The Journal of Symbolic Logic
Volume 00, Number 0, XXX 0000

PCF STRUCTURES OF HEIGHT LESS THAN ω_3

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Abstract. We show that it is relatively consistent with ZFC to have PCF structures of height δ, for all ordinals $\delta < \omega_3$.

Introduction. Computing the least upper bound on the possible values of 2^κ for strong limit cardinals κ is one of the major goals of cardinal arithmetic. The most interesting case is when κ is the least singular cardinal $\aleph_\omega$. On one hand, assuming the existence of large cardinals, one can produce models of set theory in which $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\alpha+1}$, for an arbitrary infinite countable ordinal α, see [7] for a detailed account. On the other hand, in the late 1980s Shelah proved his celebrated theorem stating that $\aleph_\omega^{\aleph_0} < \max((2^{\aleph_0})^+, \aleph_{\omega_4})$, see [13]. Assuming $\aleph_\omega$ is a strong limit cardinal this implies that $2^{\aleph_\omega} < \aleph_{\omega_4}$. Thus, the least upper bound on the possible values of $2^{\aleph_\omega}$, assuming $\aleph_\omega$ is strong limit, is somewhere between $\aleph_{\omega_1}$ and $\aleph_{\omega_4}$. In order to prove his theorem Shelah developed the theory of possible cofinalities. One of the key objects in this theory is the pcf operator which to a set A of regular cardinals assigns the set $\mathrm{pcf}(A)$ of all cofinalities of ultrapowers of A. Assuming A is an interval of regular cardinals which is *progressive*, i.e. it satisfies $|A| < \min(A)$, the pcf operator is a closure operation on subsets of $\mathrm{pcf}(A)$. The associated topological space has some important topological and combinatorial properties which allowed Shelah to show that the order type of $\mathrm{pcf}(A)$ is less than $|A|^{+4}$. Since $\mathrm{pcf}(A)$ is also an interval of regular cardinal this yields an absolute bound on $\sup \mathrm{pcf}(A)$. The statement that $|\mathrm{pcf}(A)| = |A|$ for any progressive interval of regular cardinals A is known as the PCF conjecture and is one of the major open problems in this subject.

Consider now the case $A = \{\aleph_{n+1} : n < \omega\}$ and call the associated topological space *the* PCF space. The topological properties of this space were studied in detail by Magidor [6] and various other authors. The domain of this space is the set of uncountable regular cardinals less than or equal to $\max \mathrm{pcf}(A)$, but it is convenient to identify a cardinal $\aleph_{\alpha+1}$ with α and in this way we can consider

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1991 *Mathematics Subject Classification.* 03E20, 54A25, 06E05, 54G12.

Key words and phrases. cardinal arithmetic, PCF theory, locally compact scattered spaces, forcing, Δ-functions.

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the domain of the PCF space to be an ordinal Θ which we refer to as its *height*. Thus, the PCF space is a separable locally compact scattered and countably tight topological space on an ordinal Θ. In addition, there is an interaction between the PCF topology and the usual order on Θ. We call this the *club property*. Any space together with a well ordering having these properties will be called a PCF *structure*. By Shelah's theorem any PCF structure is of height $< \omega_4$ and it is natural to ask if this part of the theorem is optimal. In this direction in [9] Jech and Shelah constructed in ZFC a PCF structure of height ω_1. Further, by developing ideas of Veličković, Ruyle showed in his PhD thesis [12] that it is relatively consistent with ZFC to have a PCF structure of height ω_2. Of course, it is not known if any of these PCF structure can be forced to be the *real* PCF space as this would provide a negative answer to the PCF conjecture. The purpose of this paper is, as the title says, to show that it is relatively consistent with ZFC to have PCF structures of height δ, for all ordinals $\delta < \omega_3$. While this does not say anything directly about the PCF conjecture it suggests that in order to prove it one would require methods significantly different from those used by Shelah in his theorem.

Our approach builds on previous constructions of thin tall locally compact scattered (LCS) topological spaces. The first major result in this area is due to Baumgartner and Shelah [5] who proved that it is consistent with ZFC to have thin LCS spaces of height ω_2. Later, Martinez [11] and Soukup in [14] proved in two different ways that it is consistent to have thin LCS spaces of height δ, for any $\delta < \omega_3$. In §1 we present another way to prove this result. In §2 we modify the construction of the first section in order to add the additional properties of the PCF space. We also present some previously unpublished results of the second author which were used by Ruyle [12].

Our notation is standard and everything undefined can be found in [10] or [8]. The reader interested in PCF theory should read [13], [6] or [2] for a complete overview of this theory.

Acknowledgment The authors would like to thank J. C. Martinez and the anonymous referee for many useful remarks and corrections.

§1. About LCS spaces. A topological space X is called *scattered* if every non-empty subspace of X has an isolated point. In the following, an LCS space will denote a locally compact scattered space. For $Y \subseteq X$, let $I(Y)$ the set of isolated points of Y. For every ordinal α, we define the α-th *Cantor-Bendixson* level of X as:

$$I_\alpha(X) = I(X \setminus \cup\{I_\beta(X) : \beta < \alpha\}).$$

If X is scattered, then there exists α such that $I_\alpha = \emptyset$. The minimal such α is called the *height* of X and is denoted by $ht(X)$. The *cardinal sequence* of X is the sequence of the cardinalities of its levels in the Cantor-Bendixson process. More precisely, we have:

$$CS(X) = \langle |I_\alpha(X)| : \alpha < ht(X)\rangle.$$

The *width* of X is the maximum cardinality of its Cantor-Bendixson levels. The following definition is a useful tool for the purpose of forcing such spaces.

In fact, it arises from the result of Baumgartner-Shelah in [5] which says it is consistent with ZFC to have an LCS space of height ω_2 and width ω. This is exactly the same definition as in [4].

DEFINITION 1.1. Given a cardinal sequence $\theta = \langle \kappa_\alpha : \alpha < \lambda \rangle$, where each κ_α is an infinite cardinal, we say that a poset $(T, \leq, i)$ is a *θ-poset* if

1. $T = \bigcup \{T_\alpha : \alpha < \lambda\}$, where each T_α is of the form $\{\alpha\} \times Y_\alpha$, and Y_α is a set of cardinality κ_α.
2. i is a function from $[T]^2$ to $[T]^{<\omega}$ with the following properties:
   (a) If $u \in i\{s,t\}$, then $u \leq s,t$
   (b) If $u \leq s,t$, then there exists $v \in i\{s,t\}$ such that $u \leq v$.
3. If $s \in T_\alpha$, $t \in T_\beta$ and $s < t$, then $\alpha < \beta$.
4. For every $\alpha < \beta < \lambda$, if $t \in T_\beta$ then the set $\{s \in T_\alpha : s < t\}$ is infinite.

Let $\theta = \langle \kappa_\alpha : \alpha < \lambda \rangle$ be a sequence of infinite cardinals and $(T, \leq)$ is a θ poset. Then λ is called the *height* of T and is denoted by $ht(T)$. The *width* of T is $wd(T) = \sup(\kappa_\alpha : \alpha < ht(T))$. For $\alpha < ht(T)$, T_α is called the α-th level of T. For every $t \in T$, we denote by $C(t)$ the cone $\{s \in T : s < t\}$ of t and by $C_\alpha(t)$ the intersection of $C(t)$ with T_α, the α-th level of T. If $s = (\alpha, x)$ is an element of T then we call α the *height* of s and denote it by $\pi_1(s)$ and we denote x by $\pi_2(s)$. A θ-poset is called *thin* if $wd(T) = \omega$. Thus, in this case θ is simply a constant sequence of length λ with all entries equal to ω. We denote such a sequence by $\Omega(\lambda)$.

The following proposition is implicitly due to Baumgartner. We reproduce the simple proof for completeness, see also [4].

PROPOSITION 1.2. *Let $\theta = \langle \kappa_\alpha : \alpha < \lambda \rangle$ be a sequence of cardinals. If there exists a θ-poset, then there exists a locally compact scattered, Hausdorff, space X with $CS(X) = \theta$.*

PROOF. Suppose $(T, \leq, i)$ is a θ-poset. For each t let

$$\mathcal{B}_t = \{C(t) \setminus (C(s_1) \cup \ldots C(s_n)) : n < \omega, s_1, \ldots, s_n \in T, s_1, \ldots, s_n < t\}.$$

Using (2) it is easy to check that $\mathcal{B} = \bigcup \{\mathcal{B}_t : t \in T\}$ is a clopen basis for a Hausdorff topology $\mathcal{T}$ on T. Let $X = (T, \mathcal{T})$. For each $t \in T$, $C(t)$ is a compact neighborhood of t (use (3)). Hence, X is locally-compact. Further, if Z is a non-empty closed subspace of X, by (3) we can always find $t \in Z$ with $C(t) \cap Z = \{t\}$. i.e., t is an isolated point of Z. Hence, X is scattered. Finally, using (4), it is clear that for each $\alpha < \lambda$, $X^\alpha \setminus X^{\alpha+1} = T_\alpha$. So, X has height λ and $CS(X) = \theta$. ⊣

The idea in this paper is to extend an $\Omega(\omega_2)$-poset to an $\Omega(\delta)$-poset of height δ, for every $\delta < \omega_3$. Our plan is to show this by induction on δ. For the successor case, the idea is to add another level above all the previous ones. The next theorem will show how to do this. For the limit case, things are more complicated since for technical reasons we can't take the union of what we constructed before without disturbing the size of the levels. The idea is to construct it in two steps. Imagine what we try to obtain as a body. To build it, we obviously first need a skeleton. And then, we can add some flesh between the bones to obtain the full body. In case of a limit ordinal α, the skeleton will naturally be a club of order type $\mathrm{cof}(\alpha)$. And the flesh is all the missing ordinals between consecutive points of the club. So, we intend to have an $\Omega(\mathrm{cof}(\alpha))$-poset with some additional properties and we want to add the missing levels between consecutive levels of the skeleton. We will now begin with a simple proposition about how to extend an $\Omega(\lambda)$-poset to an $\Omega(\lambda+1)$-poset.

