A note on Tsirelson type ideals

Boban Velickovic

Introduction

Given Borel equivalence relations E and F on Polish spaces X and Y respectively we say that E is *Borel reducible* to F and write $E \leq_{Bor} F$ if there is a Borel function $f : X \to Y$ such that for every x and y in X

$$xEy \text{ iff } f(x)Ff(y).$$

For such f let $f^* : X/E \to Y/F$ be defined by $f^*([x]_E) = [f(x)]_F$. Then f^* is an injection of X/E to Y/F which has a Borel lifting f. We write

$$E \sim_{Bor} F \text{ iff } E \leq_{Bor} F \& F \leq_{Bor} E.$$

By an ideal $\mathcal{I}$ on $\mathbb{N}$ we mean an ideal of subsets of $\mathbb{N}$ which is *nontrivial*, i.e. $\mathbb{N} \notin \mathcal{I}$, and *free*, i.e. $\{n\} \in \mathcal{I}$, for all $n \in \mathbb{N}$. We say that $\mathcal{I}$ is Borel if it is a Borel subset of $\mathcal{P}(\mathbb{N})$ in the usual product topology. Given a Borel ideal $\mathcal{I}$ on $\mathbb{N}$ we define an equivalence relation $E_{\mathcal{I}}$ on $\mathcal{P}(\mathbb{N})$ by letting:

$$XE_{\mathcal{I}}Y \text{ if and only if } X\Delta Y \in \mathcal{I}$$

Finally, we write $\mathcal{I} \leq_{Bor} \mathcal{J}$ iff $E_{\mathcal{I}} \leq_{Bor} E_{\mathcal{J}}$. The class $(\mathcal{E}, \leq_{Bor})$ of all Borel ideals with this notion of reducibility was studied by several authors. Here we identify two ideals which are $\sim_{Bor}$-equivalent. In [LV] Louveau and the author showed that this structure is very rich by embedding into it the partial ordering $(\mathcal{P}(\mathbb{N}), \subseteq^*)$ (where $X \subseteq^* Y$ iff $X \setminus Y$ is finite). The ideals constructed in this proof are all $F_{\sigma\delta}$ and P-ideals (recall that an ideal $\mathcal{I}$ is a P-ideal iff for every sequence $\{A_n : n \in \mathbb{N}\}$ of members of $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subseteq^* A$, for all n). The interest of looking for P-ideals is that in this case by a result of Solecki [So] $(\mathcal{I}, \Delta)$ is a Polish group under a suitable topology. The construction from [LV] was later modified by Mazur [Ma1] to obtain F_σ ideals. However Mazur's ideals are not P-ideals.

By $\leq_{RK}$ we denote the Rudin-Keisler ordering on ideals, i.e.

$$\mathcal{I} \leq_{RK} \mathcal{J} \text{ iff there is } f : \mathbb{N} \to \mathbb{N}(X \in \mathcal{I} \leftrightarrow f^{-1}(X) \in \mathcal{J}).$$

The Rudin-Blass ordering $\leq_{RB}$ is obtained by requiring in the above definition that f be finite-to-one. It is clear that $\mathcal{I} \leq_{RB} \mathcal{J}$ implies that $\mathcal{I} \leq_{RK} \mathcal{J}$ and this in turn implies that $\mathcal{I} \leq_{Bor} \mathcal{J}$. It is an open question whether $\mathcal{I} \leq_{Bor} \mathcal{J}$ iff there is a set

$A \in \mathcal{J}^+$ such that $\mathcal{I} \leq_{RB} \mathcal{J} \restriction A$, the restriction of $\mathcal{J}$ to $\mathcal{P}(A)$. In all known cases, this seems to be true. Mathias [Mat], Jalali-Naini [JN], and Talagrand [Ta] showed that FIN $\leq_{RB} \mathcal{I}$, for any Borel (in fact, Baire measurable) ideal $\mathcal{I}$, where FIN is the ideal of finite subsets of $\mathbb{N}$. Thus, in a way, the 'Borel cardinality' of $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$ is the smallest among all $\mathcal{P}(\mathcal{I})/\mathcal{I}$, for $\mathcal{I}$ a Borel ideal.

