Bootstrapping (first part+"S. 3")

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Dedicated to the famous bootstrapper:
Cyrano von Münchhausen-Bergerac

Abstract

We construct models of the integers, to yield: witnessing, independence and separation results for weak systems of bounded induction.

Summary

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INTRODUCTION

As one of our referees observed it: "bootstrapping denotes the tedious and mostly boring process of providing the necessary definitional extensions to allow working smoothly in a weak theory". But this aspect of bootstrapping hardly appears in our paper because we assume that the reader is sufficiently acquainted with the process to allow us to go over it quickly - otherwise see for instance $[B], [P1]$. On the other hand, bootstrapping taken literally gives a colorful image of what one would like to do when all methods available to achieve a goal miserably fail. Such is the situation in the P versus NP issue, hence in the present paper which deals with the collapse of the polynomial hierarchy. This situation together with the modesty of our results suggested to us the title of paper. The present Introduction goes into the matter somewhat more in detail than it is customary, because the work has unfamiliar aspects and because it prepares for a sequel.

An apology of counterfeiting - 98% of nonstandard studies in Arithmetic first use completeness/compactness to get a **strong** nonstandard model N; and they study N, or rather some well chosen submodel A. Models obtained that way have a shortcoming: the use of completeness/compactness is brutal and non effective, it gives you little control on N and even on A. Another defect is that you hardly can explore the universal theory of the integers in this way, because you cannot **vary** it: A inherits from N. For instance a prime in N is prime in A too and forever, preventing the application of model theory to study significant questions such as: which arithmetical axioms are needed to prove that primality is NP and that it is in P. Let us call **true** such models N and their parts – for the use of completeness/compactness makes it immediate to have N keep any first order property true in the **standard** model ω.

The 2% remaining studies use a model B which is extracted from some suitable nonstandard model R **of the reals**. Example (and first construction of this kind): Shepherdson [Shep] observed that in the real closed field of Puiseux series, the finite series with an integral constant coefficient form a model of Open Induction. In that model we have (for instance): square root of 2 is rational. Thus Open Induction does not prove irrationality of 2.

2% is little ; hence such constructions are in need of **advertisement** and to begin with, of a popular name. To fulfil this need let us call **counterfeit** all such models B, their theories and their elements (except the ones that happen to be "true integers" in the above sense: they always include ω but may also include nonstandard individuals of N where (R,N) is a strong model of second order Arithmetic).

After the start given by Shepherdson, here is a sampling of the kind of models created in this setting. i) Models constructed by iterated realization of a type inside a real closed field, used to show that every Z-ring can be

extended to a model of Open Induction [W] ; and (among others) that primality is not diophantine in the theory of Exponential Open Induction $OI(2^x)$ [B3]. ii) Puiseux series models, used to show among others that (Normal) Open Induction is not finitely axiomatizable and does not follow from existence of euclidean quotients plus the existence of all approximate n^{th} roots, $n < \omega$ [B1][B2]. iii) Models built from "transseries" (= transfinite generalizations of Puiseux series), used to show Σ_1^b witnessing by Polytime functions, for the theory $OI(x^y)$ [R1][R2][R3][B-R].
So far the counterfeiting activity only affected the study of **open** induction axioms. The present paper is a modest beginning of extension to axioms of **bounded** induction ; our main contribution can be said to lie in the methods we introduce rather than in the results. Still, the virtues of the results are: to stand not far from the threshold where they would have significant consequences in Structural Complexity ; and to give an interesting partial explanation of why the missing last step to this threshold is difficult – although it would establish conjectures of the most expected kind.

B. Independence results

Notations

$\mathcal{L}_B$:= language of Arithmetic, including the functions x (:= identity), $x+y$, $x-y$, $x \cdot y$, $\lfloor x/2 \rfloor$, $|x|$ (:= binary length of x), $2^{|x||y|}$
Base := the axioms *BASIC* of Buss [Bu] for this language, or in later cases the strengthening *EBASIC* introduced by Pollett [P1], with the function MSP$(i,x) :=\ \lfloor x/2^i \rfloor$ added to $\mathcal{L}_B$
basic term := term of $\mathcal{L}_B$
Let τ denote a basic term with the single variable x; for any formula $\varphi = \varphi(x)$ with additional parameters $\bar{u}$ and any set Γ of formulas

$$\varphi - ind^{\tau} := \forall x[\varphi(x) \to \varphi(x+1)] \to [\varphi(0) \to \forall x \leq \tau \varphi(x)]$$

$$\Gamma - ind^{\tau} := \{\forall \bar{u}(\varphi - ind^{\tau}); \varphi \in \Gamma\}.$$

The main cases will be :

- $\tau \geq x$; then $\Gamma - ind^{\tau}$ is just the Γ-induction scheme.
- $\tau = |x|$; $BASIC + \Sigma_n^b - ind^{|x|}$ is the system denoted S_2^n by Buss [Bu], which for $n = 1$ "captures" the class of Polytime functions.
- $\tau = |x|_p$ where $|x|_0 := x$ and $|x|_{p+1} := ||x|_p|$.

