in the conclusion of the lemma for $\mathfrak{A} \upharpoonright n$. The set $\mathfrak{B} = \{A \in \mathfrak{A} : A \upharpoonright n \in \mathfrak{A}'\}$ is unbounded, hence $X = \{A(n) : A \in \mathfrak{B}\}$ is also unbounded. By Lemma 4, we have that T_X is superperfect, so let b be the least ∞ -splitting node of T_X . We can assume without loss of generality that $|b| < k$. Take any $a_n \sqsupseteq b$ in T_X such that $|a_n| = k$ and $a_n(k-1) > \max\{a_i(k-1) : i < n\}$. Then $a_n \neq a_i$, for all $i < n$. Recall that $T_X = \{s \in (\omega)^{<\omega} : \{f \in X : s \sqsubseteq f\} \text{ is unbounded}\}$, thus $\mathfrak{A}^* = \{B \in \mathfrak{B} : a_n \sqsubseteq B(n)\}$ is unbounded. This completes the proof. \square

Consider all finite functions $t : D \times E \rightarrow \omega$ where $D, E \subseteq (\omega)^k$ and k is an integer. Let $\{(t_n, D_n, E_n, k_n)\}_{n < \omega}$ be any enumeration of such functions. We define $c : [\mathcal{F}]^2 \rightarrow \omega$ as follows: given $f, g \in \mathcal{F}$ and letting $n = o(f, g)$, we define

$$c(f, g) = \begin{cases} t_n(f \upharpoonright k_n, g \upharpoonright k_n) & \text{if } f \upharpoonright k_n \in D_n \text{ and } g \upharpoonright k_n \in E_n \\ 0 & \text{otherwise} \end{cases}$$

Theorem 14 ([9]). *Given an unbounded family $\mathfrak{A} \subseteq [\mathcal{F}]^n$ of pairwise disjoint sets, and an arbitrary $u : n \times n \rightarrow \omega$, there are two different $A, B \in \mathfrak{A}$ such that $c(A(i), B(j)) = u(i, j)$, for all $i, j < n$.*

Proof. Take $k < \omega$, \mathfrak{A}^* and $\{a_i\}_{i < n}$ as in the conclusion of Lemma 8 and let $D = \{a_i : i < n\}$. Consider the function $t : D \times D \rightarrow \omega$ defined by $t(a_i, a_j) = u(i, j)$, for all $i, j < n$. Assume that (t_m, D_m, E_m, k_m) is the corresponding sequence in the previous enumeration. By Lemma 7 there exist different $A, B \in \mathfrak{A}^*$ such that $o(A(i), B(j)) = m$, for all $i, j < n$. It follows that $u(i, j) = t(a_i, a_j) = t_m(A(i) \upharpoonright k_m, B(j) \upharpoonright k_m) = c(A(i), B(j))$. This completes the proof. \square

Corollary 5. *There is a \mathfrak{b} – c.c. partial order whose square is not \mathfrak{b} – c.c. \square*

The following question is still open.

Question 2. Can we do the same for some other cardinal invariant such as \mathfrak{t} or \mathfrak{p} ?

5 Partitions of countable ordinals

Oscillations provide the main tool for constructing partitions of pairs of countable ordinals with very strong properties. The goal of this section is to present the construction of an L -space due to Moore [7] which uses oscillations in an ingenious way. In order to motivate this construction we start with a simple example.

For each limit $\alpha < \omega_1$, fix $c_\alpha \subseteq \alpha$ cofinal of order type ω . As before, we will view c_α both as a set and as an ω -sequence which enumerates it in increasing order. Thus, we will write $c_\alpha(n)$ for the n -th element of c_α . The sequence $\langle c_\alpha : \alpha < \omega_1, \lim(\alpha) \rangle$ is called a \mathfrak{c} -sequence.

