Chapter 1

On the arithmetization of real fields with exponentiation

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Shepherdson proved that a discrete unitary commutative semi-ring $A^+$ satisfies $IE_0$ (induction scheme restricted to quantifier free formulas) iff $A$ is integral part of a real closed field; and Berarducci asked about extensions of this criterion when exponentiation is added to the language of rings and fields. Let $T$ range over axiom systems for ordered fields with exponentiation; for three values of $T$ we provide a theory $\mathbf{T}_T$ in the language of rings plus exponentiation such that the models $(A, \exp_A)$ of $\mathbf{T}_T$ are all integral parts $A$ of models $M$ of $T$ with $A^+$ closed under $\exp_M$ and $\exp_A = \exp_M|A^+$. Namely $T=\mathbf{EXP}$, the basic theory of real exponential fields; $T=\mathbf{EXP}+$ the Rolle and the intermediate value properties for all exp-polynomials; and $T = T_{exp}$, the complete theory of the field of reals with exponentiation.

Introduction

Let $R$ be a model of the axioms $OF$ of ordered field; an integral part of $R$ is a subring $A$ such that for every element $x$ of the field there is a unique element $\lfloor x \rfloor$ of the ring such that $\lfloor x \rfloor < x \leq \lfloor x \rfloor + 1$; $\lfloor x \rfloor$ is called the integral part of $x$ (in $A$). In general $A$ is not unique; in fact as soon as $R$ is real closed and non archimedean the number of integral parts of $R$ is infinite and large. Nevertheless we sometimes write $A = \lfloor R \rfloor$ to mean that $A$ is an integral part of $R$. Note that $A$ then satisfies the axioms $DUCR + ED$ of discrete unitary commutative ring + euclidean division (for $\lfloor x/y \rfloor$ is the euclidean quotient of $x$ by $y$). The converse
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is true: every model \( A \) of \( \text{DUCR} + \text{ED} \) is integral part of a model \( R \) of \( \text{OF} \) - we can take \( R \) to be any field in between the fraction field \( \mathbb{Q}(A) \) and its Cauchy completion \( \mathbb{Q}(A)^\text{c} \). We are interested in results of this type, relating extensions of \( \text{OF} \) with extensions of \( \text{DUCR} \). We denote \( \mathcal{L} \) the language \( \{ \leq, +, \times, -1 \} \) of \( \text{DUCR} \), tacitly considering \( \mathbb{R}, \mathbb{Z} \) as \( \mathcal{L} \)-structures and not only as sets; henceforth, \( A \) tacitly ranges over all models of \( \text{DUCR} \). We write \( \text{rel} \) for real closed or real closure and \( \text{RCF} \) for the theory of rcl fields; remember that \( \text{RCF} \) axiomatizes the complete theory of \( \mathbb{R} \), and is axiomatized by the theory \( \mathcal{R} \) of real fields plus the intermediate value scheme \( \text{IV} \) for all polynomials. \( \text{IE}_0 \) denotes (the extension of \( \text{DUCR} \) by) the quantifier free induction scheme of \( \mathcal{L} \).

Shepherdson [9] proved that \( A \) is a model of \( \text{IE}_0 \) iff \( A = \llbracket R \rrbracket \) for some rcl field \( R \) - we can take for \( R \) the rcl of \( \mathbb{Q}(A) \). And Mourgues-Ressayre [5] proved that every rcl field has an integral part. Together these results establish a kind of weak duality \( \mathcal{L}(\text{rel}) \to \mathcal{R}(\text{rel}) \) from \( \text{IE}_0 \) to \( \text{RCF} \) and back. We introduce a convenient terminology to discuss results of this kind: if \( T \) extends \( \text{OF} \) then \( \llbracket \text{mod } T \rrbracket \) denotes the class \( \{ \llbracket R \rrbracket ; \text{ } R \text{ satisfies } T \} \); \( \llbracket T \rrbracket \) denotes the (first order \( \mathcal{L} \)-theory of \( \llbracket \text{mod } T \rrbracket \). Thus the preceding result are expressed by: \( \llbracket \text{mod } T \rrbracket = \llbracket \text{mod } T \rrbracket \) for \( T = \text{OF} \) and \( T = \text{RCF} \); and by: \( \llbracket \text{OF} \rrbracket \equiv \text{DUCR} + \text{ED}, \llbracket \text{RCF} \rrbracket \equiv \text{IE}_0 \).

Let \( \mathcal{L}(\ldots) \) denote \( \mathcal{L} \) extended by all function and relation symbols written inside (\ldots); when \( \text{exp} \) is \( x^y \) or \( a^x \) (\( a > 1 \) some constant) we call \( \text{exp} \)-polynomials the terms of \( \mathcal{L}(\text{exp}) \). Berarducci [3] asked for extensions of Shepherdson’s criterion when \( \mathcal{L}(\text{exp}) \) replaces \( \mathcal{L} \). We partially answer his question, keeping the above definition of \( \llbracket \text{mod } T \rrbracket \) and \( \llbracket \text{mod } T \rrbracket \) when \( \mathcal{L}(2^x) \) is the language of \( T \) while \( \mathcal{L}(x^y) \) is the language of \( \llbracket T \rrbracket \). The reason for choosing \( 2^x \) in the first place but \( x^y \) in the second one is that for every expansion \( (\mathbb{R}, 2^x) \) of \( \mathbb{R} \) which satisfies some basic properties of exponentiation we simply set \( x^y := 2^{y \log(x)} \); whereas this kind of relation between \( x^y \) and \( 2^x \) is not to be expected in \( (\mathbb{R}, 2^x) \). Granted this we have to define integral parts \( \llbracket (\mathbb{R}, 2^x) \rrbracket \) so that they come equipped with a function \( x^y \):

We say that \( A \) is an \( x^y \)-integral part of \( (\mathbb{R}, 2^x) \) (also denoted \( A, x^y = \llbracket (\mathbb{R}, 2^x) \rrbracket \) iff \( A = \llbracket R \rrbracket \) and \( A^+ \) is closed under \( x^v \); then \( x^y_A := x^v_{R^+}A^+ \).

Let \( T_{\text{exp}} \) denote the complete theory of \( (\mathbb{R}, 2^x) \); we prove that \( \llbracket \text{mod } T_{\text{exp}} \rrbracket \) equals \( \llbracket T_{\text{exp}} \rrbracket \) and we axiomatize \( \llbracket T_{\text{exp}} \rrbracket \). Since every model of \( T_{\text{exp}} \) has an \( x^y \)-integral part (see [7]) this establishes the same amount of duality between \( \llbracket T_{\text{exp}} \rrbracket \) and \( T_{\text{exp}} \) as do Shepherdson and Mourgues-Ressayre between \( \text{IE}_0 \) and \( \text{RCF} \). One would like \( \llbracket T_{\text{exp}} \rrbracket \) to
reduce to \( LE_0(x^n) \) (least element scheme for quantifier free formulas of \( \mathcal{L}(x^n) \)), in analogy with Shepherdson’s criterion. Alas \( \mathcal{T}_{exp} \) implies \( LE_0(x^n) \) but the reciprocal is beyond reach; furthermore our axiomatization of \( \mathcal{T}_{exp} \) is natural and simple in some ways, but in one other way it is ad hoc: namely it expresses natural properties of reals, not of integers. And of course, properties of reals are natural in the context of \( T_{exp} \), whereas here in the context of \( \mathcal{T}_{exp} \) it is natural properties of integers that one would expect. In contrast Shepherdson’s criterion is not ad hoc in any respect: \( IE_0 \) is as natural a theory for integers as is \( IV \) for reals. But we shall prove two other extensions of Shepherdson’s criterion with nothing ad hoc; they characterize the \( x^y \)-integral parts of models of \( T \) for \( T = EXP \) - the basic axioms of (real) exponential fields, and for \( T = EXP + IVR(2^x) \) - which denotes the intermediate value and Rolle properties for all \( 2^x \)-polynomials. Any way we are interested in axiomatizations \( \mathcal{T}_{exp} \) even if they have ad hoc features: at least they prove that the class of integral parts of models of \( T \) is first order expressible; and they are a useful step towards a better axiomatization. In particular, we think that the axiomatization of \( \mathcal{T}_{exp} \) which is proved here prepares for an axiomatization of the form \( \mathcal{L} EXP + I + \Delta \) where \( I \) is some natural and syntactic induction scheme, while \( \Delta \) is the atomic diagram of the definable constants of \( T_{exp} \) (which is recursive under Shanuel’s conjecture, see [12]).

The present paper is the complete version of a part of the extended abstract [2] - which contained other parts, the complete versions of which are to come.

Section I - \( EXP \) and \( \mathcal{L} EXP \)

An exponential field - here with exponentiation of base 2 - is a real field \( R \) together with a function \( 2^x \) which satisfies the axioms \( EXP \):

i) \( 2^1 = 2 \) and \( 2^x \) is a homomorphism of \( + \) on (the restriction to positive elements of) \( \times \)

ii) \( 2^x \) is an ordermorphism such that \( 2 < x \rightarrow x^2 < 2^x \)

iii) \( \forall x > 0 \log x \) exists (s.t. \( 2^{\log x} = x \)).

Let \( R_{exp} := (R, 2^x) \) be any model of \( EXP \); \( \log x \) is unique and \( x^y \) denotes \( 2^{y\log x} \).

Proposition I.1.

a) \( EXP \) implies: \( y^n < 2^y \) as soon as \( 2 \leq n \leq \log(y) \).

b) \( EXP \) implies that \( 2^x \) and \( \log \) are continuous.
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c) If $A_{\text{exp}}$ is an $x^y$-integral part of $R_{\text{exp}}$ the relation $a/b < 2^p/q$ is definable in $A_{\text{exp}}$.

Proof.

(a) Indeed, by (ii) $n \leq \log(y)$ implies $y^n \leq y^{\log(y)}$ which for $y = 2^x$ equals $(2^x)x = 2^x$; then (ii) implies $x^2 < y$ and again (ii) implies $2^{x^2} < 2^{1/y}$. By transitivity of order, $y^n < 2^{1/y}$.

(b) Elementary.

(c) It can be written $a^q < 2^p b^q$. 

We now prove that $\text{mod EXP} = \text{mod EXP}$ by providing an axiomatization over $DUCR$ of $\text{mod EXP}$; it has all variables tacitly ranging over the positive integers. The axioms are written informally and it would be cumbersome to write them strictly in the form of formulas of $L(x^y)$; but using P I.1.c and using the standard interpretation of fractions as pairs of integers the formulas are a routine to write down and it is left as an exercise. Here are the axioms for $\text{mod EXP}$:

1) $\exists a, b \mid 2^{p/q} - a/b < 1/x$ and $\exists p, q \mid 2^{p/q} - a/b < 1/x$

2) (i+ii) is true when its variables range over fractions

3) From fractions to reals, $2^x$ and $\log$ are continuous functions.

In the sequel $R_{\text{exp}} = (R, 2^x)$ always denotes an expansion of a model of $RF$ and $A_{\text{exp}} = (A, x^y)$ denotes an expansion of a model of $DUCR$.

→) From $R_{\text{exp}}$ to $A_{\text{exp}}$ - Assume that $R_{\text{exp}}$ satisfies $EXP$ and $A_{\text{exp}}$ is $x^y$-integral part of $R_{\text{exp}}$, then: (1) is true because $2^x$ is continuous and $Q(A)$ is dense in $R$; (2) is inherited from $R$ because (i+ii) are universal; and (3) is inherited from $R_{\text{exp}}$ by P.I.b. We proved that $\text{mod EXP}$ includes the axioms (1+2+3).

←) From $A_{\text{exp}}$ to $Q(A)^c_{\text{exp}}$ - Conversely we assume that $A_{\text{exp}}$ satisfies these axioms and we enrich $Q(A)^c$ to a model $Q(A)^c_{\text{exp}} := (Q(A)^c, 2^x)$ of $EXP$ such that $x^y_{\text{exp}}$ is the restriction to $A^+$ of $2^{y\log(x)}$. To begin with, (1) implies that inside $Q(A)$ the cut $2^{p/q} := \{a/b \mid (a/b)^q < 2^p\}$ is a Cauchy cut; so that it defines $2^{p/q}$ as an element of $Q(A)^c$. The function $2^x|Q(A)$ is thus defined as a map from $Q(A)$ into its Cauchy completion; in addition from (3) follows that this map is continuous on $Q(A)$. Then by the usual argument, it has a unique continuous extension $2^x$ sending the totality of $Q(A)^c$ to $Q(A)^c$. Now that $2^x$ is continuous, the truth of (2) on $Q(A)$ implies the truth of (i+ii) in $(Q(A)^c, 2^x)$. Finally the continuity of $\log$ on $Q(A)$ which is asserted by (3) implies that the
extension by continuity of log to \( \mathbb{Q}(A)^c \) continues to satisfy (iii). Thus (1+2+3) guarantees that \((\mathbb{Q}(A)^c, 2^x)\) is a model of EXP.

Together (\(-\)) and (\(-\)) establish for any model \(A\) of DUCR that \(A_{exp}\) is a model of (1+2+3) iff it is \(x^y\)-integral part of some model \(R_{exp}\) - one can take \(R_{exp} = \mathbb{Q}(A)_{exp}\); thus \(\text{EXP}_A\) exists and is axiomatized over DUCR by (1+2+3).

