CHAPTER 1

Introduction

The technique of forcing, which was invented by Cohen in 1963 [**4**], is a very powerful tool for proving independency results in mathematics. Starting from a ground model V, where the Zermelo-Fraenkel's axioms plus the axiom of choice ZFC hold, a partial order is fixed, which should code the desired properties of the model one wishes to construct. Then forcing with this partial order over V yields the generic extension $V[G]$, where all this information is decoded, and ZFC holds as well. A very good introduction to forcing can be found in [**9**], and our notation will follow this book.

A forcing construction gives relative consistency results. When starting with a ground model V, then in particular the theory of V is supposed to be consistent. Thus, the theory of any model $V[G]$ which is obtained by a forcing construction from V is consistent if the theory of V is consistent. The classical example is the undecidability of the continuum hypothesis CH from ZFC: By an easy diagonalisation argument Cantor showed that the cardinality of $\mathbb{R}$ is greater than the cardinality of the natural numbers, in other words, the cardinality of $\mathbb{R}$ is uncountable. However, how much greater is it? CH is the statement that the cardinality of $\mathbb{R}$ is equal to the first uncountable cardinal, $\aleph_1$, and for a long time it was unknown whether CH follows from ZFC. The first result in this context was achieved by Gödel: His constructible universe L is a model for ZFC and CH (see, e.g., [**5**]). In finding this model he showed that ZFC and CH are consistent. Later, Cohen developed the technique of forcing to find a model of ZFC where the continuum hypothesis does not hold, in adding $\aleph_2$ many reals to L without collapsing cardinals. Thus, CH is neither provable nor refutable from ZFC.

The Cohen forcing is the following partial order: $C = (2^{<\omega}, \subseteq^*)$, where $\subseteq^*$ stands for reverse inclusion. Here finite approximations are used to create a new real, where, as usual in set theory, $\mathbb{R}$ is identified with the Baire space ω^ω, and the interval $[0;1]$ is identified with the Cantor space 2^ω. This forcing has the *countable chain condition* (ccc), i.e. there are no uncountable antichains. C has the ccc even trivially, since the poset itself is countable. This property prevents any cardinal κ from being collapsed, The property of having ccc is preserved by iterations with finite support, and so the above described construction yields a universe where there are $\aleph_2$-many reals.

Another important forcing is the random real forcing $R = (\mathcal{B}(2^\omega)/\mathcal{N}, \subseteq)$ where $\mathcal{B}(2^\omega)$ denotes the set of all Borel sets and $\mathcal{N}$ the ideal of measure zero sets. It can be shown that the Cohen forcing C is equivalent to forcing with the Borel sets modulo the ideal of meager sets, odered by inclusion [**2**]. Both have ccc, but they are quite different. While forcing with C makes the set of reals in the ground model to a measure zero set in the extension, R preserves the outer measure. R is *ω^ω-bounding*, i.e. for every R-name $\dot{x}$ and condition p forcing that $\dot{x}$ is the name for a real, there is $q \leq_R p$ and $f \in \omega^\omega$ in the ground model such that $q \Vdash_R \forall n \; \dot{x}(n) \leq f(n)$. For C this is not true, it adds an *unbounded real*.

A forcing notion can also produce something stronger, a *dominating real*. Here $x \in V[G] \cap \omega^\omega$ is called dominating iff for any $f \in \omega^\omega$ from the ground model V there is some n such that for any $m \geq n$ we have $f(m) \leq x(m)$, and we denote it by $f \leq^* x$. An example of such a forcing notion is the Hechler forcing H: the forcing conditions are pairs (s, f) where $f \in \omega^\omega$ and s is a finite initial segment of f. The ordering is given by $(s_1, f_1) \leq_H (s_0, f_0)$ iff $s_0 \sqsubseteq s_1$ and for each n $f_1(n) \geq f_0(n)$. If G is a generic filter, then $\bigcup\{s \mid \exists f \; (s, f) \in G\}$ is a dominating real. The first entry of a condition builds the generic real, while the second one makes sure that from some point on every real from the ground model will be dominated.

On the other end of the scale we find the *Sacks property*. This is a strenghthening of the ω^ω-boundedness. The latter just means that for each real x in the generic extension there is a cover $C : \omega \to \mathbb{P}(\omega)$ in the ground model such that each $C(n)$ is finite and $x(n) \in C(n)$. The Sacks property demands moreover that $|C(n)| \leq 2^n$. The random real forcing does not have this property, and it is neither provable nor refutable from ZFC whether there is a ccc forcing with the Sacks property (see section 3).

However, if we drop the requirement of ccc, then there are forcing notions with the Sacks property, namely the Sacks forcing S which consists of perfect trees in $2^{<\omega}$, ordered by inclusion. It was constructed by Gerald Sacks in [**12**] to find a model with exactly two constructibility degrees, i.e. there is some set x such that any element y from the universe is constructible in x, and if it is not in L, then x is also constructible in y.

The aim of this thesis is to examine the relationship between ccc forcings and the reals. Von Neumann asked in [**11**] whether every ccc ω^ω-bounding forcing notion is equivalent to a measure algebra. Nowadays, many models of ZFC are known where this conjecture fails, e.g. every Souslin tree is a counterexample. However, it remains open whether there is a model where von Neumann's question is answered posotively. Related to this problem is Prikry's question whether any ccc forcing notion adds a Cohen real or a random real.

In Chapter 2 we present the basic ideas of forcings and give some basic definitions which will be used later.

Chapter 3 deals with the relationship between the Sacks property and ccc forcings, which can be seen as an approximation to von Neumann's question. First we introduce the Sacks property and some of its properties. Then we continue with presenting various results dealing with ccc forcings and the Sacks property, before we show the proof that it is consistent with CH to assume that no ccc forcing has the Sacks property.

Chapter 4 shows another approach to von Neumann's question: here we ask whether any ccc ω^ω-bounding forcing adds a splitting real. Later we turn to splitting properties for ω_1, and finally we show that under certain assumptions that for a ccc forcing ”adding a splitting real” is the same as ”destroying any ultrafilter”.

CHAPTER 2

Preliminaries

1. A short overview over forcing

We begin with some basic definitions and facts concerning forcing.

DEFINITION 2.1. *A forcing notion* $(P, \leq_P)$ *is a partial ordering, i.e.* $\leq_P$ *is reflexive and transitive.*

Ocasionally, a forcing notion is also called poset. Sometimes we will simply write P instead of $(P, \leq_P)$ when referring to a forcing notion. The elements of P are called *forcing conditions,* and if $p, q \in P$, then we say that p is *stronger* than q iff $p \leq_P q$. If two conditions p_1, p_2 have a common stronger condition $q \leq p_1, p_2$, they are called *compatible,* else they are called *incompatible*. A set of pairwise incompatible elements is called *antichain,* and a forcing notion has the ccc (countable chain condition) if all the antichains are at most countable.

DEFINITION 2.2. *Let* P *be a forcing notion.* $D \subseteq P$ *is called dense if for every* $p \in P$ *there is some* $q \leq_P p$ *in* P *which is an element of* D*. If moreover,* D *is closed downwards,* D *is called dense open.*

DEFINITION 2.3. *Let* P *be a forcing notion,* M *be a collection of sets.* G *is called* P*-generic over* M *if*

(1) G *is a filter, i.e. if* $p \in G$ *and* $q \geq_P p$*, then* $q \in G$*, and if* $p, q \in G$*, then there is some* $r \in G$ *where* $r \leq_P p, q$
(2) G *meets any dense set which is in* M.

If M is equal to the universe and P is nontrivial, i.e. under each $p \in P$ there are two incompatible elements, then any G which is P-generic over V is a new element, an element outside

V. If such a G is added to V, and moreover, everything else is added which is definable from V and G, we obtain the model $V[G]$. V is called *ground model*, and $V[G]$ is called *forcing extension*.

Every element x of $V[G]$ has a P-name $\dot{x}$ in V which is interpreted by G as x, and we write $\dot{x}^G = x$. Names and how they will be interpreted by a generic filter or a generic branch are defined by recursion, more precisely

$$\dot{x}^G = \{\dot{y}^G \mid \dot{y} \text{ is a } P\text{-name and for some } p \in G\ (\dot{y}, p) \in \dot{x}\}.$$

$V[G]$ consits of exactly these $\dot{x}^G$, and since it is possible to construct a P-name for the generic filter G and for each $x \in V$ a canonical P-name $\check{x}$ which is interpreted by any generic filter as x, we have $V \subseteq V[G]$ and $G \in V[G]$. Moreover, it turns out that $V[G]$ is the smallest model of ZFC containing G and all elements from V. Let us make the following

DEFINITION 2.4. *For φ a formula we say that $p \in P$ forces φ, or $p \Vdash_P \varphi$, iff φ holds in any $V[G]$ where $p \in G$ and G is P-generic over V.*

Moreover, we have the following very important

THEOREM 2.5. *If $V[G] \models \varphi$, then there is some $p \in G$ such that $p \Vdash_P \varphi$.*

Nothing in $V[G]$ depends on chance, everything is forced at some point.

2. Important forcing notions

Sometimes we will refer to well-known forcing notions. Here, we will give a short overview:
Cohen forcing $\mathbb{C}$: this is the poset $(\mathcal{B}(2^\omega)/\mathcal{M}, \subseteq)$ where $\mathcal{B}(2^\omega)$ denotes the set of all Borel sets and $\mathcal{M}$ the σ-ideal of meager sets.

Random real forcing $\mathbb{R}$: R is equal to $(\mathcal{B}(2^\omega)/\mathcal{N}, \subseteq)$ where $\mathcal{N}$ denotes the σ-ideal of measure zero sets.

Sacks forcing $\mathbb{S}$: consists of all perfect subtrees of $2^{<\omega}$, ordered by inclusion.

Laver forcing $\mathbb{L}$: consists of all subtrees of $\omega^{<\omega}$ such that almost all nodes are infinite splitting nodes, ordered by inclusion.

Mathias forcing $\mathbb{M}$: consists of pairs (s, A) such that $s \subseteq \omega$ finite, $A \subseteq \omega \setminus \bigcup s$ infinite, where $(s, A) \leq (t, B)$ iff $t \subseteq s$, $A \subseteq B$ and $s \setminus t \subseteq B$.

3. Important trees

A tree is a partial order $(T, \leq_T)$ such that for any $t \in T$ the set of predecessors $\{s \in T \mid s \leq_T t\}$ is well-ordered. For an ordinal α, the level α, $T(\alpha)$, consists of those $t \in T$ whose set of predecessors has order type α, and the height of T is the supremum of all α such that $T(\alpha)$ is not empty. When given a tree T, then we denote by $[T]$ the set of all branches of T, i.e. the set of all maximal well-ordered subsets of T. For $t \in T$, we denote the cone of T with stem t as $T[t] = \{s \in T \mid s \leq_T t \vee s \geq_T t\}$.

Let us mention two famous trees:

DEFINITION 2.6. *An Aronszajn tree is a tree of height ω_1 with no uncoutable branch whose level are countable.*

It is provable in ZFC that an Aronszajn tree always exists [**9**]. However, whether it has further structural properties depends on the model.

DEFINITION 2.7. *An Aronszajn tree is special if it is the union of ω_1 many antichains.*

The question whether an Aronszajn tree is special is undecidable in ZFC, we will turn to this fact in 3.4.

DEFINITION 2.8. *A Souslin tree is an Aronszajn tree with the ccc.*

The existence of a Souslin tree is independent from ZFC as well [**9**]. It has some influence on forcing notions, as we will see in 3.5.

Our notation is standard and can be found in [**7**] or [**9**], as well as any basic facts which we have not mentioned here.

CHAPTER 3

Around the Sacks property

1. The Sacks property and the Laver property

The aim of this chapter is to present some results concerning ccc posets and the Sacks property. Thus, let us start with

DEFINITION 3.1. *Let $(P, \leq)$ be a forcing notion. Then P has the Sacks property iff for every new real $r \in \omega^\omega$ in the generic extension there is a small cover $C : \omega \to [\omega]^{<\omega}$ in the ground model such that $|C(n)| \leq 2^n$ for each n and $r \in \Pi_{n<\omega} C(n)$.*

The Sacks property allows a very strong control on the behaviour of the reals in the generic extension. Let us remark that it is not essential that we bound the size of $C(n)$ by 2^n, any other increasing function would do the job as well. This can be easily seen as follows: We fix an increasing $f : \omega \to \omega \setminus \{0\}$, a P-name $\dot{r}$ for a real and starting from the Sacks property, we try to find a cover for $\dot{r}$ whose size is bounded by f. To this aim, we fix $x \in \omega^\omega$ such that $x(n)$ is the minimal m such that $f(m) > 2^n$, and apply the Sacks property to the name $\dot{s} : \omega \to \omega^{<\omega}$, $\dot{s}(n) = \dot{r} \restriction x(n)$. This gives the desired small cover for $\dot{r}$. The same argumet yields that any P having the Sacks property with respect to an increasing function f has the Sacks property in the form stated above.