We can generalize the definition of osc as follows: for $f, g \in (\omega_1)^{<\omega}$,

$$\text{osc}(f, g) = |\{n < \omega : f(n) \leq g(n) \wedge f(n+1) > g(n+1)\}|.$$

Given a subset S of ω_1 consisting of limit ordinals, let

$$U_S = \{s \in [\omega_1]^{<\omega} : \{\alpha \in S : s \sqsubseteq c_\alpha\} \text{ is stationary}\}.$$

Lemma 9. *Assume $S \subseteq \omega_1$ is stationary. Then U_S is an ω_1 -superperfect tree.*

Proof. Given $s \in U_S$ let $(U_S)_s = \{t \in U_S : s \sqsubseteq t\}$ and let $\alpha_{s,n} = \sup\{t(n) : t \in (U_S)_s\}$. Then there is n such that $\alpha_{s,n} = \omega_1$. To see this, assume otherwise and let $\alpha = \sup\{\alpha_{s,n} : n < \omega\}$. Then $\alpha < \omega_1$. For each $\delta \in S \setminus (\alpha + 1)$ such that $s \sqsubseteq c_\delta$ let n_δ be the least integer such that $c_\delta(n_\delta) > \alpha$. By the Pressing Down Lemma, there is $t \in [\omega_1]^{<\omega}$ such that $s \sqsubseteq t$ and the set $\{\delta \in S : c_\delta \upharpoonright (n_\delta + 1) = t\}$ is stationary. It follows that $s \sqsubseteq t \in U_S$ and $\max(t) > \alpha$, a contradiction. \square

Lemma 10. *Given two stationary sets $S, T \subseteq \omega_1$, there is $n_0 < \omega$ such that for all $k < \omega$ there exist $\alpha \in S$ and $\beta \in T$ such that $\text{osc}(c_\alpha, c_\beta) = n_0 + k$.*

Proof. By Lemma 9 both U_S and U_T are ω_1 -superperfect. Let s and t be the least ω_1 -splitting nodes of U_S and U_T respectively. We may assume that $|s| \leq |t|$ and $s(|s|-1) \leq t(|t|-1)$. Let $n_0 = \text{osc}(s, t) + 1$. Now, as in the proof of Lemma 5, given an integer k we can find ω_1 -splitting nodes s' and t' of U_S and U_T respectively, such that $s \sqsubseteq s'$, $t \sqsubseteq t'$ and $\text{osc}(s', t') = n_0 + k - 1$. Moreover, we can arrange that $|s'| \leq |t'|$ and $s'(|s'|-1) \leq t'(|t'|-1)$. Now, pick any $\beta \in T$ such that $t' \sqsubseteq c_\beta$. Since s' is an ω_1 -splitting node of U_S , there is $\gamma > \beta$ such that $s' \hat{\wedge} \gamma \in U_S$. Pick $\alpha \in S$ such that $s' \hat{\wedge} \gamma \sqsubseteq c_\alpha$. It follows that $\text{osc}(c_\alpha, c_\beta) = \text{osc}(s', t') + 1 = n_0 + k$, as desired. \square

We can then improve osc as before to get some better coloring. We know that our coloring cannot be as strong as in the case of \mathfrak{b} , since MA_{\aleph_1} implies that the countable chain condition is productive, so we have to give up some of the properties of our coloring.

We now present a construction of Moore [7] of a coloring of pairs of countable ordinals witnessing $\omega_1 \not\rightarrow [\omega_1; \omega_1]_\omega^2$ and use it to construct an L -space. As before we fix a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ such that

- if $\alpha = \xi + 1$, then $C_\alpha = \{\xi\}$;
- if α is limit, then $C_\alpha \subseteq \alpha$ is cofinal and of order type ω .

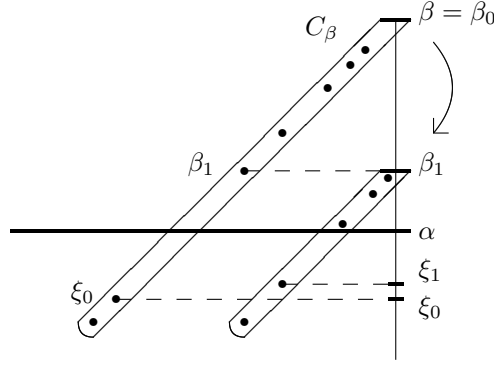
Given $\alpha < \beta$ we define the *walk* from β to α . We first define a sequence $\beta_0 > \beta_1 \cdots > \beta_l = \alpha$ as follows:

- $\beta_0 = \beta$;
- $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$.