We have been a bit quick and sketchy and the axioms are ad hoc; but it was meant as an introduction to the main part of the section, which now gives and proves a better axiomatization:

**Theorem I.2.** \(\text{EXP}_A\) is axiomatized by the following system \(\mathcal{A}_0\) (where all variables tacitly range over the positive integers)

- DUCR
- \(x^{y+z} = x^y x^z, x^{-y} x^y = 1; x^{yz} = (x^y)^z; x^y\) strictly increasing with respect to \(x, y > 1\)
- \(2 < x \rightarrow x^2 < 2^x\)
- \(\exists y 2^y \leq z < 2^{y+1}; \exists x x^z \leq 2^y < (x + 1)^z; \exists y 2(x^y) < (x + 1)^y; \exists x (x + 1)^y < 2(x^y).\)

**Proof.** We consider a field with a function \(R_{exp} = (R, 2^x)\) and an integral part \(A\) of \(R\); in I.A we assume that \(A\) is closed \(x^y\) and we outline the proof that \(A_{exp} := (A, x^y|A)\) satisfies \(\mathcal{A}_0\). In I.B we outline the proof of a reciprocal: if \(A_{exp}\) satisfies \(\mathcal{A}_0\) then \(A_{exp}\) is \(x^y\)-integral part of a model \(R_{exp}\) of EXP with \(R = \mathbb{Q}(A)^c\). Readers may find these outlines sufficient, but if not then full detail can be looked at in Section V.1.

**I.A**

The exponential \(x^y\) of \(A\) inherits from \(R\) the right properties:

1. \(A \models \exists y 2^y \leq z < 2^{y+1}\) (take \(y = 2^y|A\))
2. \(A \models \exists x x^z \leq 2^y < (x + 1)^z\) (take \(x = 2^x|A\))
3. \(A \models \exists y 2x^y < (x + 1)^y\) (for \(\log (x + 1) - \log x > 0\) so we can find \(y \in A\) such that \(\log (x + 1) - \log x > \frac{1}{y}\), which is equivalent to \(2x^y < (x + 1)^y\))
4. \(A \models \exists x (x + 1)^y < 2x^y\) (for \(\log\) continuous at 1 and the limit of \(\log (1 + t)\) has the value 0 as \(t\) approaches 0; hence for all \(y \in A\) there exists \(t > 0\) such that \(\log (1 + t) < \frac{1}{y}\). Take \(x \in A\) such that \(x > \frac{1}{t}\), then \(\log (1 + \frac{1}{t}) < \frac{1}{y}\) - which is equivalent to \((x + 1)^y < 2x^y\).
We suppose that $A$ satisfies $A_0$ and we expand its Cauchy closure $R$ to an exponential field such that $A$ is $x^y$-integral part of $R$.

We first define $\log x$ for an “integer” $x \in A^*_+$, we prove that it is a Cauchy cut, and that the $\log$ has good properties over the integers.

**Definition I.3.** Given $x \in A^*_+$ set $\log x = \{ \frac{p}{q} : p, q \in A \text{ and } 2^p \leq x^q \}$. It is sometimes denoted by $\log_{I0} x$.

**Fact I.4.**
1. $\log x$ is a cut.
2. $\log x \in R$.
3. $\log 2^x = x$ and $\log 1 = 0$.
4. $\log x^y = y \log x$.
5. $\log xy = \log x + \log y$.
6. $\frac{p}{q} \rightarrow \log p - \log q = \log p' - \log q'$.

Now we define $\log$ and $x^y$ over $\mathbb{Q}(A)$ and $R$. We prove that $\log$ has good properties over fractions and that it is strictly increasing on $R$.

**Definition I.5.** $(x, y \in A^*_+, c \in R^*_+ \text{ and } c' \in R)$
1. $\log \frac{x}{y} = \log_{I0} x - \log_{I0} y$.
   It is sometimes denoted by $\log_{I1} \frac{x}{y}$.
2. $\log c = \{ \frac{p}{q} : p, q \in A \text{ and } \exists x, y \in A \frac{x}{y} \leq c \text{ and } \frac{x}{y} \leq \log_{I1} c \}$.
   It is sometimes denoted by $\log_{I2} c$.
3. $2^{c'} = \{ \frac{p}{q} : p, q \in A \text{ and } \log \frac{p}{q} \leq c' \}$.
4. $x^y = 2^{y \log x}$.

**Fact I.6.** $(x, y \in A^*_+ \text{ and } r, r' \in \mathbb{Q}(A)^*_+)$
1. $\log \frac{x}{y} \in R$.
2. $\log rr' = \log r + \log r'$.
3. $\log \frac{x}{y} = \log r - \log r'$.
4. $x \leq y \rightarrow \log x \leq \log y$.
5. $r \leq r' \rightarrow \log r \leq \log r'$.

6. The value of the log of a fraction given by the Definition I.5.1 is the same as that given by the Definition 1.5.2 for reals (in the sense:
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\[ \log_{11} r = \log_{12} r \text{ for all } r \in \mathbb{Q}(A), \] and more precisely

\[ \log_{11} r = \{ \rho ; \exists s \leq r \rho \leq \log_{11} s \}. \]

7. \( \exists y \in A \log (x + 1) - \log x > \frac{1}{y} \).
8. \( x < y \iff \log x < \log y. \)
9. \( r < r' \iff \log r < \log r'. \)
10. For all \( a \in A: r \leq 2^a \iff \log r \leq a \iff r \in 2^a. \)
    Then, for all \( a \in A \) the value of \( 2^a \) in \( A \) is the same as the cut \( 2^a \)
given by the Definition 1.5.3 for reals.
11. Let \( c \in R^*_+ \). Then \( \log c = \sup \{ \log r ; r \leq c \} \) **.
    So: \( c' < \log c \iff \exists r \leq c \ c' < \log r. \)
12. For all \( c \in R^*_+ \), \( r < c \to \log r < \log c. \)
13. For all \( c \in R^*_+ \), \( r > c \to \log r > \log c. \)
    Hence: \( \log r = \log c \to r = c. \)
14. For all \( c \) and \( c' \) in \( R^*_+ \), \( c < c' \iff \log c < \log c'. \)
    So: \( c = c' \iff \log c = \log c'. \)

The next step proves that \( R \) is closed under \( \log \) and \( 2^x \), which are
involutive.

**Proposition I.7. \( (c \in R) \)**
1. \( c > 0 \to \log c \in R. \)
2. \( 2^c \in R. \)
3. \( c > 0 \to 2^{\log c} = c. \)
4. If \( x \) and \( y \) are in \( A \) then the value of \( x^y \) in \( A \) is equal to the value
   of \( x^y \) defined on \( R. \)
5. \( \log 2^c = c. \)

At last, we prove that \( 2^x \) is an increasing homomorphism of + on \( \times \)
(restricted to positive elements).

*The first \( 2^a \) is the value given by the exponential function \( 2^x \) defined in \( A, \)
the last \( 2^a \) is the cut \( \{ \frac{p}{q} ; p, q \in A \text{ and } \log \frac{p}{q} \leq a \} \) given by the Definition
1.5.3 for a reals.
**sup X = \{ r \in \mathbb{Q}(A); \exists x \in X r \leq x \}. **
Proposition I.8. \((x, y, z \in R)\)
1. \(2^x\) is strictly increasing on \(R\).
2. \((x > 0 \land y > 0) \rightarrow \log xy = \log x + \log y\).
3. \((x > 0 \land y > 0) \rightarrow \log x^y = y \log x\).
4. \(2^{x+y} = 2^x2^y\).
5. \((y > 1 \land z > 1) \rightarrow \log y^z = \log y \log z\).
6. \((y > 1 \land z > 1) \rightarrow \log x^{yz} = y \log x^z\).
7. \(x > 1 \rightarrow (y < z \leftrightarrow x^y < x^z)\).
   And thus: \(y = z\) iff \(x^y = x^z\).
8. \(\exists y \forall z \quad 2^y > x^n\)

In section V.1 the full proofs of Fact I.4, I.6 and Proposition I.7, I.8 are provided - but they are renumbered Fact V.1.1, V.1.2 and Proposition V.1.5, V.1.6.

Section II: \(IVR(2^x)\) and \(\langle IVR(2^x) \rangle\)

Proposition II.1.

a) \(LE_0(x^y)\) holds in every \(x^y\)-integral part of every model of \(T_{exp}\)

b) \(LE_0(2^x)\) holds in every \(2^x\)-integral part of every model of \(EXP + IVR(2^x)\).

Proof. (a) Let \((R, 2^x) = R_{exp}\) be a model of \(T_{exp}\): A. Wilkie \[12\] proved that \(T_{exp}\) is an o-minimal theory hence \(R_{exp}\) is an o-minimal structure. That is: if \(\Phi(x)\) is any formula of \(\mathcal{L}(2^x)\) with parameters in \(R\) then the interpretation of \(\Phi(x)\) in \(R_{exp}\) is a finite union of intervals with endpoints \(a_i, b_i \in R \cup \{-\infty, +\infty\}\). Thus if \(A = R_{exp}\) then \(\min x \in A: \Phi(x)\) can only be one of the following elements:

\(a_i\) (in a case where \(a_i = R, 2^x\), \(a_i = 1\)); \(\mathbf{l}a_i\) \(+ 1\); or \(\mathbf{l}b_i\) .

Assume that \((A, x^y) = A_{exp}\) is an \(x^y\)-integral part of \(R_{exp}\) and consider \(\theta(x) \in E_0(x^y)\); with the help of \(2^{y\log(x)}\) one finds a formula \(\Phi(x) \in \mathcal{L}(2^x)\) which inside \(R_{exp}\) expresses \(\theta(x)\). We just showed that \(\min x \in A: \Phi(x)\) exists; hence also \(\min x : \theta(x)\) inside \(A_{exp}\).

(b) Here \(R_{exp}\) only satisfies \(EXP + IVR(2^x)\) and is not o-minimal; but by a result of van den Dries \[10\] every non trivial \(2^x\)-polynomial has only a finite number of roots in \((R, 2^x)\). This allows to prove for every quantifier free formula \(\Phi(x) \in \mathcal{L}(2^x)\) that the interpretation of \(\Phi(x)\) in
$R_{\text{exp}}$ again is a finite union of intervals with endpoints in $R \cup \{-\infty, +\infty\}$. Thus whenever $A$ is an integral part we obtain as before the existence of $\min x \in A : R_{\text{exp}}$ satisfies $\Phi(x)$; and this $\min$ is also $\min x \in A : A_{\text{exp}}$ satisfies $\Phi(x)$, provided in addition $A_{\text{exp}}$ is $2^x$-integral part.

**Remark** - 1) The proof of P II.1 works when we allow the formula $\Phi(x)$ to have parameters in $A$ in addition to the induction variable $x$; and actually it is this case which shows $LE_0(x^y)$ in (a), $LE_0(2^x)$ in (b).

2) But the proof works unchanged if the induction formula $\Phi(x)$ is allowed to have parameters from $R$, not only from the integral part $A$. Thus stronger induction schemes can be proved - only they are not entirely expressed inside $A_{\text{exp}}$. We now introduce two ways to handle this problem: the first one relies on second order Arithmetic, the second one on translations into first order Arithmetic.

**The second order theory of integral parts**

- Let $IP \subset L(A(x))$ with $A(x)$ predicate symbol denote the obvious axioms which are satisfied by $(R, A)$ iff $A = R^+_1$.
- More generally, for any function $f = f(\bar{x})$ over the reals $IP(f)$ adds to IP that $A^+$ is closed under $f$.

We can regard $OF + IP(f)$ as a second order Arithmetic of some kind: the elements $X$ of the field are the “reals” or second order objects; among them the “integers” or first order objects are the elements of $A$ - these integers form an $f$-integral part of these reals. In the sequel a formula of $L(A(x), \ldots)$ is called **first order** if all its quantifiers are restricted to $A(x)$; whereas a second order formula has also quantifiers ranging over the whole field. Note that formulas with no occurrence of the symbol $A(x)$ have all their quantifiers ranging over the reals - we call them **pure** second order; and we denote by $L^2$ the least element scheme asserted for all formulas of the form $[A(x) \text{ and } \phi(x, U)]$ where $\phi(x, U)$ is pure second order. The theory $T_{\text{exp}} + IP(x^y)$ axiomatizes the class of pairs $(R_{\text{exp}}, A)$ of a model of $T_{\text{exp}}$ with an $x^y$-integral part; it looks rather limited, and the more so $L_{\text{exp}}$ which is the first order counterpart. But any way one kind or another of drastic restriction is necessary on a theory $L_{\text{exp}}$ of Arithmetics if we want it to correspond to a well behaved theory like $T = T_{\text{exp}}$: for $T_{\text{exp}}$ has excellent algebraic properties while Arithmetics cannot avoid Gödel’s incompleteness theorem and Tennenbaum’s theorem of non existence of recursive non-standard models. And notwithstanding the limited character of $IP(x^y)$, the whole scheme $L^2$ is a consequence of $T_{\text{exp}} + IP(x^y)$ - as established by the same proof as P II.1.a.
In the theory \( OF + IP(x^y) \) the treatment of first order formulas with fractional or real parameters is trivial: there are first order variables ranging over \( A \) and second order variables ranging over \( R \); and the parameters are free second order variables assigned with a fraction \( a/b \in \mathbb{Q}(A) \) or with an element of \( R \). Below we express the same notions in the first order language of \( A_{\text{exp}} \); of course it gets more roundabout since already from parameters in \( \mathbb{Q}(A) \) the function \( 2^x \) leads to real parameters which \( A \) can only handle with the help of quantifiers.

**fc-Translations between \( A_{\text{exp}} \) and \( \mathbb{Q}(A)_{\text{exp}} \)** - We assume that \( A_{\text{exp}} \) is \( x^y \)-integral part of \( R_{\text{exp}} \) and \( R = \mathbb{Q}(A)^c \). Let \( \Phi(\vec{x}) \) be a formula of \( L(2^x) \); an fc-translation of \( \Phi \) is a formula \( \Phi^{fc} \in L(x^y) \) such that for all fractions \( \vec{x} = a_1/b_1, \ldots, a_k/b_k, (A, x^y) \) satisfies \( \Phi^{fc}(\vec{x}) \) iff \( R_{\text{exp}} \) satisfies \( \Phi(\vec{x}) \) (the superscript fc is chosen in reference to “Completion of the Fraction field of \( A \)” because \( R = \mathbb{Q}(A)^c \); note that the sequence of variables \( \vec{x} \) of \( \Phi \) must be doubled in \( \Phi^{fc} \)). This notion of fc-translation is relative to a given \( A_{\text{exp}} \); but we say that the fc-translation holds over \( A \) if it holds whenever \( A_{\text{exp}} \) satisfies the theory \( A \).

