Chapter 1

On the arithmetization of real fields with exponentiation

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Shepherdson proved that a discrete unitary commutative semi-ring A^+ satisfies IE_0 (induction scheme restricted to quantifier free formulas) iff A is integral part of a real closed field; and Berarducci asked about extensions of this criterion when exponentiation is added to the language of rings and fields. Let T range over axiom systems for ordered fields with exponentiation; for three values of T we provide a theory $\llcorner T \lrcorner$ in the language of rings plus exponentiation such that the models (A, exp_A) of $\llcorner T \lrcorner$ are all integral parts A of models M of T with A^+ closed under exp_M and $exp_A = exp_M|A^+$. Namely T=EXP, the basic theory of real exponential fields; T=EXP+ the Rolle and the intermediate value properties for all exp-polynomials; and $T = T_{exp}$, the complete theory of the field of reals with exponentiation.

Introduction

Let R be a model of the axioms OF of ordered field; an **integral part** of R is a subring A such that for every element x of the field there is a unique element $\llcorner x \lrcorner$ of the ring such that $\llcorner x \lrcorner < x \leq \llcorner x \lrcorner + 1$; $\llcorner x \lrcorner$ is called the integral part of x (in A). In general A is not unique; in fact as soon as R is real closed and non archimedean the number of integral parts of R is infinite and large. Nevertheless we sometimes write $A = \llcorner R \lrcorner$ to mean that A is an integral part of R. Note that A then satisfies the axioms $DUCR + ED$ of discrete unitary commutative ring + euclidean division (for $\llcorner x/y \lrcorner$ is the euclidean quotient of x by y). The converse is true: every model A of $DUCR + ED$ is integral part of a model R of OF - we can take R to be any field in between the fraction field $\mathbb{Q}(A)$ and its Cauchy completion $\mathbb{Q}(A)^c$. We are interested in results of this type, relating extensions of OF with extensions of $DUCR$. We denote $\mathcal{L}$ the language $\{\leq, +, \times, -1\}$ of $DUCR$, tacitly considering $\mathbb{R}, \mathbb{Z}$ as $\mathcal{L}$-structures and not only as sets; henceforth, A tacitly ranges over all models of $DUCR$. We write **rcl** for real closed or real closure and RCF for the theory of rcl fields; remember that RCF axiomatizes the complete theory of $\mathbb{R}$, and is axiomatized by the theory RF of real fields plus the intermediate value scheme IV for all polynomials. IE_0 denotes (the extension of $DUCR$ by) the quantifier free induction scheme of $\mathcal{L}$.

Shepherdson [9] proved that A is a model of IE_0 iff $A = \lfloor R \rfloor$ for some rcl field R - we can take for R the rcl of $\mathbb{Q}(A)$. And Mourgues-Ressayre [5] proved that **every** rcl field has an integral part. Together these results establish a kind of weak duality $A :\longmapsto rcl(\mathbb{Q}(A))$ from IE_0 to RCF and back. We introduce a convenient terminology to discuss results of this kind: if T extends OF then $\lfloor mod\ T \rfloor$ denotes the class $\{\lfloor R \rfloor\,;\, R \text{ satisfies } T\}$; $\lfloor T \rfloor$ denotes the (first order $\mathcal{L}$-)theory of $\lfloor mod\ T \rfloor$. Thus the preceding result are expressed by: $\lfloor mod\ T \rfloor = mod\ \lfloor T \rfloor$ for $T = OF$ and $T = RCF$; and by: $\lfloor OF \rfloor \equiv DUCR + ED$, $\lfloor RCF \rfloor \equiv IE_0$.

Let $\mathcal{L}(...)$ denote $\mathcal{L}$ extended by all function and relation symbols written inside (...); when exp is x^y or a^x ($a > 1$ some constant) we call exp-polynomials the terms of $\mathcal{L}(exp)$. Berarducci [3] asked for extensions of Shepherdson's criterion when $\mathcal{L}(exp)$ replaces $\mathcal{L}$. We partially answer his question, keeping the above definition of $\lfloor mod\ T \rfloor$ and $mod\ \lfloor T \rfloor$ when $\mathcal{L}(2^x)$ is the language of T while $\mathcal{L}(x^y))$ is the language of $\lfloor T \rfloor$. The reason for choosing 2^x in the first place but x^y in the second one is that for every expansion $(R, 2^x)$ of R which satisfies some basic properties of exponentiation we simply set $x^y_R := 2^{ylog(x)}$; whereas this kind of relation between x^y and 2^x is not to be expected in $\lfloor (R, 2^x) \rfloor$. Granted this we have to define integral parts $\lfloor (R, 2^x) \rfloor$ so that they come equipped with a function x^y:

> *We say that A is an x^y-integral part of $(R, 2^x)$ (also denoted $(A, x^y) = \lfloor (R, 2^x) \rfloor$) iff $A = \lfloor R \rfloor$ and A^+ is closed under x^y_R; then $x^y_A := x^y_R | A^+$.*

Let T_{exp} denote the complete theory of $(\mathbb{R}, 2^x)$; we prove that $\lfloor mod\ T_{exp} \rfloor$ equals $mod\ \lfloor T_{exp} \rfloor$ and we axiomatize $\lfloor T_{exp} \rfloor$. Since every model of T_{exp} has an x^y-integral part (see [7]) this establishes the same amount of duality between $\lfloor T_{exp} \rfloor$ and T_{exp} as do Shepherdson and Mourgues-Ressayre between IE_0 and RCF. One would like $\lfloor T_{exp} \rfloor$ to reduce to $LE_0(x^y)$ (least element scheme for quantifier free formulas of $\mathcal{L}(x^y)$), in analogy with Shepherdson's criterion. Alas $\llcorner T_{exp} \lrcorner$ implies $LE_0(x^y)$ but the reciprocal is beyond reach; furthermore our axiomatization of $\llcorner T_{exp} \lrcorner$ is natural and simple in some ways, but in one other way it is ad hoc: namely it expresses natural properties of reals, not of integers. And of course, properties of reals are natural in the context of T_{exp}, whereas here in the context of $\llcorner T_{exp} \lrcorner$ it is natural properties of integers that one would expect. In contrast Shepherdson's criterion is not ad hoc in any respect: IE_0 is as natural a theory for integers as is IV for reals. But we shall prove two other extensions of Shepherdson's criterion with nothing ad hoc; they characterize the x^y-integral parts of models of T for $T = EXP$ - the basic axioms of (real) exponential fields, and for $T = EXP$ plus $IVR(2^x)$ - which denotes the intermediate value and Rolle properties for all 2^x-polynomials. Any way we are interested in axiomatizations $\llcorner T \lrcorner$ even if they have ad hoc features: at least they prove that the class of integral parts of models of T is first order expressible; and they are a useful step towards a better axiomatization. In particular, we think that the axiomatization of $\llcorner T_{exp} \lrcorner$ which is proved here prepares for an axiomatization of the form $\llcorner EXP \lrcorner + I + \llcorner \Delta \lrcorner)$ where I is some natural and syntactic induction scheme, while Δ is the atomic diagram of the definable constants of T_{exp} (which is recursive under Shanuel's conjecture, see [12]).

The present paper is the complete version of a part of the extended abstract [2] - which contained other parts, the complete versions of which are to come.

Section I - EXP and $\llcorner EXP \lrcorner$

An exponential **field** - here with exponentiation of base 2 - is a real field R together with a function 2^x which satisfies the axioms EXP:

i) $2^1 = 2$ and 2^x is a homomorphism of $+$ on (the restriction to positive elements of) $\times$

ii) 2^x is an ordermorphism such that $2 < x \longrightarrow x^2 < 2^x$

iii) $\forall x > 0$ $log\ x$ exists (s.t. $2^{logx} = x$).

Let $R_{exp} := (R, 2^x)$ be any model of EXP; $log\ x$ is unique and x^y denotes $2^{ylog\ x}$.

Proposition I.1.

a) *EXP implies: $y^n < 2^y$ as soon as $2 \leq n \leq log(y)$.*

b) *EXP implies that 2^x and log are continuous.*

c) *If A_{exp} is an x^y-integral part of R_{exp} the relation $a/b < 2^{p/q}$ is definable in A_{exp}.*

Proof.

(a) Indeed, by (ii) $n \leq log(y)$ implies $y^n \leq y^{log(y)}$ which for $y = 2^x$ equals $(2^x)^x = 2^{x^2}$; then (ii) implies $x^2 < y$ and again (ii) implies $2^{x^2} < 2^y$. By transitivity of order, $y^n < 2^y$.

(b) Elementary.

(c) It can be written $a^q < 2^p b^q$. □

We now prove that $\llcorner mod\ EXP \lrcorner = mod\ \llcorner EXP \lrcorner$ by providing an axiomatization over $DUCR$ of $\llcorner mod\ EXP \lrcorner$; it has all variables tacitly ranging over the positive integers. The axioms are written informally and it would be cumbersome to write them strictly in the form of formulas of $\mathcal{L}(x^y)$; but using P I.1.c and using the standard interpretation of fractions as pairs of integers the formulas are a routine to write down and it is left as an exercize. Here are the axioms for $\llcorner mod\ EXP \lrcorner$:

1) $\exists a, b\ |2^{p/q} - a/b| < 1/x$ and $\exists p, q |2^{p/q} - a/b| < 1/x$
2) (i+ii) is true when its variables range over fractions
3) From fractions to reals, 2^x and log are continuous functions.

In the sequel $R_{exp} = (R, 2^x)$ always denotes an expansion of a model of RF and $A_{exp} = (A, x^y)$ denotes an expansion of a model of $DUCR$.

→) From R_{exp} to A_{exp} - Assume that R_{exp} satisfies EXP and A_{exp} is x^y-integral part of R_{exp}, then: (1) is true because 2^x is continuous and $\mathbb{Q}(A)$ is dense in R; (2) is inherited from R because (i+ii) are universal; and (3) is inherited from R_{exp} by P.I.b. We proved that $\llcorner EXP \lrcorner$ includes the axioms (1+2+3).

←) From A_{exp} to $\mathbb{Q}(A)^c_{exp}$ - Conversely we assume that A_{exp} satisfies these axioms and we enrich $\mathbb{Q}(A)^c$ to a model $\mathbb{Q}(A)^c_{exp} := (\mathbb{Q}(A)^c, 2^x)$ of EXP such that x^y_A is the restriction to A^+ of $2^{y log(x)}$. To begin with, (1) implies that inside $\mathbb{Q}(A)$ the cut $2^{p/q} := \{a/b | (a/b)^q < 2^p\}$ is a Cauchy cut; so that it defines $2^{p/q}$ as an element of $\mathbb{Q}(A)^c$. The function $2^x | \mathbb{Q}(A)$ is thus defined as a map from $\mathbb{Q}(A)$ into its Cauchy completion; in addition from (3) follows that this map is continuous on $\mathbb{Q}(A)$. Then by the usual argument, it has a unique continuous extension 2^x sending the totality of $\mathbb{Q}(A)^c$ to $\mathbb{Q}(A)^c$. Now that 2^x is continuous, the truth of (2) on $\mathbb{Q}(A)$ implies the truth of (i+ii) in $(\mathbb{Q}(A)^c, 2^x)$. Finally the continuity of log on $\mathbb{Q}(A)$ which is asserted by (3) implies that the extension by continuity of log to $\mathbb{Q}(A)^c$ continues to satisfy (iii). Thus (1+2+3) guarantees that $(\mathbb{Q}(A)^c, 2^x)$ is a model of EXP.

Together ($\rightarrow$) and ($\leftarrow$) establish for any model A of $DUCR$ that A_{exp} is a model of (1+2+3) iff it is x^y-integral part of some model R_{exp} - one can take $R_{exp} = \mathbb{Q}(A)^c_{exp}$; thus $\llcorner EXP \lrcorner$ exists and is axiomatized over $DUCR$ by (1+2+3).

We have been a bit quick and sketchy and the axioms are ad hoc; but it was meant as an introduction to the main part of the section, which now gives and proves a better axiomatization:

Theorem I.2. *$\llcorner EXP \lrcorner$ is axiomatized by the following system $\mathcal{A}_0$ (where all variables tacitly range over the positive integers)*

- $DUCR$
- $x^{y+z} = x^y x^z, x^{-y} x^y = 1$; $x^{yz} = (x^y)^z$; x^y *strictly increasing with respect to* $x, y > 1$
- $2 < x \longrightarrow x^2 < 2^x$
- $\exists y\ 2^y \leq z < 2^{y+1}$; $\exists x\ x^z \leq 2^y < (x+1)^z$; $\exists y\ 2(x^y) < (x+1)^y$; $\exists x\ (x+1)^y < 2(x^y)$.

Proof. We consider a field with a function $R_{exp} = (R, 2^x)$ and an integral part A of R; in I.A we assume that A is closed x^y and we outline the proof that $A_{exp} := (A, x^y|A)$ satisfies $\mathcal{A}_0$. In I.B we outline the proof of a reciprocal: if A_{exp} satisfies $\mathcal{A}_0$ then A_{exp} is x^y-integral part of a model R_{exp} of EXP with $R = \mathbb{Q}(A)^c$. Readers may find these outlines sufficient, but if not then full detail can be looked at in Section V.1.

I.A

The exponential x^y of A inherits from R the right properties:

1. $A \models \exists\, y\ 2^y \leq z < 2^{y+1}$ (take $y = \llcorner log\ z \lrcorner_A$)
2. $A \models \exists\, x\ \ x^z \leq 2^y < (x+1)^z$ (take $x = \llcorner 2^{\frac{y}{z}} \lrcorner_A$)
3. $A \models \exists\, y\ \ 2x^y < (x+1)^y$ (for $log\ (x+1) - log\ x > 0$ so we can find $y \in A$ such that $log\ (x+1) - log\ x > \frac{1}{y}$, which is equivalent to $2x^y < (x+1)^y$)
4. $A \models \exists\, x\ \ (x+1)^y < 2x^y$ (for log continuous at 1 and the limit of $log\ (1+t)$ has the value 0 as t approaches 0; hence for all $y \in A$ there exists $t > 0$ such that $log\ (1+t) < \frac{1}{y}$. Take $x \in A$ such that $x > \frac{1}{t}$, then $log\ (1 + \frac{1}{x}) < \frac{1}{y}$ - which is equivalent to $(x+1)^y < 2x^y$)

I.B

We suppose that A satisfies $\mathcal{A}_0$ and we expand its Cauchy closure R to an exponential field such that A is x^y-integral part of R.