To illustrate the strong control of the behaviour of the reals given by the Sacks property, we will show that the ideal $\mathcal{N}^V$ of measure zero sets in the ground model V generates $\mathcal{N}^{V[G]}$, i.e. every new measure zero set is covered by a measure zero set in V, and the same holds also for nowhere dense sets. These properties were used in examining cardinal invariants [**2**]. Besides, if you would like to force a statement concerning e.g. measure, then it might prove useful to have such a control over the ideal $\mathcal{N}$.

CLAIM 3.2. *The Sacks property implies that measure zero sets in the generic extension are covered by Borel measure zero sets from the ground model.*

PROOF: Let P be a forcing notion having the Sacks property, $p \in P$ and $\dot{X}$ be a P-name for a measure zero set. Furthermore, let $\{C_m \mid m < \omega\}$ be an enumeration of all clopen subsets of the reals. Thus, for any measure zero set N and for any sequence $\langle \varepsilon_m \mid m < \omega \rangle$ we find a function $f \in \omega^\omega$ such that $N \subseteq \bigcap_{n<\omega} \bigcup_{m>n} C_{f(m)}$ where $\mu(C_{f(m)}) \leq \varepsilon_m$.
We fix a sequence $\langle \delta_i \mid i < \omega \rangle$ converging to zero, a bijection $e : \omega \times \omega \to \omega$, and for each i a sequence $\langle \varepsilon_{ij} \mid j < \omega \}$ such that $2^{e(i,j)} \cdot \varepsilon_{ij} \leq \delta_i$. In the generic extension there will be some $\dot{f}_i$ for each i such that $\dot{X} \subseteq \bigcap_{n<\omega} \bigcup_{j>n} C_{\dot{f}_i(j)}$ and $\mu(C_{\dot{f}_i(j)}) \leq \varepsilon_{ij}$.

Now consider the following name of a real: $\dot{r}(n) = m$ iff $m = \dot{f}_i(j)$ where $e(i,j) = n$. The Sacks property yields some $p' \leq p$ and a small cover C such that $|C(n)| \leq 2^n$ and $p \Vdash \dot{r} \in \Pi_{n<\omega} C(n)$, which gives us the desired cover for $\dot{X}$. $\square$

CLAIM 3.3. *The Sacks property implies that every nowhere dense set from the generic extension is covered by a Borel nowhere dense set from the ground model.*

PROOF: Recall that $A \subseteq 2^\omega$ is nowhere dense iff all $s \in \bigcup_{m<n<\omega} 2^{[m,n]}$ have an extension $t \sqsupseteq s$ such that $\{x \in 2^\omega \mid x \restriction \text{dom}(t) = t\}$ is disjoint from A. Thus, if we fix a forcing notion P having the Sacks property, and $\dot{A}$ is a name for a nowhere dense set, then for each n there is a name $\dot{s}_n$ for an element of $\bigcup_{k>n} 2^{[n,k]}$ such that $\dot{B} = \{x \mid \exists n\ x \restriction \text{dom}(\dot{s}_n) = \dot{s}_n$ is disjoint from $\dot{A}$. By the Sacks property, we find $p \in P$ and $n_0 < k_0 < n_1 < k_1 \ldots$ such that p forces that $\text{dom}(\dot{s}_{n_i}) \subseteq [n_i, k_i]$. Let $j_0 < j_1 < \ldots$ be a partition such that $j_i - j_{i-1} > 2^i$, and define $\dot{t}_i = \dot{s}_{n_{j_i}}^\frown \dot{s}_{n_{j_i+1}}^\frown \ldots^\frown \dot{s}_{n_{j_{i+1}-1}}$. Applying the Sacks property to the function $f : i \mapsto \dot{t}_i$ yields a cover C for f and some $q \leq p$ which forces that $f(i) \in C(i)$. Since $|C(i)| \leq 2^i$, we find for each i some $u_i \in 2^{[n_{j_i}, n_{j_{i+1}-1}]}$ such that for any $t \in C(i)$ there is some $k \in [j_i, j_{i+1}-2]$ where $t \restriction [n_k, n_{k+1} - 1] = u_i \restriction [n_k, n_{k+1} - 1]$. If we set $D = \{x \in 2^\omega \mid \forall i\ x \restriction \text{dom}(u_i) \neq u_i\}$, then D is a nowhere dense set which is forced by q to be a cover for $\dot{A}$. $\square$

The last two claims imply immediately that neither Cohen reals (i.e. reals which are generic for the poset of the Borel sets modulo the meager sets) nor random reals (i.e. reals which are generic for the poset of the Borel sets modulo the measure zero sets) are added, since every real can be covered by a nowhere dense set and by a measure zero set from the ground model.

A close relative to the Sacks property is the Laver property, which simply states that if a new real is bounded by a real from the ground model, then we can find a small cover as demanded for the Sacks property, more precisely:

DEFINITION 3.4. *A forcing notion P has the Laver property iff the following holds: for every $g \in \omega^\omega \cap V$ and $r \in \Pi_{n<\omega} g(n) \cap V[G]$ there is a small cover $C : \omega \to [\omega]^{<\omega}$ in V such that $|C(n)| \leq 2^n$ for each n and $r \in \Pi_{n<\omega} C(n)$.*

Although this looks quite similar, the Laver property is much weaker. It does not imply that any nowhere dense set from the generic extension can be covered by a nowhere dense set from the ground model, and it does not imply neither that any measure zero set from the generic extension can be covered by a measure zero set from the ground model. For example, the Mathias forcing has the Laver property [**2**], but the reals from the ground model form a meager measure zero set in the generic extension. However, the Laver property implies as well that no Cohen or Random reals are added, as the following short proof shows [**2**].

LEMMA 3.5. *Let P be a forcing notion with the Laver property, and $\dot{x}$ a P-name of an element of 2^ω. Then there is a $p \in P$ and a nowhere dense measure zero set M in the ground model such that $p \Vdash_P \dot{x} \in M$.*

PROOF: Fix P and $\dot{x}$ as stated in the lemma, and consider the P-name $\dot{y}$ with $\dot{y}(n) = \dot{x} \restriction 2^{2n}$. After identifying $2^{<\omega}$ with ω, the Laver property yields some $p \in P$ and a cover C with $|C(n)| \leq 2^n$ and $p \Vdash_P \forall n \ \dot{y}(n) \in C(n)$. Clearly, p forces that $\dot{x}$ will be an element of the set $M = \{r \in 2^\omega \mid \forall n \ \dot{r} \restriction 2^{2n} \in C(n)\}$. Clearly, M is nowhere dense, and $\mu(M) \leq \lim_{n\to\infty} \frac{2^n}{2^{2n}} = 0$. □

There are forcing notions related to the Sacks property and the Laver property, namely the Sacks forcing and the Laver forcing. As already mentioned in the introduction, the Sacks forcing consists of perfect trees in $2^{<\omega}$, ordered by inclusion, and the Laver forcing consists of trees in $\omega^{<\omega}$ such that almost any node is an infinite splitting node, again ordered by inclusion. It is easy to prove that the Sacks forcing has the Sacks property and the Laver forcing has the Laver property [[**2**]].

2. Analytic sets and Souslin forcings

When studying properties of sets of reals, the axiom of choice provides us very often with sets of a very chaotic structure, to nearly any property it allows us to construct a set of reals as a counterexample, not having this property. For example, if CH does not hold, then AC gives a set of reals which is uncountable, but not of size continuum, or if CH holds, then one can construct a set of reals not containing a perfect subset, and nor does its complement. However, there is a quite natural hierarchy of sets of reals, the projective hierarchy, and examining the question up to which level of this hierarchy certain properties or dichotomies hold makes suddenly sense. The bottom of the projective hierarchy is formed by analytic sets, also named Souslin sets or $\mathbf{\Sigma}^1_1$-sets, which are projections of closed sets in the plane. In general, the hierarchy for an arbitrary Polish space $\mathcal{X}$ is defined as follows:

$\mathbf{\Sigma}^1_1$: projections of closed sets in $\mathcal{X} \times \mathbb{R}$
$\mathbf{\Pi}^1_1$: complements of $\mathbf{\Sigma}^1_1$-sets
...
$\mathbf{\Sigma}^1_{n+1}$: projections of $\mathbf{\Pi}^1_n$ sets in $\mathcal{X} \times \mathbb{R}$
$\mathbf{\Pi}^1_{n+1}$: complements of $\mathbf{\Sigma}^1_{n+1}$-sets
...

There is also a syntactical definition of the projective hierarchy of the reals. Given a real r, a set S is said to be $\Sigma^1_n(r)$ iff the relation $x \in S$ is expressible by a formula of the form $\exists x_0 \ \forall x_1 \ldots \exists x_n \forall m$ (or: $\forall x_n \ \exists m$, if n is even) $(x_0 \upharpoonright m, x_1 \upharpoonright m, ldots, x_n \upharpoonright m) \in R$ where the x_i range over reals, m over ω and R is a tree in $(\omega^{<\omega})^n$ definable in the structure $\langle HF, \in, r\rangle$ where HF stands for the set of hereditarily definable sets. This formula is called a $\Sigma^1_n(r)$-formula. Analogously, a $\Pi^1_n(r)$-formula is a formula as above, just starting with $\forall$ as a quantifyer, and they define all $\Pi^1_n(r)$-sets. We get $\mathbf{\Sigma}^1_n = \bigcup_{r\in\omega^\omega} \Sigma^1_n(r)$ and

$\mathbf{\Pi}^1_n = \bigcup_{r \in \omega^\omega} \Pi^1_n(r)$.

It is well-known that Borel sets are analytic. Moreover, analytic sets carry a lot of structure. E.g., if an analytic set is uncountable, then it contains a perfect set, and in particular it has size $2^{\aleph_0}$. Furthermore, analytic sets are Lebesgue measurable and have the Baire property. The question whether all projective sets share all these properties involves large cardinals, but so far it seems that the axiom of choice is consistent with the statement that all projective sets have the above mentioned properties. A very good introduction into this subject can be found in [**7**].

The concept of the projective hierarchy has also some impact on forcing. First, there are absoluteness results like

THEOREM 3.6. *(Shoenfield's absoluteness theorem) Any $\mathbf{\Sigma}^1_2$-statement is absolute for all trasitive models $\mathcal{M}$ which contain the parameters defining the relation, all ordinals, and in which ZF+DC holds.*

The proof of the theorem can be found in [**7**].
DC is the axiom of dependent choice which is a weaker instance of the axiom of choice AC. It states that if $\mathcal{R}$ is a relation on a set A, and if for every $a_0 \in A$ there is some $a_1 \in A$ such that $a_1 \mathcal{R} a_0$, then there exists a countable sequence $\{a_i \mid i < \omega\}$ of elements of A such that for each i we have $a_{i+1} \mathcal{R} a_i$.

By Shoenfield's absoluteness theorem, $\mathbf{\Sigma}^1_2$-statements cannot be changed by forcing. Moreover, the forcing notion itself might be an analytic set, which leads us to

DEFINITION 3.7. *A Souslin forcing is a forcing $(P, \leq_P)$ such that P is analytic, $\leq_P$ is analytic, and the incompatibility relation $\perp_P$ is analytic.*

The most prominent examples of ccc Souslin forcings are certainly Cohen forcing and random real forcing. Both can be viewed as forcing with binary trees, ordered under inclusion. The Cohen forcing simply consists of the trees $\{2^{<\omega}[s] \mid s \in 2^{<\omega}\}$, while the random real forcing consists of binary trees whose branches form a subset of the reals of positive measure. Let us check that the random real forcing is a Souslin forcing:

A binary tree T is a condition iff $\exists m < \omega \; \forall \; n < \omega \;\; 2^{-n} \cdot |T(n)| \geq \frac{1}{m}$. The set of all trees of measure $\geq \frac{1}{m}$ is closed, thus the set of all trees of positive measure is an F_σ-set, or, as it is also called, a $\mathbf{\Sigma}^0_2$-set. Since Borel sets are analytic, the set of all forcing conditions is analytic. Clearly, inclusion is a closed relation, thus inclusion restricted to $P \times P$ is Souslin. Similar to above, the fact that the intersection of two trees does not contain a tree of positive measure is $\mathbf{\Pi}^0_2$, therefore the incompatibility relation is Souslin as well. Thus, the random real forcing belongs to the Souslin forcings. It is even easier to show that the Cohen forcing is also Souslin.

