A Silver like perfect set theorem with an application to Borel Model Theory

Joël COMBASE
Univ Paris I,
JP Ressayre and S Boughattas

June 15, 2011

Abstract. Silvers's perfect set theorem says that every Π^1_1 equivalence relation over a polish space either has countably many equivalence classes or has a perfect set of inequivalent elements. We prove an analog where the equivalence relation is replaced by a Π^1_1 dependence relation, and the perfect set of inequivalent elements is replaced by a perfect independent set. The dependence relation should satisfy a weak form of the exchange property ; and such a dependence relation can be canonically defined on any model of a superstable theory. Thus we can apply the preceding result to such models ; we use it prove the following theorem : every totally Borel model of a superstable theory is saturated as soon as it is ω_1-saturated. Where a totally Borel model is any quotient structure $\mathfrak{M}_0/E$ such that

- $\mathfrak{M}_0$ is a structure in a language without equality symbol, the domain of which is the Baire space ω^ω
- E is a Borel equivalence relation on $\mathfrak{M}_0$ which satisfies the axioms of equality with respect to the primitives of $\mathfrak{M}_0$
- every first order definable relation of $(\mathfrak{M}_0, E)$ is Borel
- the language of $\mathfrak{M}_0/E$ is the language of $\mathfrak{M}_0$ plus equality.

Nota-Bene – Classification theory allows to show that many superstable theories have totally Borel ω_1-saturated models.

Definitions and recalls

Silvers's perfect set theorem says that every coanalytic equivalence relation over the Baire space which has uncountably many equivalence classes has a perfect set of inequivalent elements. Our analog will say that for every coanalytic "dependence relation" over the Baire space, every analytic set which has an uncountable independent set has a perfect one. This result is subject to a condition on the dependence relation, denoted EFSW and which we define below.

A notion of closure over a set X is a relation $x \in cl(A)$ where $x \in X, A \subseteq X$ such that $cl(A) := \{x : x \in cl(A)\}$ is the union of the sets $cl(F)$ where F ranges over the finite subsets of A. The closure relation is said to be a notion of dependence on X, if for all $A \subseteq X$

- $A \subseteq cl(A)$
- if $x, y \notin cl(A)$, then
$$x \in cl(A \cup \{y\}) \text{ iff } y \in cl(A \cup \{x\}).$$

The dependence relation is a pregeometry if it satisfies the additional condition

- $cl(A) = cl(cl(A))$.

We then use the familiar vocabulary, saying that x is dependent on A just in case $x \in cl(A)$, that x is independent of y over A if $x \notin cl(A \cup \{y\})$, etc. As is well known, independent sets behave nicely in a pregeometry. More precisely, every set $A \subseteq X$ has a base, that is an independent generating subset $B : A \subseteq cl(B)$ and any two elements of B are independent (over the empty set). And any two bases of the whole space have the same cardinality, which is called the dimension of the space. We now introduce a weakening of the concept of pregeometry, where the dimension is no longer well defined.

We say that our notion of dependence satisfies EFSW if

- for every finitely spanned set $A \subseteq X$, there exist a positive integer N such that every independent subset of A has cardinality $\leq N$.

The smallest such integer in called the weight of A and, when it exists, A is said to have weight. The full strength of this condition will be made use of when we investigate the Borel model theory of superstable theories.

Here we assume that our dependence relation $x \in cl(A)$ is on the Baire space ($X = \omega^\omega$), and we say that it is Π^1_1 if it is so via any recursive coding of its restriction to finite sets A. Here is our Silver like perfect set theorem in refined form.

Theorem 1 - If our Π^1_1 dependence relation satisfies $EFSW$, then there exists a largest Π^1_1 set X containing no uncountable independent subset ; and any Σ^1_1 set Y not included in X has a perfect independent subset.

The proof is easily be seen to relativize, thus yielding the following version in the classical descriptive theoretic framework.

Theorem 1' - If a coanalytic notion of dependence on a polish space X satisties $EFSW$, then every analytic subset of X containing an uncountable independent set has a perfect one.