Proposition 1.3. *For every ordinal λ, if there exists an $\Omega(\lambda)$-poset then there exists an $\Omega(\lambda+1)$-poset.*

Proof. Let $(T, \leq, i)$ be an $\Omega(\lambda)$-poset. The α-th level T_α of T is of the form $\{\alpha\} \times Y_\alpha$, for some Y_α. We define an $\Omega(\lambda+1)$-poset T' as follows. We first define the Y'_α, for all $\alpha \leq \lambda$, by

$$Y'_\alpha = \begin{cases} \omega \times Y_\alpha & \text{if } \alpha < \lambda \\ \omega \times \{0\} & \text{if } \alpha = \lambda \end{cases}$$

Let $T'_\alpha = \{\alpha\} \times Y'_\alpha$, for all α, and let $T' = \bigcup\{T'_\alpha : \alpha \leq \lambda\}$. We define the ordering $\leq'$ on T' as follows. Let s and t be two elements of T' and suppose $s = (\alpha, (m, x))$ and $t = (\beta, (n, y))$. If $\alpha < \lambda$ and $\beta = \lambda$, say $s \leq' t$ if and only if $m = n$. If $\alpha,\ \beta < \lambda$, say $s \leq' t$ if and only if $m = n$ and $(\alpha, x) \leq (\beta, y)$.

So, what we did here is simply put a single point above all the points in T and then make ω copies of what we obtain. Now, it is obvious how to define the function i'. Suppose s and t are two elements of T', say $s = (\alpha, (m, x))$ and $t = (\beta, (n, y))$. If $m \neq n$ we let $i'\{s, t\} = \emptyset$. If $m = n$ and $\alpha < \lambda$ while $\beta = \lambda$ we let $i'\{s, t\} = \{s\}$. Finally, if $m = n$ and $\alpha, \beta < \lambda$ we let

$$i'\{s, t\} = \{(\xi, (m, u)) : (\xi, u) \in i\{(\alpha, x), (\beta, y)\}\}.$$

It should be clear now that $(T', \leq', i')$ is an $\Omega(\lambda+1)$-poset. ⊣

So, this theorem allows us to handle the successor case easily. However, one should note that we cannot hope to extend this construction to limit ordinals of uncountable cofinality since in this case the size of the levels will no longer be countable. Thus, we need something more. Let us begin with a new definition.

Definition 1.4. Let $(T^1, \leq_1, i_1)$ be an $\Omega(\alpha+1+\beta)$-poset and $(T^2, \leq_2, i_2)$ an $\Omega(\rho)$-poset. We assume that ξ-th level, T^j_ξ, of T^j is of the form $\{\xi\} \times Y^j_\xi$, for $j = 0, 1$ and all ξ. We define a new poset $(T, \leq, i)$, denoted by $T^2 \hookrightarrow^\alpha T^1$ as follows. Let height of T will be $\lambda = \alpha + 1 + \rho + \beta$. We first define

$$Y_\xi = \begin{cases} Y^1_\xi & \text{if } \xi \leq \alpha \\ \omega \times Y^2_\eta & \text{if } \xi = \alpha + 1 + \eta \text{ for } \eta < \rho \\ Y^1_{\alpha+1+\eta} & \text{if } \xi = \alpha + 1 + \rho + \eta \text{ for } \eta < \beta \end{cases}$$

Let $T_\xi = \{\xi\} \times Y_\xi$ and $T = \bigcup\{T_\xi : \xi < \lambda\}$.

So basically, $T^2 \hookrightarrow^\alpha T^1$ is T^1 with ω copies of T^2 added between the α-th and $\alpha+1$-th level of T^1. The partial order on T will be defined naturally, but first we need some preparation. Let $\{y_n\}_n$ be an enumeration of $Y^1_{\alpha+1}$. For each n, the set C_n of predecessors of $(\alpha+1, y_n)$ on level α in T^1 is infinite, so we can find infinite sets $C_{n,k}$, for $n, k < \omega$, such that $C_{n,k}$ is a subset of C_n, and $C_{m,k} \cap C_{n,l} = \emptyset$, whenever $(m, k) \neq (n, l)$. Finally, let $\{x_k\}_k$ be an enumeration of Y^2_0. We put the n-th element of $(\alpha+1)$-th level of T^1 above the n-th copy of T^2 and the k-th element of the 0-th level of the n-th copy of T^2 above the elements of $C_{n,k}$ and then extend the ordering by transitivity. More precisely, for an ordinal $\xi \in (\alpha+1) \cup [\alpha+1+\rho, \lambda)$ let $\varphi(\xi)$ be defined by:

$$\varphi(\xi) = \begin{cases} \xi & \text{if } \xi \leq \alpha \\ \alpha+1+\eta & \text{if } \xi = \alpha+1+\rho+\eta \text{ for } \eta < \beta \end{cases}$$

Let s and t be two elements of T and suppose $s = (\nu, x)$ and $t = (\xi, y)$. We have several cases.

CASE 1. If $\nu, \xi \in (\alpha+1) \cup [\alpha+1+\rho, \lambda)$ we let $s \leq t$ if and only if $(\varphi(\nu), x) \leq_1 (\varphi(\xi), y)$.

CASE 2. Suppose $\nu, \xi \in [\alpha+1, \alpha+1+\rho)$. Then we have $\nu = \alpha+1+\eta$ and $\xi = \alpha+1+\theta$, for some $\eta, \theta < \rho$. We have that x is of the form (n, u) for some $u \in Y^2_\eta$ and y is of the form (m, v) for some $v \in Y^2_\theta$. In this case we let $s \leq t$ if and only if $n = m$ and $(\eta, u) \leq_2 (\theta, v)$.

CASE 3. Suppose $\nu \leq \alpha+1$ and $\xi \in [\alpha+1, \alpha+1+\rho)$. We have that $\xi = \alpha+1+\theta$, for some $\theta < \rho$, and $y = (n, v)$ for some n and $v \in Y^2_\theta$. In this case we let $s \leq t$ if and only if there is k such that $(0, x_k) \leq_2 (\theta, v)$ and there is $z \in C_{n,k}$ such that $s \leq_1 (\alpha, z)$.

CASE 4. Suppose $\nu \in [\alpha+1, \alpha+1+\rho)$ and $\xi \in [\alpha+1+\rho, \lambda)$. Then we have $\nu = \alpha+1+\eta$, for some $\eta < \rho$ and $x = (n, u)$ for some $u \in Y^2_\eta$. We also have that $y \in Y_{\varphi(\xi)}$. We let $s \leq t$ if and only $(\alpha+1, y_n) \leq_1 (\varphi(\xi), y)$.

Let $e : T^1 \to T$ be defined by setting $e((\eta, x)) = (\varphi^{-1}(\eta), x)$, for all $(\eta, x) \in T^1$. For each n let $f_n : T^2 \to T$ be defined by $f_n((\theta, y)) = (\alpha+1+\theta, (n, y))$, for $(\theta, y) \in T^2$. It is clear that e and the f_n are embeddings and

$$T = e[T^1] \cup \bigcup_n f_n[T^2].$$

We now define the function i on $[T]^2$. If $s, t \in e[T^1]$ let $i\{s,t\} = \{e(u) : u \in i_1\{e^{-1}(s), e^{-1}(t)\}$. If $s, t \in f_n[T^2]$ let $i\{s,t\} = \{f_n(u) : u \in i_2\{f_n^{-1}(s), f_n^{-1}(t)\}\}$. For every $s \in \bigcup_n f_n[T^2]$ let $C_\alpha(s)$ be the set of predecessors of s on level α of T. If $s \in f_m[T^2]$ and $t \in f_n[T^2]$, for $m \neq n$, let

$$i\{s,t\} = \bigcup \{i_1\{u, v\} : u \in C_\alpha(s) \text{ and } v \in C_\alpha(t)\}.$$

Suppose now $s \in f_n[T^2]$, for some n, and $t \in e[T^1]$. If $s \leq t$ we let $i\{s,t\} = \{s\}$ and if $t \leq s$ we let $i\{s,t\} = \{t\}$. Finally, suppose s and t are incomparable. Then we let

$$i\{s,t\} = \bigcup \{i_1\{u, t\} : u \in C_\alpha(s)\}.$$

Notice that we cannot hope that $i\{s,t\}$ will always be finite since if $s \in f_n[T^2]$, for some n, then $C_\alpha(s)$ is infinite. It is at least clear that $i\{s,t\}$ is a basis, e.g it verifies properties $2(a)$ and $2(b)$ of Definition 1.1. We now isolate a condition which guarantees that $i\{s,t\}$ will be finite, for all $s, t \in T$.

DEFINITION 1.5. Let T be an $\Omega(\lambda)$-poset for some λ and let $\gamma < \lambda$. We say that T_γ, the γ-th level of T, is a *bone level* if:

1. If $s, t \in T_\gamma$ and $s \neq t$ then $i\{s,t\} = \emptyset$,
2. If $t \in T_{\gamma+1}$ and $s < t$ then there exists $u \in T_\gamma$ such that $s \leq u < t$

T is called an $ht(T)$-*skeleton* if every level of T is a bone level.

Now we show that a bone-level can be used to fix the previous problem and thus obtain a new $\Omega(\lambda)$-poset.