However, a Souslin forcing does not necessarily consist of certain trees which are ordered by inclusion, and sometimes one wishes to extract such trees. If a forcing P adds a new real $\dot{r} \in 2^\omega$, and if $p \in P$, then define the tree of possibilities below p as

$$T_p[\dot{r}] = \{s \in 2^{<\omega} \mid \exists \; q \leq p \; q \Vdash s \sqsubseteq \dot{r}\},$$

and let Tree(P) be the poset consisting of all the trees $T_p[\dot{r}]$, ordered by inclusion.

LEMMA 3.8. *If P is a ccc Souslin forcing and $\dot{r}$ a name for a new real, then the set of forcing conditions of Tree(P) is analytic, and Tree(P) has the ccc.*

PROOF: Clearly, if $T_p[\dot{r}] \perp T_q[\dot{r}]$, then $p \perp_P q$. Thus, Tree(P) must have the ccc, and it remains to show that the conditions form an analytic set.

For $s \in 2^{<\omega}$ we choose an antichain $A_s = \{p_{s,i} \mid i < \omega\}$ which is a maximal antichain among those conditions which force that s is an initial segment of $\dot{r}$. Thus, we have $T \in$ Tree(P) iff

$$\exists \; p \in P \; \forall \; s \in 2^{<\omega} \; (s \in T \rightarrow \exists \; q \leq_P p \; \exists \; i < \omega \; q \leq_P p_{s,i}) \vee (s \notin T \rightarrow \forall \; i < \omega \; p \perp p_{s,i}).$$

Since there is an arithmetical bijection between $2^{<\omega}$ and ω, and quantifiers over natural numbers do not harm, the formula is $\Sigma^1_1(r)$ where r is a real coding the antichains A_s and the parameter in which P is defined. □

In general, Tree(P) is not completely embeddable into P, and from this point of view it cannot be taken for a sort of coarser version of P. However, we will see later that nevertheless it inherits some properties from P, and it is quite helpful to look at Tree(P) when studying dichotomies.

Of course, in order to transform the poset into a set of perfect trees, we need a name for a new real. Concerning ccc Souslin forcings, Shelah showed in [**15**]:

LEMMA 3.9. *Any nontrivial ccc Souslin forcing adds a new real.*

While this is a ZFC result for Souslin forcings, there might be models where nontrivial ccc forcings exist which do not add reals, e.g. Souslin trees. On the other hand, it is consistent with ZFC to assume that any nontrivial ccc forcing adds a new real, e.g. MA($\aleph_1$) will do the job.

3. The Sacks property and combinatorical principles

In the past many people investigated the question how close or how far away a ccc forcing can be from the Cohen forcing and the random real forcing. 'Being close' here simply means whether the Cohen algebra or the random real algebra can be completely embedded into the given forcing. If so, then such a forcing creates also a generic filter for the Cohen forcing, or random real forcing respectively. Von Neumann conjectured that any ccc ω^ω-bounding forcing [**11**] is equivalent to a measure algebra, and its version for Souslin forcings remained undecided till today, as well as the consistency of this conjecture with ZFC.

Since the Sacks property implies that neither Cohen nor random reals are added, it is a good approximation to von Neumann's question to ask when a ccc forcing can have the Sacks property. Concerning analytic posets, Shelah showed in [**13**] that no ccc Souslin forcing has the Laver property, so in particular the Sacks property fails. Moreover, in the same paper he proved that if a Souslin forcing adds an unbounded real, then it adds a Cohen real. It is unknown whether there is a model of ZFC where any ccc forcing adding an unbounded real adds a Cohen real.

4. Some results about ccc forcings and their properties

Concerning arbitrary ccc posets, there are some combinatorical statements which imply or negate the existence of ccc posets with the Sacks property. We will recall the involved combinatorical statements, and give a summary of results.

$\diamondsuit$: There is a sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ such that $A_\alpha \subseteq \alpha$ and for any $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$ is stationary.

The principle $\diamondsuit$ implies that Souslin trees exist. Souslin trees have the ccc, and forcing with them does not add new reals, thus they trivially have the Sacks property. Moreover, Jensen constructed in [**8**] a ccc forcing having the Sacks property and adding new reals. The principle $\diamondsuit$, which implies CH, is used to guess antichains in advance and to freeze them, such that they cannot become uncountable.

CCC(S): For any family $\mathcal{D}$ of dense sets in the Sacks forcing S having at most size $2^{\aleph_0}$ there is a ccc subposet P of S such that $D \cap P$ is dense in P for each $D \in \mathcal{D}$.

This principle was introduced and shown to be consistent with ZFC by Veličković in [**22**] together with a large continuum, and from this one easily extracts a ccc subposet of the Sacks forcing having the Sacks property.

OCA: Let S be a subset of the reals, $[S]^2 = K_0 \dot{\cup} K_1$ be a coloring where K_0 is open in the product topology. Then either there is some uncountable $Y \subseteq S$ such that $[Y]^2 \subseteq X_0$, or $S = \bigcup_{i<\omega} Y_i$ where $[Y_i]^2 \subseteq K_1$.

Using this principle which makes the continuum of size at least $\aleph_2$, Veličković deduced in [**21**] that no ccc poset can have the Sacks property.

principle ($\star$) for κ: Let $\mathcal{I}$ be a p-ideal on $[\kappa]^{\leq\omega}$. Then either

(1) there is some uncountable $Y \subseteq \kappa$ such that $[Y]^\omega \subseteq \mathcal{I}$, or
(2) $\kappa = \bigcup_{i<\omega} Y_i$ such that $[Y_i]^\omega \cap \mathcal{I} = \emptyset$.

We will show in the next section that this principle introduced by Todorčević also implies that no ccc poset can have the Sacks property. Since ($\star$) restricted to $\kappa = \aleph_1$ is consistent with CH, as it was shown by Abraham and Todorčević [**1**], it is also proved that CH is not strong enough to decide the existence of ccc forcings with the Sacks property.

The case of the Laver property is different: CH already implies the existence of a ccc poset with the Laver property; just take Mathias forcing defined relatively to a Ramsey ultrafilter $\mathcal{U}$, i.e. the poset consists of pairs (s, A) where $s \in [\omega]^{<\omega}$, $\max s < \min A$ and $A \in \mathcal{U}$. Recall that a Ramsey ultrafilter is an ultrafilter $\mathcal{U}$ with the following additional property: if $\{U_i \mid i < \omega\} \subseteq \mathcal{U}$ and $U_{i+1} \subseteq U_i$, then there is some $U = \{u_i \mid i < \omega\} \in \mathcal{U}$ such that for each $j \geq i$ we have $u_j \in U_i$. There are several possibilities how to define a Ramsey ultrafilter. Using this version the proof of the fact that Mathias forcing has the Laver property can be immediately transferred to the Mathias forcing relatively to a Ramsey ultrafilter [**2**]. The fact that CH implies the existence of Ramsey ultrafilters is very easy. Besides, there is a proper forcing which adds no real, but adds a Ramsey ultrafilter: the poset which consists of infinite subsets of ω, ordered by $\subseteq^*$ (see, e.g., [**2**]).

A model where ccc forcings with the Laver property do not exist was found by Shelah in [**16**]: He showed that iterating Mathias forcing $\aleph_2$-times yields such a model. It is not clear whether in this model any ccc forcing adding an unbounded real adds a Cohen real, as it is the case with ccc Souslin forcings. However, at least one possible reason for not adding a Cohen real is eliminated, the Laver property. In particular, the Sacks property fails for ccc forcings in this model, it can be taken as an alternativity to show that the negation of CH is consistent with ”no ccc forcing has the Sacks property”.

It is always a good strategy to look for a suitable dichotomy when trying to prove statements about ccc forcings: one alternative of the dichotomy should give an uncountable antichain, and the other should take care about the property one wishes to show. Todorčević’s p-ideal principle $(\star)$ is such an example. He also found a model where the dichotomy holds for any ideal living on ω_1 provided it is generated by at most ω_1 sets [**19**]. However, this is inconsistent with CH. This dichotomy can only be used to examine forcing notions of size $\aleph_1$, and not of size $2^{\aleph_0}$. In the spirit of this dichotomy, Shelah and Zapletal constructed a model of ZFC + $2^{\aleph_0} = \aleph_2$ in which any forcing of size $\aleph_1$ adds a Cohen real [**17**].

There are also other kinds of interaction between ccc forcings and dichotomies. E.g., Hirschorn stated in his Ph.D. thesis [**6**] the following variation of the p-ideal principle $(\star)$:

principle $(\star_c)$ for κ: Let $\mathcal{I}$ be a p-ideal on $[\kappa]^{\leq\omega}$ and the cofinality of κ be uncountable. Then either

(1) there is a closed unbounded $Y \subseteq \kappa$ such that $[Y]^\omega \subseteq \mathcal{I}$, or
(2) there is a stationary subset S of κ such that $[S]^\omega \cap \mathcal{I} = \emptyset$.

Like $(\star)$, $(\star_c)$ restricted to $\kappa = \aleph_1$ together with CH is consistent with ZFC. With the aid of this principle Hirschorn showed that it is consistent with CH to assume that all Aronszajn trees are special in any forcing extension by a measure algebra. Recall that an Aronszajn tree is a tree of height ω_1 whose level are countable, and which does not have an uncountable branch. An Aronszajn tree is special if it is the countable union of antichains.

When turning to forcing notions of larger size, Blaszczyk and Shelah [**3**] have shown that it is consistent with ZFC that any σ-centered forcing notion adds a Cohen real. More precisely, they show that this statement is equivalent with the non-existence of nowhere dense ultrafilters. A nowhere dense ultrafilter is an ultrafilter with the following additional property: for every function $f : \omega \to 2^{<\omega}$ there is a set A in the ultrafilter such that the image $f''A$ is nowhere dense in the sense of the Cohen forcing, i.e. for each $t \in 2^{<\omega}$ there is some $t' \sqsupseteq t$ which is not extendible to an element of $f''A$. Together with Shelah's construction of a model with no nowhere dense ultrafilters [**14**] the result follows.

In the model Shelah constructed in [**14**], for every ultrafilter $\mathcal{U}$ there is some $f : \omega \to 2^{<\omega}$ witnessing that $\mathcal{U}$ is not nowhere dense. As Shelah mentioned in the same paper, the construction moreover guanrantees the existence of such an f for every $\mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I}$ is an ideal on ω such that the quotient space has ccc. It looks like as if a similar construction might yield a model where any ccc forcing adding an unbounded real adds a Cohen real. However, there are much more difficulties to overcome as in the case of filters.

5. The principle ($\star$) and the Sacks property

This section deals with the proof that the p-ideal dichotomy ($\star$) for $\kappa \leq 2^{\aleph_0}$ implies that no ccc forcing has the Sacks property. First we introduce some definitions and easy facts which will be used later. In the following P always denotes a collection of trees in $2^{<\omega}$, ordered by inclusion.

DEFINITION 3.10. *Let $T \in P$, $X \subseteq P$. Then we call*

$$tr(T, X) = \{T' \in X \mid T, T' \textit{ are compatible}\}$$

the trace of T in X.

$X \subseteq P$ is called large iff for any maximal antichain A there is some $T \in X$ which has a finite trace in A. As we will see later, large sets provide us with enough witnesses that a forcing notion P is ω^ω-bounding.

CLAIM 3.11. *Let $X \subseteq P$ be large, A a maximal antichain. Then*

$$X_A = \{T \in X \mid \textit{tr}(T, A) \textit{ is finite}\}$$

is also large.