We first define $log\ x$ for an "integer" $x \in A^+_*$, we prove that it is a Cauchy cut, and that the log has good properties over the integers.

Definition I.3. *Given* $x \in A^+_*$ *set* $log\ x = \{ \frac{p}{q}\ ;\ p, q \in A \text{ and } 2^p \leq x^q \}$. *It is sometimes denoted by* $log_{|0}\ x$.

Fact I.4. *(*$p,\ q,\ p',\ q',\ x,\ y \in A^*_+$*)*

1. $log\ x$ *is a cut.*
2. $log\ x \in R$.
3. $log\ 2^x = x$ *and* $log\ 1 = 0$.
4. $log\ x^y = y\, log\ x$.
5. $log\ xy = log\ x + log\ y$.
6. $\frac{p}{q} = \frac{p'}{q'} \rightarrow log\ p - log\ q = log\ p' - log\ q'$.

Now we define log and x^y over $\mathbb{Q}(A)$ and R. We prove that log has good properties over fractions and that it is strictly increasing on R.

Definition I.5. *(*$x, y \in A^*_+$*,* $c \in R^*_+$ *and* $c' \in R$*)*

1. $log\ \frac{x}{y} = log_{|0}\ x - log_{|0}\ y$.
   It is sometimes denoted by $log_{|1}\ \frac{x}{y}$.
2. $log\ c = \{ \frac{p}{q}\ ;\ p, q \in A \text{ and } \exists\, x, y \in A\ \ \frac{x}{y} \leq c \quad \text{and} \quad \frac{p}{q} \leq log_{\ |1} \frac{x}{y} \}$.
   It is sometimes denoted by $log_{\ |2}\ c$.
3. $2^{c'} = \{ \frac{p}{q}\ ;\ p, q \in A \text{ and } log\ \frac{p}{q} \leq c' \}$.
4. $x^y = 2^{y log\ x}$.

Fact I.6. *(*$x, y \in A^*_+$ *and* $r, r' \in \mathbb{Q}(A)^*_+$*)*

1. $log\ \frac{x}{y} \in R$.
2. $log\ rr' = log\ r + log\ r'$.
3. $log\ \frac{r}{r'} = log\ r - log\ r'$.
4. $x \leq y \rightarrow log\ x \leq log\ y$.
5. $r \leq r' \rightarrow log\ r \leq log\ r'$.
6. *The value of the log of a fraction given by the Definition I.5.1 is the same as that given by the Definition 1.5.2 for reals (in the sense:*

$log_{|1}\, r = log_{|2}\, r$ for all $r \in \mathbb{Q}(A)$), and more precisely

$$log_{|1}\, r = \underbrace{\{\, \rho\, ;\ \exists\, s \leq r \quad \rho \leq log_{|1}\, s \,\}}_{log_{|2}\, r}).$$

7. $\exists\, y \in A \;\; log\,(x+1) - log\; x > \frac{1}{y}$.
8. $x < y$ *iff* $log\; x < log\; y$.
9. $r < r'$ *iff* $log\; r < log\; r'$.
10. *For all $a \in A$: $r \leq 2^a$ iff $log\; r \leq a$ iff $r \in 2^a$*.*
    Then, for all $a \in A$ the value of 2^a in A is the same as the cut 2^a given by the Definition 1.5.3 for reals.
11. *Let $c \in R^*_+$. Then $log\; c = sup\,\{log\; r;\; r \leq c\}$**.*
    So: $c' < log\; c$ iff $\exists\, r \leq c \;\; c' < log\; r$.
12. *For all $c \in R^*_+$, $r < c \rightarrow log\; r < log\; c$.*
13. *For all $c \in R^*_+$, $r > c \rightarrow log\; r > log\; c$.*
    Hence: $log\; r = log\; c \rightarrow r = c$.
14. *For all c and c' in R^*_+, $c < c'$ iff $log\; c < log\; c'$.*
    So: $c = c'$ iff $log\; c = log\; c'$

The next step proves that R is closed under log and 2^x, which are involutive.

Proposition I.7. *($c \in R$)*

1. *$c > 0 \rightarrow log\; c \in R$.*
2. *$2^c \in R$.*
3. *$c > 0 \rightarrow 2^{log\; c} = c$.*
4. *If x and y are in A then the value of x^y in A is equal to the value of x^y defined on R.*
5. *$log\; 2^c = c$.*

At last, we prove that 2^x is an increasing homomorphism of $+$ on $\times$ (restricted to positive elements).

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* The first 2^a is the value given by the exponential function 2^x defined in A, the last 2^a is the cut $\{\, \frac{p}{q}\, ;\ p, q \in A$ and $log\, \frac{p}{q} \leq a \,\}$ given by the Definition 1.5.3 for a reals.

** $sup\, X = \{r \in \mathbb{Q}(A);\ \exists\, x \in X \;\; r \leq x\}$.

Proposition I.8. *$(x, y, z \in R)$*

1. *2^x is strictly increasing on R.*
2. *$(x > 0 \wedge y > 0) \rightarrow log\ xy = log\ x + log\ y$.*
3. *$(x > 0 \wedge y > 0) \rightarrow log\ x^y = y\, log\ x$.*
4. *$2^{x+y} = 2^x 2^y$.*
5. *$x > 1 \rightarrow (y < z \leftrightarrow x^y < x^z)$.*
   And thus: $y = z$ iff $x^y = x^z$.
6. *$(y > 1 \wedge z > 1) \rightarrow (y < z \leftrightarrow y^x < z^x)$.*
7. *$x^{y+z} = x^y\, x^z$.*
8. *$(x^y)^z = x^{yz}$.*
9. *log , 2^x are continuous.*
10. *$\exists y \forall x > y \ \ 2^x > x^n$*

In section V.1 the full proofs of Fact I.4, I.6 and Proposition I.7, I.8 are provided - but they are renumbered Fact V.1.1, V.1.2 and Proposition V.1.5, V.1.6.

Section II: $IVR(2^x)$ and $\llcorner IVR(2^x) \lrcorner$

Proposition II.1.

a) *$LE_0(x^y)$ holds in every x^y-integral part of every model of T_{exp}*
b) *$LE_0(2^x)$ holds in every 2^x-integral part of every model of $EXP + IVR(2^x)$.*

Proof. **(a)** Let $(R, 2^x) = R_{exp}$ be a model of T_{exp}; A. Wilkie [12] proved that T_{exp} is an o-minimal theory hence $\mathbb{R}_{exp}$ is an o-minimal structure. That is: if $\Phi(x)$ is any formula of $\mathcal{L}(2^x)$ with parameters in R then the interpretation of $\Phi(x)$ in R_{exp} is a finite union of intervals with endpoints $a_i, b_i \in R \cup \{-\infty, +\infty\}$. Thus if $A = \llcorner R \lrcorner$ then $min\ x \in A : \Phi(x)$ can only be one of the following elements:

a_i (in a case where $a_i = \llcorner a_i \lrcorner \in A$); $\llcorner a_i \lrcorner + 1$; or $\llcorner b_i \lrcorner$.

Assume that $(A, x^y) = A_{exp}$ is an x^y-integral part of R_{exp} and consider $\theta(x) \in E_0(x^y)$; with the help of $2^{y log(x)}$ one finds a formula $\Phi(x) \in \mathcal{L}(2^x)$ which inside R_{exp} expresses $\theta(x)$. We just showed that $min\ x \in A : \Phi(x)$ exists; hence also $min\ x :\ \theta(x)$ inside A_{exp}.

(b) Here R_{exp} only satisfies $EXP + IVR(2^x)$ and is not o-minimal; but by a result of van den Dries [10] every non trivial 2^x-polynomial has only a finite number of roots in $(R, 2^x)$. This allows to prove for every quantifier free formula $\Phi(x) \in \mathcal{L}(2^x)$ that the interpretation of $\Phi(x)$ in

R_{exp} again is a finite union of intervals with endpoints in $R \cup \{-\infty, +\infty\}$. Thus whenever A is an integral part we obtain as before the existence of $min\ x \in A:\ R_{exp}$ satisfies $\Phi(x)$; and this min is also $min\ x \in A:\ A_{exp}$ satisfies $\Phi(x)$, provided in addition A_{exp} is 2^x-integral part. $\square$

Remark - 1) The proof of P II.1 works when we allow the formula $\Phi(x)$ to have parameters in A in addition to the induction variable x; and actually it is this case which shows $LE_0(x^y)$ in **(a)**, $LE_0(2^x)$ in **(b)**.

2) But the proof works unchanged if the induction formula $\Phi(x)$ is allowed to have parameters from R, not only from the integral part A. Thus stronger induction schemes can be proved - only they are not entirely expressed inside A_{exp}. We now introduce two ways to handle this problem: the first one relies on second order Arithmetic, the second one on translations into first order Arithmetic.

The second order theory of integral parts

- Let $IP \subset \mathcal{L}(A(x))$ with $A(x)$ predicate symbol denote the obvious axioms which are satisfied by (R, A) iff $A = \llcorner R \lrcorner$.
- More generally, for any function $f = f(\bar{x}))$ over the reals $IP(f)$ adds to IP that A^+ is closed under f.

We can regard $OF + IP(f)$ as a second order Arithmetic of some kind: the elements X of the field are the "reals" or second order objects; among them the "integers" or first order objects are the elements of A - these integers form an f-integral part of these reals. In the sequel a formula of $\mathcal{L}(A(x), ...)$ is called **first order** if all its quantifiers are restricted to $A(x)$; whereas a second order formula has also quantifiers ranging over the whole field. Note that formulas with no occurrence of the symbol $A(x)$ have all their quantifiers ranging over the reals - we call them **pure** second order; and we denote by L^2 the least element scheme asserted for all formulas of the form $[A(x)\ and\ \phi(x, \bar{U})]$ where $\phi(x, \bar{U})$ is pure second order. The theory $T_{exp} + IP(x^y)$ axiomatizes the class of pairs (R_{exp}, A) of a model of T_{exp} with an x^y-integral part; it looks rather limited, and the more so $\llcorner T_{exp} \lrcorner$ which is the first order counterpart. But any way one kind or another of drastic restriction is necessary on a theory $\llcorner T \lrcorner$ of Arithmetics if we want it to correspond to a well behaved theory like $T = T_{exp}$; for T_{exp} has excellent algebraic properties while Arithmetics cannot avoid Gödel's incompleteness theorem and Tennenbaum's theorem of non existence of recursive non-standard models. And notwithstanding the limited character of $IP(x^y)$, the whole scheme L^2 is a consequence of $T_{exp} + IP(x^y)$ - as established by the same proof as P II.1.a.

In the theory $OF + IP(x^y)$ the treatment of first order formulas with fractional or real parameters is trivial: there are first order variables ranging over A and second order variables ranging over R; and the parameters are free second order variables assigned with a fraction $a/b \in \mathbb{Q}(A)$ or with an element of R. Below we express the same notions in the first order language of A_{exp}; of course it gets more roundabout since already from parameters in $\mathbb{Q}(A)$ the function 2^x leads to real parameters which A can only handle with the help of quantifiers.

fc-Translations between A_{exp} and $\mathbb{Q}(A)^c_{exp}$ - We assume that A_{exp} is x^y-integral part of R_{exp} and $R = \mathbb{Q}(A)^c$. Let $\Phi(\bar{x})$ be a formula of $\mathcal{L}(2^x)$; an fc-translation of Φ is a formula $\Phi^{fc} \in \mathcal{L}(x^y)$ such that for all fractions $\bar{x} = a_1/b_1, ..., a_k/b_k$, (A, x^y) satisfies $\Phi^{fc}(\bar{x})$ iff R_{exp} satisfies $\Phi(\bar{x})$ (the superscript fc is chosen in reference to "**C**ompletion of the **F**raction field of A" because $R = \mathbb{Q}(A)^c$; note that the sequence of variables $\bar{x}$ of Φ must be doubled in Φ^{fc}). This notion of fc-translation is relative to a given A_{exp}; but we say that the fc-translation holds **over** $\mathcal{A}$ if it holds whenever A_{exp} satisfies the theory $\mathcal{A}$.

Example - a) To say that $\llcorner EXP \lrcorner$ exists and is axiomatized by $\mathcal{A}_0$ is equivalent to say that $\mathcal{A}_0$ is an fc-translation of (the conjunction of) EXP over the empty theory. But over an empty or weak theory the notion of fc-translation is perilous; it is over $\llcorner EXP \lrcorner$ that it becomes robust enough for allowing some systematic and simple syntactic fc-translations - see P II.2 below.

b) Unsystematic, tentative fc-translations play a heuristic role: ad hoc fc-translations $EXP^{fc}, IVR(2^x)^{fc}$ suggested our first axiomatizations of $\llcorner EXP \lrcorner$ and $\llcorner IVR(2^x) \lrcorner$; once these translations had quickly proved $\llcorner T \lrcorner$ to exist for $T = EXP, IVR(2^x)$ we started looking for the finer results of T I.2 and T II.4.

Nota Bene - a) If $\psi^{fc}(\bar{x})$ is an fc-translation of $\psi(\bar{x})$ it does not guarantee that $\forall \bar{x} \psi^{fc}(\bar{x})$ be an fc-translation of $\forall \bar{x} \psi(\bar{x})$.

b) On the other hand fc-translation permutes with boolean operations.