Some well known results in descriptive set theory are direct corollaries of these results, and in a similar way one can also obtain new results of the same type. But we are after a new result which applies Th.1' to model theory ; the assumption EFSW on the dependence relation of Th.1' suggests to consider models of a theory T which is stable in the sense of [Sh 2], because then every model $\mathfrak{M}$ of T has a natural notion of dependence $x \in cl(A)$, based on "forking" and which satisfies EFSW whenever T is superstable. (We suggest [La] and [KP] for an updated treatment of stability theory plus important new results. More friendly approaches can be found in several textbooks). In order to apply Th.1' we need to ensure in addition that $x \in cl(A)$ is a coanalytic relation. This

is easily seen to be true whenever $\mathfrak{M}$ is totally Borel in the following sense : the domain of $\mathfrak{M}$ is the Baire space, and every definable relation in $\mathfrak{M}$ is Borel. We can make it be true even more generally : we assume that $\mathfrak{M}$ is the quotient of $\mathfrak{M}_0$ under E, where E satisfies the equality axioms with respect to the primitives $\mathfrak{M}_0$ and $(\mathfrak{M}_0, E)$ is totally Borel.

Theorem 2 - Under these assumtions on $\mathfrak{M}$ we have that $\mathfrak{M}$ is saturated as soon as it is ω_1-saturated

For this result to be interesting, we want the existence of many superstable theories to have totally Borel, ω_1-saturated models. Indeed, Shelah proved that many superstable theories are classifiable, in the sense that their models are characterized up to isomorphism by some countable tree of cardinal invariants ; then the ω_1-saturated models are those for which all these invariants are uncountable, and due to the extension of "totally Borel" to quotient structures it is easy to see that up to isomorphism, the model in which all invariants have power the continuum is totally Borel.

Conclusion – 1. A number of results have been obtained concerning Borel structures starting with SILVER [Si] and FRIEDMAN [F] followed by HARRINGTON - SHELAH [HS], SHELAH [Sh 1], HARRINGTON - SHELAH - MARKER [HSM] and LOUVEAU [Lo 1]. FRIEDMAN [F] also initiated the model theory of Borel (in fact totally Borel) structures. Here we have started to mix this subject with stability theory ...

2. We want to give to the real author of this work a final "adieu".

Joel Combase, 1947-2010

Joel was accepted as a graduate student at Berkeley University, due to his brilliant undergraduate career in Paris University. In 1984 he obtained a PhD under the guidance of J. Barwise, S. Feferman, and returned to France where he became a teacher in the Philosophy department at La Sorbonne. He taught Philosophy with emphasis on Mathematical Logic to his students, all his life except in the long periods where he was too ill to work. His lessons in Logic where fascinating... for those who were good enough to follow. One of these students, now professor at the University of Lausanne, is certain that he would have missed his love and vocation for mathematical research, if Combase had not been his teacher in Philosophy. Joel used to explain to the members of the Equipe de Logique his mathematical ideas, which always bore the distinctive mark of his personal approach to Logic. To those who became his friends, he has made an unforgettable impression by his intelligence, melancholy and kindness. JP Ressayre

References

[F] H. FRIEDMAN, *Borel Structures in Mathematics, I. Manuscript, Ohio State University, Colombus, Ohio, 1979.*

[HS] L. HARRINGTON AND S. SHELAH. *Counting equivalence classes of Co-κ-Souslin equivalence relations; Logic Colloquim'80 (D. Van Dalen, D. Lascar, J. Smiley .ed.) 1982*

[HSM] L. HARRINGTON, D. MARKER AND S. SHELAH. *Borel Orderings; Transaction of the American Mathematical Society, Vol. 310, 1, 1988.*

[Ko] S. KOPPELBERG. *Part I of Handbook of Boolean Algebra, Vol I (D. MONK and R. BONNET eds) North Holland, New York 1989.*

[KP] KIM, BYUNGHAN; PILLAY, A. *Around stable forking. Dedicated to the memory of Jerzy Los. Fund. Math. 170 (2001), no. 1-2, 107–118.*

[La] D. LASCAR. *Stability in Model Theory; Longman Scientific and Technical.*

[Lo 1] A. LOUVEAU. *Relations d'équivalence dans les espaces Polonais, in Séminaire Général de Logique 1982-83 (F. DELON, D. LASCAR, A. LOUVEAU, G. SABBAGH, eds) U.E.R. de Mathématiques et L.A. 212 du C.N.R.S., Paris.*

[Mo] I. MOSCHOVAKIS DESCRIPTIVE SET THEORY, NORTH-HOLLAND, AMSTERDAM, 1980.

[Si] J. SILVER. *Counting the number of equivalence classes of Borel and co analytic equivalence relation; Annals of Mathematical Logic, Vol. 18, 1980.*

[Sh 1] S. SHELAH. *On Co-κ-Souslin Relations; Israel Journal of Mathematics Vol. 47, 2-3, 1984.*

[Sh 2] S. SHELAH. *Classication Theory and The Number of non-isomorphic Models, 2nd. ed, North-Holland, Amsterdam, 1990.*