PROOF: Let B be a maximal antichain. Select a maximal antichain B' such that any member of B' is below an element of B and below an element of A. Since X is large, there is some $T \in X$ with a finite trace in B'. Then T is clearly in X_A and tr(T, B) is finite. □

CLAIM 3.12. *Let P be ccc and ω^ω-bounding, and $P = \bigcup_{n<\omega} P_n$ a countable partition. Then for some n the set P_n is large.*

PROOF: Assume not. For each n choose A_n witnessing the non-largeness of P_n. By ccc and ω^ω-boundedness, there is some $T \in P$ having a finite trace in any A_n. Therefore, $T \notin P_n$ for any n, contradiction. □

Now we have all done all the preliminary work we need, and we can turn to the proof of the main statement. Let us remind the reader the p-ideal principle ($\star$) for κ:

($\star$) Let $\mathcal{I}$ be a p-ideal on $[\kappa]^{\leq\omega}$. Then exactly one of the two alternatives holds:

(a) There is an uncountable $X \subseteq \kappa$ such that $[X]^\omega \subseteq \mathcal{I}$.

(b) $\kappa = \bigcup_{n<\omega} X_n$ where each X_n is orthogonal to $\mathcal{I}$.

Recall that a set $X \subseteq \kappa$ is orthogonal to an ideal I provided that the intersection of X with any element of I is finite.

THEOREM 3.13. *Assume the principle* $(\star)$ *holds for* $\kappa \leq 2^{\aleph_0}$. *Then no ccc forcing has the Sacks property.*

PROOF: We work in a model where the principle ($\star$) holds for $\kappa \leq 2^{\aleph_0}$.

Assume that in this model we find a ccc forcing notion P having the Sacks property. We will have to transform the poset a little bit, and to do so, we will need a P-name for a new real. In some models of ZFC there are ccc forcings not adding reals, namely Souslin trees. Recall that a Souslin tree is a tree T of size and height ω_1 with no uncountable branches and having the ccc with respect to the tree ordering. We do not have to deal with them in our model, since it can be derived from ($\star$) for $\kappa = \aleph_1$ that there are no Souslin trees (see, e.g. [**1**]). There is even a more general connection between arbitrary ccc forcings and Souslin trees: if there is a non-trivial ccc forcing not adding a real, then a Souslin tree exists [**16**]. For the reader's convenience we will provide a proof of this fact.

LEMMA 3.14. *If a non-trivial ccc forcing exists which does not add reals, then a Souslin tree exists.*

PROOF: Let Q be a non-trivial ccc forcing not adding a real. We may assume that 1_Q already forces that the generic filter will be no element from the ground model. Thus, there will be some minimal ordinal α such that Q will add a new function with domain α, with values in the ground model. Again, we may assume that 1_Q already knows this minimal ordinal, and we can choose a Q-name $\dot{f}$ for such a new function. By minimality, for any $\beta < \alpha$ the function $\dot{f} \restriction \beta$ will be an element from the ground model. Thus, we can build the tree of possible initial segments T for $\dot{f}$:

$$T = \{g \mid \exists\, \beta < \alpha\ \exists\, q \in Q\ \ q \Vdash \dot{f} \restriction \beta = g\}$$

Clearly, if ordered by extension, T has the ccc, and after arguing that α must be equal to ω_1, we get that T is a Souslin tree.

First of all, observe that α must be a regular cardinal. Else there would be some $g : \gamma \to \alpha$ where $\gamma < \alpha$ and the range of g is cofinal in α. Thus, there would be a Q-name $\dot{h}$ for a new function with domain γ such that for $\beta < \gamma$ we would have $\dot{h}(\beta) = \dot{f} \upharpoonright g(\beta)$, a contradiction to the minimality of α.

Since Q adds no new reals, α must be uncountable. By ccc, we find an increasing sequence $\{\gamma_i \mid i < \alpha\}$ such that every node in T of height γ_i has at least two successors on level γ_{i+1}. If $\alpha > \omega_1$, then on level γ_{ω_1} there would be a branch b from which there have been at least ω_1 ramifications before, a contradiction to ccc. Thus, $\alpha = \omega_1$, and by the same argument T has no branches of height ω_1. Therefore, T is a Souslin tree. $\square$

Thus, in our model any ccc forcing adds new reals, and we are able to fix a P-name $\dot{r}$ for a new element in 2^ω. For any $p \in P$ recall the definition

$$T_p[\dot{r}] = \{s \in 2^{<\omega} \mid \exists q \leq p \; q \Vdash s \sqsubset \dot{r}\}$$

the tree of possibilities below p. Let us further remind you that Tree(P) is the forcing notion consisting of these $T_p[\dot{r}]$, ordered by inclusion. Note that this collection of trees has the following property: For each tree T and each node $s \in T$ there is a subtree $S \subseteq T[s]$ which belongs to Tree(P).

CLAIM 3.15. *If P has the ccc and the Sacks property, then so does Tree(P).*

PROOF: First, observe that whenever $T_p[\dot{r}] \perp_{\text{Tree}(P)} T_q[\dot{r}]$, then $p \perp_P q$, therefore Tree(P) must have the ccc.

Now we will prove the Sacks property: let $\{A_i \mid i < \omega\}$ be a set of maximal antichains in Tree(P), and let $x \in \omega^\omega$ be increasing. We have to find some tree $T_p[\dot{r}]$ which is compatible with at most $x(i)$ members of A_i for any i.

For each $T_{ij} \in A_i$ choose p_{ij} such that $T_{ij} = T_{p_{ij}}[\dot{r}]$. Let $P_i \supseteq \{p_{ij} \mid j < \omega\}$ be a maximal antichain such that for any $p \in P_i$ the tree $T_p[\dot{r}]$ is contained in some T_{ij}. Note that this is possible by ω^ω-boundedness:

Assume there is some $p \in P$ such that for any $q \leq p$ the tree $T_q[\dot{r}]$ is not contained in any T_{ij}. Then consider the following name for a real: $\dot{s}(j) = \min\{n \mid \dot{r} \upharpoonright n \notin T_{ij}\}$. Fix $q \leq p$, $h \in \omega^\omega$ such that $q \Vdash \dot{s} \leq h$. Then $T_q[\dot{r}] \cap T_{ij}$ is finite, which contradicts the maximality of

A_i in Tree(P).

Now choose an enumeration $P_i = \{p^i_j \mid j < \omega\}$. Since P has the Sacks property, there is some $p \in P$ such that for any i p is compatible with at most $x(i)$ members of P_i, say $\{p_0, \ldots, p_m\}$. Therefore, $T_p[\dot{r}]$ is covered by the union of the trees $T_{p_i}[\dot{r}]$ where $i \leq m$. Since each of these trees is included in one tree of the maximal antichain A_i, we may assume for simplicity that $T_{p_i}[\dot{r}]$ is an element of A_i. But then $T_p[\dot{r}]$ cannot be compatible with any other tree in A_i, because if we assume some $T \subseteq T_p[\dot{r}] \cap T_q[\dot{r}]$ where $T_q[\dot{r}]$ is incompatible with any $T_{p_i}[\dot{r}]$, then there is some $i \leq m$ and some s such that the cone $T[s]$ is included in $T_q[\dot{r}] \cap T_{p_i}[\dot{r}]$, which is impossible. Hence, $T_p[\dot{r}]$ is compatible with at most $x(i)$ many members of A_i. $\square$

Therefore, it suffices to show the statement for ccc collections of trees in $2^{<\omega}$, ordered by inclusion and having the Sacks property. Furthermore, the trees should represent the possibilities for a new real, in other words, for any tree T and $s \in T$ the cone $T[s]$ contains a condition. For the rest of the proof, fix such a forcing notion P.

Now consider the following ideal:

Let $\mathcal{I} \subseteq [P]^{\leq\omega}$ consist of all those X for which there is a maximal antichain A such that for any $T \in A$ the trace of T in X is finite.

CLAIM 3.16. *$\mathcal{I}$ is a p-ideal.*

PROOF: $\mathcal{I}$ is clearly an ideal, so it remains to show that for any countable collection of elements of $\mathcal{I}$ there is some member of $\mathcal{I}$ which almost contains any element of this countable collection.

Let $\{X_i \mid i < \omega\} \subseteq \mathcal{I}$, and let A_i be a maximal antichain witnessing $X_i \in \mathcal{I}$. By ω^ω-boundedness, select a maximal antichain A consisting of trees which have finite traces in any A_i. Therefore, A is a common witness for $X_i \in \mathcal{I}$. Fix an enumeration $A = \{T_i \mid n < \omega\}$. So, if we set $\tilde{X}_i = X_i \setminus \{tr(T_j, X_i) \mid j \leq i\}$, then $Y = \bigcup_{i<\omega} \tilde{X}_i$ fulfills our requirements, since any $T \in A$ has a finite trace in Y. $\square$

According to the principle ($\star$) we now have two alternatives:

Assume (a) holds, i.e. there is an uncountable $X \subseteq P$ such that any countable subset belongs to $\mathcal{I}$. By ccc, 1_P cannot force that the intersection of the generic filter G with X is countable, because if so, the intersection would be covered by a countable set Y in the ground model and any generic filter G containing an element of $X \setminus Y$ would be a counterexample to this statement. Hence, we can select $T \in P$ forcing that the intersection is uncountable.
Again by ccc, choose for $i < \omega$ a countable subset $X_i \subseteq X$ such that

$$T \Vdash \forall i \, \exists T \in X_i \; T \in \dot{G}.$$

Since the intersection of X with the generic filter G will be uncountable whenever $T \in G$, we can choose the X_i in a way that they are pairwise disjoint. In particular,

$$T \Vdash \dot{G} \cap \bigcup_{i<\omega} X_i \text{ is infinite.}$$

Therefore, any tree below T has an infinite trace in $\bigcup_{i<\omega} X_i$, although the latter set is supposed to be in the ideal by the properties of X, contradiction.

Assume now (b) holds, i.e. $P = \bigcup_{n<\omega} P_n$ such that for each n the intersection of P_n with any element of $\mathcal{I}$ is finite.

By Claim 3.12 fix some P_n which is large.

CLAIM 3.17. *For any maximal antichain A there is some finite $F_A \subseteq A$ such that each $T \in P_{n,A} = \{T \in P_n \mid \text{tr}(T, A) \text{ is finite}\}$ has an infinite intersection with $\bigcup F_A$.*

PROOF: Assume not. Then there is some maximal antichain A and $\{T_i \mid i < \omega\} \in P_{n,A}$ such that whenever $i \neq j$, then $\text{tr}(T_i, A) \cap \text{tr}(T_j, A) = \emptyset$. In particular, the T_i are pairwise disjoint. Now take some maximal antichain B containing all the T_i. Thus, B witnesses $\{T_i \mid i < \omega\} \in \mathcal{I}$, although no countable subset of P_n is an element of the ideal. $\square$

If T is a tree in $2^{<\omega}$, then we define the growth function of T, g_T, as follows: $g_T(n) = \min\{m \mid |T \cap 2^m| > n\}$.
We will use Claim 3.17 to construct an unbounded family of trees, i.e. a family F such that $\{g_T \mid T \in F\}$ is unbounded in ω^ω.

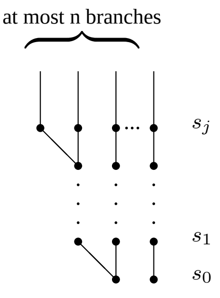
For each increasing $x \in \omega^\omega$ choose a maximal antichain A_x consisting of trees T where $g_T > x$. Note that this is possible by the Sacks property. By Claim 3.17, we can choose finite sets $F_x \subseteq A_x$ such that $\bigcup F_x$ intersects every tree in P_{n,A_x}. Then an easy argument shows that $F = \{\bigcup F_x \mid x \in \omega^\omega \text{ increasing}\}$ is an unbounded family of trees:

Assume there is some $g \in \omega^\omega$ which bounds the family F. In increasing g if necessary we may assume that $g(i) > i!$ for all $i < \omega$, and that g is strictly increasing. Thus, for any $k < \omega$ we have $g(i) > i \cdot k$ eventually which implies $g(i \cdot k) < g \circ g(i)$ eventually. Define h by $h(i) = g \circ g(i)$. Select some maximal antichain A_h consisting of trees T whose growth function g_T dominates h everywhere. Let $F_h \subseteq A_h$ be the already fixed finite subset with the property that each tree in P_{n,A_h} has an infinite intersection with one of the elements of F_h. In setting $k = |F_h|$ and $H = \bigcup F_h$ we compute the growth function of H as follows:

$$g_H(n) \geq \max\{g_T(\frac{n}{k}) \mid T \in F_h\} > h(\frac{n}{k}) = g \circ g(\frac{n}{k})$$

Thus, g_H dominates g eventually, a contradiction to the choice of g.