Recently, Kechris [Ke2] addressed the issue of finding minimal ideals above FIN under $\leq_{Bor}$. He was motivated by the well-known dichotomy results on Borel equivalence relations. He identified two ideals related to FIN denoted by $\emptyset \times \mathrm{FIN}$ and $\mathrm{FIN} \times \emptyset$ (in fact, these ideals are defined on $\mathbb{N}^2$ but they can be moved to $\mathbb{N}$ by some fixed bijection). Define:

$$X \in \emptyset \times \mathrm{FIN} \text{ iff } \forall m(\{n : (m,n) \in X\} \text{ is finite}),$$

$$X \in \mathrm{FIN} \times \emptyset \text{ iff } \exists m(X \subseteq m \times \mathbb{N}).$$

Thus, it is known and fairly easy to see that $\emptyset \times \mathrm{FIN}$ and $\mathrm{FIN} \times \emptyset$ are incomparable under $\leq_{Bor}$ and strictly above FIN (see [Ke2] for complete references). Say that $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* iff there is a permutation π of $\mathbb{N}$ such that $X \in \mathcal{I}$ iff $\pi(X) \in \mathcal{J}$. Finally say that $\mathcal{I}$ is a *trivial variations* of FIN iff there is an infinite set A such that $\mathcal{I} = \{X \subseteq \mathbb{N} : X \cap A \text{ is finite }\}$. Kechris then showed that both $\emptyset \times \mathrm{FIN}$ and $\mathrm{FIN} \times \emptyset$ are minimal above FIN, in the following strong sense.

Theorem 1 ([Ke2]) *If $\mathcal{I}$ is a Borel ideal and $\mathcal{I} \leq_{Bor} \emptyset \times FIN$ ($FIN \times \emptyset$, respectively) then it is either isomorphic to $\emptyset \times FIN$ ($FIN \times \emptyset$, respectively) or it is a trivial variation of FIN.*

By another result of Solecki [So], if $\mathcal{I}$ is a Borel ideal then $\mathrm{FIN} \times \emptyset \leq_{RB} \mathcal{I}$ iff $\mathcal{I}$ is not a P-ideal. Moreover, if $\mathcal{I}$ is a P-ideal then $\emptyset \times \mathrm{FIN} \leq_{RB} \mathcal{I}$ iff $\mathcal{I}$ is not F_σ. Thus, any ideal which is incomparable with both $\mathrm{FIN} \times \emptyset$ and $\emptyset \times \mathrm{FIN}$ is an F_σ P-ideal. One way of obtaining such ideals is from classical Banach spaces. Fix any $(\alpha_n)_n \in c_0^+ \setminus l_1$, where c_0^+ is the space of all *nonnegative* sequences of reals converging to zero, for concreteness let us say $\alpha_n = \frac{1}{n+1}$, for all n. Define the ideal $\mathcal{I}_0$ by:

$$X \in \mathcal{I}_0 \quad \text{iff} \quad \sum_{n \in X} \alpha_n < \infty.$$

Then clearly, $\mathcal{I}_0$ is an F_σ P-ideal. It is known that $\mathcal{I}_0$ is incomparable in the sense of $\leq_{Bor}$ with both $\mathrm{FIN} \times \emptyset$ and $\emptyset \times \mathrm{FIN}$ (this follows from results of Kechris-Louveau [KL], Hjorth [Hj], and has also been shown independently by Mazur [Ma2]). Moreover, Hjorth showed (unpublished) that if $\mathcal{I} \leq_{Bor} \mathcal{I}_0$, then either $\mathcal{I} \sim_{Bor} \mathcal{I}_0$, or else $\mathcal{I}$ is a trivial variation of FIN. In the light of these results Kechris conjectured that the following trichotomy holds.