Then we define $\xi_0 \leq \xi_1 \cdots \leq \xi_{l-1}$ by setting

$$\xi_k = \max \bigcup_{j=0}^k (C_{\beta_j} \cap \alpha),$$

for all $k \leq l - 1$. We call $\text{Tr}(\alpha, \beta) = \{\beta_0, \dots, \beta_l\}$ the *upper trace* and $L(\alpha, \beta) = \{\xi_0, \dots, \xi_{l-1}\}$ the *lower trace* of the walk from β to α .



Lemma 11. Suppose that $\alpha \leq \beta \leq \gamma$ and $\max(L(\beta, \gamma)) < \min(L(\alpha, \beta))$, then $L(\alpha, \gamma) = L(\alpha, \beta) \cup L(\beta, \gamma)$.

Proof. Since $\max(L(\beta, \gamma)) < \min(L(\alpha, \beta))$, we have $C_\xi \cap \alpha = C_\xi \cap \beta$ whenever ξ is in $\text{Tr}(\beta, \gamma)$ and $\xi \neq \beta$. It follows that $\beta \in \text{Tr}(\alpha, \gamma)$ and $\text{Tr}(\alpha, \gamma) = \text{Tr}(\alpha, \beta) \cup \text{Tr}(\beta, \gamma)$. Assume $\text{Tr}(\alpha, \gamma) = \{\gamma_0, \dots, \gamma_l\}$ and $L(\alpha, \gamma) = \{\xi_0, \dots, \xi_{l-1}\}$, there is $l_0 \leq l$ such that $\gamma_{l_0} = \beta$. Therefore, $\{\xi_k\}_{k \leq l_0-1} = L(\beta, \gamma)$. On the other hand $\max(C_{\gamma_{l_0}} \cap \alpha) > \xi_{l_0-1}$ because $\xi_{l_0-1} \in L(\beta, \gamma)$ and $\max C_{\gamma_{l_0}} \in L(\alpha, \beta)$, hence if $k \geq l_0$, then

$$\xi_k = \max \bigcup_{j=0}^k (C_{\gamma_j} \cap \alpha) = \max \bigcup_{j=l_0}^k (C_{\gamma_j} \cap \alpha),$$

and so $L(\alpha, \beta) = \{\xi_k\}_{k=l_0}^{l-1}$. \square

Lemma 12. If $\alpha < \beta$, then $L(\alpha, \beta)$ is a non empty finite set and, for every limit ordinal β , $\lim_{\alpha \rightarrow \beta} \min(L(\alpha, \beta)) = \beta$.

Proof. The first statement is trivial, let us prove that $\lim_{\alpha \rightarrow \beta} \min(L(\alpha, \beta)) = \beta$, for every limit ordinal β . Given $\alpha < \beta$, one can take $\alpha' \in C_\beta \setminus (\alpha + 1)$. Then $\alpha < \alpha' = \max(C_\beta \cap (\alpha' + 1)) = \min L(\alpha' + 1, \beta) \leq \lim_{\alpha \rightarrow \beta} \min(L(\alpha, \beta))$. It follows that $\beta \leq \lim_{\alpha \rightarrow \beta} \min(L(\alpha, \beta)) \leq \beta$, and this completes the proof. \square

Fix a sequence $\langle e_\alpha : \alpha < \omega_1 \rangle$ satisfying the following conditions:

1. $e_\alpha : \alpha \rightarrow \omega$ is finite-to-one;
2. $\alpha < \beta$ implies $e_\beta \upharpoonright \alpha = {}_* e_\alpha$, i.e. $\{\xi < \alpha : e_\beta(\xi) \neq e_\alpha(\xi)\}$ is finite.