Proposition II.2. *Every quantifier free formula $\varphi = \varphi(\bar{x}) \in \mathcal{L}(2^x)$ has an fc-translation.*

Proof. For φ atomic let us only give two examples and admit the general case: $[0 < x]^{fc}$ is $\exists y y^2 x = 1$; $[y = 2^{2^x}]^{fc} = [\exists z\ 2^x = z \ and\ 2^z = y]^{fc} :=$ $\forall p > 0 \exists a/b [|2^x - a/b| < 1/p \ and\ |2^{a/b} - y| < 1/p$ - where P I.1.c is used to express the latter formula. For boolean combinations one uses the above Nota Bene (b). □

Convention - For any letter x we use $\bar{x}$ as an abbreviation for the sequence $x_1, ..., x_k$, where the value of $k < \omega$ depends on the context: if the context says nothing then k ranges freely over ω; and $\bar{x}, \bar{y}$ need not have same k - except if the context substitutes one for the other.

Let $\lambda \in R$ determine a cut of $\mathbb{Q}(A)$ which is definable in A_{exp}: for some formula $F \in \mathcal{L}(x^y)$ we have $p/q < \lambda \longleftrightarrow F(p/q)$. We extend the notion of fc-translation to any "formula with real parameter" $\varphi(\lambda, \bar{x})$ where $\varphi(u, \bar{x}) \in \mathcal{L}(2^x)$: the fc-translation is a formula $\psi \in \mathcal{L}(x^y)$ such that for all fractions $\bar{a}$

$$A_{exp} \text{ satisfies } \psi(\bar{a}) \text{ iff } R_{exp} \text{ satisfies } \varphi(\lambda, \bar{a}).$$

Fact II.3. *Every quantifier free formula with real parameters has an fc-translation.*

Proof. For any term t of $\mathcal{L}(2^x)$ we set $[t(\lambda, \bar{x}) \leq 0]^{fc} := \forall p > 0 \exists r [F(r) \text{ and } \neg F(r + 1/p) \text{ and } (t(u, \bar{x}) \leq 1/p)^{fc}]$. The truth of $(t(u_p, \bar{x}) \leq 1/p)^{fc}$ for fractions u_p tending to λ and when $1/p$ tends to 0 is equivalent to the truth of $t(u_p, \bar{x}) \leq 1/p$, which implies $t(\lambda, \bar{x}) \leq 0$ at the limit; and using the continuity of t we can reverse this argument to obtain the reciprocal. Thus the fc-translation is valid for formulas of this form; all other cases follow by applying preservation of translations under boolean operations. It is clear how to extend this proof to the case of several real parameters. □

We shall apply F II.3 to a unique real parameter $\lambda = log'(1) := \{p/q : \exists r \; p/q < \frac{log\ (1+r)}{r}\}$. We can ensure that λ is a Cauchy cut with the following axiom A_λ (which due to P I.1.c can be expressed in $\mathcal{L}(x^y)$):

$$0 < r < s \rightarrow (1+s)^r < (1+r)^s \wedge \forall x > 0 \exists \, s < \frac{1}{x} \, \forall r \; \frac{(1+r)^{xs}}{(1+s)^{xr}} < 2^{rs}.$$

We let $IE_0(2^x)^{fc}$ consist of A_λ and of the scheme of induction for the fc-translation of every formula $\psi(\lambda, \bar{x}, X) \in E_0(2^x, \lambda)$ - where the parameters $\bar{x}$ range over fractions. We let $I\mathbf{E}_0(2^x)^{fc}$ denote A_λ plus the scheme which expresses in $\mathcal{L}(x^y)$ that if $F_i(u, v)$ defines for each $i < k$ inside A_{exp} a Cauchy cut λ_i of $\mathbb{Q}(A)$ and if $\phi = \phi(\bar{\lambda}, \bar{x}, X) \in \mathcal{L}(2^x)$ is a quantifier free formula with these parameters λ_i, then induction holds for the fc-translation of ϕ. The schemes $LE_0(2^x)^{fc}$ and $L\mathbf{E}_0(2^x)^{fc}$ are defined in the same way except that minimization replaces induction.

Theorem II.4. *$\llcorner IVR(2^x)\lrcorner$ exists and is axiomatized by $IE_0(2^x)^{fc}$ over $\llcorner EXP \lrcorner$.*

Corollary II.5. *Over $\llcorner EXP \lrcorner$ the theory $IE_0(2^x)^{fc}$ implies $L\mathbf{E}_0(2^x)^{fc}$ (least element scheme for quantifier free formulas of $\mathcal{L}(2^x)$ with arbitrary definable real parameters). Hence the four theories above T II.4 are equivalent.*

Proof. **T II.4 $\longrightarrow$ C II.5 -** Assume that A_{exp} satisfies $\llcorner EXP \lrcorner +$ $IE_0(2^x)^{fc}$ and fix a formula $\psi = \psi(\bar{\lambda}, \bar{x}, X) \in E_0(2^x, \bar{\lambda})$ where $\bar{\lambda}$ is any finite sequence of Cauchy cuts definable inside A_{exp}; let ψ^{fc} denote its fc-translation: for all fractions $\bar{a}, X$ $\mathbb{Q}(A)^c_{exp}$ satisfies $\psi(\bar{\lambda}, \bar{a}, X)$ iff A_{exp} satisfies $\psi^{fc}(\bar{a}, X)$. Since by T. II.4 $\mathbb{Q}(A)^c_{exp}$ satisfies $IVR(2^x)$, the proof of P. II.1.b applied to ψ^{fc} proves $minX : \psi^{fc}(\bar{a}, X)$ to exist. Thus A_{exp} satisfies $L\mathbf{E}_0(2^x)^{fc}$. The reciprocal is true because the minimization scheme for the negation of ϕ implies the induction scheme for ϕ. □

Proof. **T II.4 -** That $\llcorner IVR(2^x)\lrcorner$ implies $IE_0(2^x)^{fc}$ and even $L\mathbf{E}_0(2^x)^{fc}$ has just been proved; we now look for the reciprocal. We consider a model A_{exp} of $EXP + IE_0(2^x)^{fc}$ and we set up to prove that $R_{exp} :=$ $\mathbb{Q}(A)^c_{exp}$ satisfies $IV(2^x)$. Given a 2^x-polynomial $P(\bar{x}, Y)$ and given $\bar{x}$ in R suppose $P(\bar{x}, a) > 0 > P(\bar{x}, b)$; we want a zero of P between a and b. We first assume $\bar{x}$ in $\mathbb{Q}(A)$; an induction on the length of the 2^x-polynomial $P(\bar{x}, Y)$ derives from EXP the uniform continuity of $P(\bar{x}, Y)$ for fixed $\bar{x}$ and when Y ranges over $[a, b]$. Thus given $\epsilon > 0$ in $\mathbb{Q}(A)$ we can find $N \in A$ such that the variation of $P(\bar{x}, Y)$ is less than ϵ on every subinterval of $[a, b]$ of length $(b - a)/N$; hence on $[c_i, c_{i+1}]$ where $c_i := a+(b-a)i/N$. We can be sure that on one of these intervals $P(\bar{x}, Y)$ changes its sign - otherwise a contradiction with $IE_0(2^x)^{fc}$ is easily reached. Assume that A is countable and choose a sequence $(\epsilon_n), n < \omega$ with limit 0 in $\mathbb{Q}(A)$. By iterating for each $n < \omega$ the preceding fact applied with $1/N \leq \epsilon_n$ and with $[a_n, b_n]$ in place of $[a, b]$ we obtain a decreasing chain of subintervals $[a_n, b_n]$ of $[a, b]$ on which $P(\bar{x}, Y)$ changes sign and its variation is less than ϵ_n, moreover the intersection defines an element $r \in R$. Thus r is a root of $P(\bar{x}, Y)$ and $IV(2^x)$ is proved - for fractional parameters only; but by proving the uniform continuity of $P(\bar{x}, Y)$ on every finite $k + 1$-dimensional box we extend the result to arbitrary "real" parameters. All this is easy, but it only gives a hint for the proof of $IVR(2^x)$ which we now really start and which has the following added features: given $P(\bar{a}, c_0) > 0 > P(\bar{a}, c_1)$, with $\overline{a}, c_0, c_1 \in$ R, we first find elements $t \in]c_0, c_1[$ for which $|P(\bar{a}, t)|$ is arbitrarily small; this is proved in Fact II.6. Thus for each $N > 0$ in A there is $\rho_N \in]c_0, c_1$

such that $|P(\bar{a}, \rho_N)| < 1/N$. Then we prove a weak version of Rolle and of Differentiation lemma (see Proposition II.7 and see Fact II.8). Then we use the latter lemma together with a notion of ordinal degree for exponential polynomials (this degree denoted ord was introduced by L.van den Dries [10]); we thus show how to extract from the sequence ρ_N a zero of $P(\bar{a}, x)$. It will be done in the "Approximation theorem" which proves $IV(2^x)$. Finally we obtain $IVR(2^x)$, by combining $IV(2^x)$ with the weak version of the Rolle lemma. The rest of this section is sketchy but full detail shall be given in Section V.2: Proposition II.7, II.9 and Fact II.6, II.8 below are exactly Proposition V.2.6, V.2.8 and Fact V.2.2, V.2.7 there.

Fact II.6. *($\rho, \sigma, \overline{r} \in \mathbb{Q}(A)$ and $c_0, c_1, \overline{a} \in R$)*

1. *Assume that $\rho < \sigma$ and $f(\overline{r}, \rho) < 0 < f(\overline{r}, \sigma)$, then $\forall N > 0$ $\exists t \in [\rho, \sigma]$ such that $|f(\overline{r}, t)| < \frac{1}{N}$.*
2. *Suppose that $c_0 < c_1$ and $f(\overline{a}, c_0) < 0 < f(\overline{a}, c_1)$, then $\exists\, [\rho, \sigma] \subset\,]c_0, c_1[$ such that $\forall N > 0\, \exists t \in [\rho, \sigma] \;\; |f(\overline{a}, t)| < \frac{1}{N}$.*

Now comes a weak version of Rolle ; below $f = f(\bar{u}, x)$ is a 2^x-polynomial.

Proposition II.7. *($\rho_0, \rho_1, \overline{r} \in \mathbb{Q}(A)$)*

$$f(\overline{r}, \rho_0) = f(\overline{r}, \rho_1) \;\rightarrow\; \forall N > 0\, \exists\, t \in]\rho_0, \rho_1[\cap \mathbb{Q}(A) \;\; |f'(\overline{r}, t)| < \frac{1}{N}.$$

Fact II.8 is a weak version of the Differentiation lemma.

Fact II.8. *($s_0, s_1, \overline{r} \in \mathbb{Q}(A)$ and $c_0, c_1, \overline{a} \in R$)*

1. *$\exists\, t \in]s_0, s_1[\;\; |\frac{f(\overline{r}, s_1) - f(\overline{r}, s_0)}{s_1 - s_0} - f'(\overline{r}, t)| < \frac{1}{N}$.*
2. *$\exists\, t \in]c_0, c_1[\;\; |\frac{f(\overline{a}, c_1) - f(\overline{a}, c_0)}{c_1 - c_0} - f'(\overline{a}, t)| < \frac{1}{N}$.*
3. *$f' = 0$ on $[c_0, c_1] \rightarrow f$ is constant - even for real parameters $\bar{u}$.*

Proposition II.9. *Given $c_0 < c_1$ assume that there exists a sequence $(\rho_N)_{N \in A}$ in a closed subinterval of $]c_0, c_1[$ such that $\forall N\, |f(\overline{a}, \rho_N)| < \frac{1}{N}$.*

*Consider the cut $c = inf\, \{c' \;;\; \forall B\, \exists N > B \;\; \rho_N < c'\}$ *. Then we have*

1. *$c = \{r \;;\; \forall c' \; ((\forall B\, \exists N > B \; \rho_N < c') \rightarrow r < c')\}$ and $c_0 \leq c \leq c_1$.*

* $inf\, X = \{r;\; \forall x \in X \;\; r < x\}$.

2. *Either* $\forall\, r < c \,\forall\, B \,\exists\, N > B \;\; r < \rho_N < c$, or $\forall\, r > c \,\forall\, B \,\exists\, N > B \;\; c \leq \rho_N < r$. *Hence*

$$\forall\, r, r' \;\; (r < c < r' \rightarrow \forall\, N \exists\, \rho \in [r, r'] \; |f(\overline{a}, \rho)| < \frac{1}{N}).$$

3. *If* $c \in R$, *then* $f(\overline{a}, c) = 0$.

The continuation of this Proposition shows that if an exponential polynomial f has arbitrarily small values arbitrarily close to some non Cauchy cut, then the same is true for the derivative and for the product of f by another exponential polynomial.

Assume that $c \notin R$ $(\delta(c) > 0^{**})$.

Assume that $\exists\, \mu > 0 \,\forall\, r < c \,\forall\, N \,\exists\, \rho \;\; r < \rho < c$ *and* $|f(\overline{a}, \rho)| < \frac{\mu}{N}$, *then:*

4. *for all* δ_0, $0 < \delta_0 < \delta(c)$ *we have:* $\forall\, r < c \,\forall\, N \,\exists\, \rho, \rho' \;\; r < \rho < \rho' < c, \;\; \rho' - \rho > \delta_0, |f(\overline{a}, \rho)| < \frac{\mu}{N}$, *and* $|f(\overline{a}, \rho')| < \frac{\mu}{N}$
5. $\exists\, \mu' > 0 \,\forall\, r < c \,\forall\, N \,\exists\, \rho \;\; r < \rho < c$ *and* $|f'(\overline{a}, \rho)| < \frac{\mu'}{N}$
6. *for all* g *there exists* $\mu' > 0$ *such that* $\forall\, r < c \,\forall\, N \,\exists\, \rho \;\; r < \rho < c$ *and* $|f(\overline{a}, \rho)\, 2^{g(\overline{a}, \rho)}| < \frac{\mu'}{N}$.