**Example - a)** To say that \( \text{EXP}_0 \) exists and is axiomatized by \( A_0 \) is equivalent to say that \( A_0 \) is an fc-translation of (the conjunction of) \( \text{EXP} \) over the empty theory. But over an empty or weak theory the notion of fc-translation is perilous; it is over \( \text{EXP}_0 \) that it becomes robust enough for allowing some systematic and simple syntactic fc-translations - see P II.2 below.

**b)** Unsystematic, tentative fc-translations play a heuristic role; ad hoc fc-translations \( \text{EXP}^{fc}, \text{IVR}(2^x)^{fc} \) suggested our first axiomatizations of \( \text{EXP}_0 \) and \( \text{IVR}(2^x) \); once these translations had quickly proved \( T = \text{EXP}, \text{IVR}(2^x) \) we started looking for the finer results of T I.2 and T II.4.

**Nota Bene - a)** If \( \psi^{fc}(\vec{x}) \) is an fc-translation of \( \psi(\vec{x}) \) it does not guarantee that \( \forall \vec{x}\psi^{fc}(\vec{x}) \) be an fc-translation of \( \forall \vec{x}\psi(\vec{x}) \).

**b)** On the other hand fc-translation permutes with boolean operations.

**Proposition II.2.** Every quantifier free formula \( \varphi = \varphi(\vec{x}) \in L(2^x) \) has an fc-translation.

**Proof.** For \( \varphi \) atomic let us only give two examples and admit the general case: \( [0 < x]^{fc} = \exists y[x = 1]; [y = 2^x]^{fc} = [\exists z \ 2^x = z \ and \ 2^z = y]^{fc} := \forall p > 0 \exists a/b[|2^x - a/b| < 1/p \ and \ |2^{a/b} - y| < 1/p] \) - where P I.1.c is used
On the arithmetization of real fields with exponentiation to express the latter formula. For boolean combinations one uses the above Nota Bene (b).

**Convention** - For any letter $x$ we use $\bar{x}$ as an abbreviation for the sequence $x_1, ..., x_k$, where the value of $k < \omega$ depends on the context: if the context says nothing then $k$ ranges freely over $\omega$; and $\bar{x}, \bar{y}$ need not have same $k$ - except if the context substitutes one for the other.

Let $\lambda \in R$ determine a cut of $Q(A)$ which is definable in $A_{\exp}$: for some formula $F \in L(x^y)$ we have $p/q < \lambda \iff F(p/q)$. We extend the notion of fc-translation to any “formula with real parameter” $\varphi(\lambda, \bar{x})$ where $\varphi(u, \bar{x}) \in L(2^x)$: the fc-translation is a formula $\psi \in L(x^y)$ such that for all fractions $\bar{a}$

$$A_{\exp} \text{ satisfies } \psi(\bar{a}) \iff R_{\exp} \text{ satisfies } \varphi(\lambda, \bar{a}).$$

**Fact II.3.** Every quantifier free formula with real parameters has an fc-translation.

**Proof.** For any term $t$ of $L(2^x)$ we set $[t(\lambda, \bar{x}) \leq 0]^{fc} := \forall p > 0 \exists r [F(r) \text{ and } \neg F(r + 1/p) \text{ and } (t(u, \bar{x}) \leq 1/p)]^{fc}$. The truth of $(t(u_p, \bar{x}) \leq 1/p)^{fc}$ for fractions $u_p$ tending to $\lambda$ and when $1/p$ tends to $0$ is equivalent to the truth of $t(u_p, \bar{x}) \leq 1/p$, which implies $t(\lambda, \bar{x}) \leq 0$ at the limit; and using the continuity of $t$ we can reverse this argument to obtain the reciprocal. Thus the fc-translation is valid for formulas of this form; all other cases follow by applying preservation of translations under boolean operations. It is clear how to extend this proof to the case of several real parameters.

We shall apply F II.3 to a unique real parameter $\lambda = \log'(1) := \{p/q : \exists r p/q < \frac{\log(1+r)}{r}\}$. We can ensure that $\lambda$ is a Cauchy cut with the following axiom $A_\lambda$ (which due to P I.1.c can be expressed in $L(x^y)$):

$$0 < r < s \to (1+s)^r < (1+r)^s \land \forall x > 0 \exists s < \frac{1}{x} \forall r \frac{(1+r)^sx}{(1+s)^{sr}} < 2^{sx}.$$ 

We let $IE_0(2^x)^{fc}$ consist of $A_\lambda$ and of the scheme of induction for the fc-translation of every formula $\psi(\lambda, \bar{x}, X) \in E_0(2^x, \lambda)$ - where the parameters $\bar{x}$ range over fractions. We let $IE_0(2^x)^{fc}$ denote $A_\lambda$ plus the scheme which expresses in $L(x^y)$ that if $F_i(u, v)$ defines for each $i < k$ inside $A_{\exp}$ a Cauchy cut $\lambda_i$ of $Q(A)$ and if $\phi = \phi(\lambda, \bar{x}, X) \in L(2^x)$ is a quantifier free formula with these parameters $\lambda_i$, then induction holds for the fc-translation of $\phi$. The schemes $LE_0(2^x)^{fc}$ and $LE_0(2^x)^{fc}$ are defined in the same way except that minimization replaces induction.
Theorem II.4. \( \mathcal{IVR}(2^x) \) exists and is axiomatized by \( I_E(2^x)^f/c \) over \( \mathcal{EXP} \).

Corollary II.5. Over \( \mathcal{EXP} \) the theory \( I_E(2^x)^f/c \) implies \( LE(2^x)^f/c \) (least element scheme for quantifier free formulas of \( \mathcal{L}(2^x) \) with arbitrary definable real parameters). Hence the four theories above T II.4 are equivalent.

**Proof.** T II.4 \( \rightarrow \) C II.5 - Assume that \( A_{\text{exp}} \) satisfies \( \mathcal{EXP} + I_E(2^x)^f/c \) and fix a formula \( \psi = \psi(\bar{x}, \bar{y}, \bar{z}) \in E_0(2^x, \lambda) \) where \( \lambda \) is any finite sequence of Cauchy cuts definable inside \( A_{\text{exp}} \); let \( \psi_f/c \) denote its fc-translation: for all fractions \( \bar{a}, \bar{X} \) \( Q(A)_{\text{exp}} \) satisfies \( \psi(\bar{\lambda}, \bar{a}, \bar{X}) \) iff \( A_{\text{exp}} \) satisfies \( \psi_f/c(\bar{a}, \bar{X}) \). Since by T. II.4 \( Q(A)_{\text{exp}} \) satisfies \( IV(R(2^x)) \), the proof of P. II.1.b applied to \( \psi_f/c \) proves \( \min X : \psi_f/c(\bar{a}, X) \) to exist. Thus \( A_{\text{exp}} \) satisfies \( LE(2^x)^f/c \). The reciprocal is true because the minimization scheme for the negation of \( \phi \) implies the induction scheme for \( \phi \).

**Proof.** T II.4 - That \( \mathcal{IVR}(2^x) \) implies \( I_E(2^x)^f/c \) and even \( LE(2^x)^f/c \) has just been proved: we now look for the reciprocal. We consider a model \( A_{\text{exp}} \) of \( \mathcal{EXP} + I_E(2^x)^f/c \) and we set up to prove that \( R_{\text{exp}} := Q(A)_{\text{exp}} \) satisfies \( IV(2^x) \). Given a \( 2^x \)-polynomial \( P(\bar{x}, \bar{y}) \) and given \( \bar{x} \) in \( R \) suppose \( P(\bar{x}, a) > 0 > P(\bar{x}, b) \); we want a zero of \( P \) between \( a \) and \( b \). We first assume \( \bar{x} \) in \( Q(A) \); an induction on the length of the \( 2^x \)-polynomial \( P(\bar{x}, \bar{y}) \) derives from \( \mathcal{EXP} \) the uniform continuity of \( P(\bar{x}, \bar{y}) \) for fixed \( \bar{x} \) and when \( \bar{y} \) ranges over \( [a, b] \). Thus given \( \epsilon > 0 \) in \( Q(A) \) we can find \( N \in A \) such that the variation of \( P(\bar{x}, \bar{y}) \) is less than \( \epsilon \) on every subinterval of \( [a, b] \) of length \( (b - a)/N \); hence on \( [c_i, c_{i+1}] \) where \( c_i := a + (b - a)/N \). We can be sure that on one of these intervals \( P(\bar{x}, \bar{y}) \) changes its sign - otherwise a contradiction with \( I_E(2^x)^f/c \) is easily reached. Assume that \( A \) is countable and choose a sequence \( (\epsilon_n) , n < \omega \) with limit 0 in \( Q(A) \). By iterating for each \( n < \omega \) the preceding fact applied with \( 1/N \leq \epsilon_n \) and with \( \bar{[a_n, b_n]} \) in place of \( [a, b] \) we obtain a decreasing chain of subintervals \( [a_n, b_n] \) of \( [a, b] \) on which \( P(\bar{x}, \bar{y}) \) changes sign and its variation is less than \( \epsilon_n \), moreover the intersection defines an element \( r \in R \). Thus \( r \) is a root of \( P(\bar{x}, \bar{y}) \) and \( IV(2^x) \) is proved - for fractional parameters only; but by proving the uniform continuity of \( P(\bar{x}, \bar{y}) \) on every finite \( k + 1 \)-dimensional box we extend the result to arbitrary “real” parameters. All this is easy, but it only gives a hint for the proof of \( IV(R(2^x)) \) which we now really start and which has the following added features: given \( P(\bar{a}, c_0) > 0 > P(\bar{a}, c_1) \), with \( \bar{a}, c_0, c_1 \in R \), first find elements \( t \in [c_0, c_1] \) for which \( P(\bar{a}, t) \) is arbitrarily small; this is proved in Fact II.6. Thus for each \( N > 0 \) in \( A \) there is \( \rho_N \in [c_0, c_1] \).
such that $|P(\bar{a}, \rho_N)| < 1/N$. Then we prove a weak version of Rolle and of Differentiation lemma (see Proposition II.7 and see Fact II.8). Then we use the latter lemma together with a notion of ordinal degree for exponential polynomials (this degree denoted $\text{ord}$ was introduced by L. van den Dries [10]); we thus show how to extract from the sequence $\rho_N$ a zero of $P(\bar{a}, x)$. It will be done in the “Approximation theorem” which proves $IV(2^x)$. Finally we obtain $IVR(2^x)$, by combining $IV(2^x)$ with the weak version of the Rolle lemma. The rest of this section is sketchy but full detail shall be given in Section V.2: Proposition II.7, II.9 and Fact II.6, II.8 below are exactly Proposition V.2.6, V.2.8 and Fact V.2.2, V.2.7 there.

Fact II.6. $(\rho, \sigma, \mathfrak{r} \in \mathbb{Q}(A) \text{ and } c_0, c_1, \mathfrak{r} \in R)$

1. Assume that $\rho < \sigma$ and $f(\mathfrak{r}, \rho) < 0 < f(\mathfrak{r}, \sigma)$, then $\forall N > 0 \exists t \in [\rho, \sigma]$ such that $|f(\mathfrak{r}, t)| < \frac{1}{N}$.
2. Suppose that $c_0 < c_1$ and $f(\mathfrak{r}, c_0) < 0 < f(\mathfrak{r}, c_1)$, then $\exists [\rho, \sigma] \subset ]c_0, c_1[$ such that $\forall N > 0 \exists t \in [\rho, \sigma] |f(\mathfrak{r}, t)| < \frac{1}{N}$.

Now comes a weak version of Rolle ; below $f = f(\bar{u}, x)$ is a $2^x$-polynomial.

Proposition II.7. $(\rho_0, \rho_1, \mathfrak{r} \in \mathbb{Q}(A))$

$$f(\mathfrak{r}, \rho_0) = f(\mathfrak{r}, \rho_1) \rightarrow \forall N > 0 \exists t \in [\rho_0, \rho_1] \cap \mathbb{Q}(A) |f'(\mathfrak{r}, t)| < \frac{1}{N}.$$  

Fact II.8 is a weak version of the Differentiation lemma.

Fact II.8. $(s_0, s_1, \mathfrak{r} \in \mathbb{Q}(A) \text{ and } c_0, c_1, \mathfrak{r} \in R)$

1. $\exists t \in ]s_0, s_1[ \frac{|f(\mathfrak{r}, s_1) - f(\mathfrak{r}, s_0)|}{s_1 - s_0} - f'(\mathfrak{r}, t)| < \frac{1}{N}$.
2. $\exists t \in ]c_0, c_1[ \frac{|f(\mathfrak{r}, c_1) - f(\mathfrak{r}, c_0)|}{c_1 - c_0} - f'(\mathfrak{r}, t)| < \frac{1}{N}$.
3. $f' = 0$ on $[c_0, c_1] \rightarrow f$ is constant - even for real parameters $\bar{u}$.

Proposition II.9. Given $c_0 < c_1$ assume that there exists a sequence $(\rho_N)_{N \in A}$ in a closed subinterval of $]c_0, c_1[$ such that $\forall N |f(\mathfrak{r}, \rho_N)| < \frac{1}{N}$.

Consider the cut $c = \inf \{c' \ ; \ \forall B \exists N > B \ \rho_N < c' \}$. Then we have
1. $c = \{r \ ; \ \forall c' ((\forall B \exists N > B \ \rho_N < c') \rightarrow r < c') \} \text{ and } c_0 \leq c \leq c_1$.

* $\inf X = \{r \ ; \ \forall x \in X \ r < x \}$. 