Given $\alpha < \beta < \omega_1$ let $\Delta(\alpha, \beta)$ be the least $\xi < \alpha$ such that $e_\alpha(\xi) \neq e_\beta(\xi)$, if it exists, and α otherwise. We define $\text{osc}(\alpha, \beta)$ as follows

$$\text{osc}(\alpha, \beta) = |\{i \leq l-1 : e_\alpha(\xi_i) \leq e_\beta(\xi_i) \wedge e_\alpha(\xi_{i+1}) > e_\beta(\xi_{i+1})\}|$$

where $L(\alpha, \beta) = \{\xi_0 < \dots < \xi_{l-1}\}$.

It will be convenient also to use the notation $\text{Osc}(e_\alpha, e_\beta, L(\alpha, \beta))$ for the set $\{\xi_i \in L(\alpha, \beta) : e_\alpha(\xi_i) \leq e_\beta(\xi_i) \wedge e_\alpha(\xi_{i+1}) > e_\beta(\xi_{i+1})\}$.

Our aim is to prove the following theorem due to J. Moore (see [7]).

Theorem 15 ([7]). *Let $A, B \subseteq \omega_1$ be uncountable, then for all $n < \omega$ there exist $\alpha \in A$, $\beta_0, \beta_1, \dots, \beta_{n-1} \in B$ and k_0 such that $\alpha < \beta_0, \dots, \beta_{n-1}$ and $\text{osc}(\alpha, \beta_m) = k_0 + m$, for all $m < n$.*

This means that we can get arbitrary long intervals of oscillations with a fixed lower point $\alpha \in A$. We can generalize this to get even more:

Theorem 16 ([7]). *Given $\mathfrak{A} \subseteq [\omega_1]^k$ and $\mathfrak{B} \subseteq [\omega_1]^l$ uncountable and pairwise disjoint, and given $n < \omega$, we can find $A \in \mathfrak{A}$ and $B_0, \dots, B_{n-1} \in \mathfrak{B}$ such that $\max A < \min B_i$, for all $i < n$, and $\text{osc}(A(i), B_m(j)) = \text{osc}(A(i), B_0(j)) + m$ for all $i < k$, $j < l$ and $m < n$.*

In order to prove Theorem 15 we demonstrate the following lemma.

Lemma 13. *Let $A, B \subseteq \omega_1$ be uncountable. There exists a club $C \subseteq \omega_1$ such that if $\delta \in C$, $\alpha \in A \setminus \delta$, $\beta \in B \setminus \delta$, and $R \in \{=, >\}$, then there are $\alpha' \in A \setminus \delta$ and $\beta' \in B \setminus \delta$ satisfying the following properties:*

1. $\max L(\alpha, \beta) < \Delta(\alpha, \alpha'), \Delta(\beta, \beta')$;
2. $L(\delta, \beta) \sqsubseteq L(\delta, \beta')$;
3. for all $\xi \in L^+ = L(\delta, \beta') \setminus L(\delta, \beta)$, we have $e_{\alpha'}(\xi) R e_{\beta'}(\xi)$.

Proof. Fix a sufficiently large regular cardinal θ . We will show that if $M \prec H_\theta$ is a countable elementary substructure containing all the relevant objects, then $\delta = M \cap \omega_1$ satisfies the conclusion of the lemma. Since the set of such δ contains a club in ω_1 this will be sufficient. Thus, fix M and δ as above and let α and β be as in the hypothesis of the lemma. We first suppose that R is $=$. Since δ is a limit ordinal, we can take $\gamma_0 < \delta$ such that:

1. $\max(L(\delta, \beta)) < \gamma_0$;
2. for all $\xi \in (\gamma_0, \delta)$, $e_\alpha(\xi) = e_\beta(\xi)$.