Assume that $\exists\, \mu \,\forall\, r > c \,\forall\, N \,\exists\, \rho \;\; c < \rho < r$ *and* $|f(\overline{a}, \rho)| < \frac{1}{N}$, *then:*

7. *for all* δ_0, $0 < \delta_0 < \delta(c)$ *we have:* $\forall\, r > c \,\forall\, N \,\exists\, \rho, \rho' \;\; c < \rho < \rho' < r, \;\; \rho' - \rho > \delta_0, |f(\overline{a}, \rho)| < \frac{\mu}{N}$, *and* $|f(\overline{a}, \rho')| < \frac{\mu}{N}$
8. $\exists\, \mu' \,\forall\, r > c \,\forall\, N \,\exists\, \rho \;\; c < \rho < r$ *and* $|f'(\overline{a}, \rho)| < \frac{\mu'}{N}$
9. *for all* g *there exists* $\mu' > 0$ *such that* $\forall\, r > c \,\forall\, N \,\exists\, \rho \;\; c < \rho < r$ *and* $|f(\overline{a}, \rho)\, 2^{g(\overline{a}, \rho)}| < \frac{\mu'}{N}$.

We recall from [10] the application *ord* defined on the exponential polynoms as an analog of the degree of polynomials.

$R[x]^E$ the set of exponential polynoms is union of the R_k, $-1 \leq k$, defined by induction: $R_{k+1} = R_k \oplus I_{k+1}$, for all $k \geq -1$.

- $R_{-1} = R$, $I_0 =$ the ideal $(x)\, R[x]$. From which follows: $R_0 = R_{-1} \oplus I_0 = R[x]$.

** $\delta(c) = \{r \in \mathbb{Q}(A);\; \forall x\; (x \in c \rightarrow x + r \in c)\}$; $\delta(c) > 0$ means $\exists r > 0\; r \in \delta(c)$.

- $R_{k+1} = R_k[2^{I_k}]$, and $I_{k+1} = R_k$-sub-module of R_{k+1} generated by the 2^a, $a \in I_k$, $a \neq 0$.
  It is clear that $R_{k+1} = R_k \oplus I_{k+1}$.
- Notice that $R_k = R_0 \oplus I_1 \cdots \oplus I_k$, $k \geq 0$.

Definition II.10. *(height)*
$p(x) \in R[x]^E$ is of height k if $p \in R_k \backslash R_{k-1}$, $k > 0$, and it is of height 0 if $p \in R_0 = R[x]$.

Definition II.11. *(ord)*

- *if $p \in I_k$, $k > 0$, $p = \sum_{i=1}^{h} r_i\, 2^{a_i}$, where a_i are distinct membres of $I_{k-1} \backslash \{0\}$, and r_i are non-zero elements of R_{k-1}, we put $t(p) = h$.*
- *if $p \in R_0 = R[x]$, we put $t(p) = 0$ if $p = 0$, and $t(p) = d + 1$ if $deg_x P = d \geq 0$.*

  Then we define an ordinal $ord(p) < \omega^\omega$ for $p \in R[x]^E$.

  Note that: $R_k = R_0 \oplus I_1 \cdots \oplus I_k$, $k \geq 0$. So any $p \in R[x]^E$ of height $\leq k$ can be written uniquely as: $p = p_0 + p_1 + \cdots + p_k$, $p_0 \in R_0$, $p_i \in I_i$, for $i > 0$.

  We put $ord(p) = \omega^k \cdot t(p_k) + \cdots + \omega \cdot t(p_1) + t(p_0)$.
- *Note that: $ord(p) = 0$ iff $p = 0$.*

The following result is proved in [10].

Lemma II.12. *Let $p \in R[x]^E$ non-zero, then either $ord(2^q\, p) < ord(p)$ for some $q \in R[x]^E$, or $ord(p') < ord(p)$.*

Theorem. *(Approximation) Let f be a 2^x-polynomial with parameters in R such that $\forall N > 0\; \exists \rho \in [c_0, c_1] \quad |f(\rho)| < \frac{1}{N}$; there exists $c \in [c_0, c_1]$ such that $f(c) = 0$.*

Proof. Assume that $f \neq 0$ and $(\rho_N)_{N \in A}$ in $[c_0, c_1]$ is such that $\forall N\; |f(\rho_N)| < \frac{1}{N}$; consider the cut $c = inf\,\{c'\; ;\; \forall B\, \exists N > B \quad \rho_N < c'\}$.

By Proposition II.9.3, if $c \in R$, then $f(c) = 0$. Proceeding by way of contradiction, assume that $c \notin R$. Then, by Proposition II.9.2 we have two cases:

<u>Case 1</u>: $\forall r < c\, \forall B\, \exists N > B \quad r < \rho_N < c$.

Then:

$$\exists \mu_0\, \forall r < c\, \forall N \exists \rho \quad r < \rho < c \text{ and } |f(\rho)| < \frac{\mu_0}{N} \qquad (*_0)$$

$f \neq 0$, then $ord(f) > 0$ and by Lemma II.12. and the fact that ω^ω is well ordered, there exist $n \in \mathbb{N}^*$, $f_0 = f$,..., f_n such that:

- $ord(f_0) > ord(f_1) > ... > ord(f_n) = 0$. From which follows $f_n = 0$.
- For all $i < n$ we have: $f_{i+1} = f_i'$ or there exists g_i such that $f_{i+1} = f_i\, 2^{g_i}$.

The case $f_n = f_{n-1}\, 2^{g_{n-1}}$ is not possible, otherwise $f_{n-1} = 0$ and then $0 = ord(f_{n-1}) > ord(f_n) = 0$.

Then $f_n = f_{n-1}' = 0$. And by the weak version of Differentiation Lemma (Fact II.8), there exists $b \in R$ such that $f_{n-1} = b$.

On the other hand, $b \neq 0$, otherwise $f_{n-1} = 0$ and $0 = ord(f_{n-1}) > ord(f_n) = 0$.

- Since f satisfies $(*_0)$ one applies Proposition II.9.5 or Proposition II.9.6 to f, depending on $f_1 = f_0'$ or $f_1 = f_0\, 2^{g_0}$; then we have:

$$\exists\, \mu_1\, \forall\, r < c\, \forall\, N \exists\, \rho \quad r < \rho < c \text{ and } |f_1(\rho)| < \frac{\mu_1}{N} \qquad (*_1)$$

- Since f_1 satisfies $(*_1)$, as for f, and using $(*_1)$ plus Proposition II.9 we have:

$$\exists\, \mu_2\, \forall\, r < c\, \forall\, N \exists\, \rho \quad r < \rho < c \text{ and } |f_2(\rho)| < \frac{\mu_2}{N} \qquad (*_2)$$

- ,..., until f_{n-1}, we have:

$$\exists\, \mu_{n-1}\, \forall\, r < c\, \forall\, N \exists\, \rho \quad r < \rho < c \text{ and } |f_{n-1}(\rho)| < \frac{\mu_{n-1}}{N} \qquad (*_{n-1})$$

$f_{n-1} = b$, then:

$$\forall\, N \quad |b| < 1/N.$$

Thus $b = 0$ which is contradiction.

Case 2: $\forall\, r > c\, \forall\, B\, \exists\, N > B \quad c < \rho_N < r$.
Similar proof. □

Theorem. *(IV) For a 2^x-polynomial f with parameters in R if $f(c_0) < 0 < f(c_1)$ then there exists $c \in]c_0, c_1[$ such that $f(c) = 0$.*

Proof. Assume that $f(c_0) < 0 < f(c_1)$; by Fact II.6.2 there exists $[\rho, \sigma] \subset]c_0, c_1[$ and a sequence $(\rho_N)_{N \in A}$ in $[\rho, \sigma]$ such that for all $N > 0$ $|f(\rho_N)| < \frac{1}{N}$. Then, by the Approximation theorem there exists $c \in [\rho, \sigma]$ such that $f(c) = 0$. □

Theorem. *(Rolle) If $c_0 < c_1$ in R there exists $c \in]c_0, c_1[$ such that*

$$f(c_0) - f(c_1) = (c_0 - c_1)\, f'(c).$$

Proof. Let us prove it for the case $f(c_0) = f(c_1)$.

- if f is constant on $[c_0, c_1]$, we have the conclusion which we want.
- Assume that f is not constant on $[c_0, c_1]$.

  Then there exists $\sigma_0 \in [c_0, c_1]$ such that $f(\sigma_0) \neq f(c_0)$; without loss of generality, we can assume that $f(\sigma_0) > f(c_0)$. Then by IV theorem there exists a, $c_0 < a < \sigma_0$ such that $f(a) = \frac{f(\sigma_0)+f(c_0)}{2}$.

  Also $f(\sigma_0) > f(c_1)$ and there exists $b, \sigma_0 < b < c_1$ such that $f(b) = \frac{f(\sigma_0)+f(c_0)}{2}$. $f(a) = f(b)$ and by the weak version of Rolle (Proposition II.7) there exists a sequence $(\rho_N)_{N \in A}$ in $[a, b]$ such that $\forall N \; |f'(\rho_N)| < \frac{1}{N}$. By the Approximation theorem applied to f', there exists $c \in [a, b]$ such that $f'(c) = 0$. Furthermore c is in $]c_0, c_1[$. □

T II.4 □

Section III : The theory $\llcorner T_{exp} \lrcorner$

The proof of P II.1.a shows that $\llcorner T_{exp} \lrcorner$ implies $L\mathbf{E}_0(x^y)^{fc}$. But the reciprocal is an open question; it could hold if a highly remarkable phenomenon took place in $(\mathbb{R}, 2^x)$: non singular **systems** of 2^x-algebraic equations should reduce to **single** such equations, in a **uniform** way (that is in every model of T_{exp} in addition to $(\mathbb{R}, 2^x)$). Although this is the analog of a basic property of real algebraic closure, it is very demanding... It is in view of this uncertainty that it was interesting to have a subtheory T of T_{exp}, as strong as we are able to find and for which we know a natural axiomatization of $\llcorner T \lrcorner$; this is what Section II provided with $T = IVR(2^x)$.

The next two theorems recall an axiomatization $\mathcal{R}$ of T_{exp} which consists of sentences simple enough for $\mathcal{R}^{fc}$ to exist - thus: $\llcorner T_{exp} \lrcorner$ exists and is axiomatized by $\llcorner EXP \lrcorner + \mathcal{R}^{fc}$. Let $e(x, y)$ stand for the **relation** ($2^x = y$ *and* $x \in [0, 1]$) and let $\mathbb{R}_e, \mathcal{L}_e, T_e$ denote $(\mathbb{R}, e(x, y))$, its language and its complete theory.

Theorem III.1. *For each axiomatization $\mathcal{R}_e$ of T_e, $EXP + \mathcal{R}_e$ is an axiomatization of T_{exp}.*

Proof. See [7] or [11] (which use a bulkier version of EXP, but P I.1.a showed its equivalence with ours. Strictly speaking the language $\mathcal{L}(e(x, y))$ of $\mathcal{R}$ is not the one of T_{exp}, so that we have to **interpret** $\mathcal{R}_e$ in EXP rather than just add it ; but this is trivial since the intented interpretation of $e(x, y)$ is $2^x|[0, 1]$). □

Now we recall the axiomatization $\mathcal{R}_e$ of T_e due to Rambaud [6], which is simple enough for $\mathcal{R}_e^{fc}$ to exist. Let $t(\bar{X}, Y) = t$ be a real function;

- assume that t is $\mathcal{C}^\infty$ with arguments ranging over $[0,1]$ and for each $\bar{x}y \in [0,1]^{k+1}$ that

$$t(\bar{x},0) > 0 > t(\bar{x},1) \textit{ and } t'_Y(\bar{x},y) < 0$$

then f_t denotes the function of domain $[0,1]^k$ defined by

$$0 < f_t(\bar{x}) < 1 \textit{ and } t(\bar{x}, f_t(\bar{x})) = 0.$$

If the above assumption on t is false we do not introduce f_t.

- We denote by $\overline{t/Y}$ the function t/Y extended by continuity to $Y = 0$ and restricted to $[0,1]^k \times [-1/2, 1/2]$ - in case this function is $\mathcal{C}^\infty$ on $[0,1]^k \times [-1/2, 1/2]$; we skip $\overline{t/Y}$ otherwise.
- We denote by $\mathcal{I}$ the closure under composition and under these operations: $t \longmapsto f_t, t \longmapsto \overline{t/Y}$ of: $\mathcal{L}$, $x^{1/p}$ $(1 < p < \omega)$ and $2^x|[0,1]$.

Theorem III.2. *T_e is axiomatized by $\mathcal{R}_e := RF \cup \{(\forall x \neq 0)\exists y\ xy = 1\} \cup IVD(\mathcal{I}) \cup \Delta$, where Δ is the atomic diagram of the definable constants of T_e and where $IVD(\mathcal{I})$ expresses in the language $\mathcal{L}_e$ the existence and domain of every function f_t or $\overline{t/Y}$ of $\mathcal{I}$, as well as a bound $B_{\overline{t/Y}} \in \omega$ for each function $\overline{t/Y}$.*

Proof. see [6]. □

Below we provide an axiomatization IVD over EXP of the interpretation of $IVD(\mathcal{I})$: IVD is satisfied in a model R_{exp} of EXP iff the interpretation of $\mathcal{L}_e$ in R_{exp} satisfies $IVD(\mathcal{I})$. Then we provide fc-translations IVD^{fc}, Δ^{fc}. Thus we obtain:

Theorem III.3. *$\llcorner T_e \lrcorner$ is axiomatized over $\llcorner EXP \lrcorner$ by $IVD^{fc} + \Delta^{fc}$; hence $\llcorner T_{exp} \lrcorner$ is axiomatized by $\llcorner EXP \lrcorner + IVD^{fc} + \Delta^{fc}$.*

Before proving the theorem we must define $IVD, IVD^{fc}, \Delta^{fc}$; using induction on $t \in \mathcal{I}$ we express its graph by a formula $G_t \in \mathcal{L}(2^x)$: in this way we develop the interpretation of $e(x,y)$ in T_{exp} because G_t uses the function 2^x instead of the relation $e(x,y)$; and IVD consists of $(\forall(\bar{x}) \in D_t \exists! y\ G_t(\bar{x}, y))$ for each $t = f_s$ and of $(\forall(\bar{x}, Y) \in D_t \exists! y \leq B_t\ G_f(\bar{x}, Y, y))$ for each $t = \overline{s/Y}$ - where $D_t := [0,1]^k$ if $t = f_s$, $D_t := [0,1]^k \times [-1/2, 1/2]$ if $t = \overline{s/Y}$.