Since F is an unbounded family of trees, we find some n such that $\{g_S(n) \mid S \in F\}$ is infinite. We define by recursion:



- $s_0 \subseteq 2$ such that for any l there is some $S \in F$ where $S \cap 2 = s_0$ and $g_S(n) \geq l$.
- $s_j \subseteq 2^{j+1}$ end-extending s_{j-1} such that for any l there is some $S \in F$ where $S \cap 2^{j+1} = s_j$ and $g_S \geq l$.

Therefore, $\bigcup_{j<\omega} s_j$ is a tree with at most n branches, say $[\bigcup_{j<\omega} s_j] = \{r_0, \ldots, r_m\}$. For each n select $S_{x_n} \in F$ such that $S_{x_n} \cap 2^n = s_n$, and let A be a maximal antichain consisting of trees which have finite traces in each A_{x_n}. Thus, $\bigcup_{j<\omega} s_j$ hits each $T \in P_{n,A}$ infinitely many times, i.e. each $T \in P_{n,A}$ has at least one of the r_i as a branch. However, by Claim 3.11 we know that $P_{n,A}$ is large, so this is not possible: take a maximal antichain B consisting of trees T where $[T] \cap \{r_0, \ldots, r_m\} = \emptyset$. Now select some $T' \in P_{n,A}$ having a finite trace in B. T' cannot have any r_i as a branch, since it is covered by finitely many elements of B, and therefore each branch of T' must be a branch of one of the trees in the cover. Therefore, T' contradicts the properties of almost all s_n. □

Thus, none of these cases are possible, and no ccc forcing can have the Sacks property. □

6. Some consequences from the principle ($\star$)

The main motivation to prove theorem 3.13 was to decide the last open case about CH and the Sacks property, i.e. the question whether it is consistent with CH that no ccc forcing has the Sacks property. When assuming CH, we only need the p-ideal principle ($\star$) restricted to subsets of ω_1, and this particular instance of ($\star$) was shown to be consistent with CH by Abraham and Todorčević [**1**]. Hence,

COROLLARY 3.18. *CH is consistent with the statement 'no ccc forcing has the Sacks property'.*

Todorčević also showed in [**20**] that the principle ($\star$) extended to any p-ideal on countable subsets of an arbitrary set is a consequence of PFA. Thus,

COROLLARY 3.19. *Assume PFA holds. Then no ccc forcing has the Sacks property.*

However, since PFA implies OCA, this result already follows from [**21**].

Observe that we needed the Sacks property only in the last part, the rest was done just by ω^ω-boundedness. Moreover, the fact that we deal with trees was only used in the part (b).

For the first part, we could have defined the trace of a condition p in a set $X \subseteq P$ by tr$(p, X) = \{q \in X \mid q$ is compatible with $p\}$ and just have dealt with the order given by the poset P. Since any antichain belongs to the defined ideal above, we can note the following

COROLLARY 3.20. *Assume the principle ($\star$) holds for κ. Then any ccc ω^ω-bounding forcing notion P of size $\leq \kappa$ has the σ-finite chain condition, i.e. P can be written as $P = \bigcup_{n<\omega} P_n$ where each P_n contains only finite antichains.*

7. The special case of Souslin forcing

Corollary 3.20 is very close to the statement that any ccc ω^ω-bounding forcing notion is σ-linked. And indeed, in restricting the posets to Souslin forcings and using another partition principle one can show in ZFC:

THEOREM 3.21. *If P is a ccc ω^ω-bounding Souslin forcing, then Tree(P) is a ccc ω^ω-bounding and σ-linked forcing.*

Thus, any ccc ω^ω-bounding Souslin forcing has something in common with the measure algebra $\mathcal{B}$, since it is well-known that $\mathcal{B}$ is σ-linked [**2**].

The dichotomy we will use now in order to show Lemma 3.21 is due to Solecki [**18**]:

THEOREM 3.22. *Let A be an analytic set and $\mathcal{F}$ be a family of closed sets. Then either A is countably coverable by elements of $\mathcal{F}$, or there is a G_δ-set $G \subseteq A$ such that whenever B is a basic open set having no empty intersection with G, $G \cap B$ is not countably coverable by elements of $\mathcal{F}$.*

PROOF: (of Theorem 3.21)
Let P be an ω^ω-bounding ccc Souslin forcing. It is well known that P adds a real (see, e.g. [**15**]), therefore we can fix a name $\dot{r}$ for a new element of 2^ω. Let Tree(P) be as defined above, and fix the following family of closed sets:

$$\mathcal{F} = \{F \mid \forall T \in F\ \forall T' \in F\ T \cap T' \text{ is infinite}\}.$$

CLAIM 3.23. *Tree(P) is countably coverable by sets from $\mathcal{F}$.*

PROOF: On $2^{<\omega}$, we fix the following well-ordering: $s \leq^* t$ iff $|s| < |t|$ or $|s| = |t|$ and $s(|s|-1) \leq t(|s|-1)$. If $\tau : 2^n \to 2$ for some n, then we denote basic open sets by $N_\tau = \{T \in \text{Tree}(P) \mid \forall s \in \text{dom}(\tau)\ \tau(s) = 1 \text{ iff } s \in T\}$. A node $s \in 2^{<\omega}$ is called an end-node of τ if $s \in 2^n$ and $\tau(s) = 1$.

Assume the claim is false, and fix a G_δ-set $G \subseteq \text{Tree}(P)$, $G = \bigcap_{l<\omega} G_l$ with G_l open, such that for any basic open set N_τ which has nonempty intersection with G this intersection $G \cap N_\tau$ is not countably coverable. In particular, G contains two trees whose intersection is finite. Therefore, we find τ_0, τ_1, such that $N_{\tau_k} \cap G$ is nonempty and inside G_0, and moreover, no end-node of τ_0 end-extends any end-node in τ_1 and conversely. Hence, $N_{\tau_0} \cap N_{\tau_1} = \emptyset$. Now assume that $\tau_{s^\frown k}$ is defined for $s <^* t$. Since $N_{\tau_t} \cap G$ is not countably coverable, we find $\tau_{t^\frown 0}$, $\tau_{t^\frown 1}$ extending τ_t, such that $N_{\tau_{t^\frown k}} \cap G$ is nonempty, included in $G_{|t|}$, and furthermore, as before, no end-node of $\tau_{t^\frown 0}$ end-extends any end-node in $\tau_{t^\frown 1}$ and vice versa.
Hence, if $x \in 2^\omega$, then $\bigcap_{t \sqsubseteq x} N_{\tau(t)}$ contains a unique element $T_x \in G$, and moreover, if $x_1 \neq x_2$, then $T_{x_1} \cap T_{x_2}$ has finite height, a contradiction to the ccc. □

With the aid of this lemma we can finish the proof of Lemma 3.21 as follows:
Fix $\text{Tree}(P) = \bigcup_{n<\omega} P_n$ such that each $P_n \in \mathcal{F}$. We may assume that each P_n is maximal, i.e. whenever $T \notin P_n$, there is $T' \in P_n$ having a finite intersection with T. For each P_n choose a maximal linked subset X_n. It remains to show that $\bigcup_{n<\omega} X_n = \text{Tree}(P)$. So, assume not, and choose a witness $T \in \text{Tree}(P)$. Since T is not added to any X_n, we can select for each n some $T_n \in X_n$ incompatible to T. Now consider a name $\dot{x}$ for a real such that

$$T \Vdash \dot{x}(n) = m \text{ iff } \dot{r} \restriction m \in T_n \wedge \dot{r} \restriction m + 1 \notin T_n.$$

By ω^ω-boundedness there is some $T' \leq T$ and some $f \in \omega^\omega$ such that $T' \Vdash \dot{x} \leq f$. In particular, $T_n \cap T'$ is finite for any n. However, this is not possible since $T' \in P_n$ for some n, so $T' \cap T_n$ must be infinite. □

CHAPTER 4

Splitting reals

1. Preliminaries

It is consitent with ZFC to assume the existence of a ccc forcing adding no splitting reals. For example, assume $CCC(\mathcal{S})$. Since the Sacks forcing preserves p-points, it adds in particular no splitting real. Thus, $CCC(\mathcal{S})$ allows us to construct a ccc forcing adding no splitting real. To start with, we show this well-known fact about the Sacks forcing. The technique of interpretations which is used here has been applied to similar cases with success, and the presentation of this technique might be interesting for the reader.

CLAIM 4.1. *The Sacks forcing preserves p-points.*

PROOF: Recall that a p-point is an ultrafilter $\mathcal{U}$ with the following additional property: for any sequence $\{U_i \mid i < \omega\} \subseteq \mathcal{U}$ there is some $U \in \mathcal{U}$ such that $U \subseteq^* U_i$ for any i.

Let $\mathcal{U}$ be a p-point, T a condition of the Sacks forcing $\mathcal{S}$, and $\dot{x}$ an $\mathcal{S}$-name for a subset of ω. We have to find some $T' \subseteq T$ and $A \in \mathcal{U}$ such that $T' \Vdash A \subseteq \dot{x}$ or $T' \Vdash A \cap \dot{x} = \emptyset$.

We may assume that T has the following property: there is some increasing $u \in \omega^\omega$ such that if $s \in T \cap 2^{u(n)}$, then $T[s]$ decides whether $n \in \dot{x}$. Thus, for each path $f \in [T]$ we can assign an interpretation $I(f)$ of $\dot{x}$ in defining

$$I(f) = \{n < \omega \mid \exists m\, T[f \restriction m] \Vdash n \in \dot{x}\}.$$

Now partition the nodes $s \in T$ in the following way: $s \in Z^+$ iff $T[s]$ contains a path f_s whose interpretation is a member of $\mathcal{U}$, and $s \in Z^-$ else. If $s \in Z^-$, then any extension of s in T is in Z^-, and in switching to the name for the complement of $\dot{x}$ we may assume that a cone T' of T is included in Z^+. Since $\mathcal{U}$ is a p-point, we find $A \in \mathcal{U}$ which is almost included in any $I(f_s)$.

Fix an increasing function $g \in \omega^\omega$ such that for every n, if $s \in T' \cap 2^{g(n)}$, then $A \setminus I(f_s) \subseteq m < g(n+1)$, and $T'[f_s \restriction g(n+1)]$ decides for every $i < m$ whether it belongs to $\dot{x}$. Furthermore, we require that $T'[s]$ splits below $g(n+1)$. Now partition ω into for parts B_0, B_1, B_2, B_3 where $B_i = \bigcup_{n<\omega} [g(4n+i), g(4n+i+1))$. Wlog, $B_2 \in \mathcal{U}$ and $A \subseteq B_2$. By recursion we construct a perfect subtree $T'' \subseteq T'$ such that for each $s \in 2^{g(4n)}$ we have $T[s] = T[f_s \restriction g(4n+3)]$. We claim that $T'' \Vdash A \subseteq \dot{x}$. Assume not, and fix $\tilde{T} \leq T''$, $i \in A$ such that $\tilde{T} \Vdash i$ is minimal such that $i \notin \dot{x}$. For some n, i is in the interval $[g(4n+2), g(4n+3))$. Select some $s \in \tilde{T} \cap 2^{g(4n)}$, so $A \setminus I(f_s) \subseteq g(4n+1)$. By construction, from s on the tree follows the path f_s till $g(4n+3)$ without any splitting. Thus, i is decided according to $I(f_s)$ by $\tilde{T}[s]$, and since $A \cap [g(4n+2), g(4n+3)) \subseteq I(f_s)$, $T[s] \Vdash i \in \dot{x}$, contradiction. □

Let us explicitely carry out the construction of a ccc poset adding no splitting real from the principle $CCC(\mathcal{S})$, and besides, having the Sacks property. Recall the notion of a canonical name for a real: it consists of countably many pairs $(p_{i,j}, \check{a}_{i,j})$ such that for each i the set $A_i = \{p_{i,j} \mid j < \omega\}$ is an antichain, and each $\check{a}_{i,j}$ is a canonical $\mathcal{S}$-name for an integer. Since $\mathcal{S}$ is proper, we find for any $\mathcal{S}$-name $\dot{r}$ and for any $p \in \mathcal{S}$ some $q \leq_P p$ and some canonical name for a real in a way that q forces that $\dot{r}$ will have the same interpretation in the generic model as this canonical name. Clearly, there are only $2^{\aleph_0}$ many canonical $\mathcal{S}$-names for reals since the Sacks forcing itself is of size $2^{\aleph_0}$. For a canonical $\mathcal{S}$-name $\dot{r}$, an infinite set $Z \subseteq \omega^{<\omega}$ and an infinite set $A \subseteq \omega$ consider the dense set $D_{\dot{r},Z,A}$ of conditions p in $\mathcal{S}$ satisfying the following:

- either p forces that $\dot{r}$ is no name for a new real or p forces that $\dot{r}$ is a name for a new real,
- if p forces that $\dot{r}$ is a name for a new real, then there is a cover $C : \omega \to [\omega]^{<\omega}$ such that the cardinality of $C(n)$ is smaller than 2^n, and p forces that for any n the value of $\dot{r}(n)$ will be inside $C(n)$,

- if p forces that $\dot{r}$ is a name for a new real, then there is some infinite $B \subseteq A$ such that either p forces that on each level $l \in B$ the finite initial segment $\dot{r} \upharpoonright l$ will be disjoint from Z, or p forces that on each level $l \in B$ the finite initial segment $\dot{r} \upharpoonright l$ will be an element of Z.

