ON THE OPTIMALITY OF A CONSERVATION THEOREM OF L.D. BEKLEMISHEV

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Abstract. Let T be an elementary presented theory containing Elementary Arithmetic EA. Relativized local reflection principles for T, denoted $\mathrm{Rfn}^n_\Gamma(T)$, are axiom schemes given by

$$[n]_T\varphi \to \varphi,$$

where $[n]_T$ stands for n–provability in T and φ ranges over all sentences in the class of formulas Γ. In his comprehensive study of reflection principles in formal arithmetic [1], among many other interesting results, L.D. Beklemishev obtained the following conservation theorem for relativized local reflection principles.

Theorem 1. *Let T be an elementary presented theory containing EA. For every $n \geq 0$, $T + \mathrm{Rfn}^n_{\Sigma_{n+2}}(T)$ is conservative over $T + \mathrm{Rfn}^n_{\Sigma_{n+1}}(T)$ with respect to $\mathcal{B}(\Sigma_{n+1})$ (= boolean combinations of Σ_{n+1}) sentences.*

In this work we prove that this conservation result is optimal with respect to the arithmetical complexity in the sense that Π_{n+2}–sentences are already not conserved. This is interesting because:

$\underline{n = 0}$: $\mathrm{Rfn}^0_\Gamma(T)$ coincides with its non–relativized analogue by definition. Thus, our result shows that, in general, $T + \mathrm{Rfn}_{\Sigma_2}(T)$ is not Π_2–conservative over $T + \mathrm{Rfn}_{\Sigma_1}(T)$. This answers a question posed by Beklemisehv in [1].

$\underline{n > 0}$: It is a theorem of Beklemishev that over EA, $I\Pi^-_{n+1} \equiv \mathrm{Rfn}^n_{\Sigma_{n+2}}(EA)$ and $I\Sigma^-_n \equiv \mathrm{Rfn}^n_{\Sigma_{n+1}}(EA)$. Thus, our result implies that $I\Pi^-_{n+1}$ is not Π_{n+2}–conservative over $I\Sigma^-_n$.

References

[1] Beklemishev, L.D. *Reflection principles and provability algebras in formal arithmetic*, Russian Math. Surveys **60**, 197–268 (2005)

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