By Lemma 12 we can fix also $\gamma < \delta$ such that $\gamma_0 < \min L(\xi, \delta)$, for all $\xi \in (\gamma, \delta)$. Let D be the set of all $\delta' < \omega_1$ such that for some $\alpha' \in A \setminus \delta'$ and $\beta' \in B \setminus \delta'$ the following properties are satisfied:

- (a) $e_{\alpha'} \upharpoonright \gamma_0 = e_\alpha \upharpoonright \gamma_0$, $e_{\beta'} \upharpoonright \gamma_0 = e_\beta \upharpoonright \gamma_0$;
- (b) $L(\delta', \beta') = L(\delta, \beta)$;
- (c) for all $\xi \in (\gamma, \delta')$, $\gamma_0 < \min L(\xi, \delta')$;

(d) for all $\xi \in (\gamma_0, \delta')$, $e_{\alpha'}(\xi) = e_{\beta'}(\xi)$.

Observe that for all $\xi \geq \gamma_0$, $e_\xi \upharpoonright \gamma_0$ is in M since, by definition, $e_\xi \upharpoonright \gamma_0 =_* e_{\gamma_0}$. This means that D is definable in M , hence $D \in M$. Moreover $D \not\subseteq M$ (since $\delta \in D$) hence D is uncountable. Choose $\delta' > \delta$ in D with $\alpha' \in A \setminus \delta'$ and $\beta' \in B \setminus \delta'$ witnessing $\delta' \in D$. By condition (a) of the definition of D ,

$$\gamma_0 \leq \Delta(\alpha, \alpha'), \Delta(\beta, \beta').$$

Put $L^+ = L(\delta, \delta')$, then $\max L(\delta, \beta) = \max L(\delta', \beta') < \min L^+$, hence

$$L(\delta, \beta') = L(\delta', \beta') \cup L^+ = L(\delta, \beta) \cup L^+.$$

Given $\xi \in L^+$, by condition (c), we have $\gamma_0 < \min L^+ \leq \xi$. It follows that $\xi \in (\gamma_0, \delta')$, so (d) implies $e_{\alpha'}(\xi) = e_{\beta'}(\xi)$.

Now assume that R is $>$. Let E be the set of all limits $\nu < \omega_1$ such that for all $\alpha_0 \in A \setminus \nu$, $\nu_0 < \nu$, $\varepsilon < \omega_1$, $n < \omega$ and finite $L^+ \subseteq \omega_1 \setminus \nu$, there exists $\alpha_1 \in A \setminus \varepsilon$ with $\nu_0 \leq \Delta(\alpha_0, \alpha_1)$ and $e_{\alpha_1}(\xi) > n$, for all $\xi \in L^+$. Since E is definable from parameters in M it follows $E \in M$, as well.

Claim 5 *The ordinal δ is in E . In particular E is uncountable.*

Proof. Let $\alpha_0, \nu_0, \varepsilon, n, L^+$ be given as in the definition of E for $\nu = \delta$. Since e_{α_0} is finite-to-one, we can assume w.l.o.g. that $\nu_0 > \sup\{\xi < \delta : e_{\alpha_0}(\xi) \leq n\}$. By the elementarity of M , there exists $\delta' > \varepsilon$, δ , $\max L^+$ and $\alpha_1 \in A \setminus \delta'$ such that the following conditions hold:

- $e_{\alpha_0} \upharpoonright \nu_0 = e_{\alpha_1} \upharpoonright \nu_0$;
- for all ξ in (ν_0, δ') , we have $e_{\alpha_1}(\xi) > n$.

Since $L^+ \subseteq \delta' \setminus \delta$, this completes the proof of the claim. \square

Now apply the elementarity of M and the fact that E is uncountable to find $\gamma_0 \in E$ such that $L(\delta, \beta) < \gamma_0 < \delta$. By Lemma 12 we can find a $\gamma < \delta$ such that if $\xi \in (\gamma, \delta)$, then $\gamma_0 < L(\xi, \delta)$. Again by the elementarity of M select limit $\delta' > \delta$ and $\beta' \in B \setminus \delta'$ such that the following conditions hold:

- $e_{\beta'} \upharpoonright \gamma_0 = e_\beta \upharpoonright \gamma_0$;
- $L(\delta', \beta') = L(\delta, \beta)$;
- $\gamma < \xi < \delta'$ implies $\gamma_0 < L(\xi, \delta')$.