Conjecture 1 *If $\mathcal{I}$ is any Borel ideal on $\mathbb{N}$ and $FIN <_{Bor} \mathcal{I}$ then either $FIN \times \emptyset \leq_{Bor} \mathcal{I}$ or $\emptyset \times FIN \leq_{Bor} \mathcal{I}$ or $\mathcal{I}_0 \leq_{Bor} \mathcal{I}$.*

As noted in [Ke2], this is equivalent to a conjecture of Mazur [Ma2] which asserts that if $\mathcal{I}$ is an F_σ ideal with FIN $<_{Bor} \mathcal{I}$, then FIN $\times \emptyset \leq_{Bor} \mathcal{I}$ or $\mathcal{I}_0 \leq_{Bor} \mathcal{I}$. In this note we disprove this conjecture by showing that an ideal associated to the Tsirelson space provides a counterexample. This is a Banach space which does not contain an isomorphic copy of the classical Banach spaces c_0, and l_p, for $1 \leq p < \infty$. In fact, the picture seems to be much more complicated then suggested by the above conjecture. Thus, it seems that there are no minimal (in the sense of $\leq_{Bor}$) ideals below the ideal $\mathcal{I}_T$ constructed in the next section, but that on the other hand $(\mathcal{P}(\mathbb{N}) \subseteq^*)$ can be embedded in $(\mathcal{E}, \leq_{Bor})$ below $\mathcal{I}_T$, etc. We plan to present these and other related results in a later paper. There is a large literature on Tsirelson's and other related Banach spaces. For a good if somewhat outdated survey, we refer the reader to [CS] and for a more recent survey to [OS].

1 Tsirelson's space

We now present the Figiel-Johnson version of Tsirelson's space (see [FJ] or [CS]). This is actually the dual of the original space constructed by Tsirelson. We start with some definitions.

(a) If E, F are finite nonempty subsets of $\mathbb{N}$ we let $E \leq F$ iff $\max(E) \leq \min(F)$. We write $n \leq E$ instead of $\{n\} \leq E$. Similarly, we define $E < F$, etc. We say that a sequence $\{E_i\}_{i=1}^k$ is *admissible* if $k \leq E_1 < E_2 < \ldots < E_k$. In general, given an increasing function $h : \mathbb{N} \to \mathbb{N}$ and an integer k we say that a sequence $\{E_i\}_{i=1}^l$ is (h,k)-*admissible* if $k \leq E_1 < E_2 < \ldots < E_l$, and $l \leq h(k)$.

(b) Let $\mathbb{R}^{<\omega}$ denote the vector space of all real scalar sequences of finite support and let $\{t_n\}_{n=1}^\infty$ be the canonical unit vector basis of $\mathbb{R}^{<\omega}$. Given a vector $x = \sum_n a_n t_n \in \mathbb{R}^{<\omega}$ we define $Ex = \sum_{n \in E} a_n t_n$, the projection of x to the coordinates in E.

(c) We define inductively a sequence of norms $(||\cdot||_m)_{m=0}^\infty$ on $\mathbb{R}^{<\omega}$ as follows. Given $x = \sum_n a_n t_n \in \mathbb{R}^{<\omega}$ let:

$$||x||_0 = \max_n |a_n|$$

For $m \geq 0$, we set:

$$||x||_{m+1} = \max\{||x||_m, \frac{1}{2}\sup[\sum_{j=1}^k ||E_j x||_m] : \{E_j\}_{j=1}^k \text{ is admissible}\}.$$

(d) One verifies that the $||\cdot||_m$ are norms on $\mathbb{R}^{<\omega}$, they increase with m, and that for all m

$$||x||_m \leq \sum_n |a_n|.$$

Thus, $\lim_m ||x||_m$ exists and is majorized by the l_1-norm of x. Therefore setting

$$||x|| = \lim_m ||x||_m$$

definies a norm on $\mathbb{R}^{<\omega}$.