We define G_t by the obvious clauses: if $t = 2^x|[0,1]$, $G_t(x,y) = (2^x = y \textit{ and } x \in [0,1])$; if $t = x^{1/p}$, $G_t = (0 \leq x \textit{ and } y^p = x)$; if $t(\bar{x}) = f(s^1, ...s^n), G_t(\bar{x}, y) = \exists y^1, ..., y^n, y[G_f(\bar{x}, y^1, ..., y^n, y) \textit{ and } \wedge_j G_{s^j}(\bar{x}, y^j)$; if $t = f_s(\bar{x})$ where $s = s(\bar{x}, \underline{Y)},$ $G_t(\bar{x}, y) = [(\bar{x}) \in D_t \textit{ and } y \in [0,1] \textit{ and } G_s(\bar{x}, y, 0)]$; if $t(\bar{x}, Y) = \overline{s/Y}$, $G_t(\bar{x}, Y) = ((\bar{x}, Y) \in D_t \textit{ and } \forall \epsilon > 0 \exists Z \exists y' \neq 0 \; [G_s(\bar{x}, Z, y') \textit{ and } |Z - Y| < \epsilon \textit{ and } |Zy' - y| < \epsilon]$. Thus IVD is defined and it is clear that R_{exp} satisfies IVD iff the interpretation of $e(x,y)$ in R_{exp} by $(2^x = y \textit{ and } x \in [0,1])$ satisfies $IVD(\mathcal{I})$. We start to fc-translate the $\mathcal{L}(2^x)$-formula $G_t(\bar{x}, y)$; so we assume that

$$R_{exp} \textit{ satisfies } EXP \textit{ and } A_{exp} = \llcorner R_{exp} \lrcorner.$$

By induction on t we define an $\mathcal{L}(x^y)$-formula denoted by $t(\bar{x}) \equiv_\epsilon y$ and which inside A_{exp} expresses that $G_t(\bar{x}, y)$ (in other words the formula $t(\bar{x}) = y$) is satisfied up to an approximation measured by ϵ. The free variables of $t(\bar{x}) \equiv_\epsilon y$ may ultimately range over R but the quantified variables are always restricted to the positive elements of A (called the integers) or to the elements of $\mathbb{Q}(A)$ (called the fractions); in fact, below p, q tacitly range over these "integers" while all other bound variables represent "fractions". We abbreviate by $t(\bar{x}) \equiv_0 y$ the formula $\forall p \exists \bar{a}, r[||(\bar{a}, r) - (\bar{x}, y)|| < 1/p \textit{ and } t(\bar{a}) \equiv_{1/p} r]$; the inductive clauses defining $t(\bar{a}) \equiv_{1/p} r$ will ensure that for all $\bar{x}, y \in R$

$$**) \; G_t(\bar{x}, y) \; \longleftrightarrow \; t(\bar{x}) \equiv_0 y$$

where of course $G_t(\bar{x}, y)$ is interpreted in R_{exp} but where the quantifiers in the formula $t(\bar{x}) \equiv_0 y$ in A_{exp} have the above said interpretation in A. If $t \in \mathcal{L}(2^x)$, then $t \equiv_{1/p} y$ is the formula $[|t - y| < 1/p]^{fc}$ provided by P II.2; if $t = x^{1/n}$, $n < \omega$ then $(t \equiv_{1/p} y) := |y^n - x| < 1/p$. If $t := f_s, s = \underline{s(\bar{x}, Y)}$ then $t(\bar{x}) \equiv_{1/p} y$ is the formula $s(\bar{x}, y) \equiv_{1/p} 0$, and if $t(\bar{x}, Y) := \overline{s/Y}$ then $t(\bar{x}, Y) \equiv_{1/p} y$ is the formula $\exists Z \exists z \neq 0 \; [s(\bar{x}, Z) \equiv_{1/p} z > 0 \textit{ and } |Z - Y| < 1/p \textit{ and } |Zz - y| < 1/p]$. Finally, if $t(\bar{x}) := f(\bar{s})$ then $t(\bar{x}) \equiv_{1/p} y$ is the formula $\exists \bar{y}[f(\bar{y}) \equiv_{1/p} y \textit{ and } \wedge_{j<\ell} (t_j(\bar{x}) \equiv_{1/p} y_j]$.

From R_{exp} to A_{exp} - We assume that R_{exp} satisfies $EXP + IVD$ and see what it implies on A_{exp}; we have that A_{exp} satisfies $\llcorner EXP \lrcorner$ and by T III.1 that R_{exp} satisfies T_{exp}. In particular in R_{exp} we have Δ and the continuity of every function of $\mathcal{I}$. Using in addition the fact that $\mathbb{Q}(A)$ is dense in R, this implies (**) by induction on $t(\bar{x}) \in \mathcal{I}$.

As a consequence A_{exp} satisfies

$$1) \; \forall p \forall (\bar{a}) \in D_t \exists r \; t(\bar{a}) \equiv_{1/p} r$$

when t is one of the functions f_s, or

$$1)\ \forall p \forall (\bar{a}) \in D_t \exists r < B_t\ t(\bar{a}) \equiv_{1/p} r$$

when t is a function $\overline{s/Y}$ whose variables $\bar{x}, Y$ are renamed $\bar{a}$. In addition inside R_{exp} the elements r of (1) converge to an element y when $p \longrightarrow \infty$; in particular they are Cauchy convergent and this is expressed in A_{exp} by

$$2)\ \forall p \exists q \forall (\bar{a}) \in D_t \forall r, r'\ [(t(\bar{a}) \equiv_{1/q} r \ \textit{and}\ t(\bar{a})_{1/q} r') \longrightarrow |r - r'| < 1/p].$$

We let (3) express in A_{exp} that the restriction of t to fractions is continuous:

$$3)\ \forall p \forall (\bar{a}) \in D_t \exists q\ [\forall r, r'(t(\bar{a}) \equiv_{1/q} r \ \textit{and}\ t(\bar{a}) \equiv_{1/q} r') \longrightarrow |r - r'| < 1/p].$$

Finally since T_{exp} implies the uniform continuity of any definable continuous function over a compact box, even uniform continuity is satisfied over D_t; and this is easy to express in A_{exp} - it simply interchanges $\exists q$ with $\forall (\bar{a})$:

$$3)'\ \forall p \exists q \forall (\bar{a}) \in D_t \forall r, r'[(t(\bar{a}) \equiv_{1/q} r \ \textit{and}\ t(\bar{a}) \equiv_{1/q} r') \longrightarrow |r - r'| < 1/p].$$

Now we express in A_{exp} the satisfaction of Δ by R_{exp}; up to equivalence the definable constants of T_e are all elements of $|\Delta| := \{t \in \mathcal{I};\ t \text{ is a function with 0 arguments}\}$; and using in addition the fact that $\mathcal{I}$ is closed under $+, -, \times, 2^x[0,1]$ we can reduce Δ to $\{t = 0; t \in |\Delta| \ \textit{and}\ t = 0\} \cup \{t' \neq 0; t' \in |\Delta| \ \textit{and}\ t' \neq 0\}$. Then $\Delta^{fc} := \{t \equiv_0 0; t \in |\Delta| \ \textit{and}\ t = 0\} \cup \{t' \not\equiv_0 0; t' \in |\Delta| \ \textit{and}\ t' \neq 0\}$ is satisfied by A_{exp}. In conclusion we proved:

Proposition III.4. *Assume that R_{exp} satisfies $EXP + IVD$*

a) For all $\bar{x}, y \in R$, R_{exp} satisfies $G_t(\bar{x}, y)$ iff $t(\bar{x}) \equiv_0 y$.

b) A_{exp} satisfies $(1 + 2 + 3')$ and Δ^{fc}.

From A_{exp} to R_{exp} - Looking for the reciprocal we assume that A_{exp} satisfies $(1+2+3) + \Delta^{fc}$; this time we assume in addition that $R^c = R$.

Nota Bene - We could assume $(3)'$; but this is not used in the proof (hence $(3)'$ is consequence of the other axioms).

We shall prove that R_{exp} satisfies $IVD + \Delta$ hence using P III.4 that $IVD^{fc} + \Delta^{fc}$ is a global fc-translation of $IVD + \Delta$ over $\llcorner EXP \lrcorner$. Hence also that R_{exp} satisfies T_{exp}; so that $\llcorner EXP \lrcorner + IVD^{fc} + \Delta^{fc}$ axiomatizes $\llcorner T_{exp} \lrcorner$ - proving T III.3.

We denote $\mathcal{I}_n$ the set of functions of $\mathcal{I}$ whose definition involves at most n nested operations $s \longmapsto f_s$ or $s \longmapsto \overline{s/Y}$; we make an inductive assumption IH_n:

a) $(**)$ holds for each $s \in \mathcal{I}_n$

b) R_{exp} satisfies IVD_n:= restriction of IVD to $\mathcal{I}_n$.

For $n = 0$ this is true because IVD_0 is empty and because $(**)$ for $\mathcal{I}_0$ follows from the continuity of every function in $\mathcal{I}_0$; so we fix an arbitrary n. For each $s \in \mathcal{I}_n$, from (1+2) for $t := f_s$ follows that for all fractions $(\bar{a}) \in D_t$ there is a unique $y \in R^c = R$ such that $t(\bar{a}) \equiv_0 y$; we set $f_s^*(\bar{a}) := y$. From (3) follows that $f_s^* =: t^*$ is continuous over D_t; by definition of t^* we have $s(\bar{a}, t^*(\bar{a})) \equiv_0 0$ and using IH_n it implies $s(\bar{a}, t^*(\bar{a})) = 0$ hence R_{exp} satisfies $G_t(\bar{a}, t^*(\bar{a}))$. In addition for any element $y \in [0,1]$ satisfying $G_t(\bar{a}, y)$ hence $s(\bar{a}, y) = 0$ the continuity of s and its subterms yields for each p a fraction r such that $|y - r| < 1/p$ *and* $s(\bar{a}) \equiv_{1/p} 0$; by axiom (2) for t these approximations r of y also converge to $t^*(\bar{a})$. Thus t^* equals t and $\exists! y G_t(\bar{a}, y)$ holds on $\mathbb{Q}(A)$; finally, the continuity of t provides a unique continuous extension of t^* to R, still denoted t^*. By continuity R_{exp} keeps satisfying $G_t(\bar{a}, t^*(\bar{x}))$ - hence $t^* = t$, and it becomes easy to prove IH_{n+1} as far as t is concerned. A similar argument works for $t = \overline{s/Y}$; then preservation of continuity under composition suffices to deal with the case $t = f(s_0(\bar{x}, ... s_k(\bar{x}))$, so that all of IH_{n+1} is proved.

T III.3 □

Section IV - Conclusion

1. For a complete picture of the situation we will have to expose the remaining topics of [2]: correspondance between the definable functions of T_{exp} and the provable functions of $\llcorner T_{exp} \lrcorner$; provable polytime witnessing for $\llcorner T_{exp} \lrcorner$; existence of integral parts in exponential fields; "blunt" arithmetics.

2. The definition of $\overline{s/Y}$ hence of $\mathcal{I}$ is not given in an effective way: for what it says, the code for the language of $\mathcal{I}$ could be a Π^0_1-complete set of integers (or even worse, because the property "$\overline{t/Y}$ is a $\mathcal{C}^\infty$ function" does not even present it as arithmetical!). So one is not satisfied with $\mathcal{I}$ hence with IVD, IVD^{fc} and one would like to replace them by **recursive** sets $\mathcal{J}, IVD(\mathcal{J}), IVD^{fc}(\mathcal{J})$. To that end the other axiomatizations of T_e, T_{exp} provided by [12], [8] and [4] could be used;

thus we should obtain an **effective** axiomatization of $\llcorner T_{exp} \lrcorner$ over $\llcorner \Delta \lrcorner$. We expect that this recursive axiomatization would no longer rely on division so that it would not use the bound B_t on $t = \overline{s/Y}$ which is asserted by IVD. This non effective bound B_t is the only part of IVD which does not seem deducible from a suitable induction scheme by using the ideas and methods of Section II ; so that it should be quite possible to provide for $\llcorner T_{exp} \lrcorner$ an almost perfect analog of T I.2 and T II.4.

3. Rambaud's and Ressayre's axiomatizations of T_e and of T_{exp} work if the restricted exponential e is replaced by a whole set E of restricted o-minimal and polynomially bounded functions; this easily implies an extension of Section III where T_{exp} is replaced by the complete theory of $(\mathbb{R}_{exp}, E)$. The most interesting case is the set E which allows to define the Γ function inside $(\mathbb{R}_{exp}, E)$. This raises the problem if we can extend the present work to (x^y, Γ)-integral parts; the problem seems accesssible.

4. Other open problems are mentioned in [2], more ambitious... and less precise. Let us add one more ambition: observe that we established some cases of a duality between an arithmetical theory and a theory of fields; and that this duality was used in the direction which takes advantage of known results about fields in order to prove new results about arithmetics. The question is: what about the opposite direction?