For $f, g : \omega \to \omega$ we write $f^{\omega} \prec g$ to mean that g (eventually) **dominates** every power of f.

Theorem 0.1 *If $\tau^\omega \prec |x|_2$ (for instance if $\tau = |x|_3$), then $BASIC + \Sigma_1^b - ind^\tau$ does not prove any of the following statements :*

1. *every factor of a power of 2 is itself a power of 2*
2. *$\lfloor x/2^i \rfloor$ exists for every $i < |x|$ (equivalently: the binary expansion of x is defined)*
3. *S_2^1 and EBASIC*
4. *the set of primes is NP*
5. *NP=co-NP.*

We denote $E_n(\Gamma)$ all formulas of the form: a boolean combination of formulas from Γ, preceded by a string $\exists \bar{x}^1 \forall \bar{x}^2 .... Q^n \bar{x}^n$ of bounded quantifications. We sometimes write $\Sigma_{n+1}^{b'}$ for the class $E_n(\Sigma_1^b)$ - in particular $\Sigma_2^{b'} = E_1(\Sigma_1^b)$.

Theorem 0.2 *Assume that $exp(\tau^\omega) < |x|_2$ (for instance $\tau = |x|_4$) then*
a) the above independence results remain true for the induction scheme $\Sigma_2^{b'} - ind^\tau$.
b) In addition, for any $p < \omega$ $BASIC + \Sigma_2^{b'} - ind^\tau$ does not prove "$\Sigma_p^b = \Sigma_{p+1}^b$" (= collapse of the polynomial hierarchy on its p-th level).

More generally,
Strong Conjecture: the same independence results hold w.r.t. $\Sigma_n^{b'} - ind^\tau$ for each $n \geq 2$, assuming $exp_{n-1}(\tau^\omega) < |x|_2$; where $exp_0(x) := x, exp_{n+1}(x) := exp(exp_n(x))$. But we plan to deal with the case $n > 2$ in a second part of this work, still to come. Our independence results and conjectures w.r.t. collapse were inspired to us by the one of Pollett [P1] and stand in close relationship with it. More precisely Pollett has $EBASIC$ in his theory while ours has only $BASIC$; on the other hand he deals with induction up to τ for $\tau = |x|_{n+3}$, to be compared with our assumption $exp_{n-1}(\tau^\omega) \prec |x|_2$ which allows τ to come close to $|x|_{n+1}$. He uses the **strict** version of Σ_n^b in the induction scheme whereas our notion $\Sigma_n^{b'}$ is more liberal in some ways; and he proves independence of "$\Sigma_p^b = \Sigma_{p+1}^b$" for the **strict** notion of Σ_p^b while we prove it as well for the **liberal** one. A corollary of the Conjecture (using an argument of Pollett [P1]) would be that the conjunction of all theories $BASIC + \Sigma_n^b - ind^{|x|_{n+2}}$ is **not** finitely axiomatizable.

C. Witnessing results (of $\forall \Sigma_1^b$ formulas by basic terms of low depth).
We set

$$D_0(x, c) = D_0(x) := \{y : y < |x| \text{ or } y = x\}; D_{i+1}(x, c) := \mathcal{L}_B D_i(x, c) \upharpoonright c$$

where

$$\mathcal{L}_B X := \{f(x, y); x, y \in X \text{ and } f \in \mathcal{L}_B\}$$

Thus $b \in D_i(x,c)$ implies that b is the value of some basic term with parameters in $D_0(x)$, of **depth** $\leq i$ and hereditarily bounded by c. Given any Σ_1^b formula φ we show for instance

Theorem 0.3 *Assume that $\forall x \exists y \varphi$ is provable in Base plus $\Sigma_1^b - ind^{|x|_3}$ or plus $\Sigma_2^{b'} - ind^{|x|_4}$; then $\forall x[\exists y \in D_\mu(x, x^{|x|^K})]\varphi$ is true (in the standard model ω) for some $K < \omega$, where $\mu := |x|_3^K$ in the first case and $\mu := exp(|x|_4^K)$ in the secund case.*