(e) Finally, Tsirelson's space T is the $||\cdot||$ completion of $\mathbb{R}^{<\omega}$.

Recall that $\{t_n\}_{n=1}^{\infty}$ is the canonical unit vector basis of $\mathbb{R}^{<\omega}$. A *block* is a vector y of the form $\sum_{n\in I} a_n t_n$ for some (finite) interval I in $\mathbb{N}$. We now record some basic properties of the space T (cf. [CS, Proposition I.2]).

Proposition 1 *1) The sequence $\{t_n\}_{n=1}^{\infty}$ is a normalized 1-unconditional Schauder basis for the space T.*

2) For each $x = \sum_n a_n t_n \in T$,

$$||x|| = \max\{\max_n |a_n|, \frac{1}{2}\sup \sum_{j=1}^{k} ||E_j x|| : \{E_j\}_{j=1}^{k} \textit{ is admissible}\}.$$

3) For any $k \in \mathbb{N}$, and any k normalized blocks $\{y_i\}_{i=1}^{k}$ such that for some integers $k-1 \leq p_1 < p_2 < \ldots < p_{k+1}$ y_i is the linear combination of the base vectors t_n, for $p_i < n \leq p_{i+1}$ we have:

$$\frac{1}{2}\sum_{i=1}^{k} |b_i| \leq ||\sum_{i=1}^{k} b_i y_i|| \leq \sum_{i=1}^{k} |b_i|,$$

for all scalars $\{b_i\}_{i=1}^{k}$.

We are now ready to define a Tsirelson type ideal $\mathcal{I}_T$. Fix a vector $\alpha = \sum_n \alpha_n t_n \in c_0^+ \setminus T$, for instance we could take again $\alpha_n = \frac{1}{n+1}$. For a finite subset E of $\mathbb{N}$ let us define $\tau(E) = ||E\alpha||$, and for an arbitrary $X \subseteq \mathbb{N}$ let

$$\tau(X) = \sup_n \tau(X \cap n).$$

It is now clear from Proposition 1 that τ is a lower semi-continuous submeasure on $\mathcal{P}(\mathbb{N})$ and that for any X

$$\tau(X) < \infty \text{ iff } \lim_{n\to\infty} \tau(X \setminus n) = 0.$$

It follows now that the ideal

$$\mathcal{I}_T = \{X : \tau(X) < \infty\}$$

is an F_σ P-ideal. The main result of this note is the following.

Theorem 2 *$\mathcal{I}_T$ and $\mathcal{I}_0$ are incomparable under $\leq_{Bor}$.*

PROOF: It suffices to show that $\mathcal{I}_0 \not\leq_{Bor} \mathcal{I}_T$. Assume otherwise and fix a Borel function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ witnessing that $\mathcal{I}_0 \leq_{Bor} \mathcal{I}_T$. We first prove the following.

Lemma 1 *There is an infinite increasing sequence $F_0 < F_1 < F_2 < \ldots$ of finite sets and a sequence $(\beta_n)_n$ of positive reals converging to zero such that for every $X \subseteq \mathbb{N}$*

$$\tau(\bigcup_{n\in X} F_n) < \infty \qquad \textit{iff} \qquad \sum_{n\in X} \beta_n < \infty.$$

PROOF: First we show that we may assume that f is continuous. To this end, fix a dense G_δ set G such that $f \restriction G$ is continuous. Then, by a standard fact (see [Ke1, §8.9]), there is a partition $\mathbb{N} = X_0 \cup X_1$ and sets $Z_0 \subseteq X_0$, $Z_1 \subseteq X_1$ such that, for any $i \in \{0,1\}$, if $X \cap X_i = Z_i$ then $X \in G$. Fix now i such that $X_i \in \mathcal{I}_0^+$. It now follows that the function $g : \mathcal{P}(X_i) \to \mathcal{P}(\mathbb{N})$ defined by:

$$g(X) = f(X \cup Z_{1-i})$$

is continuous and witnesses $\mathcal{I}_0 \restriction X_i \leq_{Bor} \mathcal{I}_T$. Moreover, it is easily seen that for any $X \in \mathcal{I}_0^+$ we have $\mathcal{I}_0 \leq_{RB} \mathcal{I}_0 \restriction X$. Therefore, by composing we can obtain a continuous function witnessing $\mathcal{I}_0 \leq_{Bor} \mathcal{I}_T$.