Therefore, the collection

$$\mathcal{D} = \{D_{\dot{r},Z,A} \mid \dot{r} \text{ is a canonical } \mathcal{S}\text{-name for a real}, Z \subseteq \omega^{<\omega} \text{ infinite}, A \subseteq \omega \text{ infinite}\}$$

has the appropriate size of $2^{\aleph_0}$, and we can apply $CCC(\mathcal{S})$ to it to obtain a ccc forcing which is a subposet of $\mathcal{S}$, has the Sacks property and adds no splitting real.

2. Absoluteness of adding a splitting real

Thus, the question arises whether it is consistent that any ccc forcing adds a splitting real. Shelah showed in [**15**] that any ccc Souslin forcing adding an unbounded real adds a Cohen real. Clearly, a Cohen real is a splitting real, as well as a Random real. Therefore, the famous question whether there is a ccc ω^ω-bounding Souslin forcing which is not the measure algebra, can be weakend as follows: Is there a ccc Souslin forcing not adding a splitting real? By Shelah's result, such a forcing must be ω^ω-bounding, and would be a counterexample to the version of von Neumann's conjecture relativized to Souslin forcings. We will show now that for a ccc Souslin forcing the statement 'adding a splitting real' is absolute. Therefore, if there is a model of ZFC where any ccc ω^ω-bounding forcing adds a splitting real, then ZFC implies that any ccc Souslin forcing adds a splitting real.

THEOREM 4.2. *For a ccc Souslin forcing P the statement 'P adds a splitting real' is absolute.*

PROOF: Shelah already showed in [**15**] that any ccc Souslin forcing must add a new real. We show now that '$\dot{r}$ is the name of a splitting real' is absolute, in fact a $\mathbf{\Pi}^1_1$-statement. Then the existence of such an $\dot{r}$ is expressible with the complexity $\mathbf{\Sigma}^1_2$, and by Shoenfield's absoluteness theorem (Theorem 3.6) the result follows.

Fix a ccc Souslin forcing P. Each canonical name for a new element consists of countably many antichains of size $\aleph_0$, $\dot{r} = \{(p_{i,j}, \check{a}_{i,j}) \mid i, j \in \omega\}$ such that for each i, $A_i = \{p_{i,j} \mid j < \omega\}$ is a maximal antichain and $\check{a}_{i,j}$ is either the canonical P-name for 0 or the canonical P-name for 1. We only concentrate on names $\dot{r}$ such that for no i the maximal condition 1_P forces a value to $\dot{r}(i)$. Moreover, in switching to a finer maximal antichain we may assume that for each i there are infinitely many conditions inside A_i forcing that $\dot{r}(i) = 0$, and there are infinitely many conditions inside A_i forcing that $\dot{r}(i) = 1$. Thus, for simplicity we can assume that if j is uneven, then $\check{a}_{i,j} = \check{0}$, and if j is even, then $\check{a}_{i,j} = \check{1}$.

Since the forcing conditions are reals, we can clearly code the P-name $\dot{r}$ with one single real $\tilde{r}$. E.g., fix the canonical bijection $e : \omega \to \omega \times \omega \times \omega$ and let $\tilde{r}(l) = p_{i,j}(k)$ where $e(l) = (i, j, k)$. From $\tilde{r}$ we can win back all the components from $\dot{r}$.
Now we have to express several properties:

- A_i is a maximal antichain,
- For any forcing condition $p \in P$ there is some integer n such that for any $i \geq n$ p does neither force $\dot{r}(i) = 0$ nor $\dot{r}(i) = 1$.

The complexity of the first statement is already computed in [**15**], and to be complete we repeat this calculation. A_i is a maximal antichain iff

$$\forall j\ \forall k\ j \neq k \to p_{i,j} \perp p_{i,k}$$

and

$$\forall p\ \exists j\ p \notin P \vee p \parallel p_{i,j}.$$

Since the incompatibility relation and membership of P do both have the complexity $\mathbf{\Sigma}^1_1$, the first part is a $\mathbf{\Sigma}^1_1$-statement, and the second part is a $\mathbf{\Pi}^1_1$-statement. In using that $\leq_P$ is a $\mathbf{\Sigma}^1_1$-relation, we even get that the first part is equivalent to the $\mathbf{\Pi}^1_1$-statement

$$\forall p\ \forall j\ \forall k\ (j \neq k) \to (p \notin P \vee \neg(p \leq_P p_{i,j} \wedge p \leq_P p_{i,k})).$$

We do not need to say explicitely that $p_{i,j} \in P$, since the incompatibility relation is only defined on members of P.

The second statement holds iff

$$\forall p\ \exists n\ \forall i \geq n\ \exists j, k\ p \parallel p_{i,2j} \wedge p \parallel p_{i,2k+1}.$$

Here we get a $\mathbf{\Pi}^1_1$-statement, which finishes our proof. □

3. Transforming posets into sets of perfect trees

Let P be a σ-linked ω^ω-bounding forcing notion. Notice that P must add a new real: if $P = \bigcup_{i<\omega} X_i$ where each X_i is linked, then select for each i a maximal antichain $A_i = \{p_{i,j} \mid j < \omega\}$ of elements incompatible with something in X_i. In denoting the canonical P-name for the ordered pair of integers (i, j) as op(i, j), we let $\dot{x} = \{(p_{i,j}, \text{op}(i, j)) \mid i, j \in \omega\}$. Clearly, $\dot{x}$ is a P-name for a new real, since no condition forcing $\dot{x}$ to be equal to some real from the ground model could be in any X_i. Since P must add a new element in the Baire space ω^ω, it adds a new real in the Cantor space 2^ω, and we can choose a name $\dot{r}$ for such a real. Recall that to any forcing condition $p \in P$ we can associate the tree of possibilities for $\dot{r}$, $T_p[\dot{r}]$, by $T_p[\dot{r}] = \{s \in 2^{<\omega} \mid \exists q \leq_P p\ q \Vdash \dot{r} \restriction |s| = s\}$. Recall further that each $T_p[\dot{r}]$ is a perfect tree, and we call the poset which consists of all these associated trees, ordered by inclusion, Tree(P). Although Tree(P) might not be completely embeddable into P, we have already seen with Claim 3.15 that Tree(P) inherits some properties from P. The same is true in the current context:

CLAIM 4.3. *Let P be a σ-linked ω^ω-bounding forcing notion adding no splitting real. Then Tree(P) is σ-linked, ω^ω-bounding and does not add a splitting real neither.*

PROOF: A slight modification of the proof of Claim 3.15 yields that Tree(P) is σ-linked and ω^ω-bounding. We only have to care about the existence of splitting reals.

Assume that Tree(P) adds a splitting real. Thus, there are maximal antichains A_i for each $i < \omega$ and disjoint partitions A^0_i, A^1_i such that $A_i = A^0_i \cup A^1_i$. Moreover, for any tree $T \in \text{Tree}(P)$ there is some n such that for each $i \geq n$ the tree T is compatible to some tree in A^0_i and to some tree in A^1_i. Now we will lift up this Tree(P)-name for a splitting real to a

P-name for a splitting real. Define a P-name $\dot{x}$ such that

$$1_P \Vdash \dot{x}(i) = 1 \longleftrightarrow \exists p \in \dot{G}\ \exists T \in A^1_i\ T_p[\dot{r}] \subseteq T.$$

Note that if $T_p[\dot{r}]$ and $T_q[\dot{r}]$ are incompatible in Tree(P), then p and q are incompatible in P. Therefore, the definition of $\dot{x}$ makes sense, and we claim that $\dot{x}$ is a P-name for a splitting real. Because if not, then there is some $p \in P$ and some infinite $A \subseteq \omega$ such that p forces that $\dot{x} \restriction A$ is constant. Wlog, let x force that the constant value is 1. However, there is some n such that for each $i \geq n$ there are trees in A^0_i which have an infinite tree as intersection with $T_p[\dot{r}]$. Let us enumerate $\bigcup_{i \in A \setminus n} A^0_i = \{T_j \mid j < \omega\}$. This enumeration will serve us to construct the following P-name for a real $\dot{y}$:

$$x \Vdash \dot{y}(j) = \min\{k \mid \dot{r} \restriction k \notin T_j\}$$

Since we assume that p forces that no condition will be in the generic filter whose tree of possibilities is included in some of these T_j, $\dot{y}$ is well-defined. Moreover, by ω^ω-boundedness, we find some $q \leq_P p$ and some $f \in \omega^\omega$ such that q forces that $\dot{y}$ will be bounded by f. In particular, the intersection of $T_q[\dot{r}]$ with any T_j is smaller than $f(j)$, a contradiction to the fact that the antichains A_i together with the partitions A^0_i, A^1_i describe a splitting real in Tree(P). □

4. Splitting reals and σ-linked ω^ω-bounding forcing notions

The following can be seen as a weak variant of von Neumann's conjecture:

CONJECTURE 4.4. *It is consistent with ZFC that any σ-linked ω^ω-bounding forcing notion adds a splitting real.*

Assume the conjecture is false. By the preceeding section, we can pick some P which consists of perfect trees in $2^{<\omega}$ ordered by inclusion, is σ-linked, ω^ω-bounding and adds no splitting real. The aim is to work out some properties which such a P must have.

Call a set $\{p_i \mid i < \omega\} \subseteq P$ small iff 1_P forces that $\{p_i \mid i < \omega\}$ will be almost inside or almost outside the generic filter. Observe that any infinite subset of P contains an infinite small subset. To see this, consider the following name for a real $\dot{x}$: $\dot{x}(i) = 1$ iff p_i is in the generic filter. By ccc, and since no splitting real is added, we find some infinite $A \subseteq \omega$ where $\dot{x}$ is forced by 1_P to be almost constant. Thus, $\{p_i \mid i \in A\}$ is a small set.

Fix a decomposition $P = \bigcup_{i<\omega} X_i$ such that each X_i is a maximal linked set, i.e. if $p \notin X_i$, then there is some $q \in X_i$ incompatible with p. For $s \in 2^{<\omega}$ set $Y_s = \{T \in P \mid \forall i \in |s|\ T \in X_i \text{ iff } s(i) = 1\}$. By Claim 3.12 we may assume that X_0 is large, and all the Y_s will help to partition X_0 in a very useful way. Let B consist of all those $s \in 2{<}\ \omega$ such that $s(0) = 1$ and Y_s is a large set, in fact a large subset of X_0. The largeness guarantees that B is a tree without finite endnodes.

CLAIM 4.5. *For each branch $b \in [B]$ there is a condition p such that p belongs to no X_i where $b(i) = 0$.*

PROOF: Since for $m < n$ we have $Y_{b\restriction m} \supseteq Y_{b\restriction n}$, we can find a small set $\{p_i \mid i < \omega\}$ which is almost included in any $Y_{b\restriction n}$. By largeness of X_0 we can argue that there is some $p \in P$ and some integer j such that p is below any p_i for $i > j$: because if not, then 1_P forces that the small set $\{p_i \mid i < \omega\}$ will be almost outside the generic filter G. Therefore, there must be some $q \in X_0$ having just a finite trace in the maximal antichain determining from which point on this sequence will be outside G. This simply means that q is incompatible with almost all p_i, contradicting linkedness of X_0.