Remark

- If $\tau^\omega < |x|_2$ then eventually $D_{\tau(x)}(x, x^{|x|^K})$ is of cardinal less than x for every $K < \omega$. This prevents the above results from following by Parikh's well known result that for some $k < \omega$, $\forall x(\exists y < x^{|x|^k})\varphi$.
- We shall strengthen T 0.3.a by extending the theory $\Sigma_1^b - ind^{|x|_3}$ to a theory T^τ which satisfies **provable** witnessing:= the **same** theory T^τ which occurs in the assumption also proves the witnessing conclusion.
- These witnessing results remain true if instead of $BASIC$, $Base$ denotes the strengthening $EBASIC$; but the base language hence the set of witnesses then are enlarged in a way that weakens the conclusion - so that the weakening of the assumption is not for free. The same remark applies to other still enriched versions of $Base$.
- For $n > 2$ we strongly conjecture the same result extended to $Base +$ *strict* $\Sigma_n^{b'} - ind^\tau$ whenever $exp_{n-1}(\tau^\omega) < |x|_2$. But this will appear in the second part.

D. Don't laugh ! The separation theorems of (B) **would** be more significant **if** about stronger theories - such as $\Sigma_1^b - ind^{|x|_2}$. Alas, here they are proved precisely for **any** Σ_1^b induction that is not **quite** that long.
We offer no less than 4 excuses :
i) propagation of collapse tends to attribute the same difficulty to significant results as to apparently ridiculous results. For instance the statement "T does not prove P=NP" is more significant for T=$\Sigma_1^b - ind$ than for a theory such as T=$\Sigma_1^b - ind^{|x|_{1001}}$, which has no physical meaning because $|x|_{1000} = 1$ for any physical value of x. But as long as you did not prove any of the two statements, you cannot exclude that Σ_{1001}^b reduces to Σ_1^b, in which case (with the help of "divide and conquer" iterated 1000 times) the two statements are equivalent. This state of affairs tends to make the statement "T does not prove P=NP" difficult to prove even in the ridiculous case...
ii) The method which proves the witnessing results (C) also proves the independence results (B)) ; so if **still using our present method** one could

improve say by one exponential the length of the induction in the latter results then one could also improve the witnessing results to the same extent - which would be surprising. Thus improving the independence results requires either a substantial change in the methods or an unexpected positive result...
iii) Our independence results are of the "don't laugh" type only w.r.t. the P versus NP issues ; but there exists smaller complexity classes (namely certain uniform families of bounded depth circuits with suitable doors) which form a natural hierarchy ; we are preparing substantial results on this hierarchy which show its importance. And for this smaller hierarchy our results are no longer of the don't laugh type: they entail or they come very close to actual separation and effective witnessing results.
iv) We have medium term plans to introduce a new complement to our methods: the use of the Vapnik Tchervonenkis property inside o-minimal structures. This complement should enable to... bootstrap our present results up to a level of strength where they no longer would be "don't laugh", **even** w.r.t. the polynomial hierarchy...

E. About the methods – So much for the results ; as stated at the beginning of this section we expect the methods introduced here to be the main contribution of the paper. We already introduced one of the methods: "counterfeiting" ; below we discuss other methods which we develop - they can be applied both to true models and to counterfeit ones.

We build non standard models of the integers inside a set $D_i(a, c)$ – as defined for the above T.3 but when a, i, c are **nonstandard.** This **"nonstandard resource in depth"** $D_i(a, c)$ is to be compared with the **"nonstandard resource in time"** $\{x : x$ is computable from a in time $< a^{|a|^c}$ by a program$< c\}$ inside which Wilkie (in an unpublished manuscript around 1990 ; see also Pudlak [HP]) constructed a model of S_2^1 proving (after Buss but by model theory) that S_2^1 "captures" polynomial time. Our resource in depth is of quite a different nature, a fact that suggests to look for still other kinds of resources.

We develop the use of **indiscernibles** and of **"quantifier control"**, to build models of the integers and in particular counterfeit ones, from a resource. These tools are inspired by Jeff Paris's proof of the Harrington-Paris theorem [L] (where the "resource" is a very large non standard interval of the integers). But many complementary notions are needed to make the idea work in the present context of weak arithmetic. The case of quantifier control used by Jeff's proof is the only one we knew so far – except for the familiar notion of a structure that is a (partially) **elementary** substructure of another. We think that the new cases we provide are steps to move **non**elementary quantifier control, from the status of a device used once or twice in mathematics, towards the status of a useful general concept. One of our referees mentioned still other works related to these methods and which are in some cases older, in some others at least independent from the present one: Adamowicz-Kolodziejczyk [AK], Ratajczyk [R], Pudlak [Pu], Solovay-Kettonen [KS].