PROPOSITION 1.6. *Let $(T, \leq, i)$ be an $\Omega(\lambda)$ poset and let $\gamma < \lambda$. If T_γ is a bone level, then for every $s \in T_{\gamma+1}$ and every t incomparable with s, the set $I(s,t) = \{u \in T_\gamma : u < s \text{ and } i\{u,t\} \neq \emptyset\}$ is finite.*

PROOF. Note that if $x \leq s, t$ then by (2) of Definition 1.5 there is $u \in I(s,t)$ such that $x \leq u$. On the other hand, if $u, v \in I(s,t)$ then by (1) of Definition 1.1 $i\{u,v\} = \emptyset$. It follows that the sets $i\{u,t\}$, for $u \in I(s,t)$, are pairwise disjoint and $i\{s,t\} = \bigcup\{i\{u,t\} : u \in I(s,t)\}$. Since $i\{s,t\}$ is finite it follows that $I(s,t)$ is finite, as well. ⊣

We now have the following immediate consequence.

LEMMA 1.7. *Let $(T^1, \leq_1, i_1)$ be an $\Omega(\alpha+1+\beta)$-poset and $(T^2, \leq_2, i_2)$ an $\Omega(\rho)$-poset. Assume T^1_α, the α-th level of T^1, is a bone level. Then $T^2 \hookrightarrow^\gamma T^1$ is an $\Omega(\alpha+1+\rho+\beta)$-poset.* ⊣

In the next theorem we show that a skeleton of height κ can be stretched to a skeleton of height δ, for any $\delta < \kappa^+$.

THEOREM 1.8. *Let κ be an infinite cardinal. Assume there is a κ-skeleton. Then there is an $\Omega(\delta)$-poset, for any $\delta < \kappa^+$.*

PROOF. We show by induction that for every $\delta < \kappa^+$ there is an $\Omega(\delta)$-poset $(T^\delta, \leq_\delta, i_\delta)$. We start the induction at $\delta = \kappa$ for which by the assumption of the theorem we know that there is a δ-skeleton, which is of course an $\Omega(\delta)$-poset. If $\delta = \gamma + 1$ is a successor, Proposition 1.3 allows us to build an $\Omega(\delta)$-poset from an $\Omega(\gamma)$-poset. Assume now δ is a limit ordinal and let $\mu = \mathrm{cof}(\delta)$. Then $\mu \leq \kappa$ and we know that there is an μ-skeleton S. We could, for instance, take the first μ levels of a κ-skeleton. Let $C = \{\gamma_\nu : \nu < \mu\}$ be a club in δ of order type μ and let $\gamma_\mu = \delta$ by convention. Let $\delta_\nu = o.t.(\gamma_{\nu+1} \setminus (\gamma_\nu + 1))$. The idea is to simply insert an $\Omega(\delta_\nu)$-poset between the ν-th and the $\nu+1$-st level of S as in Proposition 1.7. More precisely, let $E_\nu = \gamma_\nu \cup (C \setminus \gamma_\nu)$. Then the order type of E_ν is $\gamma_\nu + (\mu - \nu)$. Let us define the function $e_\nu : \mu \to \gamma_\nu + (\mu - \nu)$ by

$$e_\nu(\eta) = \begin{cases} \gamma_\eta & \text{if } \eta \leq \gamma_\nu \\ \gamma_\nu + \xi & \text{if } \eta = \nu + \xi \text{ for some } \xi < \mu - \eta \end{cases}$$

and let φ_ν be defined on S by $\varphi_\nu(\eta, y) = (e_\nu(\eta), y)$. We will construct by induction on $\nu \leq \mu$ an $\Omega(\gamma_\nu)$-poset R^ν with the following properties.

1. R^ν is an $\Omega(\gamma_\nu + (\mu - \nu))$-poset.
2. If $\nu < \xi$ the $R^\xi \restriction \gamma_\nu = R^\nu \restriction \gamma_\nu$.
3. φ_ν is an isomorphism between S and $R^\nu \restriction e_\nu[\mu]$.
4. $R^{\nu+1} = T^{\delta_\nu} \hookrightarrow^{\gamma_\nu} R^\nu$.

If $\nu = \eta + 1$, then γ_η is a bone-level of R^η so we can let $R^\nu = T^{\delta_\eta} \hookrightarrow^{\gamma_\eta} R^\eta$. If ν is a limit ordinal we let

$$R^\nu = \bigcup\{R^\eta \restriction \gamma_\eta : \eta < \nu\} \cup \varphi_\nu[S].$$

The ordering $\leq_\nu$ is defined in the natural way. On $\bigcup\{T^\eta \restriction \gamma_\eta : \eta < \nu\}$ we take $\bigcup\{\leq_\eta : \eta < \nu\}$. On $\varphi_\nu[S]$ we copy the ordering of S and then extend to an ordering of all of T^ν by transitivity. The definition of the function i_ν is similar. Now, it should be clear that R^ν has the required properties. Then T^δ is just R^μ. ⊣

Our next goal is to show that it is relatively consistent with ZFC to have an ω_2-skeleton. Since the proof is a mild modification of an argument of Baumgartner and Shelah from [5] we will be rather sketchy. We begin with a key definition from [5].

DEFINITION 1.9. Let $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ be a function with $f\{\alpha, \beta\} \subseteq \alpha \cap \beta$ for $\{\alpha, \beta\} \in [\omega_2]^2$. We say that two finite subsets x and y of ω_2 are *good* for f if for $\tau \in x \cap y$, $\alpha \in x \setminus y$ and $\beta \in y \setminus x$ we have:

- if $\tau < \alpha, \beta$ then $\tau \in f\{\alpha, \beta\}$
- if $\tau < \beta$ then $f\{\alpha, \tau\} \subseteq f\{\alpha, \beta\}$
- if $\tau < \alpha$ then $f\{\beta, \tau\} \subseteq f\{\alpha, \beta\}$

We say that f is a Δ*-function* if every uncountable family of finite subsets of ω_2 contains two elements which are good for f. We say that f is a *strong* Δ*-function* if every uncountable family A of finite subsets of ω_2 contains an uncountable subfamily B such that any two elements of B are good for f.

A Δ-function is the key technical tool used in [5] to generically add a thin very tall LCS space by a ccc forcing notion. Originally Baumgartner and Shelah first added a Δ-function by Shelah's method of historical forcing. It was later shown by the second author in [16] that a Δ-function can be obtained from Jensen's principle $\square_{\omega_1}$ using Todorcevic's method of minimal walks. For the convenience of the reader we reproduce the definitions from [16]. Let $\langle C_\alpha : \alpha < \omega_2 \text{ and } \lim(\alpha)\rangle$ be a $\square_{\omega_1}$-sequence, i.e. a sequence satisfying the following properties:

1. C_α is a club in α, for all α.
2. If α is a limit point of C_β then $C_\alpha = C_\beta \cap \alpha$.
3. $o.t.(C_\alpha) \leq \omega_1$, for all α.

If α is a successor ordinal, say $\alpha = \beta + 1$, we let $C_\alpha = \{\beta\}$. We now recall the definition of Todorcevic's ρ function (see [15], page 204). First, let

$$\Lambda(\alpha, \beta) = \text{ maximal limit point of } C_\beta \cap (\alpha + 1)$$

when such a limit point exists; otherwise let $\Lambda(\alpha, \beta) = 0$. We define the function $\rho : [\omega_2]^2 \to \omega_1$ recursively by the following formula.

$$\rho(\alpha, \beta) = \max\{o.t.(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\}$$

where we define by convention $\rho(\alpha, \alpha) = 0$.

We now define a function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ by the following formula.

$$f\{\alpha, \beta\} = \{\xi < \min\{\alpha, \beta\} : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}.$$

It was established in [16] that f is a Δ-function. In fact, it can be shown that f is a strong Δ-function (see, for instance, [15], Lemma 7.4.9.) We now have the following.

PROPOSITION 1.10. *Assume $\square_{\omega_1}$. Then there exists a strong Δ-function.*

This strong Δ-function has the following property which will be useful in the next section.

PROPOSITION 1.11. *Let $\langle C_\alpha : \alpha < \omega_2 \& \lim(\alpha)\rangle$ be a $\square_{\omega_1}$-sequence and let f be the strong Δ-function constructed from this sequence. Then for every $\eta < \alpha < \beta < \omega_2$, if η is a limit point of both C_α and C_β then $\eta \in f\{\alpha, \beta\}$.*

PROOF. Recall that $f\{\alpha,\beta\} = \{\xi < \alpha : \rho(\xi,\alpha) \leq \rho(\alpha,\beta)\}$ where ρ is as defined as before. Since η is a limit point of C_α it is straightforward to check that $\rho(\eta,\alpha) = o.t.(C_\eta)$. On the other hand, we have $\rho(\alpha,\beta) \geq o.t.(C_\beta \cap \alpha)$. But since η is also a limit point of C_β we have $C_\beta \cap \eta = C_\eta$ and thus $o.t.(C_\beta \cap \alpha) \geq o.t.(C_\eta)$ that is $\rho(\eta,\alpha) \leq \rho(\alpha,\beta)$ ⊣

The proof of the following theorem is a slight modification of an argument from [5]. We present the proof since it will be used in §2.

THEOREM 1.12. *Assume there is a strong Δ-function. Then there is a property K forcing notion which adds an ω_2-skeleton.*

PROOF. Let us fix a strong Δ-function f as in Proposition 1.10. We define a forcing notion $\mathcal{P}$ which adds a required partial ordering on $\omega_2 \times \omega$. Recall that if x is of the form (α, n) then we denote α by $\pi_1(x)$ and n by $\pi_2(x)$. We say that $p \in \mathcal{P}$ if $p = (x_p, \leq_p, i_p)$ where x_p is a finite subset of $\omega_2 \times \omega$, $\leq_p$ is a partial ordering of x_p and $i_p : [x_p]^2 \to [x_p]^{<\omega}$ and the following conditions are satisfied.