Section V - APPENDIX

We prove the results admitted in Section I and II; it is assumed that the reader makes occasional use of this appendix while reading these sections. So the statements from Section I and II which we prove here are not repeated; however they are renumbered for the sake of coherence with auxiliary results introduced here (the initial numbering is recalled in parenthesis).

Section V.1

We recall $\mathcal{A}_0$ (where all variables tacitly range over the positive integers):

- $DUCR$
- $x^{y+z} = x^y x^z, x^{-y} x^y = 1$; $x^{yz} = (x^y)^z$; x^y strictly increasing with respect to $x, y > 1$
- $2 < x \longrightarrow x^2 < 2^x$

- $\exists y\ 2^y \leq z < 2^{y+1}$; $\exists x\ x^z \leq 2^y < (x+1)^z$; $\exists y\ 2(x^y) < (x+1)^y$; $\exists x\ (x+1)^y < 2(x^y)$.

We consider a model A of $\mathcal{A}_0$

Fact V.1.1. *(= Fact I.4)*

Proof.

1. Consider $\frac{p}{q} \in log\ x$ and $\frac{p'}{q'} < \frac{p}{q}$. We have: $2^p \leq x^q$ and $p'q < pq'$. Then: $2^{p'q} < 2^{pq'} \leq x^{qq'}$, and $2^{p'} < x^{q'}$ which leads to $\frac{p'}{q'} \in log\ x$.
2. By $\mathcal{A}_0$, for all $q \in A$, there exists $p \in A$ such that: $2^p \leq x^q < 2^{p+1}$. So $\frac{p}{q} \in log\ x$, $\frac{p+1}{q} > log\ x$ and $\frac{p+1}{q} - \frac{p}{q} \leq \frac{1}{q}$; thus $log\ x$ is a Cauchy cut.
3. $\frac{p}{q} \in log\ 2^x$ iff $2^p \leq 2^{xq}$ iff $p \leq xq$ iff $\frac{p}{q} \leq x$.
4. $\frac{p}{q} \in log\ x^y$ iff $2^p \leq x^{yq}$ iff $\frac{p}{yq} \in log\ x$ iff $\frac{p}{q} \in y\, log\ x$.
5.

- We prove that $\forall\, r \in log\ xy\ \forall\, r_1 > log\ x\ \ \forall r_2 > log\ y\ r < r_1 + r_2$.

For $r =: \frac{a}{d}$, $r_1 =: \frac{b}{d}$, $r_2 =: \frac{c}{d}$ we have: $2^a \leq (xy)^d$, $x^d < 2^b$ and $y^d < 2^c$. Then $2^a \leq (xy)^d < 2^b\, 2^c$, so $a < b + c$ and $r < r_1 + r_2$.

- We prove that $\forall\, r_1 \in log\ x\ \ \forall r_2 \in log\ y\ \ \forall\, r > log\ xy\ r_1 + r_2 < r$.

For $r =: \frac{a}{d}$, $r_1 =: \frac{b}{d}$, $r_2 =: \frac{c}{d}$ we have: $2^b \leq x^d$, $2^c \leq y^d$ and $(xy)^d < 2^a$. Then $2^b\, 2^c \leq (xy)^d < 2^a$, hence $b + c < a$ and $r_1 + r_2 < r$.

6. Trivial.

□

Notice that the last point (6.) legitimizes the Definition I.5.1 of the log for fractions. Furthermore $log_{|0}\, 1 = 0$, then $log_{|1}\, \frac{x}{1} = log_{|0}\, x - log_{|0}\, 1 = log_{|0}\, x$, hence the value of the log of an integer given by Definition I.3 is the same as that given by Definition I.5.1 for rationals. In the same vein Fact V.1.2 will prove that the log of a rational given by the Definition I.5.1 (for rationals) is the same as that given by Definition I.5.2 (for reals). And Proposition V.1.5 will prove that for all x, y in A, the value of x^y in A is the same as that given by the Definition I.5.4 for reals.

In Fact V.1.2 and V.1.4 below we prove properties of the cuts $log\ c$, $c \in R$ **before** knowing that they are Cauchy cuts (ie that $log\ c \in R$) - something we will prove in Proposition V.1.5.

Fact V.1.2. *(= Fact I.6)*

Proof.

1. $log\ \frac{p}{q} = log\ p - log\ q$, and $log\ p$, $log\ q$ are in R.
2. Trivial.
3. Trivial.
4. Let $\frac{p}{q} \in log\ x$. Then $2^p \leq x^q \leq y^q$, and thus $\frac{p}{q} \in log\ y$.
5. Put $r = \frac{a}{b}$ and $r' = \frac{a'}{b}$ and assume that $r \leq r'$. Then $a \leq a'$, and $log\ r' - log\ r = log\ a' - log\ b - (log\ a - log\ b) = log\ a' - log\ a \geq 0$.
6. Notice that the preceding point (5.) says: $s \leq r \rightarrow log_{|1}\ s \leq log_{|1}\ r$. We prove that the cuts $log_{|1}\ r$ and $log_{|2}\ r$ are equal.

- Assume that $\rho \leq log_{|1}\ r$. There exists $s \leq r$ such that $\rho \leq log_{|1}\ s$ - namely $s = r$; thus $\rho \in log_{|2}\ r$.
  (It shows that $log_{|1}\ r \leq log_{|2}\ r$.)
- Assume that $\rho \in log_{|2}\ r$; then there exists $s \leq r$ such that $\rho \leq log_{|1}\ s$. As $s \leq r$, $log_{|1}\ s \leq log_{|1}\ r$ and thus $\rho \leq log_{|1}\ r$.
  (It shows that $log_{|2}\ r \leq log_{|1}\ r$.)

7. By $\mathcal{A}_0\ \exists y\ \ 2x^y < (x+1)^y$. Then $log\ 2x^y \leq log\ (x+1)^y$, $log\ 2 + log\ x^y \leq log\ (x+1)^y$, $1 + y\,log\ x \leq y\,log\ (x+1)$ and thus $log\ (x+1) - log\ x > \frac{1}{y}$.
8.

- Case $x < y$. Then $x < x+1 \leq y$, $log\ x < log(x+1) \leq log\ y$ and thus $log\ x < log\ y$.
- Case $x \geq y$. If $x = y$ then $log\ x = log\ y$. If $x > y$ then $log\ x > log\ y$.

9. $\frac{p}{q} < \frac{p'}{q'}$ iff $pq' < p'q$ iff $log\ (pq') < log\ (p'q)$ iff $log\ p + log\ q' < log\ p' + log\ q$ iff $log\ \frac{p}{q} < log\ \frac{p'}{q'}$.
10. $r \leq 2^a$ iff $log\ r \leq log\ 2^a = a$ iff $r \in 2^a$ (the last 2^a is the cut defined for reals).
11. $\rho \in log\ c$ iff it exists $r \leq c$ such that $\rho \leq log\ r$ iff $\rho \in sup\,\{log\ r;\ r \leq c\}$.
12. Fix r and r' such that $r < r' < c$; then $log\ r < log\ r'$. Fix s such that $log\ r < s < log\ r'$. Then $log\ r < log\ c$ because $s \in log\ c$ and $log\ r < s$.
13. Case $r > c$: fix r' such that $r > r' > c$; then

$$log\ r > log\ r' \geq sup\{log\ s\,;\ c \geq s\} = log\ c.$$

Case $c < c'$: fix r such that $c < r < c'$. Then $log\ c < log\ r < log\ c'$.

□

Definition V.1.3. *Let X be a subset of R ; X is* **pseudo-integral** *if:*

- *X is cofinal in R,*
- *for all $x \in X$, there exists $y \in X$, $y > x$ and: $\forall z \in X \ \ x < z \to y \leq z$. y is called the successor of x in X and denoted $succ_X(x)$,*
- *for all $x \in X$, $inf\{z - y;\ y, z \in X$ and $y < z \leq x\} > 0$*.*

We will prove and use that $log(A_*^+)$ is pseudo-integral (a name chosen to point out a weak similarity with integral parts).

Fact V.1.4. *(p, q, x, y, z range over A_+^*, r over $\mathbb{Q}(A)$ and c, c' over R)*

1. *$\exists y \forall z \leq x \ \ log\ (z+1) - log\ z > \frac{1}{y}$.*
2. *$\exists x \ \ log\ x \leq \frac{p}{q} < log\ (x+1)$.*
3. *Let X be a pseudo-integral subset of R such that each positive rational is between two successive elements of X; then each positive real is between two successive elements of X.*
4. *$\forall c \geq 0 \exists x \ \ log\ x \leq c < log\ (x+1)$.*
5. *$\exists x \ \ log\ \frac{x}{y} \leq c < log\ \frac{x+1}{y}$. Then $\forall y \exists x \ \ log\ \frac{x}{y} < c < log\ \frac{x+2}{y}$.*
6. *$\exists x \ \ log\ (x+1) - log\ x < \frac{1}{y}$. And thus $\lim_{x \to +\infty} log\ (x+1) - log\ x = 0$, in other words: $\lim_{x \to +\infty} log\ (1 + \frac{1}{x}) = 0$.*
7. *$c' > log\ c$ iff it exists $r > c$ such that $c' > log\ r > log\ c$. That remains true if we replace $>$ by $<$.*

Proof.

1. By V.1.2.7 there exists y such that $log\ (x+1) - log\ x > \frac{1}{y}$. But $\forall z \leq x \ \ (z \geq 1 \to \frac{1+z}{z} \geq \frac{1+x}{x})$. Then $log\ \frac{1+z}{z} \geq log\ \frac{1+x}{x}$, and thus $log\ (z+1) - log\ z \geq log\ (x+1) - log\ x > \frac{1}{y}$
2. By $\mathcal{A}_0$, for all p, q there exists x such that $x^q \leq 2^p < (x+1)^q$; take the log of the inequalities.
3. Let $X' = \{\, x \in X;\ x < c \,\}$; X' is not empty (let $r < c$, there exists $x \in X$ such that $x \leq r \leq succ_X(x)$, then $x \in X'$).
   We prove (3.) by way of contradiction, so we assume: (H) $x < c \to succ_X(x) < c$.
   We are going to see that $sup\, X' = c$. On the one hand $sup\, X' \leq c$ (trivial). On the other hand $c \leq sup\, X'$, because if $r \in \mathbb{Q}(A)$ and $r < c$, then there exists $x \in X$ such that $x \leq r < succ_X(x)$, then $x < c$, and by (H) $succ_X(x) < c$. From which $succ_X(x) \in X'$ and as $r < succ_X(x)$, then $r \in supX'$. So $c \leq supX'$.

---

* $inf\, X > 0$ means $\exists \mu > 0\ \forall x \in X\ x > \mu$.

Besides, fix $x_0 \in X$ and $x_0 > c$ (X is cofinal in R), we have $0 = \delta(c) = \delta(sup\, X') \geq inf\,\{|z-y|;\ y, z \in X'$and distinct$\} \geq inf\,\{z - y;\ y, z \in X$ and $y < z < x_0\} > 0$ (The first inequality is true because $X' \subset X$ and $X' < x_0$; the last inequality because X discrete). We reached a contradiction hence (3.) is proved: for each $c > 0$ there exists $x \in X$, $x < c \leq succ_X(x)$.

4. Put $X = \{\, log\, x\,;\ x \geq 1\}$. X is pseudo-integral in R. Indeed:

- X is cofinal in R ($log\ 2^x = x$).
- $log\ (x+1)$ is the successor of $log\ x$.
- For all x, there exists B such that $log\ (x+1) - log\ x > \frac{1}{B}$, and thus whenever $y < z \leq x$, $log\ z - log\ y \geq log\ (y+1) - log\ y > log\ (x+1) - log\ x > \frac{1}{B}$.
- By (2.) each positive rational is between two successive elements of X.

  Then by (3.) it is the same for all positive element of R.

5. $log\, y \in R$, then $c + log\, y \in R$. Apply (4.) to $c + log\, y$: there exists x such that $log\, x \leq c + log\, y < log\ (x+1)$.
6. By $\mathcal{A}_0$, for all y there exists x such that $(x+1)^y < 2x^y$. Thus $y\, log\ (x+1) < 1 + y\, log\, x$, and $y(log\ (x+1) - log\ x) < 1$.

   Fix $z > x$. Then $\frac{z+1}{z} < \frac{x+1}{x}$, and thus: $log\ \frac{z+1}{z} < log\ \frac{x+1}{x} < \frac{1}{y}$.
7. Suppose $log\ c < c'$. Consider y such that $\frac{x}{y} < c < \frac{x+2}{y}$ and $log\ (x+2) - log\ x < c' - log\ c$. Then: $log\ \frac{x}{y} < log\ c < log\ \frac{x+2}{y} < c'$. □

Now we prove that log is a one-to-one correspondance between R^*_+ and R and 2^x is it's inverse.

Proposition V.1.5. *(= Proposition I.7)*

Proof.

1. Notice that $log\ c$ is a cut. Assume that $c \notin \mathbb{Q}(A)$; then for all y, there exists x such that $x < y\, c < x+1$.

   Then $log\ \frac{x}{y} < log\ c < log\ \frac{x+1}{y}$. Let ρ, ρ' in $\mathbb{Q}(A)$ such that: $log\ \frac{x}{y} < \rho < log\ c < \rho' < log\ \frac{x+1}{y}$. Then $\rho < log\ c < \rho'$ and $|\rho' - \rho| < log\ \frac{x+1}{y} - log\ \frac{x}{y} = log\ (x+1) - log\ x$ which limit is 0 as x approaches infinity.
2. Notice that 2^c is a cut. For all y there exists x such that $log\ \frac{x}{y} < c < log\ \frac{x+2}{y}$. Then $\frac{x}{y} \in 2^c$, $\frac{x+2}{y} \in \overline{2^c}$ and $\frac{x+2}{y} - \frac{x}{y} = \frac{2}{y}$ which limit is 0 as y approaches infinity.