To simplify notation let us now assume that f is already continuous. Now, following [Ve], we can find a strictly increasing sequence $0 = n_0 < n_1 < n_2 < \ldots$ of integers, sets $Z_i \subseteq [n_i, n_{i+1})$, and functions $f_i : \mathcal{P}(n_i) \to \mathcal{P}(n_i)$ such that:

(a) for every $X \subseteq \mathbb{N}$, if $X \cap [n_i, n_{i+1}) = Z_i$ then $f(X) \cap n_i = f_i(X \cap n_i)$,

(b) for every $X, Y \subseteq \mathbb{N}$, if $X \cap [n_i, n_{i+1}) = Y \cap [n_i, n_{i+1}) = Z_i$ and $X \Delta Y \subseteq n_i$ then

$$\tau((f(X)\Delta f(Y)) \setminus n_{i+1}) \leq \frac{1}{2^{i+1}}.$$

Now for $\epsilon = 0, 1, 2$, let:

$$X_\epsilon = \bigcup\{[n_i, n_{i+1}) : i \equiv \epsilon \bmod 3\},$$

$$W_\epsilon = \bigcup\{Z_\epsilon : i \equiv \epsilon \bmod 3\}.$$

Let us assume, for concreteness, that $X_0 \notin \mathcal{I}_0$ and define a function $g : \mathcal{P}(X_0) \to \mathcal{P}(\mathbb{N})$ by:

$$g(X) = f(X \cup W_1 \cup W_2)\Delta f(W_1 \cup W_2).$$

Then g is continuous and witnesses that $\mathcal{I} \restriction X_0 \leq_{Bor} \mathcal{I}_T$. Now, for each i, define a function $g_i : \mathcal{P}([n_{3i}, n_{3i+1})) \to \mathcal{P}([n_{3i-1}, n_{3i+2}))$ by:

$$g_i(X) = g(X) \cap [n_{3i-1}, n_{3i+2}).$$

and let

$$g^*(X) = \bigcup_i g_i(X \cap [n_{3i}, n_{3i+1})).$$

Note that (a) and (b) imply that for every $X \subseteq X_0$

$$\tau(g(X)\Delta g^*(X)) \leq 1.$$

Now since g witnesses that $\mathcal{I} \restriction X_0 \leq_{Bor} \mathcal{I}_T$ and $g(\emptyset) = \emptyset$ it follows that for any $X \subseteq X_0$:

$$X \in \mathcal{I}_0 \quad \text{iff} \quad g^*(X) \in \mathcal{I}_T.$$

Since $X_0 \notin \mathcal{I}_0$ we can find subsets B_i of $[n_{3i}, n_{3i+1})$ such that letting

$$\beta_i = \sum_{k \in B_i} \frac{1}{k+1}$$

we have that $\lim_{i\to\infty} \beta_i = 0$ and $\sum_{i=0}^{\infty} \beta_i = \infty$. Finally, let $F_i = g_i(B_i)$, for each i. Then the sequences $(\beta_i)_i$ and $(F_i)_i$ are as required.