2. Either $\forall r < c \forall B \exists N > B \ r < \rho_N < c$, or $\forall r > c \forall B \exists N > B \ c \leq \rho_N < r$. Hence

$$\forall r, r' \ (r < c < r' \rightarrow \forall N \exists \rho \in [r, r'] \ |f(\pi, \rho)| < \frac{1}{N}).$$

3. If $c \in R$, then $f(\pi, c) = 0$.

The continuation of this Proposition shows that if an exponential polynomial $f$ has arbitrarily small values arbitrarily close to some non Cauchy cut, then the same is true for the derivative and for the product of $f$ by another exponential polynomial.

Assume that $c \not\in R (\delta(c) > 0**)$. Assume that $\exists \mu > 0 \forall r < c \forall N \exists \rho \ r < \rho < c$ and $|f(\pi, \rho)| < \frac{\mu}{N}$, then:

4. for all $\delta_0$, $0 < \delta_0 < \delta(c)$ we have: $\forall r < c \forall N \exists \rho, \rho' r < \rho < \rho' < c$, $\rho' - \rho > \delta_0$, $|f(\pi, \rho)| < \frac{\mu}{N}$, and $|f(\pi, \rho')| < \frac{\mu}{N}$

5. $\exists \mu' > 0 \forall r < c \forall N \exists \rho \ r < \rho < c$ and $|f(\pi, \rho)| < \frac{\mu'}{N}$

6. for all $g$ there exists $\mu' > 0$ such that $\forall r < c \forall N \exists \rho \ r < \rho < c$ and $|f(\pi, \rho) 2^{g(\pi, \rho)}| < \frac{\mu'}{N}$.

Assume that $\exists \mu \forall r > c \forall N \exists \rho \ c < \rho < r$ and $|f(\pi, \rho)| < \frac{1}{N}$, then:

7. for all $\delta_0$, $0 < \delta_0 < \delta(c)$ we have: $\forall r > c \forall N \exists \rho, \rho' c < \rho < \rho' < r$, $\rho' - \rho > \delta_0$, $|f(\pi, \rho)| < \frac{\mu}{N}$, and $|f(\pi, \rho')| < \frac{\mu}{N}$

8. $\exists \mu' \forall r > c \forall N \exists \rho \ c < \rho < r$ and $|f(\pi, \rho)| < \frac{\mu'}{N}$

9. for all $g$ there exists $\mu' > 0$ such that $\forall r > c \forall N \exists \rho \ c < \rho < r$ and $|f(\pi, \rho) 2^{g(\pi, \rho)}| < \frac{\mu'}{N}$.

We recall from [10] the application $\text{ord}$ defined on the exponential polynomials as an analog of the degree of polynomials.

$R[x]^E$ the set of exponential polynomials is union of the $R_k$, $-1 \leq k$, defined by induction: $R_{k+1} = R_k \oplus I_{k+1}$, for all $k \geq -1$.

- $R_{-1} = R$, $I_0$ = the ideal $(x) R[x]$. From which follows: $R_0 = R_{-1} \oplus I_0 = R[x]$.

---

**$\delta(c) = \{ r \in \mathbb{Q}(A); \forall x (x \in c \rightarrow x + r \in c) \}; \delta(c) > 0$ means $\exists r > 0 \ r \in \delta(c)$.
• $R_{k+1} = R_k[2I_k]$, and $I_{k+1} = R_k$-sub-module of $R_{k+1}$ generated by the $2^n$, $a \in I_k$, $a \neq 0$.

It is clear that $R_{k+1} = R_k \oplus I_{k+1}$.

• Notice that $R_k = R_0 \oplus I_1 \cdots \oplus I_k$, $k \geq 0$.

**Definition II.10. (height)**

$p(x) ∈ R[x]^E$ is of height $k$ if $p \in R_k \setminus R_{k-1}$, $k > 0$, and it is of height 0 if $p \in R_0 = R[x]$.

**Definition II.11. (ord)**

- if $p \in I_k$, $k > 0$, $p = \sum_{i=1}^{k} r_i 2^{a_i}$, where $a_i$ are distinct members of $I_k-1\setminus \{0\}$, and $r_i$ are non-zero elements of $R_{k-1}$, we put $t(p) = h$.
- if $p \in R_0 = R[x]$, we put $t(p) = 0$ if $p = 0$, and $t(p) = d + 1$ if $deg_p = d \geq 0$.

Then we define an ordinal $ord(p) < \omega^\omega$ for $p \in R[x]^E$.

Note that: $R_k = R_0 \oplus I_1 \cdots \oplus I_k$, $k \geq 0$. So any $p \in R[x]^E$ of height $\leq k$ can be written uniquely as: $p = p_0 + p_1 + \cdots + p_k$, $p_i \in R_0$, $p_i \in I_i$, for $i > 0$.

We put $ord(p) = \omega^k \cdot t(p_k) + \cdots + \omega \cdot t(p_1) + t(p_0)$.

• Note that: $ord(p) = 0$ iff $p = 0$.

The following result is proved in [10].

**Lemma II.12.** Let $p \in R[x]^E$ non-zero, then either $ord(2^tp) < ord(p)$ for some $q \in R[x]^E$, or $ord(p') < ord(p)$.

**Theorem. (Approximation)** Let $f$ be a $2^r$-polynomial with parameters in $R$ such that $\forall N > 0 \exists \rho \in [c_0, c_1] \ |f(\rho)| < \frac{1}{N}$; there exists $c \in [c_0, c_1]$ such that $f(c) = 0$.

**Proof.** Assume that $f \neq 0$ and $(\rho_N)_{N \in A}$ in $[c_0, c_1]$ is such that $\forall N \ |f(\rho_N)| < \frac{1}{N}$; consider the cut $c = \inf \{c' : \forall B \exists N > B \ \rho_N < c'\}$.

By Proposition II.9.3, if $c \in R$, then $f(c) = 0$. Proceeding by way of contradiction, assume that $c \notin R$. Then, by Proposition II.9.2 we have two cases:

**Case 1:** $\forall r < c \exists B \ |B > B \ r < \rho_N < c$.

Then:

$\exists \mu_0 \forall r < c \exists B \ |B > B \ r < \rho_N < c$ and $|f(\rho)| < \frac{\mu_0}{N}$ \ (0)
\[ f \neq 0, \text{ then } \text{ord}(f) > 0 \] and by Lemma II.12. and the fact that \( \omega^{\omega} \) is well ordered, there exist \( n \in \mathbb{N}^* \), \( f_0 = f, ..., f_n \) such that:

- \( \text{ord}(f_0) > \text{ord}(f_1) > ... > \text{ord}(f_n) = 0 \). From which follows \( f_n = 0 \).
- For all \( i < n \) we have: \( f_{i+1} = f_i' \) or there exists \( g_i \) such that \( f_{i+1} = f_i 2^{g_i} \).

The case \( f_n = f_{n-1} 2^{g_{n-1}} \) is not possible, otherwise \( f_{n-1} = 0 \) and then \( 0 = \text{ord}(f_{n-1}) > \text{ord}(f_n) = 0 \).

Then \( f_n = f_{n-1}' = 0 \). And by the weak version of Differentiation Lemma (Fact II.8), there exists \( b \in \mathbb{R} \) such that \( f_n - 1 = b \).

- Since \( f \) satisfies \((*0)\) one applies Proposition II.9.5 or Proposition II.9.6 to \( f \), depending on \( f_1 = f_0' \) or \( f_1 = f_0 2^{g_0} \); then we have:

\[ \exists \mu_1 \forall r < c \exists \rho \ r < \rho < c \text{ and } |f_1(\rho)| < \frac{\mu_1}{N} \quad (*1) \]

- Since \( f_1 \) satisfies \((*1)\), as for \( f \), and using \((*1)\) plus Proposition II.9 we have:

\[ \exists \mu_2 \forall r < c \exists \rho \ r < \rho < c \text{ and } |f_2(\rho)| < \frac{\mu_2}{N} \quad (*2) \]

- ..., until \( f_{n-1} \), we have:

\[ \exists \mu_{n-1} \forall r < c \exists \rho \ r < \rho < c \text{ and } |f_{n-1}(\rho)| < \frac{\mu_{n-1}}{N} \quad (*_{n-1}) \]

\( f_{n-1} = b \), then:

\[ \forall N \quad \left| b \right| < 1/N. \]

Thus \( b = 0 \) which is contradiction.

\textbf{Case 2:} \( \forall r > c \exists B \exists N > B \ r < \rho_N < r \).

Similar proof.

\textbf{Theorem. (IV)} For a \( 2^x \)-polynomial \( f \) with parameters in \( R \) if \( f(c_0) < 0 < f(c_1) \) then there exists \( c \in [c_0, c_1] \) such that \( f(c) = 0 \).

\textbf{Proof.} Assume that \( f(c_0) < 0 < f(c_1) \); by Fact II.6.2 there exists \([\rho, \sigma] \subseteq [c_0, c_1] \) and a sequence \((\rho_N)_{N \in A}\) in \([\rho, \sigma] \) such that for all \( N > 0 \)

\[ |f(\rho_N)| < \frac{1}{N}. \]

Then, by the Approximation theorem there exists \( c \in [\rho, \sigma] \) such that \( f(c) = 0 \).
Theorem. (Rolle) If $c_0 < c_1$ in $R$ there exists $c \in (c_0, c_1]$ such that $f(c_0) - f(c_1) = (c_0 - c_1) f'(c)$.

Proof. Let us prove it for the case $f(c_0) = f(c_1)$.

- If $f$ is constant on $[c_0, c_1]$, we have the conclusion which we want.
- Assume that $f$ is not constant on $[c_0, c_1]$.

Then there exists $\sigma_0 \in [c_0, c_1]$ such that $f(\sigma_0) \neq f(c_0)$; without loss of generality, we can assume that $f(\sigma_0) > f(c_0)$. Then by IV theorem there exists $a, c_0 < a < \sigma_0$ such that $f(a) = \frac{f(c_0) + f(\sigma_0)}{2}$.

Also $f(\sigma_0) > f(c_1)$ and there exists $b, \sigma_0 < b < c_1$ such that $f(b) = \frac{f(c_0) + f(\sigma_0)}{2}$, $f(a) = f(b)$ and by the weak version of Rolle (Proposition II.7) there exists a sequence $(\rho_N)_{N \in \mathbb{A}}$ in $[a, b]$ such that $\forall N \ |f'(\rho_N)| < \frac{1}{N}$. By the Approximation theorem applied to $f'$, there exists $c \in [a, b]$ such that $f'(c) = 0$. Furthermore $c$ is in $]c_0, c_1[$. 

Section III : The theory $\mathbf{I}^\mathbf{T}_{\exp}$

The proof of P II.1.a shows that $\mathbf{I}^\mathbf{T}_{\exp}$ implies $\mathbf{LE}_0(x^y)^{f_e}$. But the reciprocal is an open question; it could hold if a highly remarkable phenomenon took place in $(\mathbb{R}, 2^x)$: non singular systems of $2^x$-algebraic equations should reduce to single such equations, in a uniform way (that is in every model of $T_{\exp}$ in addition to $(\mathbb{R}, 2^x)$). Although this is the analog of a basic property of real algebraic closure, it is very demanding... It is in view of this uncertainty that it was interesting to have a subtheory $T$ of $T_{\exp}$, as strong as we are able to find and for which we know a natural axiomatization of $\mathbf{I}_{\exp}$; this is what Section II provided with $T = IV R(2^x)

The next two theorems recall an axiomatization $\mathcal{R}$ of $T_{\exp}$ which consists of sentences simple enough for $\mathcal{R}^{f_e}$ to exist - thus $\mathcal{I}^\mathcal{R}_{\exp}$ exists and is axiomatized by $\mathbf{I}^\mathbf{EXP} + \mathcal{R}^{f_e}$. Let $e(x, y)$ stand for the relation ($2^x = y$ and $x \in [0, 1]$) and let $\mathcal{R}_e, \mathcal{L}_e, T_e$ denote $(\mathbb{R}, e(x, y))$, its language and its complete theory.

Theorem III.1. For each axiomatization $\mathcal{R}_e$ of $T_e$, $\mathbf{EXP} + \mathcal{R}_e$ is an axiomatization of $T_{\exp}$.

Proof. See [7] or [11] (which use a bulkier version of EXP, but P I.1.a showed its equivalence with ours. Strictly speaking the language $\mathcal{L}(e(x, y))$ of $\mathcal{R}$ is not the one of $T_{\exp}$, so that we have to interpret $\mathcal{R}_e$ in $\mathbf{EXP}$ rather than just add it ; but this is trivial since the intended interpretation of $e(x, y)$ is $2^x[0, 1]$.)
Now we recall the axiomatization \( R_e \) of \( T_e \) due to Rambaud [6], which is simple enough for \( R_e^{fc} \) to exist. Let \( t(\bar{X}, Y) = t \) be a real function:

- assume that \( t \) is \( C^\infty \) with arguments ranging over \([0, 1]\) and for each \( \bar{xy} \in \[0, 1]\) \( k+1 \) that
  
  \[ t(\bar{x}, 0) > 0 \quad \text{and} \quad t(\bar{x}, 1) > 0 \]

then \( f_t \) denotes the function of domain \([0, 1]^k\) defined by

\[ 0 < f_t(\bar{x}) < 1 \quad \text{and} \quad t(\bar{x}, f_t(\bar{x})) = 0. \]

If the above assumption on \( t \) is false we do not introduce \( f_t \).

- We denote by \( t/\bar{Y} \) the function \( t/\bar{Y} \) extended by continuity to \( \bar{Y} = 0 \) and restricted to \([0, 1]^k \times [-1/2, 1/2]\); in case this function is \( C^\infty \) on \([0, 1]^k \times [-1/2, 1/2]\); we skip \( t/\bar{Y} \) otherwise.

- We denote by \( I \) the closure under composition and under these operations: \( t \mapsto -\to f_t \), \( t \mapsto -\to t/\bar{Y} \) of:
  
  \( L, \bar{x}^1/p \ (1 < p < \omega) \) and \( 2^x[0, 1] \).