3. $r \in 2^{log\ c}$ iff $log\ r \leq log\ c$ iff $r \leq c$.
4. Given $x, y \in A$. Put e the value of x^y in A ($e = x^y$). By Fact V.1.1.4 $log_{|1}\, e = y\, log_{|1}\, x$. Put $c = 2^{y\, log_{|2}\, x}$. By V.1.2.6, $log_{|1}\, z = log_{|2} z$, for all $z \in A$.
   Then $e = 2^{log_{|2} e} = 2^{log_{|1}\, e} = 2^{y\, log_{|1}\, x} = 2^{y\, log_{|2}\, x} = c$.
5.

- Let $r \in log\ 2^c$. Then there exists $r' \leq 2^c$ such that $r \leq log\ r'$; and $r' \in 2^c$ (otherwise $log\ r' \geq c$). Then there exists ρ such that $log\ r' > log\ \rho > c$, then $r' > \rho$ and $\rho \notin 2^c$ and thus $r' > 2^c$. Since $r' \in 2^c$, $log\ r' \leq c$, and thus $r \leq c$.
- Let $r < c$. Then, for all y rather big, there exists x such that $r < log\ \frac{x}{y} < c < log\ \frac{x+2}{y}$. Then $\frac{x}{y} \in 2^c$, and $\frac{x}{y} \leq 2^c$. But $r < log\ \frac{x}{y}$, then $r \in log\ 2^c$. □

At last, we prove that 2^x is an increasing homomorphism of $+$ on $\times$ (restricted to positive elements), continuous...

Proposition V.1.6. *(= Proposition I.8)*

Proof.

1. 2^x is the inverse of log which is strictly increasing.
2. In this proof x and y are in R and the other variables are in $\mathbb{Q}(A)$.

- Let $\rho \in log\ xy$ and $\rho_1 > log\ x$ and $\rho_2 > log\ y$. let us prove that $\rho < \rho_1 + \rho_2$.
  Consider $r \leq xy$ such that $\rho \leq log\ r$; given $s > x$ and $s' > y$ such that $\rho_1 > log\ s > log\ x$ and $\rho_2 > log\ s' > log\ y$, $r \leq xy$ implies $r < ss'$ and $\rho \leq log\ r < log\ ss' = log\ s + log\ s' < \rho_1 + \rho_2$.
- Fix $\rho_1 < log\ x$ and $\rho_2 < log\ y$ and $\rho > log\ xy$; Let's prove that $\rho_1 + \rho_2 < \rho$.
  Assume $s < x$ and $s' < y$ such that $\rho_1 < log\ s$ ($< log\ x$) and $\rho_2 < log\ s'$ ($< log\ y$). Consider $r > xy$ such that $\rho > log\ r$ ($> log\ xy$); then $r > ss'$ and $\rho > log\ r > log\ ss' = log\ s + log\ s' > \rho_1 + \rho_2$.

3. $log\ x^y = log\ 2^{y\, log\ x} = y\, log\ x$.
4. $log\ 2^{x+y} = x + y = log\ 2^x + log\ 2^y = log\ 2^x 2^y$. log one-to-one, then $2^{x+y} = 2^x\, 2^y$.
5. $x^y < x^z$ iff $log\ x^y < log\ x^z$ iff $y\, log\ x < z\, log\ x$ iff $y < z$, because $log\ x > 0$.
6. $y^x < z^x$ iff $log\ y^x < log\ z^x$ iff $x\, log\ y < x\, log\ z$ iff $y < z$.
7. $log\ x^{y+z} = (y + z)\, log\ x = y\, log\ x + z\, log\ x = log\ x^y + log\ x^z = log\ x^y\, x^z$. Since log is injective $x^{y+z} = x^y\, x^z$.

8. $log\ (x^y)^z = z\,log\ x^y = zy\,log\ x = log\ x^{yz}$. Since log is one-to-one $(x^y)^z = x^{yz}$.
9. log , 2^x are strictly increasing and one-to-one correspondance.
10. Exercice. □

Section V.2

We consider a model A of $IE_0(2^x)^{fc}$ - including the axiom A_λ:

$$(0 < r < s \rightarrow \quad (1+s)^r < (1+r)^s) \wedge (\exists\, s < \frac{1}{B}\ \forall\, r\ \ \frac{(1+r)^{Bs}}{(1+s)^{Br}} < 2^{rs}).$$

Fact V.2.1.

1. *$log\,'(1)$ is a Cauchy cut definable in A (hence log and exponential polynoms are differentiable).*
2. *f exponential polynom and X bounded subset of $R^m \longrightarrow f(X)$ is bounded.*
3. *(Uniform continuity) $X \subset R^m$ and $Y \in R^n$ bounded set imply:*

$$\forall \epsilon > 0 \exists \alpha > 0 \forall \overline{x} \in X \forall \overline{\xi} \in X (|\overline{\xi} - \overline{x}| < \alpha \rightarrow \forall \overline{y} \in Y |f(\overline{\xi}, \overline{y}) - f(\overline{x}, \overline{y})| < \epsilon).$$

Proof.

1. We consider the cut $c = \{r, \exists s > 0 \quad r < \frac{log\ (1+s)}{s}\}$ and we prove that $c \in R$ and $c = log\,'(1)$.

- c is bounded.
  By A_λ, for all B there exists $s_B < \frac{1}{B}$ such that: $\forall\, r\ \ \frac{(1+r)^{Bs}}{(1+s_B)^{Br}} < 2^{rs}$; or equivalently: for all $r > 0$ $\frac{log\ (1+r)}{r} < \frac{1}{B} + \frac{log\ (1+s_B)}{s_B}$. From which follows that $c \leq \frac{1}{B} + \frac{log\ (1+s_B)}{s_B}$.
- $c \in R$.
  Otherwise fix ρ such that $0 < \rho < \delta(c)$. Consider B such that $\frac{1}{B} < \rho$ and fix $s_B < \frac{1}{B}$ such that:
  $\forall\, r\ \ \frac{log\ (1+r)}{r} < \frac{1}{B} + \frac{log\ (1+s_B)}{s_B}$ (*),
  Then $\frac{log\ (1+s_B)}{s_B} < c$ hence $\frac{log\ (1+s_B)}{s_B} + \rho < c$ and there exists r $(< s_B)$ such that:
  $(\ c >)\ \frac{log\ (1+r)}{r} > \frac{log\ (1+s_B)}{s_B} + \rho > \frac{log\ (1+s_B)}{s_B} + \frac{1}{B}$, from which follows that:
  $\frac{log\ (1+r)}{r} > \frac{log\ (1+s_B)}{s_B} + \frac{1}{B}$, in contradiction to (*).

- By A_λ, for all r and s in $\mathbb{Q}(A)$ we have: $0 < r < s \rightarrow \quad (1+s)^r < (1+r)^s$; thus $0 < r < s \rightarrow \quad \frac{log\,(1+s)}{s} < \frac{log\,(1+r)}{r}$.
  Since log is continuous, for all x and y in R we have $0 < x < y \rightarrow \frac{log\,(1+y)}{y} < \frac{log\,(1+x)}{x} < c$.
- c is the right derivative of log.
  Indeed, fix $\epsilon > 0$; there exists $s \in \mathbb{Q}(A)$ such that $0 < c - \frac{log\,(1+s)}{s} < \epsilon$.
  But $\forall\, r \in\ R\ \ (0 < r < s \rightarrow \frac{log\,(1+s)}{s} < \frac{log\,(1+r)}{r} < c)$.
  Then $\forall\, r \in\ R\ \ (0 < r < s \rightarrow 0 < c - \frac{log\,(1+r)}{r} < \epsilon)$.
- c is the left derivative of log.
  Indeed, $\frac{log\,(1-r)}{-r} = \frac{log\,(\frac{1}{1-r})}{r} = (1-r)\,\frac{log\,(1+\frac{r}{1-r})}{\frac{r}{1-r}}$.
  Then $\lim_{r\rightarrow 0^+} \frac{log\,(1-r)}{-r} = \lim_{r\rightarrow 0^+} (1-r)\,\frac{log\,(1+\frac{r}{1-r})}{\frac{r}{1-r}} = \lim_{t\rightarrow 0^+} \frac{log\,(1+t)}{t} = c$.
- $log'\,(x) = \frac{c}{x}$.
  Indeed, $\frac{log\ y - log\ x}{y-x} = \frac{1}{x}\frac{log\ \frac{y}{x}}{\frac{y}{x}-1}$.
  Then $\lim_{y\rightarrow x} \frac{log\ y - log\ x}{y-x} = \frac{1}{x}\ \lim_{y\rightarrow x} \frac{log\ \frac{y}{x}}{\frac{y}{x}-1} = \frac{c}{x}$.
- $(2^x)' = \frac{1}{c}\,2^x$; 2^x is inverse to log .

2. By induction on f.
3. Idem. □

The next Fact states that if an exponential polynom changes sign on a given interval, then it takes arbitrary small values.

Fact V.2.2. *(= Fact II.6)*

Proof.

1. Put $I = [\rho, \sigma]$. Fix $\epsilon < \frac{1}{N}$. By uniform continuity there exists α such that: $\forall\, t,\, t' \in I\ \ (|t'-t| < \alpha \rightarrow\ |f(\overline{r}, t') - f(\overline{r}, t)| < \epsilon)$.
   Consider N_0 such that $\frac{1}{N_0} < \alpha$.
   Put $\rho =: \frac{p_0}{q}$, $\sigma =: \frac{p_1}{q}$ and $g(x) := f(\overline{r}, \frac{x}{N_0 q})$.
   Then, $g(N_0 p_0) < 0 < g(N_0 p_1)$. A satisfies $IE_0(2^x)^{fc}$ hence there exists $x \in A$ such that $N_0 p_0 \le x < N_0 p_1$ and $g(x) < 0 < g(x+1)$.
   Put $t = \frac{x}{N_0 q}$ and $t' = \frac{x+1}{N_0 q}$. Then:

- t and t' are in I because $\rho = \frac{N_0 p_0}{N_0 q} \le \frac{x}{N_0 q} < \frac{x+1}{N_0 q} \le \frac{N_0 p_1}{N_0 q} = \sigma$
- $|t'-t| < \alpha$ because $t' - t = \frac{1}{N_0 q} < \alpha$.
  Then $|f(\overline{r}, t') - f(\overline{r}, t)| < \epsilon$.

$|f(\overline{r},t)| < \frac{1}{N}$. Indeed, $f(\overline{r},t) < 0 < f(\overline{r},t')$, $|f(\overline{r},t') - f(\overline{r},t)| < \epsilon$, and thus $|f(\overline{r},t)| < |f(\overline{r},t') - f(\overline{r},t)| < \epsilon < \frac{1}{N}$.

2. Fix $\epsilon < \frac{1}{2N}$; by uniform continuity there exists α such that:
$\forall \overline{b}\ (|\overline{b} - \overline{a}| < \alpha \rightarrow \forall t \in [c_0, c_1]\ |f(\overline{b},t) - f(\overline{a},t)| < \epsilon)$.
By continuity there exists $\overline{r}$ such that $|\overline{r} - \overline{a}| < \alpha$ and $f(\overline{r}, c_0) < 0 < f(\overline{r}, c_1)$, and there exists $[\rho,\sigma] \subset]c_0, c_1[$ such that $f(\overline{r},\rho) < 0 < f(\overline{r},\sigma)$.
By 1 there exists $t \in [\rho,\sigma]$ such that $|f(\overline{r},t)| < \frac{1}{2N}$.
But $|\overline{r} - \overline{a}| < \alpha$, so $|f(\overline{r},t) - f(\overline{a},t)| < \epsilon < \frac{1}{2N}$.
Hence $|f(\overline{a},t)| \leq |f(\overline{r},t) - f(\overline{a},t)| + |f(\overline{r},t)| < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N}$. □

Fact V.2.3 and Propositions V.2.4, V.2.5 contribute to the proof of Proposition V.2.6 which is a weak version of Rolle.

Fact V.2.3. *Let f be a 2^x-polynomial with parameters in $\mathbb{Q}(A)$; given $r_0 < r_1$ in $\mathbb{Q}(A)$ s.t. $f(r_0) < f(s_0)$, for all $N > 0$ in A there exists $r < s$ such that $|r - s| < \frac{1}{N}$, $r_0 < r < s < s_0$ and $f(r) < f(s)$.*

Proof.

Let us assume that $f(r_0) < f(s_0)$. By continuity and density, there exists $r_1 < s_1$ in $\mathbb{Q}(A)$ s.t $r_0 < r_1 < s_1 < s_0$ and $f(r_1) < f(s_1)$.

Consider $N \in A$, $N > 0$, and $\ell \in A$ s.t $|s_1 - r_1| < \ell$. Set

$$g(x) = f(r_1 + x\frac{s_1 - r_1}{\ell N}), h(x) = g(x) - g(0)$$

We have: $h(0) \leq 0$ and $h(\ell N) > 0$.

Then according to the axiom of induction (in $IE_0(2^x)^{fc}$) there exists i between 0 and $\ell N - 1$ s.t. $h(i) \leq 0$ and $h(i+1) > 0$.

Therefore $g(i) \leq g(0)$ and $g(i+1) > g(0)$, and then

$$g(i) < g(i+1).$$

Let us put $r = r_1 + i\frac{s_1 - r_1}{\ell N}$ and $s = r_1 + (i+1)\frac{s_1 - r_1}{\ell N}$; then we have:
$r_0 < r1 \leq r < s \leq s_1 < s_0$, $|r - s| < \frac{1}{N}$ and $f(r) < f(s)$. □

Proposition V.2.4. *Assume that A is countable. Given $x_0 < x_1$ in R and a 2^x-polynomial f with parameters in $\mathbb{Q}(A)$,*

1. *if $f(x_0) < f(x_1)$ there exists $c \in]x_0, x_1[$ s.t $f'(c) \geq 0$*
2. *if $f(x_0) > f(x_1)$ there exists $c \in]x_0, x_1[$ s.t $f'(c) \leq 0$.*

Proof.