1. If $s,t \in x_p$ and $s <_p t$ then $\pi_1(x) < \pi_1(t)$.
2. If $s,t \in x_p$ with $s <_p t$ and $\pi_1(t)$ is a successor ordinal then there is $u \in x_p$ such that $s \leq_p u <_p t$ and $\pi_1(t) = \pi_1(u) + 1$.
3. If s and t are incomparable then $i_p\{s,t\} \subseteq (f\{\pi_1(s), \pi_1(t)\} \times \omega) \cap x_p$.
4. If $s \leq_p t$ then $i_p\{s,t\} = \{s\}$.
5. If $s \neq t$ and $\pi_1(s) = \pi_1(t)$ then $i_p\{s,t\} = \emptyset$.
6. If $u \in i_p\{s,t\}$ then $u \leq_p s,t$.
7. For every $u \leq_p s,t$ there is $v \in i_p\{s,t\}$ such that $u \leq_p v$.

We let $p \leq q$ if and only if $x_p \supseteq x_q$, $\leq_p \restriction x_q = \leq_q$ and $i_p \restriction [x_q]^2 = i_q$. In order to verify that $\mathcal{P}$ satisfies Knaster's chain condition, suppose $\mathcal{A}$ is an uncountable subset of $\mathcal{P}$. By extending if necessary, we can assume that the domain x_p of each $p \in \mathcal{A}$ is of the form $E_p \times n_p$. By shrinking $\mathcal{A}$ we may assume that the sets E_p, for $p \in \mathcal{A}$, form a Δ-system with root R, that there is an integer n such that $n_p = n$, for all $p \in \mathcal{A}$ and the conditions in $\mathcal{A}$ generate isomorphic structures over $R \times n$.

Using the fact that f is a strong Δ-function we can find an uncountable subset $\mathcal{B}$ of $\mathcal{A}$ such that if $p, q \in \mathcal{B}$ and $p \neq q$ then E_p and E_q are good for f. Consider now two conditions p and q from $\mathcal{B}$. We will show that they are compatible. Let $r = (x_r, \leq_r, i_r)$ be defined as follows: $x_r = x_p \cup x_q$ and $s \leq_r t$ if and only if $s \leq_p t$, or $s \leq_q t$, or there is $u \in R \times n$ such that $s \leq_p u \leq_q t$ or $s \leq_q u \leq_p t$. One can verify easily that $\leq_r$ is a partial ordering on x_r and $\leq_r \restriction x_p = \leq_p$

and $\leq_r\restriction x_q = \leq_q$. We define $i_r : [x_r]^2 \to [\omega_2]^{<\omega}$ by letting $i_r \restriction [x_p]^2 = i_p$, $i_r \restriction [x_q]^2 = i_q$, $i_r\{s,t\} = \{s\}$ if $s \leq_r t$ and if $s \in x_p \setminus R \times n$ and $t \in x_q \setminus R \times n$ are $\leq_r$-incomparable then

$$i_r\{s,t\} = \{u \in (f\{\pi_1(s), \pi_1(t)\} \times n) \cap x_r : u \leq_r s,t\}.$$

We need to check that $r \in \mathcal{P}$. Condition (1) follows from the way we have defined the ordering $\leq_r$. To see that (2) is satisfied assume $s \in x_p \setminus x_q$ and $t \in x_q \setminus x_p$. If $s \leq_r t$ then there is $s' \in x_p \cap x_q$ such that $s \leq_p s' \leq_q t$. Then $s', t \in x_q$ and we can apply (2) for s' and t in q to obtained the desired u.

Conditions (3)-(6) are straightforward and we leave them to the reader. It is nontrivial to check that r satisfies condition (7). To see this, assume $u \leq_r s,t$. If $s,t \in x_p \setminus x_q$ and $u \in x_q \setminus x_p$ then there are $z_s, z_t \in R \times n$ such that $u \leq_q z_a \leq_p s$ and $u \leq_q z_b \leq_p t$. By (7) for q there is $v \in i_q\{z_s, z_t\}$ such that $u \leq_q v$. since $i_q\{z_a, z_b\} \subseteq R \times n$ we conclude that $v \in R \times n$. Thus $v \leq_p s,t$ so by (7) for p there is $w \in i_p\{s,t\}$ such that $v \leq_p w$ and therefore $u \leq_r w$. The case $s \in R \times n$, $\beta \in x_p \setminus x_q$ and $u \in x_q \setminus x_p$ is similar.

Suppose now $s \in x_p \setminus x_q$, $t \in x_q \setminus x_p$ and they are $\leq_r$-incomparable. Suppose $u \leq s,t$ and for concreteness $u \in x_p$. Then $u \leq_p s$ and $u \leq_p v \leq_q t$, for some $v \in R \times n$. By (7) for p there is $w \in i_p\{s,v\}$ such that $u \leq_p w \leq_p s$. We need to check that w was put in $i_r\{s,t\}$. Let $\alpha = \pi_1(s)$, $\beta = \pi_1(t)$, $\tau = \pi_1(v)$ and $\xi = \pi_1(w)$. We need to check that $\xi \in f\{\alpha,\beta\}$. Note that by (1) for p and q we must have $\tau < \beta$. If $v \leq_p s$ then $w = v$ and we must have $\tau < \alpha, \beta$ and since E_p and E_q are good for f then $\tau \in f\{\alpha,\beta\}$. If v and s are incomparable then by property (3) for p we must have $i_p\{s,v\} \subseteq f\{\alpha,\tau\} \times n$. Now, again since E_p and E_q are good for f we must have that $f\{\alpha,\tau\} \subseteq f\{\alpha,\beta\}$, i.e. $\xi \in f\{\alpha,\beta\}$. It follows that $w \in i_r\{s,t\}$. The remaining cases are similar.

A simple density argument shows that if p is any condition in $\mathcal{P}$, $t \in \omega_2 \times \omega$, $\alpha < \pi_1(t)$ and n is an integer then there is a condition $q \leq p$ such that $t \in x_q$ and the set $\{s \in x_q : \pi_1(s) = \alpha \text{ and } s \leq_q t\}$ has at least n elements.

Let now G be a V-generic filter on $\mathcal{P}$ and let $\leq_G$ be the union of $\leq_p$, for $p \in G$, and let i_G be the union of i_p, for $p \in G$. It is now straightforward to check that $(\omega_2 \times \omega, \leq_G, i_G)$ is an ω_2-skeleton. ⊣

Now, putting Theorem 1.8 and Theorem 1.12 we have the following immediate corollary.

Corollary 1.13. *It is relatively consistent with ZFC that there are $\Omega(\delta)$-posets, for all $\delta < \omega_3$.* ⊣

§2. PCF structures. In this section we deal with the main topic of this paper, that is PCF structures. The idea is to isolate the properties of the PCF space used by Shelah to prove his celebrated theorem in [13] and see how far can we go with these. We give a simplified definition which only deals with thin PCF structures and we do not investigate all possible cardinal sequences. First, we recall the basic properties of the PCF operator.

If A is a set of regular cardinals and if U is an ultrafilter over A, the set $\prod A/U$ is linearly ordered, and so it has some cofinality κ. Define

$$\mathrm{pcf}(A) = \{\mathrm{cof}(\prod A/U) \ : \ U \text{ ultrafilter on } A\}.$$

If A is an interval of regular cardinals such that $|A| < \min(A)$ then $\mathrm{pcf}(A)$ is also an interval of regular cardinals and the pcf operator has the following properties for any X, $Y \subseteq \mathrm{pcf}(A)$.

(a) $X \subseteq \mathrm{pcf}(X)$, $\mathrm{pcf}(X) \cup \mathrm{pcf}(Y) = \mathrm{pcf}(X \cup Y)$, $\mathrm{pcf}(\mathrm{pcf}(X)) = \mathrm{pcf}(X)$.
(b) If $\gamma \in \mathrm{pcf}(X)$, then there exists $X' \subseteq X$ with $|X'| = |A|$ such that $\gamma \in \mathrm{pcf}(X')$.
(c) $\mathrm{pcf}(X)$ has a maximal element.
(d) If $\nu < \max\mathrm{pcf}(A)$ is a singular cardinal of uncountable cofinality then there exists a club C in ν such that $\max\mathrm{pcf}(\{\lambda^+ : \lambda \in C\}) = \nu^+$.

All these properties together implies that $|\mathrm{pcf}(A)| < |A|^{+4}$. A proof of this fact can be found in Shelah's book [13] or in [6]. We remark that by the first property we can view the pcf operator as a topological closure operator.

In our case we will work with $A = \{\aleph_{n+1} : n \in \omega\}$. Then $\max(\mathrm{pcf}(A))$ exists and is equal to some $\aleph_{\rho+1}$ with $\rho < \omega_4$. Since $\mathrm{pcf}(A)$ is an interval of regular cardinals the map $\alpha \mapsto \aleph_{\alpha+1}$ is a bijection from $\rho + 1$ to $\mathrm{pcf}(A)$. We can then transfer the PCF topology to $\rho + 1$. Note that in this case A is identified with ω and Property (d) becomes

(d') For every limit $\nu \leq \rho$ of uncountable cofinality there is a club C in ν such that $\mathrm{pcf}(C) \subseteq \nu + 1$.

We now give here a simple definition of a PCF structure for the purpose of forcing.

Definition 2.1. A *thin PCF structure* of height λ is a tuple $(T, \leq_T, i_T, \lhd_T)$ such that $(T, \leq_T, i_T)$ is an $\Omega(\lambda)$-poset and $\lhd_T$ is a well-ordering on T such that:

1. If $\pi_1(s) < \pi_1(t)$ then $s \lhd_T t$.
2. T_α is ordered by $\lhd_T$ in order type ω, for every $\alpha < ht(T)$.
3. For every $s \in T$ of uncountable $\lhd_T$-cofinality there is a $\lhd_T$-club C of $\lhd_T$-predecessors of s such that $x \leq_T s$, for every $x \in C$.

We remark here that the $C(s)$, for $s \in T$, (recall that $C(s) = \{t \in T : t \leq_T s\}$) are closely related to the generators introduced in [13]. So, the next proposition is in fact a slight modification of arguments from [13] and [6].