**Theorem III.2.** \( T_e \) is axiomatized by \( R_e := RF \cup \{(\forall x \neq 0) \exists y \ xy = 1\} \cup IVD(I) \cup \Delta, \) where \( \Delta \) is the atomic diagram of the definable constants of \( T_e \) and where \( IVD(I) \) expresses in the language \( L_e \) the existence and domain of every function \( f_t \) or \( t/\bar{Y} \) of \( I \), as well as a bound \( B_{t/\bar{Y}} \in \omega \) for each function \( t/\bar{Y} \).

**Proof.** see [6]. \( \square \)

Below we provide an axiomatization \( IVD \) over \( EXP \) of the interpretation of \( IVD(I) \): \( IVD \) is satisfied in a model \( R_{exp} \) of \( EXP \) iff the interpretation of \( L_e \) in \( R_{exp} \) satisfies \( IVD(I) \). Then we provide fc-translations \( IVD^{fc}, \Delta^{fc} \). Thus we obtain:

**Theorem III.3.** \( T_{exp} \) is axiomatized over \( EXP \) by \( IVD^{fc} + \Delta^{fc} \); hence \( T_{exp} \) is axiomatized by \( EXP + IVD^{fc} + \Delta^{fc} \).

Before proving the theorem we must define \( IVD, IVD^{fc}, \Delta^{fc} \); using induction on \( t \in I \) we express its graph by a formula \( G_t \in L(2^x) \): in this way we develop the interpretation of \( e(x, y) \) in \( T_{exp} \) because \( G_t \) uses the function \( 2^x \) instead of the relation \( e(x, y) \); and \( IVD \) consists of \( (\forall \bar{x} \in D_t) \exists y \ G_t(\bar{x}, y) \) for each \( t = f_s \) and of \( (\forall (\bar{x}, Y) \in D_t) \exists y \leq B_t \ G_t(\bar{x}, Y, y) \) for each \( t = s/\bar{Y} \) - where \( D_t := [0, 1]^k \) if \( t = f_s \), \( D_t := [0, 1]^k \times [-1/2, 1/2] \) if \( t = s/\bar{Y} \).
We define $G_t$ by the obvious clauses: if $t = 2^x[0,1]$, $G_t(x,y) = (2^x = y \text{ and } x \in [0,1])$; if $t = x^{1/p}$, $G_t = (0 \leq x \text{ and } y^p = x)$; if $t(\bar{x}) = f(s^1, \ldots, s^n)$, $G_t(\bar{x}, y) = \exists y^1, \ldots, y^n, y|G_f(\bar{x}, y^1, \ldots, y^n, y)$ and $\wedge_j G_{v_j}(\bar{x}, y_j)$; if $t = f_s(\bar{x})$ where $s = s(\bar{x}, y)$, $G_t(\bar{x}, y) = [(\bar{x}) \in D_t \text{ and } y \in [0,1] \text{ and } G_s(\bar{x}, y, 0)]$; if $t(\bar{x}, \bar{Y}) = s/Y$. $G_t(\bar{x}, \bar{Y}) = ((\bar{x}, \bar{Y}) \in D_t \text{ and } \forall \epsilon > 0 \exists Z \exists y' \neq 0 [G_s(\bar{x}, Z, y') \text{ and } |Z - Y| < \epsilon \text{ and } |Zy' - y| < \epsilon]$.

Thus $IVD$ is defined and it is clear that $R_{exp}$ satisfies $IVD$ iff the interpretation of $e(x, y)$ in $R_{exp}$ by $(2^x = y \text{ and } x \in [0,1])$ satisfies $IVD(I)$. We start to fc-translate the $L(2^x)$-formula $G_t(\bar{x}, y)$; so we assume that

$$R_{exp} \text{ satisfies } EXP \text{ and } A_{exp} = \downarrow R_{exp}\downarrow$$

By induction on $t$ we define an $L(x^n)$-formula denoted by $t(\bar{x}) \equiv e y$ and which inside $A_{exp}$ expresses that $G_t(\bar{x}, y)$ (in other words the formula $t(\bar{x}) = y$) is satisfied up to an approximation measured by $\epsilon$. The free variables of $t(\bar{x}) \equiv e y$ may ultimately range over $R$ but the quantified variables are always restricted to the positive elements of $A$ (called the integers) or to the elements of $Q(A)$ (called the fractions); in fact, below $p, q$ tacitly range over these “integers” while all other bound variables represent “fractions”. We abbreviate by $t(\bar{x}) \equiv_0 y$ the formula $\forall p \exists x, r(||(\bar{a}, r) - (\bar{x}, y)|| < 1/p \text{ and } t(\bar{a}) \equiv_1 1/p \circ r)$; the inductive clauses defining $t(\bar{a}) \equiv_1 1/p \circ r$ will ensure that for all $\bar{x}, y \in R$

$$** G_t(\bar{x}, y) \iff t(\bar{x}) \equiv_0 y$$

where of course $G_t(\bar{x}, y)$ is interpreted in $R_{exp}$ but where the quantifiers in the formula $t(\bar{x}) \equiv_0 y$ in $A_{exp}$ have the above said interpretation in $A$. If $t \in L(2^x)$, then $t \equiv_1 1/p \circ y$ is the formula $||t - y|| < 1/p |||\circ$ provided by P II.2; if $t = x^{1/n}$, $n < \omega$ then $(t \equiv_1 1/p) := |y^n - x| < 1/p$. If $t := f_s, s = s(\bar{x}, y)$ then $t(\bar{x}) \equiv_1 1/p \circ y$ is the formula $s(\bar{x}, y) \equiv_1 1/p \circ 0$, and if $t(\bar{x}, \bar{Y}) := s/y \text{ then } t(\bar{x}, \bar{Y}) \equiv_1 1/p \circ y$ is the formula $\exists Z \exists z \neq 0 [s(\bar{x}, Z) \equiv_1 1/p \circ z > 0 \text{ and } |Z - Y| < 1/p \text{ and } |Zy' - y| < 1/p]$. Finally, if $t(\bar{x}) := f(\bar{s})$ then $t(\bar{x}) \equiv_1 1/p \circ y$ is the formula $\exists y[f(\bar{y}) \equiv_1 1/p \circ y \text{ and } \wedge_j < t_j(\bar{x}) \equiv_1 1/p \circ y_j]$.

**From $R_{exp}$ to $A_{exp}$** - We assume that $R_{exp}$ satisfies $EXP + IVD$ and see what it implies on $A_{exp}$; we have that $A_{exp}$ satisfies $\downarrow EXP\downarrow$ and by T III.1 that $R_{exp}$ satisfies $T_{exp}$. In particular in $R_{exp}$ we have $\Delta$ and the continuity of every function of $\mathcal{I}$. Using in addition the fact that $Q(A)$ is dense in $R$, this implies (***) by induction on $t(\bar{x}) \in \mathcal{I}$.

As a consequence $A_{exp}$ satisfies

1) $\forall p \forall(\bar{a}) \in D_t \exists r t(\bar{a}) \equiv_1 1/p \circ r$
when \( t \) is one of the functions \( f_s \), or

1) \( \forall p \forall (\bar{a}) \in D \exists r < B \ t(\bar{a}) \equiv_{1/p} r \)

when \( t \) is a function \( s/Y \) whose variables \( \bar{x}, Y \) are renamed \( \bar{a} \). In addition inside \( R_{exp} \) the elements \( r \) of (1) converge to an element \( y \) when \( p \to \infty \); in particular they are Cauchy convergent and this is expressed in \( A_{exp} \) by

2) \( \forall p \exists q \forall (\bar{a}) \in D \ \exists r \ [t(\bar{a}) \equiv_{1/q} r \text{ and } t(\bar{a}) \equiv_{1/q} r' \to |r - r'| < 1/p] \).

We let (3) express in \( A_{exp} \) that the restriction of \( t \) to fractions is continuous:

3) \( \forall p \forall (\bar{a}) \in D \exists q \ [\forall (r, r') \ [t(\bar{a}) \equiv_{1/q} r \text{ and } t(\bar{a}) \equiv_{1/q} r' \to |r - r'| < 1/p] \).

Finally since \( T_{exp} \) implies the uniform continuity of any definable continuous function over a compact box, even uniform continuity is satisfied over \( D_1 \); and this is easy to express in \( A_{exp} \) - it simply interchanges \( \exists q \) with \( \forall (\bar{a}) \):

3′) \( \forall p \exists q \forall (\bar{a}) \in D \ \exists r \ [t(\bar{a}) \equiv_{1/q} r \text{ and } t(\bar{a}) \equiv_{1/q} r' \to |r - r'| < 1/p] \).

Now we express in \( A_{exp} \) the satisfaction of \( \Delta \) by \( R_{exp} \); up to equivalence the definable constants of \( T_e \) are all elements of \( |\Delta| := \{ t \in I; \ t \text{ is a function with 0 arguments} \} \); and using in addition the fact that \( I \) is closed under \(+, -, \times, 2^\mathbb{Z}[0,1] \) we can reduce \( \Delta \) to \( \{ t = 0; t \in |\Delta| \text{ and } t = 0 \} \cup \{ t' \neq 0; t' \in |\Delta| \text{ and } t' \neq 0 \} \). Then \( \Delta^{fc} := \{ t \equiv_0 0; t \in |\Delta| \text{ and } t = 0 \} \cup \{ t' \neq_0 0; t' \in |\Delta| \text{ and } t' \neq 0 \} \) is satisfied by \( A_{exp} \). In conclusion we proved:

**Proposition III.4.** Assume that \( R_{exp} \) satisfies \( EXP + IVD \)

a) For all \( \bar{x}, y \in R \), \( R_{exp} \) satisfies \( G_t(\bar{x}, y) \iff t(\bar{x}) \equiv_0 y \).

b) \( A_{exp} \) satisfies \( (1 + 2 + 3') \) and \( \Delta^{fc} \).

**From \( A_{exp} \) to \( R_{exp} \) -** Looking for the reciprocal we assume that \( A_{exp} \) satisfies \( (1 + 2 + 3') + \Delta^{fc} \); this time we assume in addition that \( R^c = R \).

**Nota Bene -** We could assume \( (3') \); but this is not used in the proof (hence \( (3') \) is consequence of the other axioms).

We shall prove that \( R_{exp} \) satisfies \( IVD + \Delta \) hence using P III.4 that \( IVD^{fc} + \Delta^{fc} \) is a global fc-translation of \( IVD + \Delta \) over \( \mathbb{L} EXP \). Hence
also that $R_{exp}$ satisfies $T_{exp}$; so that $\text{EXP} + IV D^{fc} + \Delta^{fc}$ axiomatizes $T_{exp}$: proving T III.3.

We denote $I_n$ the set of functions of $\mathcal{I}$ whose definition involves at most $n$ nested operations $s \mapsto f_\sigma$ or $s \mapsto s/Y$; we make an inductive assumption $IH_n$:

a) $(\ast \ast)$ holds for each $s \in I_n$

b) $R_{exp}$ satisfies $IV D_n :=$ restriction of $IV D$ to $I_n$.

For $n = 0$ this is true because $IV D_0$ is empty and because $(\ast \ast)$ for $I_0$ follows from the continuity of every function in $I_0$: so we fix an arbitrary $n$. For each $s \in I_n$, from $(1+2)$ for $t := f_s$ follows that for all fractions $(\bar{a}) \in D_t$ there is a unique $y \in R^{c} = R$ such that $t(\bar{a}) \equiv_0 y$; we set $f_\sigma^*(\bar{a}) := y$. From $(3)$ follows that $f_s^* = t^*$ is continuous over $D_t$; by definition of $t^*$ we have $s(\bar{a}, t^*(\bar{a})) \equiv_0 0$ and using $IH_n$ it implies $s(\bar{a}, t^*(\bar{a})) = 0$ hence $R_{exp}$ satisfies $G_t(\bar{a}, t^*(\bar{a}))$. In addition for any element $y \in [0,1]$ satisfying $G_t(\bar{a}, y)$ hence $s(\bar{a}, y) = 0$ the continuity of $s$ and its subterms yields for each $p$ a fraction $r$ such that $|y - r| < 1/p$ and $s(\bar{a}) \equiv_{1/p} 0$; by axiom $(2)$ for $t$ these approximations $r$ of $y$ also converge to $t^*(\bar{a})$. Thus $t^*$ equals $t$ and $\exists y G_t(\bar{a}, y)$ holds on $Q(A)$; finally, the continuity of $t$ provides a unique continuous extension of $t^*$ to $R$, still denoted $t^*$. By continuity $R_{exp}$ keeps satisfying $G_t(\bar{a}, t^*(\bar{x}))$ - hence $t^* = t$, and it becomes easy to prove $IH_{n+1}$ as far as $t$ is concerned.

A similar argument works for $t = s/Y$; then preservation of continuity under composition suffices to deal with the case $t = f(s_0(\bar{x}, ..., s_k(\bar{x})))$, so that all of $IH_{n+1}$ is proved.

T III.3

Section IV - Conclusion

1. For a complete picture of the situation we will have to expose the remaining topics of [2]: correspondence between the definable functions of $T_{exp}$ and the provable functions of $T_{exp}^{PA}$; provable polytime witnessing for $T_{exp}^{PA}$; existence of integral parts in exponential fields; “blunt” arithmetics.

2. The definition of $s/Y$ hence of $\mathcal{I}$ is not given in an effective way: for what it says, the code for the language of $\mathcal{I}$ could be a $\Pi^0_1$-complete set of integers (or even worse, because the property "$t/Y$ is a $C^\infty$ function" does not even present it as arithmetical!). So one is not satisfied with $\mathcal{I}$ hence with $IVD, IVD^{fc}$ and one would like to replace them by recursive sets $\mathcal{J}, IV D(\mathcal{J}), IV D^{fc}(\mathcal{J})$. To that end the other axiomatizations of $T_e, T_{exp}$ provided by [12], [8] and [4] could be used;
thus we should obtain an effective axiomatization of $\mathbb{L}T_{\text{exp}}$ over $\mathbb{L}^\Delta$.