□

Let us fix for the remainder of the proof sequences $(F_n)_n$ and $(\beta_n)_n$ as in Lemma 1. For a subset X of $\mathbb{N}$ let us define:

$$\varphi(X) = \sum_{n \in X} \beta_n.$$

Then we have that for every such X

$$\varphi(X) < \infty \quad \text{iff} \quad \tau(\bigcup_{n \in X} F_n) < \infty. \tag{1}$$

Given a finite subset a of $\mathbb{N}$ let $E_a = \bigcup_{n \in a} F_n$. For a sequence $S = \{a_n\}_{n=1}^{\infty}$ of finite subsets of ω let $FU(S)$ denote the family of finite unions of members of S. Call such an S *acceptable* iff $a_1 < a_2 < \ldots$ and

$$\lim_{n\to\infty} \tau(E_{a_n}) = 0 \quad \text{and} \quad \tau(\bigcup_{n=1}^{\infty} E_{a_n}) = \infty$$

Given an acceptable sequence $S = \{a_n\}_{n=1}^{\infty}$ let us define:

$$K(S) = \sup_n \frac{\tau(E_{a_n})}{\varphi(a_n)},$$

Note that if $S^* \subseteq FU(S)$ is also acceptable then $K(S^*) \leq K(S)$. Finally let

$$K = \inf\{K(S) : S \text{ acceptable }\}.$$

We first prove the following.

Lemma 2 *$K = 0$ or $K = \infty$.*

PROOF: We show that if there is an acceptable S such that $K(S)$ is finite then there is another acceptable $S^* \subseteq FU(S)$ such that

$$K(S^*) \leq \frac{119}{120} K(S).$$

The proof of this follows closely the proof of Lemma 2.1 of [FJ] or Proposition 1.3 of [CS]. To begin, let us fix an acceptable $S = \{a_n\}_{n=1}^{\infty}$ such that $K(S)$ is finite. Note that since $\tau(\bigcup_{n=1}^{\infty} E_{a_n}) = \infty$ and $\lim_{n\to\infty} \tau(E_{a_n}) = 0$ we know that there exists an integer N such that for all $n \geq N$ and every integer k we can find some $b \in FU(S)$ such that $k \leq E_b$ and $\frac{15}{16n} \leq \tau(E_b) \leq \frac{17}{16n}$.

Claim 1 *For every $n \geq N$ and k there is $b \in FU(S)$ such that $k \leq E_b$,*

$$\tau(E_b) \leq \frac{119}{64n} \quad \text{and} \quad \varphi(b) \geq \frac{30}{16nK(S)}.$$

Note that using this claim we can easily produce an increasing sequence $b_1 < b_2 < \dots$ of members of $FU(S)$ such that

$$\frac{\tau(E_{b_n})}{\varphi(b_n)} \leq \frac{119}{120} K(S),$$

$$\sum_{n=1}^{\infty} \varphi(b_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \tau(E_{b_n}) = 0.$$

Then we will have that $S^* = \{b_n\}_{n=1}^{\infty}$ is acceptable and $K(S^*) \leq \frac{119}{120} K(S)$, as desired.

PROOF of Claim 1. Fix $n \geq N$ and k. First find some $b_0 \in FU(S)$ such that $k \leq E_{b_0}$ and

$$\frac{15}{16n} \leq \tau(E_{b_0}) \leq \frac{17}{16n}.$$

Set $n_0 = \max E_{b_0}$. Now let $r = 2n_0$ and find sets $b_i \in FU(S)$, for $1 \leq i \leq r$, such that $b_0 < b_1 < \dots b_r$, and for every $1 \leq i \leq r$:

$$\frac{15}{16nr} \leq \tau(E_{b_i}) \leq \frac{17}{16nr}.$$

Finally, let $b' = \bigcup_{i=1}^{r} b_i$ and $b = b_0 \cup b'$. We claim that b is as required.