Proposition 2.2. *If there exists a thin PCF structure $(T, \leq_T, i_T, \lhd_T)$ of height λ then there exists a closure operator on $\omega \cdot \lambda + 1$ with properties (a)-(c) and (d') above.*

Proof. Let $\mu = \omega \cdot \lambda$ and notice that since the levels of T are ordered in type ω then $(T, \lhd_T)$ is isomorphic to μ with the usual ordering. We can therefore identify T with μ via this isomorphism. We let $T^* = T \cup \{\mu\}$, i.e. $T^* = \mu + 1$, and extend the ordering $\leq_T$ to $\leq_{T^*}$ by letting $t \leq_{T^*} \mu$, for every $t \in T$. The ordering $\lhd_{T^*}$ is just the usual well ordering on $\mu + 1$. By Proposition 1.2 we know that $\{C(s) \setminus \bigcup_{1 \leq j \leq n} C(s_j) : s \in T^*, n \in \omega, s_j <_T s\}$ is a clopen basis for a locally compact scattered topology. Moreover, since T^* has a maximal point this topology is actually compact. We then consider the closure operator relative to this topology. So if we show Properties (a)-(c) for this closure operator, as well as the club property (d'), then we are done. Property (a) is obvious as it is a closure operator.

For Property (b), first notice that T_0 (recall that T_0 is the first level of T^*) is dense in our topology. This is a basic result from LCS spaces, as any non empty set must contain an isolated point. Now, let $X \subseteq T^*$ and let $s \in \bar{X}$ ($\bar{X}$ is the closure of X). We define $C(Y) = \bigcup\{C(s) : s \in Y\}$ for any $Y \subseteq T^*$. Now, find a countable $Y \subseteq C(s) \cap X$ such that $C(Y) \cap T_0 = (C(s) \cap C(X)) \cap T_0$. Suppose $s \notin \bar{Y}$. Then, there exists s_1, ..., $s_n <_{T^*} s$ such that

$$[C(s) \setminus \bigcup_{1 \leq j \leq n} C(s_j)] \cap Y = \emptyset.$$

By transitivity of the ordering $\leq_{T^*}$, this implies that $C(Y) \subseteq \bigcup_{1 \leq j \leq n} C(s_j)$ and so that

$$[C(s) \setminus \bigcup_{1 \leq j \leq n} C(s_j)] \cap C(Y) \cap T_0 = \emptyset.$$

But since Y has been chosen so that $C(Y) \cap T_0 = (C(s) \cap C(X)) \cap T_0$ we have

$$[C(s) \setminus \bigcup_{1 \leq j \leq n} C(s_j)] \cap C(X) \cap T_0 = \emptyset.$$

And this is a contradiction since $[C(s) \setminus \bigcup_{1 \leq j \leq n} C(s_j)] \cap C(X)$ is an nonempty open set and T_0 is dense.

For Property (c) we already known T^* is compact since it has a maximal element. For $X \subseteq T^*$, $\bar{X}$ is then compact since it is a closed subset of T^*. By compactness there exist $s_1, \ldots, s_n \in \bar{X}$ such that $\bar{X} \subseteq \bigcup_{1 \leq j \leq n} C(s_j)$. If we let s be the maximum in the well ordering $\lhd_{T^*}$ of the s_j then by Property 1 of the previous definition, s is the $\lhd_T$-maximum of $\bar{X}$.

Finally Property (d') is an immediate consequence of Property 3 in the previous definition. $\dashv$

Our goal is to construct thin PCF structures of height δ, for all $\delta < \omega_3$. The proof is quite similar to the one we saw in §1, as our construction gives the club property almost for free. But in order to do this, we need to have an ω_2-skeleton which is also a thin PCF structure. In §1, we built an ω_2-skeleton on $\omega_2 \times \omega$, so the well-ordering we will define is a natural one. We let $(\alpha, n) \lhd (\beta, m)$ if and only if $\alpha < \beta$ or $\alpha = \beta$ and $n < m$. Then for each $\alpha < \omega_2$ the α-th level of the skeleton can be identified with the interval $[\omega \cdot \alpha, \omega \cdot (\alpha + 1))$ and the whole skeleton is isomorphic to ω_2. With this ordering, it is easy to see that $(\alpha, 0)$ has $\lhd$-cofinality ω_1 if and only if α has cofinality ω_1 and that if C_α is a club in α then $C_\alpha \times \{0\}$ is a $\lhd$-club in $(\alpha, 0)$.

So, what we need to do is to find clubs C_α in ω_2 such that property 3 of Definition 2.1 is satisfied for the $\lhd$-clubs $C_\alpha \times \{0\}$. Since we want to add our space by a ccc forcing notion we need to find these clubs in the ground model. For technical reasons (namely because of the function i defined in §1) we will need that our clubs have finite pairwise intersections which is the purpose of the next two lemmas.

Lemma 2.3. *Let $\mathcal{A}$ be a family of ω_1 closed countable subsets of ω_1 such that $A \cap B$ is finite, for all distinct $A, B \in \mathcal{A}$. Then there is a proper forcing notion $\mathcal{C}$ of size $\aleph_1$ which adds a club in ω_1 which has finite intersection with all members of $\mathcal{A}$.*

Proof. Let us fix a $1-1$ enumeration $\langle A_\alpha : \alpha < \omega_1\rangle$ of $\mathcal{A}$. We define the forcing notion $\mathcal{C}$ as follows. Elements of $\mathcal{C}$ are of the form $p = \langle \mathcal{I}_p, F_p\rangle$ where $\mathcal{I}_p$ is a finite collection of countable closed disjoints intervals of ω_1 and F_p is a finite subset of ω_1. For $p \in \mathcal{C}$ let $C_p = \{\min I : I \in \mathcal{I}_p\}$. We say $p \leq q$ if $\mathcal{I}_p \supseteq \mathcal{I}_q$, $F_p \supseteq F_q$, and $(C_p \setminus C_q) \cap A_\alpha = \emptyset$, for all $\alpha \in F_q$. Now, if G is a V-generic filter over $\mathcal{C}$ then we claim that $C_G = \bigcup\{C_p : p \in G\}$ will be the required club. First note that C_G will be unbounded in ω_1. To see this note that for any $\alpha < \omega_1$ the set of $p \in \mathcal{C}$ such that $C_p \setminus \alpha \neq \emptyset$ is dense in $\mathcal{C}$. Next, notice that C_G will have finite intersection with A_α, for all $\alpha < \omega_1$. To see this note that the set of $p \in \mathcal{C}$ such that $\alpha \in F_p$ is dense in $\mathcal{C}$. Finally, note that C_G will be closed in ω_1. To see this suppose $p \in \mathcal{C}$, $\gamma < \omega_1$ and $p \Vdash \gamma \notin C_{\dot{G}}$. Then in particular $\gamma \notin C_p$. If there is $I \in \mathcal{I}_p$ such that $\gamma \in I$ then let $J = I \setminus \{\min I\}$. Then J is an open interval containing γ and $p \Vdash J \cap C_{\dot{G}} = \emptyset$. If $\gamma \notin \bigcup \mathcal{I}_p$ then we can find a closed interval J such that $J \cap \bigcup \mathcal{I}_p = \emptyset$ and $\gamma \in J^* = J \setminus \{\min J\}$. Let $q = \langle \mathcal{I}_p \cup \{J\}, F_p$. Then $q \leq p$ and $q \Vdash J^* \cap C_{\dot{G}} = \emptyset$. Thus, we have shown that for every γ the set of conditions p such that either p forces that γ is in $C_{\dot{G}}$ or there is an open interval U containing γ such that p forces that $U \cap C_{\dot{G}} = \emptyset$ is dense in $\mathcal{C}$. Therefore, C_G is closed in ω_1. It remains to establish the following.

Claim 2.4. *$\mathcal{C}$ is a proper forcing notion.*

Proof. Let θ be a sufficiently large regular cardinal and fix a countable elementary submodel M of H_θ containing all the relevant information. Let $p \in \mathcal{C} \cap M$. We need to find a $q \leq p$ which is $(M,\mathcal{C})$-generic. Let $\delta = M \cap \omega_1$. First notice that $A_\alpha \subseteq M$, for all $\alpha \in F_p$ therefore $q = \langle \mathcal{I}_p \cup \{[\alpha, \alpha]\}, F_p\rangle \in \mathcal{C}$. We show that q is the required condition. To see this let D be a dense subset of $\mathcal{C}$ with $D \in M$. We would like to show that $D \cap M$ is predense below q. Fix $r \leq q$ and assume, without loss of generality, that $r \in D$. Let $r_0 = \langle \mathcal{I}_r \cap M, F_r \cap M\rangle$. Then $r_0 \in M$. If α is a countable ordinal and $\mathcal{I}$ a finite collection of intervals let $\mathcal{I} \restriction \alpha$ denote the collection of all $I \in \mathcal{I}$ such that $\sup I < \alpha$. For a condition $s \in \mathcal{C}$ let $s \restriction \alpha$ denote $\langle \mathcal{I}_s \restriction \alpha, F_s \cap \alpha\rangle$. Let $\nu < \delta$ be sufficiently large so that $r_0 \restriction \gamma = r_0$. Let n be the number of intervals in $\mathcal{I}_r \setminus I_{r_0}$. Then for every ξ such that $\nu \leq \xi < \delta$ there exists $s \in D$ such that $s \restriction \xi = r_0$ and $|I_s \setminus I_{r_0}| = n$, namely we can take $s = r$. By elementarity such a condition s exists in M. Therefore, we can find a sequence $\langle s_i : i < \omega\rangle \in M$ such that $s_i \in D$, $s \restriction \nu = r_0$ and $|\mathcal{I}_{s_i} \setminus I_{r_0}| = n$, for all i, and such that $\max(C_{s_i} \setminus C_{r_0}) < \min(C_{s_j} \setminus C_{r_0})$, whenever $i < j$. We want to show that some s_i is compatible with r, for some i. If not then since $s_i \in M$ and $s \leq \nu = r_0$ then there there is $\xi_i \in F_r \setminus F_{r_0}$ and $J_i \in I_{s_i} \setminus I_{r_0}$ such that $\min J_i \in A_{\xi_i}$. Let $l_i < n$ be such that J_i is the l_i-th interval in the natural order of $\mathcal{I}_{s_i} \setminus \mathcal{I}_{r_0}$. Now let $\mathcal{U}$ be a nonprincipal ultrafilter on ω such that $\mathcal{U} \in M$. Since $F_r \setminus F_{r_0}$ is finite there must exist a fixed $\xi \in F_r \setminus F_{r_0}$ and $l < n$ such that $\xi_i = \xi$ and $l_i = l$, for $\mathcal{U}$-many i. Now $\xi > \delta$ and by elementarity of M there must exist $\eta < \delta$ such that the set of i such that A_η contains the minimum of the l-th interval of $\mathcal{I}_{s_i}$ is in $\mathcal{U}$. It follows that $A_\xi \cap A_\eta$ is infinite, which contradicts our assumption on $\mathcal{A}$. This finishes the proof of the claim and Lemma 2.3. ⊣

⊣

The following lemma will be used to build a poset which adds a PCF structure of size ω_2.