2. Apply (1.) to $-f$.

1. Let us assume $f(x_0) < f(x_1)$;
   by continuity and density there exists $r_0 < s_0$ in $\mathbb{Q}(A)$ s.t $x_0 < r_0 < s_0 < x_1$ and $f(r_0) < f(s_0)$.
   Given (N_k) cofinal in A, if we have $r_k < s_k$ s.t. $f(r_k) < f(s_k)$ then by Fact V.2.3 there exists r_{k+1} and s_{k+1} s.t.:
   $$r_k < r_{k+1} < s_{k+1} < s_k, |r_{k+1} - s_{k+1}| < \frac{1}{N_{k+1}}, f(r_{k+1}) < f(s_{k+1}).$$
   The diameter of the cut $c = \{r \in \mathbb{Q}(A);$ there exists k s.t. $r \leq r_k\}$ is 0; and $c \in]x_0, x_1[$. We claim that $f'(c) \geq 0$;
   otherwise $f'(c) < 0$ and there exists ϵ s.t. for all $x \in]c - \epsilon, c[$ for all $x' \in]c, c + \epsilon[$:
   $f(x) > f(c)$, $f(x') < f(c)$ hence $f(x) > f(x')$.
   Fix k_0 s.t. for all $k > k_0$ $r_k \in]c - \epsilon, c[$ and $s_k \in]c, c + \epsilon[$; then $f(r_k) > f(s_k)$ for all $k \geq k_0$. This contradicts the construction of (r_k) and (s_k). □

The next Proposition is a first order property of A_{exp} because it does not involve the Cauchy closure R of $\mathbb{Q}(A)$; consequently it is true as soon as it holds for countable A. This allows to apply V.2.4 to its proof.

Proposition V.2.5. *Given a 2^x-polynomial f and given $\overline{\sigma}$, ρ_0 and ρ_1 $\in \mathbb{Q}(A)$ with $\rho_0 < \rho_1 . f(\overline{\sigma}, x)$.*
If $f(\overline{\sigma}, \rho_0) = f(\overline{\sigma}, \rho_1)$, then we have:

- *either $\exists\, r_0, r_1 \in]\rho_0, \rho_1[\cap \mathbb{Q}(A)$ $f'(\overline{\sigma}, r_0)$ $f'(\overline{\sigma}, r_1) < 0$,*
- *or $\forall\, N\ \exists\, t \in]\rho_0, \rho_1[\cap \mathbb{Q}(A)$ $|f'(\overline{\sigma}, t)| < \frac{1}{N}$.*

Proof. We shall write $f(x)$ instead of $f(\overline{\sigma}, x)$; we can suppose that f is not constant in $[\rho_0, \rho_1]$. Fix x_0 in $]\rho_0, \rho_1[$ s.t. $f(x_0) \neq f(\rho_0)$.

Case 1: $f(x_0) > f(\rho_0)$.

Then, by continuity there exists $x_1 < x_0$, $x_1 \in]\rho_0, \rho_1[$ and there exists $x_2 > x_0$, $x_2 \in]\rho_0, \rho_1[$ s.t.

$$f(x_1) < f(x_0), f(x_2) < f(x_0).$$

By Proposition V.2.4, there exists $c_0 \in]x_1, x_0[$ s.t. $f'(c_0) \geq 0$ and there exists $c_1 \in]x_0, x_2[$ s.t. $f'(c_1) \leq 0$.

- Case 1.1: $f'(c_0) = 0$ or $f'(c_1) = 0$.

Let us assume for instance that $f'(c_0) = 0$. Since $c_0 \in]\rho_0, \rho_1[$ by continuity and density, for all $N \in A$, $N > 0$ there exists $t \in]\rho_0, \rho_1[\cap \mathbb{Q}(A)$ s.t. $|f'(t)| < \frac{1}{N}$.

- Case 1.2: $f'(c_0) \neq 0$ and $f'(c_1) \neq 0$.

Then $f'(c_0) > 0$ and $f'(c_1) < 0$. Since c_0 and c_1 are in $]\rho_0, \rho_1[$ by continuity and density there exists r_0 and r_1 in $]\rho_0, \rho_1[\cap\mathbb{Q}(A)$ s.t. $f'(r_0) > 0$ and $f'(r_1) < 0$.

Case 2: $f(x_0) < f(\rho_0)$.

Apply Case 1 to $-f$. □

The proposition which follows is a weak version of Rolle.

Proposition V.2.6. *(= Proposition II.7)*

Proof. We apply Proposition V.2.5, and if $\exists\, r_0, r_1 \in]\rho_0, \rho_1[\cap\mathbb{Q}(A)$ s.t. $f'(\overline{\sigma}, r_0)\; f'(\overline{\sigma}, r_1) < 0$ then we apply Fact V.2.2.2 to f'. □

The Fact which follows is a weak version of the Differentiation Lemma.

Fact V.2.7. *(= Fact II.8)*

Proof.

1. Put $g(\overline{r}, x) = f(\overline{r}, x) - f(\overline{r}, s_0) - \frac{f(\overline{r},s_1)-f(\overline{r},s_0)}{s_1-s_0}(x - s_0)$.
   We have: $g(\overline{r}, s_1) = g(\overline{r}, s_0)$.
   We apply the Proposition V.2.6 to g, where $g'(\overline{r}, x) = f'(\overline{r}, x) - \frac{f(\overline{r},s_1)-f(\overline{r},s_0)}{s_1-s_0}$.
2. Fix s_0, s_1, $\overline{r}$ such that $c_0 < s_0 < s_1 < c_1$, $|\frac{f(\overline{a},c_1)-f(\overline{a},c_0)}{c_1-c_0} - \frac{f(\overline{r},s_1)-f(\overline{r},s_0)}{s_1-s_0}| < \frac{1}{3N}$ and $\sup_{t\in[c_0,c_1]}\ |f'(\overline{r}, t) - f'(\overline{a}, t)| < \frac{1}{3N}$. Applying 1., consider $t \in [s_0, s_1]$ such that $|\frac{f(\overline{r},s_1)-f(\overline{r},s_0)}{s_1-s_0} - f'(\overline{r}, t)| < \frac{1}{3N}$.
3. Consequence of (2.) □

The next proposition shows that if an exponential polynomial f has arbitrarily small values arbitrarily close to some non Cauchy cut, then the same is true for the derivative and for the product of f by another exponential polynomial.

Proposition V.2.8. *(= Proposition II.9)*

Proof.

1.

- $c = \{r\ ;\ \forall c'\ ((\forall B\, \exists N > B\ \rho_N < c') \rightarrow r < c')\}$ by definition of inf.

- $c \leq c_1$ because $\forall B\, \exists N > B \;\; \rho_N \leq \sigma < c_1$.
- $c_0 \leq c$; indeed, let c' such that $\forall B\, \exists N > B \;\rho_N < c'$ and let N_0 such that $\rho_{N_0} < c'$. As $\forall N \; c_0 \leq \rho_N$, we have $c_0 \leq \rho_{N_0} < c'$ and thus $c_0 < c'$.
  Then $\forall c' \;\; (\forall B\, \exists N > B \;\rho_N < c') \rightarrow c_0 < c'$, and $c_0 \leq c$.

2. $c = \{r \;;\; \forall c' \; (\forall B\, \exists N > B \;\; \rho_N < c') \rightarrow r < c'\}$.

- **Case 1:** $\forall B\, \exists N > B \;\; \rho_N < c$.

Fix r, r' such that $r < r' < c$. Then there exists B' such that $\forall N > B' \; \rho_N \geq r'$.

Consider $B \in A$, then by Case 1 there exists $N > sup(B, B')$ such that $\rho_N < c$. As $N > B'$, then $\rho_N \geq r' > r$. From which we get $N > B$ and $r < \rho_N < c$.

- **Case 2:** $\exists B\, \forall N > B \;\; \rho_N \geq c$.

Fix B_0 such that $\forall N > B_0 \;\; \rho_N \geq c$.

Consider $r > c$. Then $r \notin c$, and there exists c', such that: $(\forall B\, \exists N > B \; \rho_N < c')$ and $(r \geq c')$.

Given $B \in A$ there exists $N > sup(B, B_0)$ such that $\rho_N < c' \; (\leq r)$. From which we get $N > B$ and $c \leq \rho_N < r$.

3. Assume $c \in R$. Let's prove that $|f(\overline{a}, c)| < \frac{1}{N}$ for all N.

   Fix N; by continuity there exists r, r' such that $r < c < r'$ and $\forall \rho \in [r, r'] \; |f(\overline{a}, c) - f(\overline{a}, \rho)| < \frac{1}{2N}$.

   And according to the remark made above in (2.) there exists $\rho \in [r, r'] \; |f(\overline{a}, \rho)| < \frac{1}{2N}$.

   Hence $|f(\overline{a}, c)| \leq |f(\overline{a}, c) - f(\overline{a}, \rho)| + |f(\overline{a}, \rho)| < \frac{1}{N}$.

4. Let $r < c$ and $N \in A$; there exists ρ, $r < \rho < c$ such that $|f(\overline{a}, \rho)| < \frac{\mu}{N}$.

   $\rho + \delta_0 < c$ since $\delta_0 < \delta(c)$ and $\rho < c$; and there exists ρ', $\rho + \delta_0 < \rho' < c$ such that $|f(\overline{a}, \rho')| < \frac{\mu}{N}$.

   Thus we found ρ, ρ', $r < \rho < \rho' < c$, $\rho' - \rho > \delta_0$ such that $|f(\overline{a}, \rho)| < \frac{\mu}{N}$ and $|f(\overline{a}, \rho')| < \frac{\mu}{N}$.

5. Fix δ_0 such that $0 < \delta_0 < \delta(c)$, and put $\mu' = \frac{\mu}{\delta_0}$. Let $r < c$, $N \in A$ and ρ, ρ' such that $r < \rho < \rho' < c$, $\rho' - \rho > \delta_0$, $|f(\overline{a}, \rho)| < \frac{\mu}{3N}$, and $|f(\overline{a}, \rho')| < \frac{\mu}{3N}$.

   By Fact V.2.7 there exists $s \in [\rho, \rho']$ such that $|\frac{f(\overline{a}, \rho') - f(\overline{a}, \rho)}{\rho' - \rho} - f'(\overline{a}, s)| < \frac{\mu}{3\delta_0\, N}$.

Then: $r < s < c$ and $|f'(\overline{a},s)| \leq |\frac{f(\overline{a},\rho')-f(\overline{a},\rho)}{\rho'-\rho} - f'(\overline{a},s)| + |\frac{f(\overline{a},\rho')-f(\overline{a},\rho)}{\rho'-\rho}| < \frac{\mu}{3\delta_0 N} + |\frac{f(\overline{a},\rho')}{\rho'-\rho}| + |\frac{f(\overline{a},\rho)}{\rho'-\rho}| < \frac{\mu}{3\delta_0 N} + \frac{\mu}{3\delta_0 N} + \frac{\mu}{3\delta_0 N}$*.

From which results: $r < s < c$ and $|f'(\overline{a},s)| < \frac{\mu'}{N}$.

6. Fix $r_0 < c$ and $B > sup\{2^{g(\rho)}\ ;\ \rho \in [r_0,c]\}$ and put $\mu_0 = B\mu$.
   Consider $r < c$, $N \in A$ and ρ such that $sup(r,r_0) < \rho < c$ and $|f(\rho)| < \frac{\mu}{N}$. Then $|f(\rho)\, 2^{g(\rho)}| < \frac{B\mu}{N}$.
   Hence $\exists\mu_0\, \forall r \forall N\, \exists \rho\ \ r < \rho < c$ and $|f(\rho)\, 2^{g(\rho)}| < \frac{\mu_0}{N}$.
7. Similar to (4.)
8. Similar to (5.)
9. Similar to (6.) □

References

[1] S. Boughattas, “Trois Théorèmes sur l’induction pour les formules ouvertes munies de l’exponentielle”, The Journal of Symbolic Logic, vol 65, No.1, 111-154 (2000).

[2] S. Boughattas; J-P. Ressayre, “Arithmetization of the field of reals with exponentiation”, RAIRO-Theor. Inf. Appl, 42, 2008, 105–119.

[3] A. Berarducci, “Proceedings of the Ravello Conference in honour of A. Macintyre”

[4] A. Berarducci; T. Servi, “An effective version of Wilkie’s theorem of the complement and some effective o-minimality results”, Ann. Pure Appl. Logic, 125 (2004), no 1-3, 43–74.

[5] M-H. Mourgues; J-P. Ressayre, “Every real closed field has an integer part”, J. Symbolic Logic, vol. 58 (1993), no. 2, 641–647.

[6] A. Rambaud, “Quasi-analycité, o-minimalité et élimination des quantificateurs”, Thèse, Université Paris 7, 2005.

[7] J-P. Ressayre, “Integer Parts of Real Closed Exponential Fields”, Arithmetic, Proof Theory and Computational Complexity, P. Clote and J. Krajicek eds, Oxford Logic Guides 23.

[8] J-P. Ressayre, ”Gabrielov’s theorem refined,” manuscript.

[9] J.C. Shepherdson, “A non-standard model for a free variable fragment of number theory”, Bulletin de l’Académie Polonaise des Sciences, vol.12 (1964), 79-86.

* $\rho' - \rho > \delta_0$

[10] L. van den Dries, “Exponential rings, exponential polynomials and exponential functions”, Pacific Journal of Mathematics, vol. 113, No.1, (1984).

[11] L. van den Dries; A. Macintyre; D. Marker, “Logarithmic exponential power series”, J. London Math. Soc. (2) 56 (1997), no. 3, 417–434.

[12] A. Wilkie, “Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function”, J. Amer. Math. Soc, vol. 9 (1996), no. 4, 1051–1094.