Put $L^+ = L(\delta, \delta')$, then $L^+ \subseteq \omega_1 \setminus \gamma_0$. Since $\gamma_0 \in E$ we can apply the definition of E with $\nu_0 = \max L(\delta, \beta) + 1$, $n = \max\{e_{\beta'}(\xi); \xi \in L^+\}$ to find $\alpha' \in A \setminus \delta$ such that for all $\xi \in L^+$, $\max L(\delta, \beta) < \Delta(\alpha, \alpha')$ and $e_{\alpha'}(\xi) > e_{\beta'}(\xi)$. This completes the proof of Lemma 13. \square

We can finally prove Theorem 15.

Proof. (of **Theorem 15**). Let $A, B \subseteq \omega_1$ be two uncountable sets and let $M \prec H_{\aleph_2}$ be a countable substructure containing everything relevant with $\delta = M \cap \omega_1$. Since M contains A and B , the club C provided by Lemma 13 is in M . Use Lemma 13 to select $\alpha_0, \alpha_1, \dots, \alpha_n, \dots$ in $A \setminus \delta$, $\beta_0, \beta_1, \dots, \beta_n, \dots$ in $B \setminus \delta$ and $\xi_0, \xi_1, \dots, \xi_n, \dots$ in δ such that for all $n < \omega$ the following conditions are satisfied:

1. $L(\delta, \beta_n) \sqsubset L(\delta, \beta_{n+1})$;
2. $\xi_n \in L(\delta, \beta_{n+1}) \setminus L(\delta, \beta_n)$;
3. $\text{Osc}(e_{\alpha_{n+1}}, e_{\beta_{n+1}}, L(\delta, \beta_{n+1})) = \text{Osc}(e_{\alpha_n}, e_{\beta_n}, L(\delta, \beta_n)) \cup \{\xi_n\}$;
4. if $m > n$, then $\xi_n < \Delta(\alpha_m, \alpha_{m+1}), \Delta(\beta_m, \beta_{m+1})$;
5. $e_{\alpha_n}(\max L(\delta, \beta_n)) > e_{\beta_n}(\max L(\delta, \beta_n))$.

Suppose α_n and β_n have been defined. We obtain α_{n+1} and β_{n+1} by applying Lemma 13 twice: first with R being $=$, second with R being $>$. If α' and β' are the two ordinals obtained by applying the lemma the first time, then $\xi_n = \min(L(\delta, \beta_{n+1}) \setminus L(\delta, \beta'))$.

Now let n be given, pick $\gamma_0 < \delta$ such that

$$\gamma_0 > \max L(\delta, \beta_n), \max\{\xi < \delta : \exists m, m' \leq n (e_{\beta_m}(\xi) \neq e_{\beta_{m'}}(\xi))\}.$$

Using the elementarity of M and Lemma 12, select $\alpha \in A \cap \delta$ such that

$$\max L(\delta, \beta_n) < \Delta(\alpha, \alpha_n) \text{ and } \gamma_0 < \min L(\alpha, \delta).$$

Now let $m < n$ be fixed. It follows from Lemma 11 that

$$L(\alpha, \beta_m) = L(\alpha, \delta) \cup L(\delta, \beta_m).$$

Finally $e_{\beta_m} \upharpoonright L(\alpha, \delta)$ does not depend on m since

$$\min L(\alpha, \delta) > \gamma_0 > \max\{\xi < \delta : \exists m, m' \leq n (e_{\beta_m}(\xi) \neq e_{\beta_{m'}}(\xi))\}.$$

Therefore

$$\text{Osc}(e_\alpha, e_{\beta_0}, L(\alpha, \delta)) = \text{Osc}(e_\alpha, e_{\beta_m}, L(\alpha, \delta)).$$