We expect that this recursive axiomatization would no longer rely on division so that it would not use the bound $B_t$ on $t = s/\gamma$ which is asserted by IVD. This non effective bound $B_t$ is the only part of IVD which does not seem deducible from a suitable induction scheme by using the ideas and methods of Section II; so that it should be quite possible to provide for $\mathbb{L}T_{\text{exp}}$ an almost perfect analog of T I.2 and T II.4.

3. Rambaud’s and Ressayre’s axiomatizations of $T_e$ and of $T_{\text{exp}}$ work if the restricted exponential $e$ is replaced by a whole set $E$ of restricted o-minimal and polynomially bounded functions; this easily implies an extension of Section III where $T_{\text{exp}}$ is replaced by the complete theory of $(\mathbb{R}_{\text{exp}}, E)$. The most interesting case is the set $E$ which allows to define the $\Gamma$ function inside $(\mathbb{R}_{\text{exp}}, E)$. This raises the problem if we can extend the present work to $(x^y, \Gamma)$-integral parts; the problem seems accessible.

4. Other open problems are mentioned in [2], more ambitious... and less precise. Let us add one more ambition: observe that we established some cases of a duality between an arithmetical theory and a theory of fields; and that this duality was used in the direction which takes advantage of known results about fields in order to prove new results about arithmetics. The question is: what about the opposite direction?

Section V - APPENDIX

We prove the results admitted in Section I and II; it is assumed that the reader makes occasional use of this appendix while reading these sections. So the statements from Section I and II which we prove here are not repeated; however they are renumbered for the sake of coherence with auxiliary results introduced here (the initial numbering is recalled in parenthesis).

Section V.1

We recall $A_0$ (where all variables tacitly range over the positive integers):

- $DU CR$
- $x^{y+z} = x^y x^z$, $x^{-y} x^y = 1$; $x^{yz} = (x^y)^z$; $x^y$ strictly increasing with respect to $x, y > 1$
- $2 < x \rightarrow x^2 < 2^x$
On the arithmetization of real fields with exponentiation

- \( \exists y \ 2^y \leq z < 2^{y+1} ; \exists x \ x^z \leq 2^y < (x + 1)^z \); \( \exists x \ (x + 1)^y < 2(x^y) \).

We consider a model \( A \) of \( A_0 \)

**Fact V.1.1.** (\( = \) Fact I.4)

**Proof.**

1. Consider \( \frac{p}{q} \in \log x \) and \( \frac{p'}{q'} < \frac{p}{q} \). We have: \( 2^p \leq x^q \) and \( p'q < pq' \).
   Then: \( 2^{p'}q < 2^{pq'} < x^{pq'} \), and \( 2^{p'}q < x^q \) which leads to \( \frac{p'}{q'} \in \log x \).

2. By \( A_0 \), for all \( q \in A \), there exists \( p \in A \) such that: \( 2^p \leq x^q < 2^{p+1} \).
   So \( \frac{p}{q} \in \log x \), \( \frac{2^p}{2^q} > \log x \) and \( \frac{2^p}{2^q} - \frac{p}{q} \leq \frac{1}{q} \); thus \( \log x \) is a Cauchy cut.

3. \( \frac{p}{q} \in \log 2^x \) iff \( 2^p \leq 2^{xq} \) iff \( p \leq xq \) iff \( \frac{p}{q} \leq x \).

4. \( \frac{p}{q} \in \log x^y \) iff \( 2^p \leq x^{yq} \) iff \( \frac{p}{yq} \in \log x \) iff \( \frac{p}{q} \in y \log x \).

5. • We prove that \( \forall r \in \log xy \ \forall r_1 > \log x \ \forall r_2 > \log y \ r < r_1 + r_2 \).
   For \( r =: \frac{a}{b}, r_1 =: \frac{b}{c}, r_2 =: \frac{d}{e} \) we have: \( 2^a \leq (xy)d, x^d < 2^b \) and \( y^d < 2^c \). Then \( 2^a \leq (xy)d < 2^b 2^c \), so \( a < b + c \) and \( r < r_1 + r_2 \).

• We prove that \( \forall r_1 \in \log x \ \forall r_2 \in \log y \ \forall r > \log xy \ r_1 + r_2 < r \).
   For \( r =: \frac{a}{b}, r_1 =: \frac{b}{c}, r_2 =: \frac{d}{e} \) we have: \( 2^b \leq x^d, 2^c \leq y^d \) and \( (xy)^d < 2^a \). Then \( 2^b 2^c \leq (xy)^d < 2^a \), hence \( b + c < a \) and \( r_1 + r_2 < r \).

6. Trivial.

We notice that the last point (6.) legitimizes the Definition I.5.1 of the log for fractions. Furthermore \( \log_{0.1} 1 = 0 \), then \( \log_{0.1} \frac{a}{b} = \log_{0.1} x - \log_{0.1} \frac{1}{b} = \log_0 x \), hence the value of the log of an integer given by Definition I.3 is the same as that given by Definition I.5.1 for rationals. In the same vein Fact V.1.2 will prove that the log of a rational given by the Definition I.5.1 (for rationals) is the same as that given by Definition I.5.2 (for reals). And Proposition V.1.5 will prove that for all \( x, y \in A \), the value of \( x^y \) in \( A \) is the same as that given by the Definition I.5.4 for reals.

In Fact V.1.2 and V.1.4 below we prove properties of the cuts \( \log c \), \( c \in R \) before knowing that they are Cauchy cuts (ie that \( \log c \in R \)) - something we will prove in Proposition V.1.5.

**Fact V.1.2.** (\( = \) Fact I.6)

**Proof.**
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1. \( \log \frac{p}{q} = \log p - \log q \), and \( \log p, \log q \) are in \( R \).
2. Trivial.
3. Trivial.
4. Let \( \frac{p}{q} \in \log x \). Then \( 2^p \leq x^q \leq y^q \), and thus \( \frac{p}{q} \in \log y \).
5. Put \( r = \frac{a}{b} \) and \( r' = \frac{a'}{b} \) and assume that \( r \leq r' \). Then \( a \leq a' \), and
   \[
   \log r' - \log r = \log a' - \log b - (\log a - \log b) = \log a' - \log a \geq 0.
   \]
6. Notice that the preceding point (5.) says: \( s \rightarrow \log_{11} s \leq \log_{11} r \).
   We prove that the cuts \( \log_{11} r \) and \( \log_{11} r' \) are equal.
   - Assume that \( \rho \leq \log_{11} r \). There exists \( s \leq r \) such that \( \rho \leq \log_{11} s \) - namely \( s = r \); thus \( \rho \in \log_{12} r \).
     (It shows that \( \log_{11} r \leq \log_{12} r \).
   - Assume that \( \rho \in \log_{12} r \); then there exists \( s \leq r \) such that \( \rho \leq \log_{11} s \). As \( s \leq r, \log_{11} s \leq \log_{11} r \) and thus \( \rho \leq \log_{11} r \).
     (It shows that \( \log_{12} r \leq \log_{11} r \).
7. By \( A_0 \exists y \ 2^y < (x + 1)^y \). Then \( \log 2x^y \leq \log (x + 1)^y \), \( \log 2 + \log x^y \leq \log (x + 1)^y \), \( 1 + y \log x \leq y \log (x + 1) \) and thus \( \log (x + 1) - \log x > \frac{1}{y} \).
8. 
   - Case \( x < y \). Then \( x < x + 1 \leq y \), \( \log x < \log(x + 1) \leq \log y \) and thus \( \log x < \log y \).
   - Case \( x \geq y \). If \( x = y \) then \( \log x = \log y \). If \( x > y \) then \( \log x > \log y \).
9. \( \frac{p}{q} < \frac{p'}{q'} \) iff \( pq' < p'q \) iff \( \log (pq') < \log (p'q) \) iff \( \log p + \log q' < \log p' + \log q \) iff \( \log \frac{p}{q} < \log \frac{p'}{q'} \).
10. \( r \leq 2^a \) iff \( \log r \leq \log 2^a = a \) iff \( r \in 2^a \) (the last \( 2^a \) is the cut defined for reals).
11. \( \rho \in \log c \) iff it exists \( \rho \leq c \) such that \( \rho \leq \log r \) iff \( \rho \in \sup \{ \log r; \ r \leq c \} \).
12. Fix \( r \) and \( r' \) such that \( r < r' < c \); then \( \log r < \log r' \). Fix \( s \) such that \( \log r < s < \log r' \). Then \( \log r < \log c \) because \( s \in \log c \) and \( \log r < s \).
13. Case \( r > c \): fix \( r' \) such that \( r > r' > c \); then
   \[
   \log r > \log r' \geq \sup \{ \log s; \ c \geq s \} = \log c.
   \]
   Case \( c < c' \): fix \( r \) such that \( c < r < c' \). Then \( \log c < \log r < \log c' \).
Definition V.1.3. Let \( X \) be a subset of \( R \); \( X \) is **pseudo-integral** if:

- \( X \) is cofinal in \( R \),
- for all \( x \in X \), there exists \( y \in X \), \( y > x \) and: \( \forall z \in X \quad x < z \rightarrow y \leq z \). \( y \) is called the successor of \( x \) in \( X \) and denoted \( \text{succ}_X(x) \),
- for all \( x \in X \), \( \inf \{ z - y; \; y, z \in X \; \text{and} \; y < z \leq x \} > 0 \).

We will prove and use that \( \log(A^+) \) is pseudo-integral (a name chosen to point out a weak similarity with integral parts).

Fact V.1.4. \((p, q, x, y, z \; \text{range over} \; A^+, \; r \; \text{over} \; \mathbb{Q}(A) \; \text{and} \; c, c' \; \text{over} \; R)\)

1. \( \exists y \forall z \leq x \quad \log \left( z + 1 \right) - \log z > \frac{1}{y} \).
2. \( \exists x \quad \log x \leq \frac{p}{q} < \log (x + 1). \)
3. Let \( X \) be a pseudo-integral subset of \( R \) such that each positive rational is between two successive elements of \( X \); then each positive real is between two successive elements of \( X \).
4. \( \forall c \geq 0 \exists x \quad \log x \leq c < \log (x + 1). \)
5. \( \exists x \; \log \frac{p}{q} \leq c < \log \frac{p+1}{q}. \) Then \( \forall y \exists x \; \log \frac{p}{q} < c < \log \frac{p+2}{q}. \)
6. \( \exists x \; \log (x+1) - \log x < \frac{1}{y}. \) And thus \( \lim_{x \to +\infty} \log(x+1) - \log x = 0 \), in other words: \( \lim_{x \to +\infty} \log \left( 1 + \frac{1}{q} \right) = 0. \)
7. \( c' > \log c \; \text{iff} \; \text{it exists} \; r > c \; \text{such that} \; c' > \log r > \log c. \) That remains true if we replace \( > \) by \( < \).

**Proof.**

1. By V.1.2.7 there exists \( y \) such that \( \log(x + 1) - \log x > \frac{1}{y}. \) But \( \forall z \leq x \; \left( z \geq 1 \rightarrow \frac{z+1}{z} \geq \frac{1+x}{x} \right) \). Then \( \log \frac{z+1}{z} \geq \log \frac{1+x}{x} \), and thus \( \log(x+1) - \log z \geq \log(x+1) - \log x > \frac{1}{y}. \)

2. By \( A_0 \), for all \( p, q \) there exists \( x \) such that \( x^q \leq 2^q \to (x + 1)^q \); take the \( \log \) of the inequalities.

3. Let \( X' = \{ x \in X; \; x < c \}; \; X' \) is not empty (let \( r < c \), there exists \( x \in X \) such that \( x < r \leq \text{succ}_X(x) \), then \( x \in X' \)).

   We prove (3.) by way of contradiction, so we assume: \( \text{(H)}: \; x < c \rightarrow \text{succ}_X(x) < c. \)

   We are going to see that \( \sup X' = c \). On the one hand \( \sup X' \leq c \) (trivial). On the other hand \( c \leq \sup X' \), because if \( r \in \mathbb{Q}(A) \) and \( r < c \), then there exists \( x \in X \) such that \( x \leq r < \text{succ}_X(x) \), then \( x < c \), and by \( \text{(H)} \) \( \text{succ}_X(x) < c. \) From which \( \text{succ}_X(x) \in X' \) and as \( r < \text{succ}_X(x) \), then \( r \in \sup X' \). So \( c \leq \sup X' \).

---

* \( \inf X > 0 \) means \( \exists \mu > 0 \; \forall x \in X \; x > \mu. \)
Besides, fix $x_0 \in X$ and $x_0 > c$ ($X$ is cofinal in $R$), we have $0 = \delta(c) = \delta(\sup X') \geq \inf \{|z - y|; \ y, z \in X'$ and distinct $\} \geq \inf \{z - y; \ y, z \in X$ and $y < z < x_0\} > 0$ (The first inequality is true because $X' \subset X$ and $X' < x_0$; the last inequality because $X$ discrete). We reached a contradiction hence (3.) is proved: for each $c > 0$ there exists $x \in X$, $x < c \leq \text{su}ce_X(x)$.

4. Put $X = \{\log x; \ x \geq 1\}$. $X$ is pseudo-integral in $R$. Indeed:
   - $X$ is cofinal in $R$ ($\log 2^x = x$).
   - $\log (x + 1)$ is the successor of $\log x$.
   - For all $x$, there exists $B$ such that $\log (x + 1) - \log x > \frac{1}{B}$, and thus whenever $y < z \leq x$, $\log z - \log y \geq \log (y + 1) - \log y > \log (x + 1) - \log x > \frac{1}{B}$.
   - By (2.) each positive rational is between two successive elements of $X$.