F. More details

Notation For any constant ℓ let $\varphi - least^{\ell}$ denote the axiom: "$\exists$ least x : $x = \ell$ or $\varphi(x)$". $\Gamma - DC^{\ell^{\omega}}$ is the scheme expressing for any set Γ of formulas θ: if $\forall x < a \exists y < a \theta(x, y)$ holds then there is a coded sequence $(z)_{i,i<\ell}$ such that $\forall i < \tau \theta((z)_i, (z)_{i+1})$. The constant ℓ may be replaced by the term $\tau = \tau(x)$ in the above notions; then an additional universal quantification on the variable x is prefixed to the axioms. Here the way chosen to encode a sequence $(z)_{i,i<\ell}$ bounded by a is to construct an integer $z < 2^{|a|\ell}$ such that $\lfloor z/2^{|a|i} \rfloor$ exists for each $i \leq \ell$ and moreover $(z)_i = \lfloor z/2^{|a|i} \rfloor - \lfloor z/2^{|a|(i+1)} \rfloor 2^{|a|}$.

Remark
a) Modulo $EBASIC + \Sigma_0^b - ind^{\ell}$ every bounded Σ_0^b definable sequence of length ℓ has a code ; and for $z < 2^{|a|\ell}, i < \ell$ the relation $y = (z)_i$ is Σ_0^b definable.
b) Modulo $BASIC + \Sigma_1^b - ind^{\ell}$ every bounded Σ_1^b definable sequence of length ℓ has a code ; and if z is such that $(z)_i$ exists for each $i < \ell$ then the relation $y = (z)_i$ is also Π_1^b. Thus if $\varphi(u) \in \Sigma_n^b, \psi(u) \in \Pi_n^b$ then $\varphi((z)_i) \in \Sigma_n^b$; and in case $(z)_i$ exists for each $i < \ell$ then $\psi((z)_i) \in \Pi_n^b$.

Proposition 0.4 *a) Modulo $E_1(\Sigma_n^b) - ind^{\ell}$, $E_1(\Sigma_n^b)$ formulas are closed under "short quantifications" – of the form $\forall i < \ell^k$ and $\exists i < \ell^k$ for any $k < \omega$.*
b) For $0 < n < \omega$ the following schemes are equivalent (modulo Base):

$$E_1(\Sigma_n^b) - ind^{\ell}, E_1(\Sigma_n^b) - ind^{\ell^{\omega}}, E_1(\Sigma_n^b) - least^{\ell}, E_1(\Sigma_n^b) - DC^{\ell}$$

c) $E_1(\Sigma_n^b) - ind^{\ell}$ implies $\Sigma_n^b - ind^{exp(\ell^{\omega})}$
d) The same results hold when the constant ℓ is replaced by $\tau(x)$.

$\square$ **P.4** The proofs are classical in case of S_2^1, below we check that they still work in the present context even if $2^{\ell^{\omega}}$ is smaller than the involved parameters
(a) Assume that $\varphi(u) \in \Sigma_n^b, \psi(u) \in \Pi_n^b$ then ω satisfies: $\exists u \forall i < \ell(\varphi$ or $\psi)$ iff $\exists u \exists \epsilon < 2^{\ell}$ [$\forall i < \ell((\epsilon)_i = 0$ or 1) and $\forall i < \ell$ ($(\epsilon)_i = 1$ or $\varphi(i)$) and $\forall i < \ell$ ($(\epsilon)_i = 1 \longrightarrow \psi(i)$)]. In addition the "if" part clearly holds in any model of *Base* ; and the "only if" part follows from $E_1(\Sigma_n^b) - ind^{\ell}$. Indeed, if we inductively assume to have the required sequence ϵ of length ℓ, then in order to have it for length $\ell + 1$ we only need to append one more digit to ϵ. The argument is easily extended to $\wedge_{j<n}(\varphi^j$ or $\psi^j)$ - it suffices

to replace $\epsilon < 2^\ell$ by $\epsilon < 2^{\ell.n}$; thus it gets rid of short quantification just before a boolean combination of Σ_n^b formulas. There remains the case of $\forall i < \ell \exists u < a\varphi$ where φ is boolean combination of Σ_n^b ; then ω satisfies: $\forall i < \ell \exists u < a\ \varphi$ iff $\exists U < 2^{|a|\ell} \forall i < \ell[\ (U)_i$ exists and $\varphi((U)_i)\ ]$. Using the preceding Remark (b) and the preceding case the latter formula reduces to $E_1(\Sigma_n^b)$; and the reduction is provable in $E_1(\Sigma_n^b) - ind^\ell$ easily: as in the preceding case one only has to append one more digit to the code U in order to do the induction step from ℓ to $\ell + 1$.