LEMMA 2.5. *Assume GCH. Then there is a $\aleph_2$-cc proper forcing notion $\mathcal{Q}$ which adds a sequence $\langle C_\alpha : \alpha < \omega_2 \,\&\, \mathrm{cof}(\alpha) = \omega_1 \rangle$ such that C_α is a club in α, for all α, and $C_\alpha \cap C_\beta$ is finite, for all $\alpha \neq \beta$.*

PROOF. The forcing notion $\mathcal{Q}$ is obtained as a countable support iteration $\langle \mathcal{Q}_\xi; \dot{\mathcal{C}}_\xi : \xi < \omega_2 \rangle$ of forcing notions as constructed in Lemma 2.3. At the ξ-th stage of the iteration we have already build C_η, for all $\eta < \xi$ such that $\mathrm{cof}(\eta) = \omega_1$. If ξ is not of cofinality ω_1 let $\dot{\mathcal{C}}_\xi$ be the $\mathcal{Q}_\xi$-name for a trivial forcing notion. Otherwise, fix a club E in ξ of order type ω_1. Let $A_\eta = C_\eta \cap E$, for $\eta < \xi$. Then the A_η are countable, closed and have pairwise finite intersections. Let $\mathcal{A} = \{A_\eta : \eta < \xi \,\&\, cof(\eta) = \omega_1\}$. By Lemma 2.3 there is a proper forcing notion $\mathcal{C}$ of size $\aleph_1$ which adds a club C in ω_1 which has finite intersection with A_η, for all $\eta < \xi$. Let then $\dot{\mathcal{C}}_\xi$ be a $\mathcal{Q}_\xi$-name for such a forcing notion and let $\dot{C}_\xi$ be a $\mathcal{Q}_{\xi+1}$-name for such a C. The fact that the iteration is proper is standard. Moreover, since each iterand is of size $\aleph_1$ the iteration satisfies the $\aleph_2$-cc (see, for instance, [1]) and therefore preserves $\aleph_2$. ⊣

We now state a simple property that the previous sequence of clubs satisfies. It will be useful to prove the ccc for our forcing notion.

PROPOSITION 2.6. *Let A be an uncountable set of finite pairwise disjoint subsets of ω_2 and let $\langle C_\alpha : \alpha < \omega_2 \rangle$ be a sequence such that C_α is a subset of α, for all α, and $C_\alpha \cap C_\beta$ is finite, for all $\alpha \neq \beta$. Then there exist distinct $x, y \in A$ such that for every $\alpha \in x$ and $\beta \in y$, $C_\alpha \cap y = C_\beta \cap x = \emptyset$.*

PROOF. Since A is countable, we may assume that there exists $n \in \omega$ such that $|x| = n$, for all $x \in A$. For $x \in A$ let $C_x = \bigcup\{C_\alpha : \alpha \in x\}$. Note that the sets C_x, for $x \in A$, have pairwise finite intersection.

We claim that if $x_0, ..., x_n \in A$ are distinct then the set

$$Y = \{y \in A : y \cap C_{x_i} \neq \emptyset, \text{ for all } i \leq n\}$$

is finite. To see this, note that for every $y \in Y$ there are distinct $i_y, j_y \leq n$ such that $y \cap C_{x_{i_y}} \cap C_{x_{j_y}} \neq \emptyset$. Since Y is infinite there are $i, j \leq n$ and an infinite subset Y_0 of Y such that $i_y = i$ and $j_y = j$, for all $y \in Y_0$. Since the elements of Y_0 are pairwise disjoint it follows that $C_{x_i} \cap C_{x_j}$ is infinite, a contradiction.

Now, we claim that there exists an uncountable $B \subseteq A$ such that for all $x \in B$,

$$\{y \in B : y \cap C_x \neq \emptyset\}$$

is countable. If not, let $A_0 = A$, and given an uncountable $A_i \subseteq A$, let $x_i \in A_i$ be distinct from the x_j for $j < i$ and such that

$$A_{i+1} = \{y \in A_i : y \cap C_{x_i} \neq \emptyset\}$$

is uncountable. But then $x_0, \ldots, x_n$ contradict the previous claim.

We can now construct a sequence $\langle x_\xi : \xi < \omega_1 \rangle$ of distinct elements of B such that if $\xi < \eta < \omega_1$ then $x_\eta \cap C_{x_\xi} = \emptyset$. We claim that there are $\xi < \eta$ such that $x_\xi \cap C_{x_\eta} = \emptyset$. Otherwise, we have that $x_\xi \cap C_{x_\eta} \neq \emptyset$, for all $\xi < \eta$. But then $\{x_i : i \in \omega\}$ and $x_\omega, ..., x_{\omega+n}$ contradict the previous claim. ⊣

We are now almost ready to define our forcing notion. Before we start we introduce some notations. Recall that $\lhd$ is the well-ordering on $\omega_2 \times \omega$ such that $(\alpha, n) \lhd (\beta, m)$ if and only if $\alpha < \beta$ or $\alpha = \beta$ and $n < m$. So $\lhd$ is simply the lexicographic ordering on $\omega_2 \times \omega$. The notions of $\lhd$-cofinality and $\lhd$-club are the natural ones related to the well-ordering $\lhd$. An ω_2-skeleton $(T, \leq_T, i_T)$ such that $T = \omega_2 \times \omega$ will be called a *candidate*. Given a candidate $U = (\omega_2 \times \omega, \leq_U, i_U)$ and $\alpha \in \omega_2$ of uncountable cofinality, α is said to be *good for* U if there exists a club C in α such that for all $\gamma \in C$ $(\gamma, 0) \leq_U (\alpha, 0)$. Obviously if every ordinal in ω_2 of uncountable cofinality is good for U then U is a PCF structure which is also an ω_2-skeleton. This is because $x \in \omega_2 \times \omega$ is of uncountable $\lhd$-cofinality if and only if $x = (\alpha, 0)$ for some $\alpha \in \omega_2$ of uncountable cofinality. The set of good points for a candidate U will be denoted by S_U. In the following S^{κ}_{λ} will be the set of ordinals less than λ of cofinality κ for any λ, κ infinite cardinals. So, our hope is to define a ccc forcing notion which adds a PCF structure but unfortunately the best we can do for is the following theorem.

Theorem 2.7. *It is relatively consistent with ZFC that there exists a candidate U such that S_U is stationary.*

Assume Theorem 2.7 for now and we will show that this result is in fact enough to achieve our goal. The next lemma is a standard result and a proof of it can be found in [3].

Lemma 2.8. *Assume $2^{\aleph_1} = \aleph_2$ and let $S \subseteq S^{\omega_1}_{\omega_2}$ be a stationnary subset of ω_2. Then there exists a forcing notion which preserves cardinals and such that in the generic extension $S \cup S^{\omega}_{\omega_2}$ contains a club.* ⊣

The following is due to Ruyle but we reproduce the proof here for completeness.

Lemma 2.9. *If there exists a candidate $U = (\omega_2 \times \omega, \leq_U, i_U)$ such that $S_U \cup S^{\omega}_{\omega_2}$ contains a club of ω_2 then there exists a PCF-structure of height ω_2.*

Proof. Fix a club $C \subseteq S_U \cup S^{\omega}_{\omega_2}$ such that no successor element of C is an ordinal of uncountable cofinality. For every α good for U, let C_α be a club witnessing it. Now, for $\alpha \in C \cap S_U$ let $D_\alpha = C_\alpha \cap C$. Obviously D_α is a club in α and since C_α witnesses that α is good for U, $(\beta, 0) \leq_U (\alpha, 0)$ for all $\beta \in D_\alpha$.

Let $Z = S^{\omega_1}_{\omega_2} \setminus C$. Choose a disjoint family $\{D_\alpha : \alpha \in Z\}$ such that $D_\alpha \cap C = \emptyset$ and D_α is a club in α, for all $\alpha \in Z$. To do this note that for each $u \in C$ if we let v be the first element of C above u, we can choose the D_α for $\alpha \in [u, v] \cap Z$ to be subsets of $[u, v]$. Since there are at most $\aleph_1$ such α in the interval $[u, v]$, we can arrange that they are disjoint.

Now we define a bijection h from $\omega_2 \times \omega$ to itself. For each $\alpha \in Z$ and each $\beta \in D_\alpha$, find some $(\beta, n) \leq_E (\alpha, 0)$ and let $h(\beta, n) = (\beta, 0)$ and $h(\beta, 0) = (\beta, n)$. For any other points, let h be the identity.