Then by (3.) it is the same for all positive element of $R$.

5. $\log y \in R$, then $c + \log y \in R$. Apply (4.) to $c + \log y$: there exists $x$ such that $\log x \leq c + \log y < \log (x + 1)$.

6. By $A_0$, for all $y$ there exists $x$ such that $(x + 1)^y < 2x^y$. Thus $y\log (x + 1) < 1 + y\log x$, and $y(\log (x + 1) - \log x) < 1$.

Fix $z > x$. Then $\frac{z + 1}{z} < \frac{x + 1}{x}$, and thus: $\log \frac{z + 1}{z} < \log \frac{x + 1}{x} < \frac{1}{y}$.

7. Suppose $\log c < c'$. Consider $y$ such that $\frac{z}{y} < c < \frac{z + 1}{y}$ and $\log (x + 2) - \log x < c' - \log c$. Then: $\log \frac{z}{y} < \log c < \log \frac{z + 1}{y} < c'$. \hfill $\Box$

Now we prove that $\log$ is a one-to-one correspondence between $R_+^*$ and $R$ and $2^x$ is its inverse.

**Proposition V.1.5.** (= Proposition I.7)

**Proof.**

1. Notice that $\log c$ is a cut. Assume that $c \not\in \mathbb{Q}(A)$; then for all $y$, there exists $x$ such that $y < x < c < x + 1$.

Then $\log \frac{x}{y} < \log c < \log \frac{x + 1}{y}$. Let $\rho$, $\rho'$ in $\mathbb{Q}(A)$ such that: $\log \frac{x}{y} < \rho < \log c < \rho' < \log \frac{x + 1}{y}$. Then $\rho < \log c < \rho'$ and $|\rho’ - \rho| < \log \frac{x + 1}{y} - \log \frac{x}{y} = \log (x + 1) - \log x$ which limit is 0 as $x$ approaches infinity.

2. Notice that $2^c$ is a cut. For all $y$ there exists $x$ such that $\log \frac{x}{y} < c < \log \frac{x + 2}{y}$. Then $\frac{x}{y} \in \mathbb{Z}$, $\frac{x + 1}{y} \in \mathbb{Z}$ and $\frac{x + 2}{y} - \frac{x}{y} = \frac{2}{y}$ which limit is 0 as $y$ approaches infinity.
3. $r \in 2^\log c$ iff $\log r \leq \log c$ iff $r \leq c$.

4. Given $x, y \in A$. Put $e$ the value of $x^y$ in $A$ ($e = x^y$). By Fact V.1.1.4 $\log_1 e = y \log_1 x$. Put $c = 2^y \log_2 x$. By V.1.2.6, $\log_1 z = \log_2 z$, for all $z \in A$.
Then $e = 2^{\log_2 c} = 2^\log_1 e = 2^y \log_1 x = 2^y \log_2 x = c$.

5. Let $r \in \log 2^c$. Then there exists $r' \leq 2^c$ such that $r \leq \log r'$; and $r' \in 2^c$ (otherwise $\log r' \geq c$). Then there exists $\rho$ such that $\log r' > \log \rho > c$, then $r' > \rho$ and $\rho \not\in 2^c$ and thus $r' > 2^c$. Since $r' \in 2^c$, $\log r' \leq c$, and thus $r \leq c$.

At last, we prove that $2^x$ is an increasing homomorphism of $+ \times$ (restricted to positive elements), continuous...

**Proposition V.1.6.** ( = Proposition I.8)

**Proof.**

1. $2^x$ is the inverse of $\log$ which is strictly increasing.

2. In this proof $x$ and $y$ are in $R$ and the other variables are in $Q(A)$.

- Let $\rho \in \log xy$ and $\rho_1 > \log x$ and $\rho_2 > \log y$. Let us prove that $\rho < \rho_1 + \rho_2$.
- Consider $r \leq xy$ such that $\rho \leq \log r$ ; given $s > x$ and $s' > y$ such that $\rho_1 > \log s > \log x$ and $\rho_2 > \log s' > \log y$, $r \leq xy$ implies $r < ss'$ and $\rho \leq \log r < \log ss' = \log s + \log s' < \rho_1 + \rho_2$.

- Fix $\rho_1 < \log x$ and $\rho_2 < \log y$ and $\rho > \log xy$; Let’s prove that $\rho_1 + \rho_2 < \rho$.
- Assume $s < x$ and $s' < y$ such that $\rho_1 < \log s ( < \log x)$ and $\rho_2 < \log s' ( < \log y)$. Consider $r > xy$ such that $\rho > \log r ( > \log xy)$; then $r > ss'$ and $\rho > \log r > \log ss' = \log s + \log s' > \rho_1 + \rho_2$.

3. $\log x^y = 2^y \log x = y \log x$.

4. $\log 2^{x+y} = x + y = \log 2^x + \log 2^y = \log 2^x 2^y$. $\log$ one-to-one, then $2^{x+y} = 2^x 2^y$.

5. $x^y < x^z$ iff $\log x^y < \log x^z$ iff $y \log x < z \log x$ iff $y < z$, because $\log x > 0$.

6. $y^x < z^x$ iff $\log y^x < \log z^x$ iff $x \log y < x \log z$ iff $y < z$.

7. $\log x^{y+z} = (y + z) \log x = y \log x + z \log x = \log x^y + \log x^z = \log x^y x^z$. Since $\log$ is injective $x^{y+z} = x^y x^z$. 

8. \( \log (x^y)^z = z \log x^y = zy \log x = \log x^{yz} \). Since \( \log \) is one-to-one \( (x^y)^z = x^{yz} \).

9. \( \log, 2^x \) are strictly increasing and one-to-one correspondance.

10. Exercice. \( \square \)

Section V.2

We consider a model \( A \) of \( IE_0(2^x)^{le} \) - including the axiom \( A_\lambda \):

\[
(0 < r < s \rightarrow (1 + s)^r < (1 + r)^s) \land (\exists s < B r \forall (1 + s)^{B r} < 2^{r s}).
\]

Fact V.2.1.

1. \( \log'(1) \) is a Cauchy cut definable in \( A \) (hence \( \log \) and exponential polynoms are differentiable).

2. \( f \) exponential polynom and \( X \) bounded subset of \( R^m \rightarrow f(X) \) is bounded.

3. (Uniform continuity) \( X \subset R^m \) and \( Y \in R^n \) bounded set imply:

\[
\forall \epsilon > 0 \exists \alpha > 0 \forall x \in X \forall \xi \in X (|\xi - x| < \alpha \rightarrow \forall y \in Y |f(\xi, y) - f(x, y)| < \epsilon).
\]

Proof.

1. We consider the cut \( c = \{ r, \exists s > 0 \quad r < \frac{\log(1+s)}{s} \} \) and we prove that \( c \in R \) and \( c = \log'(1) \).

- \( c \) is bounded.

  By \( A_\lambda \), for all \( B \) there exists \( s_B < \frac{1}{B} \) such that: \( \forall r \frac{\log(1+r)^{B r}}{1+r} < 2^{r}\); or equivalently: for all \( r > 0 \frac{\log(1+r)}{r} < \frac{1}{B} + \frac{\log(1+s_B)}{s_B} \). From which follows that \( c \leq \frac{1}{B} + \frac{\log(1+s_B)}{s_B} \).

- \( c \in R \).

  Otherwise fix \( \rho \) such that \( 0 < \rho < \delta(c) \). Consider \( B \) such that \( \frac{1}{B} < \rho \) and fix \( s_B < \frac{1}{B} \) such that:

  \[
  \forall r \quad \frac{\log(1+r)}{r} < \frac{1}{B} + \frac{\log(1+s_B)}{s_B} \quad (*),
  \]

  Then \( \frac{\log(1+s_B)}{s_B} < c \) hence \( \frac{\log(1+s_B)}{s_B} + \rho < c \) and there exists \( r \) \( (< s_B) \) such that:

  \[
  (c > ) \frac{\log(1+r)}{r} > \frac{\log(1+s_B)}{s_B} + \rho > \frac{\log(1+s_B)}{s_B} + \frac{1}{B},
  \]

  from which follows that:

  \[
  \frac{\log(1+r)}{r} > \frac{\log(1+s_B)}{s_B} + \frac{1}{B}, \quad \text{in contradiction to (*)}.
  \]
On the arithmetization of real fields with exponentiation

- By $A_\lambda$, for all $r$ and $s$ in $\mathbb{Q}(A)$ we have: $0 < r < s \rightarrow (1 + s)^r < (1 + r)^s$; thus $0 < r < s \rightarrow \frac{\log (1 + s)}{s} < \frac{\log (1 + r)}{r}$.

  Since $\log$ is continuous, for all $x$ and $y$ in $R$ we have $0 < x < y \rightarrow \frac{\log (1 + y)}{y} < \frac{\log (1 + x)}{x} < c$.

- $c$ is the right derivative of $\log$.

  Indeed, fix $\epsilon > 0$; there exists $s \in \mathbb{Q}(A)$ such that $0 < c - \frac{\log (1 + s)}{s} < \epsilon$.

  But $\forall r \in R \ (0 < r < s \rightarrow \frac{\log (1 + s)}{s} < \frac{\log (1 + r)}{r} < c)$.

  Then $\forall r \in R \ (0 < r < s \rightarrow 0 < c - \frac{\log (1 + r)}{r} < \epsilon)$.

- $c$ is the left derivative of $\log$.

  Indeed, $\frac{\log (1 - r)}{1 - r} = \frac{\log (1 + r)}{r} = (1 - r) \frac{\log (1 + r)}{1 - r}$.

  Then $\lim_{r \to 0} \frac{\log (1 - r)}{1 - r} = \lim_{r \to 0} (1 - r) \frac{\log (1 + r)}{1 - r}$

  $\Rightarrow \lim_{r \to 0} \frac{\log (1 - r)}{1 - r} = c$.

- $(2^x)' = \frac{1}{2} \cdot 2^x$; $2^x$ is inverse to $\log$.

2. By induction on $f$.

3. Idem. $\Box$

The next Fact states that if an exponential polynom changes sign on a given interval, then it takes arbitrary small values.

**Fact V.2.2.** (= Fact II.6)

**Proof.**

1. Put $I = [\rho, \sigma]$. Fix $\epsilon < \frac{1}{\rho}$. By uniform continuity there exists $\alpha$ such that: $\forall t, t' \in I \ (|t' - t| < \alpha \rightarrow |f(\bar{t}, t') - f(\bar{t}, t)| < \epsilon)$.

  Consider $N_0$ such that $\frac{1}{N_0} < \alpha$.

  Put $\rho := \frac{\rho}{N_0}$, $\sigma := \frac{\sigma}{N_0}$ and $g(x) := f(\bar{t}, \frac{x}{N_0})$.

  Then, $g(N_0 p_0) < 0 < g(N_0 p_1)$. A satisfies $IE_0(2^x)^f$ hence there exists $x \in A$ such that $N_0 p_0 \leq x < N_0 p_1$ and $g(x) < 0 < g(x + 1)$.

  Put $t = \frac{x}{N_0}$ and $t' = \frac{x + 1}{N_0}$. Then:

- $t$ and $t'$ are in $I$ because $\rho = \frac{N_0 p_0}{N_0 q} \leq \frac{x}{N_0 q} < \frac{x + 1}{N_0 q} \leq \frac{N_0 p_1}{N_0 q} = \sigma$

- $|t' - t| < \alpha$ because $t' - t = \frac{1}{N_0 q} < \alpha$.

  Then $|f(\bar{t}, t') - f(\bar{t}, t)| < \epsilon$. 

Assume that $f(\overline{\varphi}, t) < \frac{1}{N}$. Indeed, $f(\overline{\varphi}, t) < 0 < f(\overline{\varphi}, t')$, $|f(\overline{\varphi}, t') - f(\overline{\varphi}, t)| < \epsilon$, and thus $|f(\overline{\varphi}, t') - f(\overline{\varphi}, t)| < \epsilon < \frac{1}{N}$.

2. Fix $\epsilon < \frac{1}{2N}$; by uniform continuity there exists $\alpha$ such that:

$\forall \overline{b} \ (|\overline{b} - \overline{a}| < \alpha \rightarrow \forall t \in [c_0, c_1] \ |f(\overline{b}, t) - f(\overline{a}, t)| < \epsilon)$. 

By continuity there exists $\overline{\varphi}$ such that $|\overline{\varphi} - \overline{a}| < \alpha$ and $f(\overline{\varphi}, c_0) < 0 < f(\overline{\varphi}, c_1)$, and there exists $[\rho, \sigma] \subset [c_0, c_1]$ such that $f(\overline{\varphi}, \rho) < 0 < f(\overline{\varphi}, \sigma)$.

By 1 there exists $t \in [\rho, \sigma]$ such that $|f(\overline{\varphi}, t)| < \frac{1}{2N}$.

But $|\overline{\varphi} - \overline{a}| < \alpha$, so $|f(\overline{\varphi}, t) - f(\overline{a}, t)| < \epsilon < \frac{1}{2N}$.

Hence $|f(\overline{\varphi}, t)| \leq |f(\overline{\varphi}, t) - f(\overline{a}, t)| + |f(\overline{a}, t)| < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N}$.

Fact V.2.3 and Propositions V.2.4, V.2.5 contribute to the proof of Proposition V.2.6 which is a weak version of Rolle.

**Fact V.2.3.** Let $f$ be a $2^\mathbb{R}$-polynomial with parameters in $\mathbb{Q}(A)$; given $r_0 < r_1$ in $\mathbb{Q}(A)$ s.t. $f(r_0) < f(s_0)$, for all $N > 0$ in $A$ there exists $r < s$ such that $|r - s| < \frac{1}{N}$, $r_0 < r < s < s_0$ and $f(r) < f(s)$.

**Proof.**

Let us assume that $f(r_0) < f(s_0)$. By continuity and density, there exists $r_1 < s_1$ in $\mathbb{Q}(A)$ s.t. $r_0 < r_1 < s_1 < s_0$ and $f(r_1) < f(s_1)$.