By 5., $\text{Osc}(e_\alpha, e_{\beta_m}, L(\alpha, \beta_m)) = \text{Osc}(e_\alpha, e_{\beta_m}, L(\alpha, \delta)) \cup \text{Osc}(e_\alpha, e_{\beta_m}, L(\delta, \beta_m))$ so by 3., $\text{Osc}(e_\alpha, e_{\beta_m}, L(\alpha, \beta_m)) = \text{Osc}(e_\alpha, e_{\beta_0}, L(\alpha, \beta_0)) \cup \{\xi_{m'}; m' < m\}$. Hence $\text{osc}(\alpha, \beta_m) = \text{osc}(\alpha, \beta_0) + m$ and this completes the proof. \square

By using the previous results we can, finally, prove the existence of an L -space, that is, a regular Hausdorff space which is hereditarily Lindelöf but not hereditarily separable. We will work in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We fix a sequence $\langle z_\alpha : \alpha < \omega_1 \rangle$ of rationally independent elements of \mathbb{T} . It is easy to find such a sequence since given any countable rationally independent subset I of \mathbb{T} , there are only countable many z for which $I \cup \{z\}$ is rationally dependent. Consider, now, the function defined as follows:

$$o(\alpha, \beta) = z_\alpha^{\text{osc}(\alpha, \beta) + 1},$$

for all $\alpha < \beta < \omega_1$.

We will use the *Kronecker's Theorem* (see [6] or [8]) which is the following statement:

Theorem 17. *Suppose that $\langle z_i \rangle_{i < k}$ is a sequence of elements of \mathbb{T} which are rationally independent. For every $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that if $u, v \in \mathbb{T}^k$, there is $m < n_\epsilon$ such that*

$$|u_i z_i^m - v_i| < \epsilon,$$

for all $i < k$.

We can, now, define the L -space. For every $\beta < \omega_1$, we define a function $w_\beta : \omega_1 \rightarrow \mathbb{T}$ as follows:

$$w_\beta(\xi) = \begin{cases} o(\xi, \beta) & \text{if } \xi < \beta \\ 1 & \text{otherwise.} \end{cases}$$

Let $\mathcal{L} = \{w_\beta : \beta < \omega_1\}$ viewed as a subspace of \mathbb{T}^{ω_1} .

Remark 3. \mathcal{L} is not separable.

For all $X \subseteq \omega_1$, let $\mathcal{L}_X = \{w_\beta \upharpoonright X; \beta \in X\}$ viewed as a subspace of \mathbb{T}^X . We will simply write w_β for $w_\beta \upharpoonright X$ when referring to elements of \mathcal{L}_X . Our aim is to prove that \mathcal{L}_X is an L -space, for all X uncountable.

Lemma 14. *Let $\mathcal{A} \subseteq [\omega_1]^k$ and $\mathcal{B} \subseteq [\omega_1]^l$ be uncountable families of pairwise disjoint sets. For every sequence $\langle U_i \rangle_{i < k}$ of open neighborhoods in \mathbb{T} and every $\phi : k \rightarrow l$, there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\max(a) < \min(b)$ and for all $i < k$,*

$$o(a(i), b(\phi(i))) \in U_i.$$

Proof. We may assume without loss of generality that every U_i is an ϵ -ball about a point v_i , for some fixed $\epsilon > 0$. We can assume also that the integer n_ϵ of the Kronecker's Theorem for the sequence $\langle z_{a(i)} \rangle_{i < k}$ is uniform for $a \in \mathcal{A}$. Apply Theorem 16 to find $a \in \mathcal{A}$ and $\langle b_m \rangle_{m < n_\epsilon}$ a sequence of elements of \mathcal{B} such that

$$a < b_m$$

$$\text{osc}(a(i), b_m(j)) = \text{osc}(a(i), b_0(j)) + m,$$

for all $i < k, j < l$ and $m < n_\epsilon$. For each $i < k$, put $u_i = o(a(i), b_0(\phi(i)))$. There is an $m < n_\epsilon$, such that

$$|u_i z_{a(i)}^m - v_i| < \epsilon,$$

for all $i < k$ or, equivalently, $o(a(i), b_m(\phi(i))) \in U_i$. This completes the proof. \square