(b) We prove for $n > 0$ that $E_1(\Sigma_n^b) - DC^\ell$ is consequence of $E_1(\Sigma_n^b) - ind^\ell$. Let $\varphi(x, y) = \exists v < a\ \varphi_0(x, y, v)$ be a $E_1(\Sigma_n^b)$ formula, where φ_0 is a boolean combination of Σ_n^b formulas. Assume $\forall x < a \exists y < a\ \varphi(x, y)$. The desired conclusion follows from $\exists V < a^\ell \exists z < a^\ell \forall i < \ell\ [\ (V)_i, (z)_i$ exist and $\varphi_0((z)_i, (z)_{i+1}, (V)_i)\ ]$. Using (a) the latter formula can be put in $E_1(\Sigma_n^b)$ form and then is easy to prove by $E_1(\Sigma_n^b) - ind^\ell$: again the induction step is achieved just by adding one more digit - here to V and to z.

The other results are left to the reader for they are even simpler, and similar to the ones carefully proved for instance in [Bu].

(c) Via its equivalence with $E_1(\Sigma_n^b) - DC^\ell$ proved in (a), $Base + E_1(\Sigma_n^b) - ind^\ell$ allows one to define sequences $(z)_{i,i<\ell}$ by iterating ℓ times a bounded $E_1(\Sigma_n^b)$ definable application. This applies in particular to adapt the familiar divide and conquer argument deducing Σ_n^b-induction up to $exp(\ell)$ from Σ_{n+1}^b induction up to ℓ ; so as to prove (c).

P.4□

Remark 0.5 *i) The above (c) is a precise tradeoff between the quantifier level of an induction scheme (Σ_1^b, Σ_2^b, etc.) and the length of the induction (expressed by the term τ): $\Sigma_{n+1}^{b'} - ind^\tau$ implies $\Sigma_n^{b'} - ind^{exp(\tau^\omega)}$ and does not seem to imply more than that. In other words losing (slightly more than) one exponential in length seems to be the cost of winning one level in complexity ; the witnessing and independence results we stated in (B) and (C) agree with this tradeoff.*

On the other hand the results which we obtain in § 3 are much weaker than those predicted by the tradeoff ; however the method and results are interesting.

ii) Note that if $|a|_p \le \ell^\omega$ for some standard p then inside any submodel M which is bounded by $a^{|a|^\omega}$, $Base + \Gamma - ind^\ell$ implies $\Gamma - ind^{|x|_p}$. This follows from the above proposition (a).

iii) Modulo EBASIC the formula $c(z, i, a, y)$ which expresses that y is the value of the i^{th} digit of the sequence of length $|a|$ in base $2^{|a|}$ coded by z is Σ_0^b; and $(z)_i$ exists for every $z, a, i < |a|$. Modulo BASIC the formula $c(z, i, a, y)$ is only Σ_1^b and $(z)_i$ is no longer guaranteed to exist if i is non standard. Modulo $\Sigma_1^b - ind^\ell$ the formula is Δ_1^b for $i < |a|$ and for every Σ_1^b

definable sequence of length$< \ell$ bounded by a there exists a unique element z which codes that sequence.

In the present paper, assuming $exp(\tau^{\omega}) < |x|_2$ we shall construct for any $m < \omega$ a model M of $BASIC + \Sigma_2^{b'} - ind^{\tau} + \Sigma_m^b - ind^{|x|_p}$ for some $p = p_m < \omega$. And counterfeiting is used to ensure by some pathology (for instance : a power of 2 with an odd divisor > 1) that M does not satisfy S_2^1. Thus we separate $BASIC + \Sigma_2^{b'} - ind^{\tau}$ from S_2^1 and in addition the former theory does not imply "Σ_m^b-collapse", that is: $\Sigma_m^b = \Sigma_{m+1}^b$. For otherwise, $\Sigma_m^b - ind^{|x|_p}$ would inside M imply $\Delta_0 - ind^{|x|_p}$ - which implies full Δ_0-induction by p repeated divide and conquer arguments. Thus Σ_m^b-collapse would imply the absurdity that M satisfies S_2 but not S_2^1...
We use a two-fold way to refer back to results and remarks: i) they are numbered - with a unique numbering for all kinds of items. For instance "1.3" refers to the third numbered item of § 1.3 (but we omit the digit of the chapter if the reference is being made in that **same** chapter). ii) They also get a name or nickname (such as "low depth witnessing theorem") when it is useful to keep them in mind. The beginning of the proof of say theorem 1.4 is indicated by "□ T 4" and its end by "T 4 □".