In removing a finite initial segment of the small sequence $\{p_i \mid i < \omega\}$ we may assume that there is some p which is below any p_i. Let us examine to which X_i this p can belong. If $b(i) = 0$, then for any $j \geq i$ the condition p_j is not in X_i, and by maximality of X_i there is a reason for it, there is some $q \in X_i$ incompatible with p_i. Thus, p is also incompatible with q, and $p \notin X_i$. This shows that p is as wanted. □

The small set we have chosen follows the pattern of b, i.e. for any n a tail of this sequence is included in $Y_{b\restriction n}$. Let us define the Boolean value

$$Bool(b) = ||\text{There is a small set following the pattern of } b \text{ which is in } G||.$$

The above stated claim shows that $Bool(b)$ is not empty. In fact, we will show that $Bool(b)$ is the same as the Boolean value for 'any small set following the pattern of b is almost included in G'.

CLAIM 4.6. *Let b, $\{p_i \mid i < \omega\}$ and p be as in Claim 4.5, and let $\{q_i \mid i < \omega\}$ be a small set following the pattern of b. Then p forces that $\{q_i \mid i < \omega\}$ will be almost included in G.*

PROOF: Assume not. In cutting away a finite initial segment of $\{q_i \mid i < \omega\}$ we may assume that we find some $q <_P p$ forcing that no element from $\{q_i \mid i < \omega\}$ will be in the generic filter G. Again, we ask to which X_j this q can belong. Since $q <_P p$, by claim 4.5 q cannot belong to any X_j where $b(j) = 0$. Now select some j where $b(j) = 1$. As $\{q_i \mid i < \omega\}$ follows the pattern of b, $q_j \in X_j$, and q_j is incompatible with q. Thus, q belongs to no X_j, contradicting the assumptions about σ-linkedness of the forcing notion P. □

In particular, each large linked set is orthogonal to the p-ideal $\mathcal{I}^- = \{S \subseteq P \mid 1_P \Vdash S \cap \dot{G} \text{ is finite }\}$.

CLAIM 4.7. *Let b be a branch through the tree B and n some integer. Then there is some m such that $Y_{b\restriction m}$ is n-linked.*

PROOF: Assume not, and select some branch b as a counterexample. For each j choose $\{p_{j,k} \mid k < n\} \subseteq Y_{b\restriction j}$ having no common lower bound. Moreover, we can choose them in a way that $\{p_{j,k} \mid j < \omega\}$ is a small set for any $k < n$. By Lemma 4.6 there is some p forcing that all of these small sequences will be almost included in the generic filter G. Thus, we can reinforce p to some q and find m such that for each $k < n$ the condition q forces that $\{p_{j,k} \mid m < j < \omega\}$ is included in G. In particular, q is a common lower bound of $p_{m,0}, \ldots, p_{m,n-1}$ which was supposed not to exist. □

Since B is a tree with a finite splitting, compactness yields for any n some m_n such that if $s \in 2^{m_n}$ is in B, then Y_s is n-linked. Moreover, with the aid of B one could transform X_0 into a closed set of conditions such that for any n, this closed set is finitely coverable by n-linked clopen sets of trees. If one does not care about the last property, there is a more direct way to transform P into an F_σ-set. By Claim 4.5 any tree in the closure of X_0 contains a condition. Moreover, any tree $T \in P$ belongs to some large X_i, so without loss of generality we may assume that any X_i is large. Thus, $Q = \bigcup_{i<\omega} \bar{X}_i$ is an F_σ-forcing containing P as a dense subset. In particular, this shows that if there is a σ-linked ω^ω-bounding forcing notion not adding a splitting real, then there is a definable one having all these properties. Moreover, if the p-ideal-dichotomy holds for $k \leq 2^{\aleph_0}$, then any ccc ω^ω-bounding forcing is decomposable into countably many large sets which are orthogonal to $\mathcal{I}^-$, and assuming this principle, one gets that if there is a ccc ω^ω-bounding forcing notion not adding a splitting real, then there is a definable one. All this makes it very unlikely that such a forcing exists. Its construction would refute von Neumann's conjecture at the same time within ZFC.

We can refute the existence of such a forcing if we assume a further property: the covering for nowhere dense sets, i.e. each nowhere dense set in the generic extension can be covered by a nowhere dense set in the ground model.

CLAIM 4.8. *Let P be a σ-linked ω^ω-bounding forcing notion with the covering property for nowhere dense sets, and let Tree(P) be built with the aid of a P-name for a new real. Then Tree(P) is a σ-linked ω^ω-bounding forcing notion with the covering property for nowhere dense sets.*

PROOF: If the claim does not hold, we can fix such a forcing notion P which is a counterexample. By claim 4.3, for any new P-name for a real $\dot{r}$ the corresponding poset Tree(P) is σ-linked and ω^ω-bounding. Therefore, the only thing to make the claim fail is that Tree(P) does not have the covering property for nowhere dense sets. Thus, we can select a Tree(P)-name $\dot{N}$ for a nowhere dense set which cannot be covered by a set from the ground model. It is wellknown that for every nowhere dense set N there is a sequence $\{s_i \mid i < \omega\}$ such that dom(s_i)=$[i, n_i)$ for some $n_i > i$, and for any real x, if x contains some s_i, then $x \notin N$. This sequence simply describes a dense open set inside the complement of N. Thus, instead of looking at the Tree(P)-name $\dot{N}$ we consider $\{\dot{s}_i \mid i < \omega\}$ such that each $\dot{s}_i$ is a Tree(P)-name for an element of $\bigcup_{n<\omega} \omega^{[i,n)}$ and $1_{Tree(P)}$ forces that this sequence represents a dense open set in the complement of N as described above.

For each $\dot{s}_i$ we select a maximal antichain $A_i = \{T_{i,j} \mid j < \omega\}$ in Tree(P) such that each $T_{i,j}$ forces some value $s_{i,j}$ to $\dot{s}_i$. Using A_i, we transform $\dot{s}_i$ to a P-name $\dot{t}_i$ in the following way:

let $\dot{t}_i = s_{i,j}$ iff there is some $p \in P$ which is in the P-generic filter and $T_p[\dot{r}] \subseteq T_{i,j}$. Observe that for $j \neq k$ the trees $T_{i,j}$ and $T_{i,k}$ cannot both contain a common $T_p[\dot{r}]$. Thus, in order to show that this definition makes sense, it remains to prove that for each i the set of conditions p whose tree of possibilities $T_p[\dot{r}]$ is included in some $T_{i,j}$ is dense. This follows easily from ω^ω-boundedness of P. Assume some $p \in P$ such that no $q \leq_P p$ has its tree of possibilities $T_q[\dot{r}]$ included in some $T_{i,j}$. Therefore, below p we can define the following P-name $\dot{x}$ for a real: $\dot{x}(j)$ is the minimal k such that $\dot{r} \restriction k \notin T_{i,j}$. Then we find some $q \leq_P p$ and some $f \in \omega^\omega$ such that q forces that $\dot{x}$ will be dominated by f. In particular, for each $j < \omega$ the height of $T_q[\dot{r}] \cap T_{i,j}$ is at most $f(j)$, which contradicts the maximality of the antichain A_i in Tree(P). Thus, $\{\dot{t}_i \mid i < \omega\}$ describes a P-name for a dense open set.

P was chosen to have the covering property for nowhere dense sets. Therefore, there is some $p \in P$ and a sequence $\{u_i \mid i < \omega\}$ such that dom$(u_i) = [i, m_i)$, and p forces that the dense open set described by this sequence will be included in the dense open set described by $\{\dot{t}_i \mid i < \omega\}$. In other words, p forces that each u_i will contain some $\dot{t}_j$ where $j < m_i$. However, by our assumption, the interpretation of the Tree(P)-name $\dot{N}$ cannot be covered by a nowhere dense set from the ground model. In particular, $T_p[\dot{r}]$ forces that not every u_i will contain some $\dot{s}_j$. Wlog, $T_p[\dot{r}]$ forces already that u_0 contains no $\dot{s}_j$.

Below $T_p[\dot{r}]$, select some maximal antichain A consisting of trees fixing the values for $\dot{s}_0, \ldots, \dot{s}_{m_0}$. Since p forces that u_0 will contain one of the $\dot{t}_0, \ldots, \dot{t}_{m_0}$, there is no $q \leq_P p$ such that $T_q[\dot{r}]$ is contained in one of the trees in A. Again, by ccc and ω^ω-boundedness, we find some q whose tree $T_q[\dot{r}]$ just has a finite intersection with any tree in A, showing that A could not have been maximal. Thus, Tree(P) has the covering property for nowhere dense sets. $\square$

CLAIM 4.9. *Let P be a σ-linked ω^ω-bounding forcing notion. Then P has to share at least one of the following properties with the random real algebra:*

- *P adds a splitting real, or*

- *There is a name for a nowhere dense subset of the reals which cannot be covered by a nowhere dense set from the ground model.*

PROOF: If the claim does not hold, we can select a counterexample P. By Claims 4.3 and 4.8 we can assume that P is an F_σ-forcing, $P = \bigcup_{i<\omega}[X_i]$ where each X_i is a tree whose nodes are initial segments for binary trees. Moreover, since the countable union of non-large sets is non-large, we can assume that if $\tau \in X_i$, then the branches of $X_i[\tau]$ form a large set. Let $\dot{N}$ be the name the set of all branches of X_0 which will be in the generic filter. We claim that this set has to be nowhere dense in $[X_0]$.

To see that this holds, assume some condition p which forces that $\dot{N}$ will be dense in some $X_0[\tau]$. Since the trees contain a tree of possibilities for a new real $\dot{r}$, p forces that $\dot{r}$ will be in the intersection of all trees which are branches in the cone $X_0[\tau]$. Thus, the intersection must have an infinite branch b. However, by largeness this is impossible: Choose a maximal antichain A_b of elements of P which do not have b as a branch. Then, by largeness, we find some $T \in [X_0[\tau]]$ having just a finite trace in A_b. In particular, b is no branch of T, so b cannot be in the intersection of all trees which are branches in $X_0[\tau]$. Thus, we conclude that $\dot{N}$ has to be nowhere dense.

By our assumption that any nowhere dense set in the generic extension can be covered by a nowhere dense set from the ground model, we find by ccc a meager set M in the ground model such that 1_P already forces that $\dot{N}$ will be a subset of M. By the Baire category theorem, M cannot contain all branches from X_0, and we find some $T \in [X_0] \setminus M$. This means that 1_P forces that T will not be in the generic filter, which is absurd. □

Every forcing with the Sacks property does not add a new nowhere dense set which cannot be covered by a nowhere dense set from the ground model. Thus, covering nowhere dense sets is a weakening of the Sacks property. It is a proper weakening, e.g. the PP-property and the weak PP-property imply already covering for nowhere dense sets. However, it is still too strong for the random real algebra, as we will show now.

CLAIM 4.10. *Let $\mathcal{B}$ denote the random real algebra. There is a $\mathcal{B}$-name $\dot{N}$ for a nowhere dense set such that its interpretation in the generic extension cannot be covered by a nowhere dense set from the ground model.*

PROOF: We denote by $\dot{r}$ the name for the random real. For $i < \omega$ let $\dot{s}_i$ be the name for the finite sequence $\dot{r} \upharpoonright [2^i, 2^{i+1})$. With these finite sequences we can describe a name for a nowhere dense set as follows: We simply let $\dot{N}$ be the name for the subset of 2^ω containing exactly those reals x who do not contain any s_i. Now assume that there is some $p \in \mathcal{B}$ and some nowhere dense set $\tilde{N}$ such that p forces that $\dot{N}$ is a subset of $\tilde{N}$. Select a sequence $\{t_i \mid i < \omega\}$ such that dom(t_i)=$[2^i, n_i)$ for some $n_i > 2^i$, and for any real x, if x contains some t_i, then $x \notin \tilde{N}$. The dense open set described by this sequence is forced by p to be in the complement of $\dot{N}$, so any t_i must contain some s_j. Now fix some i and let us compute the Lebesgue measure of the set $Z_i = \{x \in 2^\omega \mid \exists j \ x \upharpoonright [2^j, 2^{j+1}) \subseteq t_i\}$. Since the domain of t_i starts with 2^i, we have $\mu(Z_i) < 2^{-i} + 2^{-i-1} + \ldots + 2^{\log_2 n_i} < 2^{-i+1}$. Thus, $\bigcap_{i<\omega} Z_i$ has Lebesgue measure zero, and p forces $\dot{r}$ to be inside, which is not possible. □

In a model of ZFC where the p-ideal dichotoy for $\kappa \leq 2^{\aleph_0}$ holds, there are no Souslin trees [**1**]. Thus, we know by Claim 3.14 that any ccc ω^ω-bounding forcing notion of size less or equal $2^{\aleph_0}$ adds a new real, and it can be decomposed into countably many large X_i which are orthogonal to the p-ideal $\mathcal{I}^- = \{A \in [P]^{\leq\omega} \mid 1_P \Vdash A \cap \dot{G} \text{ is finite}\}$. Therefore, any small set inside X_i has a nonempty meet, and Tree(P) is equivalent to an F_σ-forcing. Thus, we have

COROLLARY 4.11. *Assume that the p-ideal dichotomy holds for $\kappa \leq 2^{\aleph_0}$. If there is a ccc ω^ω-bounding forcing notion which adds no splitting real, then there is a ccc F_σ-forcing which adds no splitting real.*

5. Splitting up ω_1

Now we turn to splitting properties for larger cardinals.