Lemma 15. *If $X, Y \subseteq \omega_1$ have countable intersection, then there is no continuous injection from any uncountable subspace of \mathcal{L}_X into \mathcal{L}_Y .*

Proof. Suppose, by way of contradiction, that such an injection g does exist. Then there are an uncountable set $X_0 \subseteq X$ and an injection $f : X_0 \rightarrow Y$ such that $g(w_\beta) = w_{f(\beta)}$. We may assume without loss of generality that X_0 is disjoint from Y . For each $\xi < \omega_1$, let $\beta_\xi \in X_0$ and $\zeta_\xi \in Y$ be such that $f(\beta_\xi) > \zeta_\xi$

and if $\xi < \xi'$, then $\beta_\xi < \zeta_{\xi'}$. Let $\Xi \subseteq \omega_1$ be uncountable such that for every $\xi \in \Xi$ there is an open neighborhood V in \mathbb{T} , such that $g(w_{\beta_\xi})(\zeta_\xi) \notin \bar{V}$. Let $W_\xi = \{w \in \mathcal{L}_Y : w(\zeta_\xi) \notin \bar{V}\}$, for all $\xi < \omega_1$. Since g is continuous at w_{β_ξ} , there is a basic open neighborhood U_ξ of w_{β_ξ} such that $U_\xi \subseteq g^{-1}W_\xi$. By applying the Δ -system lemma and the second countability of \mathbb{T} , we can find an uncountable $\Xi' \subseteq \Xi$, a sequence of open neighborhoods $\langle U_i \rangle_{i < k}$ in \mathbb{T} , and $a_\xi \in [X]^k$ such that for all $\xi \in \Xi'$, the following conditions hold:

- $\{a_\xi\}_{\xi \in \Xi'}$ is a Δ -system with root a ;
- $w_{\beta_\xi} \in \{w \in \mathcal{L}_X : \forall i < k (w(a_\xi(i)) \in U_i)\} \subseteq U_\xi$;
- the inequality $\beta_\xi < f(\beta_\xi)$ does not depend on ξ ;
- $|\zeta_\xi \cap a_\xi|$ does not depend on ξ .

Let $\mathcal{A} = \{a_\xi \cup \{\xi\} \setminus a\}_{\xi \in \Xi'}$ and $\mathcal{B} = \{\beta_\xi, f(\beta_\xi)\}_{\xi \in \Xi'}$. By applying Lemma 14 we can find $\xi < \xi'$ in Ξ' such that for all $i < k$,

$$\begin{aligned} a_\xi \cup \{\zeta_\xi\} &< \min(\beta_{\xi'}, f(\beta_{\xi'})), \\ w_{\beta_{\xi'}}(a_\xi(i)) &= o(a_\xi(i), \beta_{\xi'}) \in U_i, \\ g(w_{\beta_{\xi'}}) &= w_{f(\beta_{\xi'})}(\zeta_\xi) = o(\zeta_\xi, f(\beta_{\xi'})) \in V. \end{aligned}$$

We have that $w_{\beta_{\xi'}} \in U_\xi$ and $g(w_{\beta_{\xi'}}) \notin W_\xi$. Contradiction. \square

Theorem 18 ([7]). *For every X , \mathcal{L}_X is hereditarily Lindelöf.*

Proof. If not, then \mathcal{L}_X would contain an uncountable discrete subspace. Moreover it would be possible to find disjoint $Y, Z \subseteq X$ such that \mathcal{L}_Y and \mathcal{L}_Z contain uncountable discrete subspaces. It is well known that any function from a discrete space to another discrete space is continuous and this contradicts Lemma 15. \square

Corollary 6 ([7]). *There exists an L -space, i.e. a hereditary Lindelöf non separable T_3 topological space.* \square

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