1 The resource of depth

In this section the ground model of our constructions is a "true" model of the integers denoted N and unless otherwise stated, i) N is a nonstandard model of Peano Arithmetic ii) "formula" means arithmetical formula with parameters in N iii) "satisfied" means satisfied in N.

Conventions We use the ordinal ω to also denote the standard part of N, identified with the standard model of Arithmetic. For any integer $\ell \in N$ we often use ℓ to denote the set $\{0, 1, \ldots, \ell - 1\}$ – letting the context indicate whether we speak of the element or of the set ℓ. This also applies when the letter ℓ is replaced by another notation x for integers. Example : the notation $|x| \cup \{x\}$ denotes the set $\{x, 0, 1, \ldots, |x| - 1\}$; and if D is a set, $D \restriction x$ denotes the set $D \cap \{0, 1, \ldots, x - 1\}$.

Recall that $D_i(a, c)$ denotes the set of (values of) closed basic terms of depth $\leq i$ and hereditarily bounded by c, with parameters from $D_0(a, c) := |a| \cup \{a\}$. Inside N and for nonstandard elements a, d, c of N, $D_d(a, c)$ is a "resource" from which we build models of the integers :

Notation

- A **cut** is an initial segment of N closed under successor.
- For any strictly increasing ω-sequence $(s_k)_{k<\omega}$ let s_ω denotes the cut $\bigcup_{k<\omega} s_k$. For instance $a^{|a|^\omega}$ as well as $2^{|a|^\omega}$ denote the cut : $\bigcup_{k<\omega} a^{|a|^k}$.
- for any cut $I < d$,

$$D_I(a, c) := \bigcup_{i \in I} D_i(a, c) \restriction a^{|a|^\omega}.$$

Remark 1.1 *a) $D_I(a, c)$ is closed under $\mathcal{L}_B$, hence is a model of Base. And since $|a|^\omega \subset D_I(a, c) \subset a^{|a|^\omega}$, it is also a model of $\Sigma_0^b - ind^{|x|}$.*
b) This is true for any cut I and our problem is to choose I so that $D_I(a, c)$ satisfies stronger properties. In order to solve this problem one may use various combinatorial properties: in subsection (A) below we use the pigeon hole principle while in (B) we use a strengthening of the (finite) Ramsey theorem. In § 3 we iterate the construction of $D_I(a, c)$ made in (B): we repeat (B) when the ground model N is replaced by $D_I(a, c)$ or by its counterfeit analog B_I which is built in § 2.
c) The relation $x \in D_\ell(a, c)$is defined by the Σ_1^0 formula: " there is a coded sequence $(C_j)_{j \leq 2^\ell}$ of elements of c such that each C_j codes a sequence of numbers$< c$ of length $2^\ell - j$ and $x = C_{2^\ell}$ and im $C_0 \subset D_0(a, c)$ and $\forall j < 2^\ell($ im $C_j \subset \mathcal{L}_B$ im C_{j+1}) ". If we restrict ℓ to range over $|c|_2.\omega$ then the above C is bounded by c^{2^ω} ; hence for $i \leq \ell$ the relation "$x \in D_i(a, c)$" becomes Σ_1^b.

d) We believe that Base proves :

$$D_{i+1}(a,c) = [\mathcal{L}_B D_i(a,c)] \upharpoonright c$$

provided the relation "$x \in D_i(a,c)$" is defined in the above Σ_1^0 way ; or provided $i < |c|_2$ and we are using the Σ_1^b way. But we did not check this point ; we shall only need the weaker and easy assertion that "$\forall i < \ell \; D_{i+1}(a,c) = [\mathcal{L}_B D_i(a,c)] \upharpoonright c$" is for $\ell < |c|_2.\omega$ provable in Base $+ \Sigma_1^b - ind^\ell$.