Let $\leq^*$ be the partial order on $\omega_2 \times \omega$ defined by $x \leq^* y$ if and only if $h(x) \leq^* h(y)$ and let i^* be defined in the same obvious way. Now we claim that $U^* = (\omega_2 \times \omega, \leq^*, i^*, \lhd)$ is a PCF-structure which is also an ω_2-skeleton. Since h is a bijection $(\omega_2 \times \omega, \leq^*, i^*)$ is clearly an ω_2-skeleton. Now for each $\alpha \in \omega_2$ of uncountable cofinality observe that α is good for U^*. This is an immediate consequence of the definition of the bijection h and the fact that the D_α's for $\alpha \in Z$ are disjoint. ⊣

Now we are ready to prove Theorem 2.7.

PROOF. Starting with a model of ZFC+GCH+$\square_{\omega_1}$ we use Lemma 2.5 to force a sequence $\langle D_\alpha : \alpha < \omega_2 \,\&\, \mathrm{cof}(\alpha) = \omega_1\rangle$ such that D_α is a club in α for all α and $D_\alpha \cap D_\beta$ is finite for all $\alpha \neq \beta$. Since the forcing in Lemma 2.5 preserves cardinals $\square_{\omega_1}$ holds in the generic extension. So by Proposition 1.10 there exists a strong Δ-function f in the generic extension. Let $\langle C_\alpha : \alpha < \omega_2\rangle$ be the $\square_{\omega_1}$-sequence from which f is constructed. Without loss of generality, we can consider that for every α of uncountable cofinality, $D_\alpha \subseteq \lim(C_\alpha)$ where $\lim(C_\alpha)$ is the set of limit points of C_α. If not then just replace D_α by $D_\alpha \cap \lim(C_\alpha)$. Observe that by Proposition 1.11 for every $\alpha \neq \beta$ of uncountable cofinality, $D_\alpha \cap D_\beta \subseteq f\{\alpha, \beta\}$ and that $\mathrm{cof}(\xi) \neq \omega_1$ for every $\xi \in D_\alpha$. We define a forcing notion $\mathcal{Q}$ which will add a candidate U such that S_U is stationary. We let $p \in \mathcal{Q}$ if $p = (x_p, \leq_p, i_p)$ where x_p is a finite subset of $\omega_2 \times \omega$, $\leq_p$ is a partial ordering of x_p and $i_p : [x_p]^2 \to [x_p]^{<\omega}$ and the following conditions are satisfied.

1. If $s, t \in x_p$ and $s <_p t$ then $\pi_1(s) < \pi_1(t)$.
2. If $s, t \in x_p$ with $s <_p t$ and $\pi_1(t)$ is a successor ordinal then there is $u \in x_p$ such that $s \leq_p u <_p t$ and $\pi_1(t) = \pi_1(u) + 1$.
3. If $s, t \in x_p$ are of $\lhd$-cofinality ω_1 then $[D_{\pi_1(s)} \cap D_{\pi_1(t)}] \times \{0\} \subseteq x_p$.
4. If $s \in x_p$ is of $\lhd$-cofinality ω_1 then there is no $t \neq s$ such that $s \leq_p t$.
5. If s and t are incomparable then $i_p\{s, t\} \subseteq [(f\{\pi_1(s), \pi_1(t)\} \times \omega) \cap x_p]$
6. If $s \leq_p t$ then $i_p\{s, t\} = \{s\}$.
7. If $s \neq t$ and $\pi_1(s) = \pi_1(t)$ then $i_p\{s, t\} = \emptyset$.
8. If $u \in i_p\{s, t\}$ then $u \leq_p s, t$.
9. For every $u \leq_p s, t$ there is $v \in i_p\{s, t\}$ such that $u \leq_p v$.

For $\alpha < \omega_2$ of uncountable cofinality and $p \in \mathcal{Q}$ say that α is *p-good* if $(\alpha, 0) \in x_p$ and for all $\beta \in D_\alpha$ if $(\beta, 0) \in x_p$ then $(\beta, 0) \leq_p (\alpha, 0)$. We let $p \leq q$ if and only if $x_p \supseteq x_q$, $\leq_p\restriction x_q = \leq_q$, $i_p \restriction [x_q]^2 = i_q$ and for all $\alpha < \omega_2$, if α is q-good then α is p-good.

CLAIM 2.10. *$\mathcal{Q}$ has the countable chain condition.*

PROOF. Let $\mathcal{A} \subseteq \mathcal{Q}$ be uncountable. If we let $\mathcal{P}$ be the forcing defined in Theorem 1.12 one can note that $\mathcal{Q} \subseteq \mathcal{P}$ (assuming the fact that we used the same Δ-function). So by repeating the proof of Theorem 1.12 and by shrinking $\mathcal{A}$ if necessary we can assume that every p, q in $\mathcal{A}$ are $\mathcal{P}$-compatible. For $p \in \mathcal{Q}$ we let $L_p = \{\alpha \in \omega_2 : (\alpha, 0) \in x_p\}$. By shrinking again if necessary, we can assume that $\{L_p : p \in \mathcal{A}\}$ is a Δ-system with root R. By applying Proposition 2.6 to $\{L_p \setminus R : p \in \mathcal{A}\}$ we can find distinct p and q in $\mathcal{A}$ such that for every $\alpha \in L_p \setminus R$ and $\beta \in L_q \setminus R$ of uncountable cofinality we have

$$D_\alpha \cap (L_q \setminus R) = D_\beta \cap (L_p \setminus R) = \emptyset.$$

Let r_0 be an amalgamation of p and q in $\mathcal{P}$ such that $x_{r_0} = x_p \cup x_q$. We define $r = (x_r, \leq_r, i_r)$ by letting

$$x_r = x_{r_0} \cup \bigcup\{(D_\alpha \cap D_\beta) \times \{0\} : \alpha \in (L_p \setminus R) \cap S^{\omega_1}_{\omega_2}; \beta \in (L_q \setminus R) \cap S^{\omega_1}_{\omega_2}\}.$$

Note that since the D_α contain no points of uncountable cofinality r will satisfy (3) in the definition of $\mathcal{Q}$. We have to define the ordering $\leq_r$. First of all set $\leq_r\restriction x_{r_0} = \leq_{r_0}$. Suppose now $s \in x_r \setminus x_{r_0}$. By the definition of x_r we know that there is $\gamma \in ((L_p \cup L_q) \setminus R) \cap S^{\omega_1}_{\omega_2}$ such that $s \in D_\gamma \times \{0\}$. We set $s \leq_r (\gamma, 0)$ for all such γ and make s incomparable with all other elements of x_r. Note that by condition (4) of our forcing $\leq_r$ is a partial order and r obviously satisfies condition (1),(2) and (4) in the definition of $\mathcal{Q}$.

We need to define the function $i_r\{s,t\}$ for all $\{s,t\} \in [x_r]^2$. Assume first $\{s,t\} = \{(\alpha,0),(\beta,0)\}$, where $\alpha \in (L_p \setminus R) \cap S^{\omega_1}_{\omega_2}$ and $\beta \in (L_q \setminus R) \cap S^{\omega_1}_{\omega_2}$. In this case let

$$i_r\{s,t\} = (D_\alpha \cap D_\beta \setminus R) \times \{0\} \cup i_{r_0}\{s,t\}.$$

Since by proposition 1.11 $D_\alpha \cap D_\beta \subseteq f\{\alpha,\beta\}$, r will satisfy condition (5) in the definition of $\mathcal{Q}$. For all other pairs $\{s,t\}$ in $[x_{r_0}]^2$ let $i_r\{s,t\} = i_{r_0}\{s,t\}$. Finally, assume at least one of s and t is in $x_r \setminus x_{r_0}$. If they are $\leq_r$-incomparable let $i_r\{s,t\} = \emptyset$. If they are $\leq_r$-comparable, say $s \leq_r t$, let $i_r\{s,t\} = \{s\}$. Then, using condition (3) for p and q and the fact that r_0 is an amalgamation of p and q in $\mathcal{P}$ it is straightforward to check that $r \in \mathcal{Q}$ and that r extends both p and q. ⊣

Our forcing will then preserve cardinals. We have to show now that if G is V-generic then $\bigcup_{p \in G} x_p = \omega_2 \times \omega$.

Claim 2.11. *For every $s \in \omega_2 \times \omega$ the set $F_s = \{p \in \mathcal{Q} : s \in x_p\}$ is dense.*

Proof. Let $s \in \omega_2 \times \omega$ and $p \in \mathcal{Q}$ such that $s \notin x_p$. We have to find $q \leq p$ such that $s \in x_q$. Several cases occur:
CASE 1. s is not of $\lhd$-cofinality ω_1 and $\pi_2(s) \neq 0$ or s is not of $\lhd$-cofinality ω_1, $\pi_2(s) = 0$ and for all α p-good $\pi_1(s) \notin D_\alpha$.

Then q can be defined in the obvious way. Let $x_q = x_p \cup \{s\}$; $\leq_q \restriction x_p = \leq_p$; $\leq_q \cap (x_p \times \{s\}) = \leq_q \cap (\{s\} \times x_p) = \emptyset$; $i_q \restriction [x_p]^2 = i_p$ and $i_q\{s,t\} = \emptyset$ for all $t \in x_p$. Then $q = (x_q, \leq_q, i_q)$ is as wished.
CASE 2. s is not of $\lhd$-cofinality ω_1, $\pi_2(s) = 0$ and for some $\alpha < \omega_2$, α is p-good and $\pi_1(s) \in D_\alpha$. Note that by condition (3) of the definition of $\mathcal{Q}$, α must be unique.