Consider an admissible sequence $l \leq H_1 < H_2 < \dots < H_l$, for some l.
If $l > n_0$ then

$$\sum_{j=1}^{l} \tau(H_j \cap E_b) = \sum_{j=1}^{l} \tau(H_j \cap E_{b'}) \leq 2 \sum_{j=1}^{l} \tau(E_{b_j}) \leq \frac{34}{16n}.$$

If $l \leq n_0$ we define:

$$A = \{i : H_j \cap E_{b_i} \neq \emptyset, \text{ for at least two values of } j\}.$$

$$B = \{i : H_j \cap E_{b_i} \neq \emptyset, \text{ for at most one value of } j\}.$$

Then since A has at most l elements we have:

$$\begin{aligned}
\sum_{j=1}^{l} \tau(H_j \cap E_b) &\leq \sum_{j=1}^{l} \tau(H_j \cap E_{b_0}) + (\sum_{i \in A} \sum_{j=1}^{l} + \sum_{i \in B} \sum_{j=1}^{l}) \tau(H_j \cap E_{b_i}) \\
&\leq 2\tau(E_{b_0}) + 2 \sum_{i \in A} \tau(E_{b_i}) + \sum_{i \in B} \tau(E_{b_i}) \\
&\leq \frac{34}{16n} + (2l + r - l) \frac{17}{16nr} \\
&\leq \frac{17}{16n}(2 + \frac{r+l}{r}) \\
&\leq \frac{17}{16n}(3 + \frac{n_0}{r}) = \frac{119}{32n}
\end{aligned}$$

From these two inequalities it now follows that

$$\tau(E_b) = \sup\{\frac{1}{2} \sum_{j=1}^{l} \tau(H_j \cap E_b) : \{H_j\}_{j=1}^{l} \text{is admissible }\} \leq \frac{119}{64n}$$

On the other hand, notice that

$$\varphi(b) = \sum_{i=0}^{r} \varphi(b_i) \geq \frac{1}{K(S)} \sum_{i=0}^{r} \tau(E_{b_i}) \geq \frac{30}{16nK(S)}.$$

This completes the proof of Claim 1 and Lemma 2.

□

We now show that (1) fails in both cases of Lemma 2, thus arriving at a contradiction.

Case 1 $K = \infty$. In this case we first show the following.

Claim 2 *For every $N \in \mathbb{N}$ and $\epsilon > 0$ there is a finite set of integers a such that $N \leq E_a$, $\tau(E_a) \geq 1$, and $\varphi(a) < \epsilon$.*

PROOF: Suppose not and fix N and $\epsilon > 0$ for which Claim 2 is false.

Subclaim For every $k \geq N$ there is N_k such that for every a if $N_k \leq E_a$ and $\frac{2}{k} \leq \tau(E_a) < \frac{4}{k}$ then $\varphi(a) \geq \frac{\epsilon}{k}$.

PROOF of Subclaim: Otherwise we could pick sets $\{a_i\}_{i=1}^{k}$ such that $\max\{k, N\} \leq E_{a_1} < E_{a_2} < \ldots E_{a_k}$, $\frac{2}{k} \leq \tau(E_{a_i}) < \frac{4}{k}$ and $\varphi(a_i) < \frac{\epsilon}{k}$, for $i = 1, \ldots, k$. But then, since the sequence $\{E_{a_i}\}_{i=1}^{k}$ is admissible, by setting $a = \bigcup_{i=1}^{k} a_i$ and using Proposition 1 we have:

$$\tau(E_a) \geq \frac{1}{2} \sum_{i=1}^{k} \tau(E_{a_i}) \geq \frac{1}{2} k \frac{2}{k} = 1.$$

On the other hand,

$$\varphi(a) = \sum_{i=1}^{k} \varphi(a_i) < k\frac{\epsilon}{k} = \epsilon.$$

Thus we have $N \leq E_a$, $\tau(E_a) \geq 1$, and $\varphi(a) < \epsilon$, contradiction.

$\square$

Now, by the Subclaim, we can produce an infinite increasing sequence $S = \{a_k\}_{k=1}^{\infty}$ of finite subsets of $\mathbb{N}$ such that $\frac{2}{k} \leq \tau(E_{a_k}) \leq \frac{4}{k}$ and $\varphi(a_k) \geq \frac{\epsilon}{k}$. It follows that S is acceptable and that $K(S) \leq \frac{4}{\epsilon}$, contradicting the fact that $K = \infty$.