A. Using the pigeonhole principle
As a first application of this principle we obtain:

Theorem 1.2 *Inside any countable N if $c > a^{|a|^\omega}$ and $b + \ell^\omega < d$ then there is a cut I between b and d such that $D_I(a,c)$ satisfies $\Sigma_1^b - ind^\ell$.*

□ T 2
Since $b+\ell^\omega < d$, by overspill there is $r > \omega$ such that $b+\ell^r < d$. And we can easily divide the interval [b,d] into $\ell+1$ successive intervals $[b^i, b^{i+1}], i \leq \ell$ of length $> \ell^\omega$ and with $b^0 = b, b^\ell = d$.

Claim
a) For any formula $\phi(u,v)$ we have:
*) $\exists j < \ell+1 \forall i < \ell [\phi(i,j) \Rightarrow \exists j' < j \phi(i,j')]$
b) For any Σ_1^b-formula $\varphi(u)$ with parameters in $D_b(a,c)$ there is $j < \ell$ such that $D_J(a,c)$ satisfies $\varphi(u) - least^\ell$ for every cut $J \in (b^{j-1}, b^j)$.
□Claim
(a) - Otherwise $\forall j < \ell+1 \; \exists f(j) < \ell \; [\phi(f(j),j)$ and $\forall j' < j \neg \phi(f(j),j')]$
Clearly $j' < j \rightarrow f(j') \neq f(j)$; and this defines an injection $f : \ell+1 \hookrightarrow \ell$. Since it contradicts the pigeonhole principle, by contradiction we proved the formula (*).
(In view of (*)'s proof it will be denoted also $\ell+1 \not\hookrightarrow_\phi \ell$; and $\tau^\omega+1 \not\hookrightarrow_{\Sigma_1^b} \tau^\omega$ denotes the scheme expressing that $\forall x \forall \ell < \tau(x)^\omega \; [\ell+1 \not\hookrightarrow_\phi \ell]$ where ϕ ranges over Σ_1^b)

(b) - We apply (a) when $\phi(i,j)$ is the Σ_1^b formula which expresses that (the $\mathcal{L}_B$-structure generated by) $D_{b_j}(a,c)$ satisfies $\varphi(i)$; the structure is not definable in N but $\phi(i,j)$ nevertheless exists because the quantifiers of $\varphi(i)$ being bounded, their interpretation in $D_{b^j}(a,c)$ is Σ_1^b definable in N. We obtain $j < \ell+1$ such that for each $i < \ell$, if $D_{b^j}(a,c)$ satisfies $\varphi(i)$ then so does already $D_{b^{j-1}}(a,c)$. For any cut $J \in]b^{j-1}, b^j[$ the least x such that ($x = \ell$ or $\varphi(x)$) exists inside $D_J(a,c)$, because it coincides with the element defined in N as least $x : x = \ell$ or $\phi(x,j)$.

Claim□

We arrive at the last step in the proof of T 2 :

Construction of the cut I. The preceding Claim showed that for every instance θ of $\Sigma_1^b - least^\ell$, there are b', d' such that $b \le b', b' + \ell^\omega < d' \le d$ and θ holds in $D_J(a,c)$ for any cut $J, b' < J < d'$. Let $(\theta_n)_{n<\omega}$ be a sequence of such instances θ (with parameters from N). By applying ω times the Claim we obtain a sequence $(b_n, d_n)_{n<\omega}$ such that for every $n < \omega$ we have $b \le b_n$, $b_n + \ell^\omega < d_n \le d$ and θ_n holds in $D_J(a,c)$ for any cut J, $b_n < J < d_n$. So let I be any cut in the intersection of the intervals $]b_n, d_n[$; $D_I(a,c)$ satisfies $Base + \Sigma_0^b - least^{|x|}$. And the freedom left so far in the choice of θ_n can be used so that every instance of $\Sigma_1^b - least^\ell$ with parameters in $D_I(a,c)$ equals θ_n for some $n < \omega$. Thus $D_I(a,c)$ satisfies $\Sigma_1^b - least^\ell$.

T 2$\square$

From now on we assume that τ satisfies $\tau(x^{|x|^\omega}) \le \tau(x)^\omega$; note that in view of the usual cases this requirement is low or even empty: for $\tau = |x|, |x|_2, |x|_3, |x|_4$ we have $\tau(x^{|x|^\omega}) = \tau(x)^\omega, \tau(x).\omega, \tau(x)+\omega, \tau(x)+1$! Nevertheless, to be sure we include this requirement on τ in the $Base$ axioms.