CLAIM 4.12. *Assume CH. Then any ccc ω^ω-bounding forcing notion of size ω_1 adds a subset of ω_1 which splits any uncountable subset of ω_1 from the ground model in two parts.*

PROOF: Let P be a forcing notion of size ω_1 which is ω^ω-bounding and has the ccc, and fix an enumeration $P = \{p_\alpha \mid \alpha < \omega_1\}$ of its elements. In order to define the appropriate name for the new subset of ω_1 we have to prepare P a little bit. In using CH, we construct an uncountable sequence $\{A_\alpha \mid \alpha < \omega_1\}$ of maximal antichains such that

- for $\alpha < \beta$, each $p \in A_\beta$ has a finite trace in A_α,
- $\bigcup_{\alpha<\omega_1} A_\alpha$ is a dense subset of P,
- For any maximal antichain B there is some $\alpha < \omega_1$ such that A_α refines B, i.e. for any $p \in A_\alpha$ there is some $q \in B$ such that $p \leq_P q$.

By CH and ccc, fix an enumeration $\{B_\alpha \mid \alpha < \omega_1\}$ of all maximal antichains in P. Assume that $\{A_\alpha \mid \alpha < \beta\}$ is already constructed such that the first condition is satisfied, such that A_α refines B_α for $\alpha < \beta$. Let $\beta = \{\alpha_i \mid i < \omega\}$ be an enumeration of the elements of β in ordertype ω. Furthermore, the ccc of P allows us to fix for each such α_i an enumeration $A_{\alpha_i} = \{p_{i,j} \mid j < \omega\}$. Then the following is a P-name for a real: $\dot{x} = \{(p_{\alpha_i}, \text{op}(\check{i}, \check{j}) \mid i < \omega\}$ where op$(\check{i}, \check{j})$ denotes the canonical P-name which will be interpreted by any P-generic filter as (i, j). By ω^ω-boundedness, we find a maximal antichain A_β refining B_β and consisting of elements which force already a bound from the ground model to this name $\dot{x}$. In other words, any $p \in A_\beta$ is compatible with only finitely many members of each A_α, and in strengthening one of the elements of A_β, we may assume that for p_β there is some $p \leq_P p_\beta$ inside A_β. Continuing like that ω_1-times, we get a sequence of maximal antichains $\{A_\alpha \mid \alpha < \omega_1\}$ fulfilling the required properties.

Now we claim that we can find for each α some A_β such that for no $\gamma < \alpha$ the condition p_γ has a finite trace in A_β. Else we could find a condition p which has a finite trace in uncountably many A_β. By construction of these antichains, this means that p would have a finite trace in any A_β. However, by construction of $\{A_\alpha \mid \alpha < \omega\}$ we find some α such that each element from A_α is either below p or incompatible to p, and moreover, A_α contains infinitely many elements which are below p. Thus, p could not have a finite trace in A_α, and we find a sequence $\{A_{\alpha_\beta} \mid \beta < \omega_1\}$ such that no p_γ has a finite trace in A_{α_β} for $\gamma < \beta$.

After this preparation we are ready to define the name for a new uncountable subset of ω_1 which meets any uncountable subset of ω_1 in the ground model, and so does its complement.

For any β we will define $C_{\beta,0}$ and $C_{\beta,1}$ as two disjoint baskets of pairwise incompatible elements such that each $C_{\beta,i}$ will contain a condition below p_γ for all $\gamma < \beta$. To this aim we use again the enumeration $\beta = \{\alpha_i \mid i < \omega\}$, and proceed by induction.

To start, choose $p_{0,0}$ and $p_{0,1}$ which are incompatible, below p_{α_0} and below some condition in A_{α_β}, and put $p_{0,0}$ in $C_{\beta,0}$ and $p_{0,1}$ in $C_{\beta,1}$. Assume the construction has been carried out to $i-1$. Since p_{α_i} has no finite trace in A_{α_β}, $\{p_{j,k} \mid j < i, k < 2\}$ is not predense below p_{α_i}. Thus, we can select below p_{α_i} two incompatible conditions $p_{i,0}$ and $p_{i,1}$ which are also below some condition in A_{α_β} and incompatible to the already chosen elements, and put $p_{i,0}$ into C_{β_0} and resp. $p_{i,1}$ into $C_{\beta,1}$. After ω many steps we obtain that for any $\gamma < \beta$ the condition p_γ has a stronger condition in $C_{\beta,0}$ and in $C_{\beta,1}$, and in order to describe a name we fill out $C_{\beta,0}$ with conditions until $C_{\beta,0} \cup C_{\beta,1}$ forms a maximal antichain.

Now we are almost done. Choose a P-name $\dot{Z}$ such that $1_P \Vdash_P \check{\beta} \in \dot{Z} \longleftrightarrow C_{\beta,1} \cap \dot{G} \neq \emptyset$ where $\dot{G}$ denotes the canonical P-name for the generic filter. Clearly, no $p \in P$ can force for a certain uncountable subset of ω_1 in the ground model that it is included in $\dot{Z}$ or disjoint from $\dot{Z}$, which finishes the proof. $\square$

6. CCC forcings and the cardinal invariant $\mathfrak{t}$

Any ccc ω^ω-bounding forcing notion adding a new real has something in common with the measure algebra: Under CH, such a forcing diagonalizes some tower (see [**6**]). Recall that a tower T consists of infinite subsets of ω which are wellordered by $\supseteq^*$, and a tower T is diagonalized by some infinite $A \subseteq \omega$ if A is almost included in any element of T. Thus, a ccc ω^ω-bounding forcing notion which adds a new real might be no candidate if one does not want to increase the cardinal invariant $\mathfrak{t}$ which is the size of the smallest tower which cannot be diagonalized.

CLAIM 4.13. *Under CH any ccc ω^ω-bounding forcing notion adding a new real diagonalizes a tower which was not diagonizable in the ground model.*

PROOF: To see this, fix such a forcing notion P, and let $\dot{r}$ be a P-name for a new real, i.e. a new element of 2^ω. Furthermore, define $\mathcal{I} = \{A \subseteq 2^{<\omega} \mid 1_P \Vdash \dot{r} \cap A \text{ is finite }\}$. $\mathcal{I}$ is clearly an ideal, and we can show that it is even a p-ideal:
Let $\{A_i \mid i < \omega\}$ be a countable collection of elements from $\mathcal{I}$. We have to find some $A \in \mathcal{I}$ which almost contains any of these A_i. To this aim, fix for each i an enumeration $A_i = \{p_{i,j} \mid j < \omega\}$ and consider the following name $\dot{x}$ for a real: $\dot{x}(i)$ is the smallest integer m such that for each $j > m$ the forcing condition $p_{i,j}$ is not in the generic filter G. By ccc and ω^ω-boundedness we find in the ground model some $f \in \omega^\omega$ such that $1_P \Vdash \dot{x} \leq^* f$. Clearly, $A = \{p_{i,j} \mid i < \omega, j > f(i)\}$ is an element from the ideal $\mathcal{I}$, and each A_i is almost contained in it.

Using CH, we can now construct the tower which will get diagonalized by P. Let $\{Z_\alpha \mid \alpha < \omega_1\}$ be an enumeration of all infinite elements of $\mathcal{I}$. Assume that for some $\beta < \omega_1$ we have constructed $\{A_\alpha \mid \alpha < \beta\}$ such that each $A_\alpha \in \mathcal{I}$, $A_\alpha \subseteq^* A_\gamma$ for each $\alpha < \gamma$, and $Z_\alpha \subseteq A_\alpha$. Since $\mathcal{I}$ is a p-ideal, we find some A_β containing Z_β which almost contains any A_α for $\alpha < \beta$. Continuing like this ω_1-times, we obtain a tower consisting of the complements $\{A_\alpha^c \mid \alpha < \omega_1\}$. Since any infinite subset of $2^{<\omega}$ contains an infinite chain or an infinite antichain with respect to extension, the p-ideal $\mathcal{I}$ is dense in $2^{<\omega}$, and therefore, the tower is maximal. Clearly, $\dot{r}[G]$ diagonalizes this tower, because for every α, $\{\dot{r} \restriction n \mid n < \omega\}$ is forced to be almost included in A_α. □

The cardinal invariant $\mathfrak{t}$ has some influence on the question whether any forcing destroying all ultrafilters on ω adds a splitting real.

CLAIM 4.14. *Let P be a ccc forcing with $|P| \leq \mathfrak{t} = 2^{\aleph_0}$. If P adds no splitting real, then there is a p-$2^{\aleph_0}$-point which still generates an ultrafilter in the generic extension.*

PROOF: By ccc, there are at most $2^{\aleph_0}$-many canonical P-names for subsets of ω. Thus we can select an enumeration $\{\dot{Z}_\alpha \mid \alpha < 2^{\aleph_0}\}$ of all these names, an select an enumeration $\{A_\alpha \mid \alpha < 2^{\aleph_0}\}$ of all infinite subsets of ω. By recursion, we build a p-$2^{\aleph_0}$-point $\mathcal{U}$ as follows:

To start with, consider the name $\dot{Z}_0 \cap A_0$. Since no splitting real is added, we find for any $p \in P$ some $q \leq_P p$ and an infinite subset $B_q \subseteq A$ auch that q either forces that B_q will be included in $\dot{Z}_0$ or it forces that B_q and $\dot{Z}_0$ will be disjoint. By ccc, we find some infinite $B_0 \subseteq A_0$ such that 1_P forces that B_0 will be either almost included in $\dot{Z}_0$ or $B_0 \cap \dot{Z}_0$ will be finite. We choose B_0 to be in $\mathcal{U}$.

On stage $\alpha < 2^{\aleph_0}$ we constructed already a tower $B_0 \supseteq^* B_1 \supseteq^* B_2 \ldots \supseteq^* B_\beta \ldots$ of length α. By our assumption, $\alpha < \mathfrak{t}$, and therefore we find some infinite B which diagonalizes the tower. Moreover, we can choose B to be a subset of A_α or disjoint from it. Now we repeat the first step with $\dot{Z}_\alpha \cap B$ and obtain some infinite $B_\alpha \subseteq B$ which is forced by 1_P to be either almost included in $\dot{Z}_\alpha$ or almost disjoint from it.

Clearly, $\{B_\alpha \mid \alpha < 2^{\aleph_0}\}$ generates a p-$2^{\aleph_0}$-point $\mathcal{U}$, and $\mathcal{U}$ will still generate a p-$2^{\aleph_0}$-point in the generic extension produced by P. $\square$

COROLLARY 4.15. *Under the assumptions of Claim 4.14, any ccc forcing of size at most* $\mathfrak{t} = 2^{\aleph_0}$ *adds a splitting real iff it destroys all ultrafilters iff it destroys all p-*$2^{\aleph_0}$*-points.*

Quite trivially, CH implies $\mathfrak{t} = 2^{\aleph_0}$. However, it is also a consequence of MA, since Mathias forcing relatively to a tower has the ccc and adds a diagonalisation. Thus, $\mathfrak{t} = 2^{\aleph_0}$ is consistent with construction principles like $\diamondsuit$ or CCC($\mathcal{S}$). If $\mathfrak{t} = 2^{\aleph_0}$, then σ-centered($\mathcal{L}$) holds where $\mathcal{L}$ stands for the Laver forcing, see [**10**]. Thus, in such a model there are ccc forcings adding neither a Random real nor a Cohen real.

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