Recursive saturation of the real closure as a Tennenbaum-like property.

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It is well-known that every real closed field has a so-called integer part, which is a discretely ordered ring satisfying the very weak arithmetic theory *IOpen*. In a recent paper, d'Aquino, Knight and Starchenko (JSL 75(1)(2010)) considered the problem which real closed fields have integer parts satisfying stronger theories of arithmetic than just *IOpen*. They proved among other things that if a nonarchimedean rcf has an integer part satisfying Peano Arithmetic, or even $I\Sigma_4$, then it is recursively saturated, but that the theorem no longer holds if $I\Sigma_4$ is replaced by $I\Delta_0$. The general question how much arithmetic in an integer part guarantees recursive saturation was left unanswered.

We study the question more systematically and show that the necessary amount of arithmetic is much smaller than $I\Sigma_4$: recursive saturation of an rcf follows for instance from the existence of integer parts satisfying PV (the canonical theory for polynomial-time reasoning) or severely restricted forms of Σ^b_1 induction studied by Boughattas and Ressayre. Moreover, the theorem also holds for $I\Delta_0$ under a rather mild assumption on cofinality ("unboundedness"). However, it fails for a number of extensions of *IOpen* by algebraic axioms, even assuming unboundedness. The status of some intermediate fragments of $I\Delta_0$, in particular IE_1, remains open.

It turns out that the crucial case to study is when the rcf is actually the real closure of the integer part in question. Our work suggests that "all unbounded models of T have recursively saturated real closures" is a Tennenbaum-like property, i.e. a structural property of models separating "algebraic fragments of arithmetic" from "theories with significant arithmetical content".