$\square$

Now by using Claim 1 and Proposition 1 again, we can easily produce and infinite set X such that $\varphi(X) < \infty$ and $\tau(\bigcup_{n\in X} F_n) = \infty$. Contradiction.

Case 2 $K = 0$. In this case we first show that for every integer N and $\epsilon > 0$ there is a finite subset a of $\mathbb{N}$ such that $N \leq E_a$, $\tau(E_a) < \epsilon$, and $\varphi(a) \geq 1$. To see this, fix an acceptable $S = \{a_n\}_{n=1}^{\infty}$ such that $K(S) < \frac{\epsilon}{2}$. Moreover, by thinning out if necessary, we may assume that $N \leq E_{a_1}$ and that $\varphi(a_n) \leq 1$, for all n. Now there is an integer k such that letting $a = \bigcup_{i=1}^{k} a_i$ we have $1 \leq \varphi(a) \leq 2$. On the other hand, using the fact that τ is subadditive and that $\frac{\tau(E_{a_i})}{\varphi(a_i)} < \frac{\epsilon}{2}$, for every i, we have that $\tau(E_a) < \epsilon$.

Now we easily produce an infinite set X such that $\varphi(X) = \infty$, but $\tau(\bigcup_{n\in X} F_n) < \infty$. Contradiction.

$\square$

References

[CS] P.G. Casazza and T. J. Shura, *Tsirelson's Space*, Lecture Notes in Mathematics, vol. 1363, Springer-Verlag, (1989)

[FJ] T. Fiegel and W.B. Johnson, "A uniformly convex Banach space which contains no l_p", *Compositio Math.*, vol.*29*(2), (1974), pp. 179-190

[Hj] G. Hjorth, "Actions of S_∞," manuscript

[JN] S.- Jalali-Naini , "The monotone subsets of Cantor space, filters and descriptive set theory", *Doctoral Dissertation*, Oxford 1976

[Ke1] A. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, 1995

[Ke2] A. Kechris, "Rigidity properties of Borel ideals on the integers", preprint

[KL] A. Kechris and A. Louveau, "The structure of hypersmooth Borel equivalence relations", *J. Amer. Math. Soc.*, vol. *10*, (1997), pp. 215-242.

[LV] A. Louveau and B. Velickovic, "A note on Borel equivalence relations", *Proc. Amer. Math. Soc.*, vol. 120, (1994), pp.255-259

[Ma1] K. Mazur, "A modification of Louveau and Velickovic construction for F_σ-ideals, preprint

[Ma2] K. Mazur, "Towards a dichotomy for F_σ-ideals", preprint

[Mat] A.R.D. Mathias, "A remark on rare filters", in *Coll. Math. Soc. Janos Bolyai*, vol. *10*, *Infinite and Finite Sets*, (A. Hajnal et al. ed.), North Holland, (1975)

[OS] E. Odell and Th. Schlumprecht "Distortion and stabilized structure in Banach spaces; new geometric phenomena for Banach and Hilbert spaces", in *Proc. Int. Congress Math.*, Zurich, Switzerland, (1995), pp. 955-965

[So] S. Solecki, "Analytic ideals", *Bull. Assoc. Symb. Logic*, vol.*2*(1996), pp. 339-348

[Ta] M. Talagrand, "Compacts de fonctions mesurables et filtres nonmesurables", *Bull. Studia Math.*, vol.*67*(1980), pp. 13-43

[Ve] B. Velickovic, "Definable automorphisms of $\mathcal{P}(\omega)/\mathrm{FIN}$ ", *Proc. Amer. Math. Society*, vol. *96* (1986), 130-135.

B. Velickovic
UFR de Math
Universite de Paris 7
2 Place Jussieu
75251 Paris, FRANCE