Let $x_q = x_p \cup \{s\}$. Let $\leq_q$ extends $\leq_p$ and for α p-good such that $\pi_1(s) \in D_\alpha$ let $s \leq_q (\alpha, 0)$. The function i_q is defined in the obvious way. Then again $q = (x_q, \leq_q, i_q)$ is as wished.
CASE 3. s is of $\lhd$-cofinality ω_1. Then s is of the form $(\alpha, 0)$ for some α of cofinality ω_1. Let

$$x_q = x_p \cup \{s\} \cup [\bigcup \{(D_\beta \cap D_\alpha) \times \{0\} : \beta \in L_p \cap S^{\omega_1}_{\omega_2}\}].$$

Define $\leq_q$ by letting $\leq_q \restriction x_p = \leq_p$ and for all β p-good, for all $\gamma \in D_\beta \cap D_\alpha$ let $(\gamma, 0) \leq_q (\beta, 0)$. The function i_q extends i_p and is defined in an obvious way for what is left. Observe that this is possible since by clause (3) of the definition of $\mathcal{Q}$ if β_1, β_2 are p-good then $(D_{\beta_1} \cap D_{\beta_2}) \times \{0\} \subseteq x_p$ and so if $t \in x_q \setminus x_p$ we cannot have $t \leq_q (\beta_1, 0), (\beta_2, 0)$. ⊣

The two previous claims show that if G is $\mathcal{Q}$-generic then in $V[G]$, if we let $\leq_G = \bigcup_{p \in G} \leq_p$ and $i_G = \bigcup_{p \in G} i_p$, $G^* = (\omega_2 \times \omega, \leq_G, i_G)$ is a candidate. So in order to complete the proof of Theorem 2.7 it suffices to show that there is a condition $p \in \mathcal{Q}$ which forces that S_{G^*}, the set of good points of G^*, is stationary in $V[G]$.

CLAIM 2.12. *There is $p \in \mathcal{Q}$ which forces that S_{G^*} is stationnary in $V[G]$.*

PROOF. Assume otherwise. Then there is a $\mathcal{Q}$-name $\dot{C}$ for a club such that $\Vdash \dot{C} \cap \dot{S}_{G^*} = \emptyset$. Since the forcing $\mathcal{Q}$ is ccc there a club E in V such that $\Vdash E \subseteq \dot{C}$. Let $\alpha \in E$ be a point of cofinality ω_1 and consider the condition $p_\alpha = (\{\alpha\}, \emptyset, \emptyset)$. Clearly α is p_α-good and therefore $p_\alpha \Vdash \alpha \in \dot{S}_{G^*}$, a contradiction. This finishes the proof of the claim and of Theorem 2.7. ⊣

⊣

So finally we can state the following theorem which is an immediate consequence of Theorem 2.7 and Lemmas 2.8 and 2.9.

THEOREM 2.13. *It is relatively consistent with ZFC that there exists a PCF structure which is also an ω_2-skeleton.*

Now using the ideas developped in §1 we can prove the following:

THEOREM 2.14. *If there exists a PCF structure which is also a κ-skeleton then there exists PCF structures of height δ, for all $\delta < \kappa^+$.*

Before we start the proof let us remark that any thin PCF structure of height, say λ, is isomorphic to a thin PCF structure T of the form $(\lambda \times \omega, \leq_T, i_T, \lhd_T)$, where $\lhd_T$ is just the lexicographical ordering on $\lambda \times \omega$ and T_α, the α-th level of T, is equal to $\{\alpha\} \times \omega$.

PROOF. As in Theorem 1.8 we construct by induction on δ, a thin PCF structure T^δ of height δ, for all $\delta < \kappa^+$. For $\delta = \kappa$ this is the hypothesis of the theorem.

For $\delta = \gamma+1$ let $(T^\gamma, \leq^\gamma, i^\gamma, \lhd^\gamma)$ be a PCF-structure of height γ. By the above remark we may assume that $T^\gamma = \gamma \times \omega$ and $\lhd^\gamma$ is the lexicographical ordering on $\gamma \times \omega$. Then Proposition 1.3 provides an $\Omega(\gamma+1)$-poset $T^{\gamma+1}$. Recall that since $T^\gamma = \gamma \times \omega$ then

$$T^{\gamma+1} = \gamma \times (\omega \times \omega) \cup \{\gamma+1\} \times \omega.$$

All that is left to do is to define a well ordering $\lhd^{\gamma+1}$ on $T^{\gamma+1}$. We can let $\lhd^{\gamma+1}$ be any well-ordering such that each level of $T^{\gamma+1}$ is of order type ω and for every $\alpha < \gamma$ and every $n, m \in \omega$ we have $(\alpha, 0, 0) \lhd^{\gamma+1} (\alpha, n, m)$. That is, we let the first point of each level in the lexicographic order be the first point of that level in the well ordering.

Now if $\alpha < \gamma$ is of uncountable cofinality then by the inductive hypothesis there exists a club C_α in α such that for all $\xi \in C_\alpha$ we have $(\xi, 0) \leq^\gamma (\alpha, 0)$. Then by definition of $\leq^{\gamma+1}$ we have $(\xi, 0, 0) \leq^{\gamma+1} (\alpha, 0, 0)$. By noticing that for $s \in T^{\gamma+1}$, s has $\lhd^{\gamma+1}$-cofinality ω_1 if and only if $s = (\alpha, 0, 0)$ for some $\alpha < \gamma$ of uncountable cofinality we are done.

Assume now δ is a limit ordinal and let $\mu = \mathrm{cof}(\delta)$. Then $\mu \leq \kappa$ and we know that there is a PCF structure S which is also a μ-skeleton. Let $C = \{\gamma_\nu : \nu < \mu\}$ be a club in δ of order type μ and let by convention $\gamma_\mu = \delta$. Let $\delta_\nu = o.t.(\gamma_{\nu+1} \setminus (\gamma_\nu + 1))$. As in Theorem 1.8, let $E_\nu = \gamma_\nu \cup (C \setminus \gamma_\nu)$. Then the order type of E_ν is $\gamma_\nu + (\mu - \nu)$. Recall that $e_\nu : \mu \to \gamma_\nu + (\mu - \nu)$ is defined by

$$e_\nu(\eta) = \begin{cases} \gamma_\eta & \text{if } \eta \leq \gamma_\nu \\ \gamma_\nu + \xi & \text{if } \eta = \nu + \xi \text{ for some } \xi < \mu - \eta \end{cases}$$

and φ_ν is defined on S by $\varphi_\nu(\eta, y) = (e_\nu(\eta), y)$. We construct by induction on $\nu \leq \mu$ an $\Omega(\gamma_\nu)$-poset R^ν with the following properties.

1. R^ν is an $\Omega(\gamma_\nu + (\mu - \nu))$-poset.
2. If $\nu < \xi$ then $R^\xi \restriction \gamma_\nu = R^\nu \restriction \gamma_\nu$.
3. φ_ν is an isomorphism between S and $R^\nu \restriction e_\nu[\mu]$.
4. $R^{\nu+1} = T^{\delta_\nu} \hookrightarrow^{\gamma_\nu} R^\nu$.

If $\nu = \eta + 1$, then γ_η is a bone-level of R^η so we can let $R^\nu = T^{\delta_\eta} \hookrightarrow^{\gamma_\eta} R^\eta$. where T^{δ_η} is a PCF structure of height δ_η. By the previous remark, we may assume that $T^{\delta_\eta} = \delta_\eta \times \omega$. If ν is a limit ordinal we let

$$R^\nu = \bigcup\{R^\eta \restriction \gamma_\eta : \eta < \nu\} \cup \varphi_\nu[S],$$

and we define $\leq^\nu$ and i^ν as in Theorem 1.8. Then, $T^\delta = R^\mu$.

Note that $T^\delta_\alpha = \{\alpha\} \times \omega$ if $\alpha = \gamma_\nu$ for some $\nu < \mu$ and $T^\delta_\alpha = \{\alpha\} \times (\omega \times \omega)$ if not. Then the well ordering $\lhd^\delta$ is defined in a natural way. If $\alpha = \gamma_\nu$ for some $\nu < \mu$ then $(\alpha, n) \lhd^\delta (\alpha, m)$ if and only if $n < m$. Else, we define it as we did in the successor case.

Now, let $\alpha < \delta$ of uncountable cofinality. Then, we have 3 possibilities:

CASE 1. $\alpha \neq \gamma_\nu$, for all $\nu < \mu$.

Let $\nu < \mu$ be the least such that $\alpha < \gamma_{\nu+1}$. Then $T^\delta \restriction \gamma_{\nu+1} = T^{\delta_\nu} \hookrightarrow^{\gamma_\nu} R^\nu \restriction c_{\nu+1}$. Since T^{δ_ν} is a PCF structure, by an argument similar to the one used in the successor case there exists a club C_α such that for all $\xi \in C_\alpha$, $(\xi, 0, 0) \leq^\delta (\alpha, 0, 0)$.

CASE 2. $\alpha = \gamma_\nu$, for some limit $\nu < \mu$.

Since S is a PCF structure, there exists a club C_ν such that $(\xi, 0) \leq_S (\nu, 0)$, for all $\xi \in C_\nu$. But then if we let $C_\alpha = \{\gamma_\xi : \xi \in C_\nu\}$ we have $(\gamma_\xi, 0) \leq^\delta (\alpha, 0)$ for all $\xi \in C_\nu$, and C_α is a club in α. This is because in our induction we did not modify anything in the skeleton.

CASE 3. $\alpha = c_{\nu+1}$, for some $\nu < \mu$.

We have $T^\delta \restriction \gamma_{\nu+1} = T^{\delta_\nu} \hookrightarrow^{\gamma_\nu} R^\nu \restriction \gamma_{\nu+1}$. Then, for all ξ such that $\gamma_\nu < \xi < \gamma_{\nu+1}$, we have $(\xi, 0, 0) \leq^\delta (\alpha, 0)$. (because of the definition of the $\hookrightarrow$ operation)

Finally, by noticing that s is of uncountable $\lhd^\delta$-cofinality if and only if s is of the form $(\alpha, 0)$ or $(\alpha, 0, 0)$ for α of uncountable cofinality we are done. ⊣

Now, putting Theorem 2.13 and 2.14 together we have the following corollary.

COROLLARY 2.15. *It is relatively consistent with ZFC that there are PCF structures of height δ, for all $\delta < \omega_3$.* ⊣

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