Consider $N \in A$, $N > 0$, and $\ell \in A$ s.t. $|s_1 - r_1| < \ell$. Set

$g(x) = f(r_1 + x \frac{s_1 - r_1}{\ell N})$, $h(x) = g(x) - g(0)$

We have: $h(0) \leq 0$ and $h(\ell N) > 0$.

Then according to the axiom of induction (in $IE_0(2^\mathbb{R})$) there exists $i$ between 0 and $\ell N - 1$ s.t. $h(i) \leq 0$ and $h(i + 1) > 0$.

Therefore $g(i) \leq g(0)$ and $g(i + 1) > g(0)$, and then

$g(i) < g(i + 1)$.

Let us put $r = r_1 + i \frac{s_1 - r_1}{\ell N}$ and $s = r_1 + (i + 1) \frac{s_1 - r_1}{\ell N}$; then we have:

$r_0 < r_1 \leq r < s < s_1 < s_0$, $|r - s| < \frac{1}{N}$ and $f(r) < f(s)$.

**Proposition V.2.4.** Assume that $A$ is countable. Given $x_0 < x_1$ in $R$ and a $2^\mathbb{R}$-polynomial $f$ with parameters in $\mathbb{Q}(A)$,

1. if $f(x_0) < f(x_1)$ there exists $c \in ]x_0, x_1[$ s.t. $f'(c) \geq 0$
2. if $f(x_0) > f(x_1)$ there exists $c \in ]x_0, x_1[$ s.t. $f'(c) \leq 0$.

**Proof.**

2. Apply (1.) to $-f$. 
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1. Let us assume \( f(x_0) < f(x_1) \); by continuity and density there exists \( r_0 < s_0 \) in \( \mathbb{Q}(A) \) s.t. \( x_0 < r_0 < s_0 < x_1 \) and \( f(r_0) < f(s_0) \).

Given \( (N_k) \) cofinal in \( A \), if we have \( r_k < s_k \) s.t. \( f(r_k) < f(s_k) \) then by Fact V.2.3 there exists \( r_{k+1} < s_{k+1} \) s.t.:

\[
r_k < r_{k+1} < s_{k+1} < s_k, |r_{k+1} - s_{k+1}| < \frac{1}{N_{k+1}}, f(r_{k+1}) < f(s_{k+1}).
\]

The diameter of the cut \( c = \{ r \in \mathbb{Q}(A) \); there exists \( k \) s.t. \( r \leq r_k \} \) is 0, and \( c \in [x_0, x_1] \). We claim that \( f'(c) \geq 0 \); otherwise \( f'(c) < 0 \) and there exists \( \epsilon \) s.t. for all \( x \in [c - \epsilon, c] \) for all \( x' \in [c, c + \epsilon] \):

\[
f(x) > f(c), f(x') < f(c) \text{ hence } f(x) > f(x').
\]

Fix \( k_0 \) s.t. for all \( k > k_0 \) \( r_k \in [c - \epsilon, c] \) and \( s_k \in [c, c + \epsilon] \); then \( f(r_k) > f(s_k) \) for all \( k \geq k_0 \). This contradicts the construction of \( (r_k) \) and \( (s_k) \).

The next Proposition is a first order property of \( A_{exp} \) because it does not involve the Cauchy closure \( R \) of \( \mathbb{Q}(A) \); consequently it is true as soon as it holds for countable \( A \). This allows to apply V.2.4 to its proof.

**Proposition V.2.5.** Given a \( 2^\sigma \)-polynomial \( f \) and given \( \sigma, \rho_0 \) and \( \rho_1 \) in \( \mathbb{Q}(A) \) with \( \rho_0 \leq \rho_1, f(\sigma, x) \).

If \( f(\sigma, \rho_0) = f(\sigma, \rho_1) \), then we have:

- either \( \exists x_0, r_1 \in [\rho_0, \rho_1] \cap \mathbb{Q}(A) \) \( f'(\sigma, r_0) f'(\sigma, r_1) < 0 \),
- or \( \forall N \exists t \in [\rho_0, \rho_1] \cap \mathbb{Q}(A) \) \( |f'(\sigma, t)| < \frac{1}{N} \).

**Proof.** We shall write \( f(x) \) instead of \( f(\sigma, x) \); we can suppose that \( f \) is not constant in \( [\rho_0, \rho_1] \). Fix \( x_0 \) in \( [\rho_0, \rho_1] \) s.t. \( f(x_0) \neq f(\rho_0) \).

**Case 1:** \( f(x_0) > f(\rho_0) \).

Then, by continuity there exists \( x_1 < x_0, x_1 \in [\rho_0, \rho_1] \) and there exists \( x_2 > x_0, x_2 \in [\rho_0, \rho_1] \) s.t.

\[
f(x_1) < f(x_0), f(x_2) < f(x_0).
\]

By Proposition V.2.4, there exists \( c_0 \in [x_1, x_0] \) s.t. \( f'(c_0) \geq 0 \) and there exists \( c_1 \in [x_0, x_2] \) s.t. \( f'(c_1) \leq 0 \).

- Case 1.1: \( f'(c_0) = 0 \) or \( f'(c_1) = 0 \).

Let us assume for instance that \( f'(c_0) = 0 \). Since \( c_0 \in [\rho_0, \rho_1] \) by continuity and density, for all \( N \in A, N > 0 \) there exists \( t \in [\rho_0, \rho_1] \cap \mathbb{Q}(A) \) s.t. \( |f'(t)| < \frac{1}{N} \).
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Case 1.2: \( f'(c_0) \neq 0 \) and \( f'(c_1) \neq 0 \).

Then \( f'(c_0) > 0 \) and \( f'(c_1) < 0 \). Since \( c_0 \) and \( c_1 \) are in \( [\rho_0, \rho_1] \) by continuity and density there exists \( r_0 \) and \( r_1 \) in \( [\rho_0, \rho_1] \cap \mathbb{Q}(A) \) s.t. \( f'(r_0) > 0 \) and \( f'(r_1) < 0 \).

**Case 2:** \( f(x_0) < f(\rho_0) \).

Apply Case 1 to \(-f\). \( \square \)

The proposition which follows is a weak version of Rolle.

**Proposition V.2.6.** (= Proposition II.7)

*Proof.* We apply Proposition V.2.5, and if \( \exists r_0, r_1 \in [\rho_0, \rho_1] \cap \mathbb{Q}(A) \) s.t. \( f''(\bar{r}, r_0) f''(\bar{r}, r_1) < 0 \) then we apply Fact V.2.2.2 to \( f' \). \( \square \)

The Fact which follows is a weak version of the Differentiation Lemma.

**Fact V.2.7.** (= Fact II.8)

*Proof.*

1. Put \( g(\bar{r}, x) = f(\bar{r}, x) - f(\bar{r}, s_0) - \frac{f(\bar{r}, s_1) - f(\bar{r}, s_0)}{s_1 - s_0}(x - s_0) \).

   We have: \( g(\bar{r}, s_1) = g(\bar{r}, s_0) \).

   We apply the Proposition V.2.6 to \( g \), where \( g'(\bar{r}, x) = f'(\bar{r}, x) - \frac{f(\bar{r}, s_1) - f(\bar{r}, s_0)}{s_1 - s_0} \).

2. Fix \( s_0, s_1, \bar{r} \) such that \( c_0 < s_0 < s_1 < c_1, |\frac{f(\bar{r}, s_1) - f(\bar{r}, c_0)}{c_1 - c_0} - \frac{f(\bar{r}, s_1) - f(\bar{r}, s_0)}{s_1 - s_0}| < \frac{1}{3N} \) and \( \sup_{t \in [c_0, c_1]} |f'(\bar{r}, t) - f'(\bar{r}, t)| < \frac{1}{3N} \). Applying 1., consider \( t \in [s_0, s_1] \) such that \( |\frac{f(\bar{r}, s_1) - f(\bar{r}, s_0)}{s_1 - s_0} - f'(\bar{r}, t)| < \frac{1}{3N} \).

3. Consequence of (2.) \( \square \)

The next proposition shows that if an exponential polynomial \( f \) has arbitrarily small values arbitrarily close to some non Cauchy cut, then the same is true for the derivative and for the product of \( f \) by another exponential polynomial.

**Proposition V.2.8.** (= Proposition II.9)

*Proof.*

1. \( c = \{ r : \forall c' ((\forall B \exists N > B \rho_N < c') \rightarrow r < c') \} \) by definition of \( \inf \).

\( r \)
• \( c \leq c_1 \) because \( \forall B \exists N > B \quad \rho_N \leq \sigma < c_1 \).

• \( c_0 \leq c \); indeed, let \( c' \) such that \( \forall B \exists N > B \quad \rho_N < c' \) and let \( N_0 \) such that \( \rho_{N_0} < c' \). As \( \forall N \quad c_0 \leq \rho_N \), we have \( c_0 \leq \rho_{N_0} < c' \) and thus \( c_0 < c' \).

Then \( \forall c' \quad (\forall B \exists N > B \quad \rho_N < c') \rightarrow c_0 < c', \) and \( c_0 \leq c \).

2. \( c = \{ r : \forall c' (\forall B \exists N > B \quad \rho_N < c') \rightarrow r < c' \} \).

- **Case 1:** \( \forall B \exists N > B \quad \rho_N < c \).

Fix \( r, r' \) such that \( r < r' < c \). Then there exists \( B' \) such that \( \forall N > B' \quad \rho_N \geq r' \).

Consider \( B \in A \), then by Case 1 there exists \( N > \text{sup}(B, B') \) such that \( \rho_N < c \). As \( N > B' \), then \( \rho_N \geq r' > r \). From which we get \( N > B \) and \( r < \rho_N < c \).

- **Case 2:** \( \exists B \forall N > B \quad \rho_N \geq c \).

Fix \( B_0 \) such that \( \forall N > B_0 \quad \rho_N \geq c \).

Consider \( r > c \). Then \( r \notin c \), and there exists \( c' \), such that: \( (\forall B \exists N > B \quad \rho_N < c') \) and \( (r \geq c') \).

Given \( B \in A \) there exists \( N > \text{sup}(B, B_0) \) such that \( \rho_N < c' \) \( (\leq r) \). From which we get \( N > B \) and \( c \leq \rho_N < r \).

3. Assume \( c \in R \). Let’s prove that \( |f(\overline{a}, c)| < \frac{1}{N} \) for all \( N \).

Fix \( N \); by continuity there exists \( r, r' \) such that \( r \leq c < r' \) and \( \forall \rho \in [r, r'] \quad |f(\overline{a}, c) - f(\overline{a}, \rho)| < \frac{1}{N} \).

And according to the remark made above in (2.) there exists \( \rho \in [r, r'] \quad |f(\overline{a}, \rho)| < \frac{1}{N} \).

Hence \( |f(\overline{a}, c)| \leq |f(\overline{a}, c) - f(\overline{a}, \rho)| + |f(\overline{a}, \rho)| < \frac{1}{N} \).

4. Let \( r < c \) and \( N \in A \); there exists \( \rho, r < \rho < c \) such that \( |f(\overline{a}, \rho)| < \frac{1}{N} \).

\( \rho + \delta_0 < c \) since \( \delta_0 < \delta(c) \) and \( \rho < c \); and there exists \( \rho', \rho + \delta_0 < \rho' < c \) such that \( |f(\overline{a}, \rho')| < \frac{1}{N} \).

Thus we found \( \rho, \rho', r < \rho < \rho' < c, \rho' - \rho > \delta_0 \) such that \( |f(\overline{a}, \rho)| < \frac{1}{N} \) and \( |f(\overline{a}, \rho')| < \frac{1}{N} \).

5. Fix \( \delta_0 \) such that \( 0 < \delta_0 < \delta(c) \), and put \( \mu' = \frac{\mu}{\delta_0} \). Let \( r < c, N \in A \) and \( \rho, \rho' \) such that \( r < \rho < \rho' < c, \rho' - \rho > \delta_0 \), \( |f(\overline{a}, \rho)| < \frac{1}{\delta_0 N} \), and \( |f(\overline{a}, \rho')| < \frac{1}{\delta_0 N} \).

By Fact V.2.7 there exists \( s \in [\rho, \rho'] \) such that \( \left| \frac{f(\overline{a}, \rho') - f(\overline{a}, \rho)}{\rho' - \rho} - f'(\overline{a}, s) \right| < \frac{\mu}{\delta_0 N} \).
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Then: \( r < s < c \) and \( |f'(\overline{t}, s)| \leq \frac{|f(\overline{t}, \rho') - f(\overline{t}, \rho)|}{\rho' - \rho} + \frac{|f(\overline{t}, \rho') - f(\overline{t}, \rho)|}{\rho' - \rho} < \frac{\mu}{3\delta_0 N} + \frac{\mu}{3\delta_0 N} + \frac{\mu}{3\delta_0 N} \ast. \)

From which results: \( r < s < c \) and \( |f'(\overline{t}, s)| < \frac{\mu}{N}. \)

6. Fix \( r_0 < c \) and \( B > \sup\{2^{g(\rho)} : \rho \in [r_0, c]\} \) and put \( \mu_0 = B\mu. \)

Consider \( r < c, N \in A \) and \( \rho \) such that \( \sup(r, r_0) < \rho < c \) and \( |f(\rho)| < \frac{\mu}{N}. \) Then \( |f(\rho) 2^{g(\rho)}| < \frac{\mu_0}{N}. \)

Hence \( \exists \mu_0 \forall r \forall N \exists \rho \ r < \rho < c \) and \( |f(\rho) 2^{g(\rho)}| < \frac{\mu_0}{N}. \)

7. Similar to (4.)
8. Similar to (5.)
9. Similar to (6.) \( \Box \)

References


\( \ast \rho' - \rho > \delta_0 \)
On the arithmetization of real fields with exponentiation