Theorem 1.3 *"low depth witnessing"*
a) Given any $\varphi \in \Sigma_1^b$, assume that $\forall x \; \exists y \; \varphi$ is provable in $T^\tau := Base + \Sigma_0^b - least^{|x|} + \tau^\omega + 1 \not\to_{\Sigma_1^b} \tau^\omega$ where $\tau^\omega \prec |x|_2$. Then for some $K < \omega$, the statement

$$\forall x [\exists y \in D_{\tau(x)^K}(x, x^{|x|^K})] \; \varphi$$

is true (in the standard model)
b) In addition the statement is provable in T^τ.

Remark T 3.a is not a surprise considering the weakness of T^τ ; but this very weakness gives a remarkable character to (b). And (b) implies that the theorem is true in a uniform way. But this type of consequence of (b) will be developed in a sequel to this work.

$\square$ **T 3** If the conclusion of (a) fails, then by compactness we can find $d < a < c \in N$ such that $d > \tau(a)^\omega$ and $c > a^{|a|^\omega}$ and $\neg\exists y \, [\varphi(a,y) \text{ and } y \in D_d(a,c)]$; then T 3.a applied with $b = 0$ provides a model $D_I(a,c)$ of the induction part of T^τ and which violates the conclusion of (a). Below we modify the construction of I from that proof so as to have $D_I(a,c)$ also satisfy the pigeonhole principle ; this will prove (a) in contrapositive form.
By our special axiom about τ we know that $\tau(a^{|a|^\omega})^\omega = \tau(a)^\omega$ and we can assume that $\ell^\omega = \tau(a)^\omega$; so we are left with proving $\lambda + 1 \not\to_\psi \lambda$ for every $\lambda < \ell^K$ ($K < \omega$) and every $\psi(i,k) \in \Sigma_1^b$. To that end we consider any stage $n < \omega$ of the construction of I ; we have $b \le b_n < b_n + \ell^\omega < d_n \le d$ and we can divide $[b_n, d_n]$ into $\lambda^2 + \lambda + 1$ intervals $[c^j, c^{j+1}]$ such that $c^j + \ell^\omega < c^{j+1}$. We abbreviate $D_c^j(a,c)$ by D^j and we apply the pigeonhole principle $\lambda^2 + \lambda + 1 \not\to_\Phi \lambda$ for the formula $\Phi(< i,k >, j) :=$ "$D^j(a,c)$ satisfies $\psi(i,k)$"

; we obtain $j < \lambda^2 + \lambda + 1$ such that whenever $D_c^j(a,c)$ satisfies $\psi(i,k)$ with $k < \lambda+1, i < \lambda$ (hence with $< i,k >< \lambda^2 + \lambda$) then so does already $D^{j-1}(a,c)$. Thus if for this fixed j we denote $\psi'(i,k)$ the formula which expresses that D^{j-1} satisfies $\psi(i,k)$ then for any cut J between c^{j-1} and c^j and for each $k < \lambda+1, i < \lambda$ we have that $D_J(a,c)$ satisfies $\psi(i,k)$ iff $\psi'(i,k)$; and the truth in $D_J(a,c)$ of the pigeonhole principle $\lambda+1 \not\leftrightarrow_\psi \lambda$ is proved for any $J \in (c^{j-1}, c^j)$ – because it is reduced to $\lambda+1 \not\leftrightarrow_{\psi'} \lambda$ inside N.
Clearly we can mix the construction of I made in the proof of T 2 with the construction just done, so as to satisfy the whole theory T^τ. This proves (a).
If the conclusion of (b) fails, then by compactness the theory "$d > \tau(a)^\omega$ and $c > a^{|a|^\omega}$ and $\neg\exists y\,[\varphi(a,y)$ and $y \in D_d(a,c)]$" is consistent with T^τ. We let N' be a model of this theory ; we are going to provide a cut $I \leq d$ such that $D_I(a,c)$ is a model of T^τ. Then $\neg\exists y \in D_I(a,c)\varphi(a,y)$ hence the assumption of (b) also fails and (b) is proved in contrapositive form. To that end we set $\ell = \tau(a)$ and we imitate inside N' the construction which made in N gave us the proof of (a) ; to begin with the division into intervals $[b_i, b_{i+1}]$ and $[c^j, c^{j+1}]$ clearly remains possible in N'. And it appears that besides this point, all properties of N that were used in (a) explicitly belong to T^τ. So that the construction of I made with N can be repeated texto with N'.

T 3$\square$

In the next subsection we go back to the situation where N satisfies Peano Arithmetic ; the pigeon hole principle $\ell + 1 \not\leftrightarrow \ell$ is replaced by another combinatorial principle: " $d \rightarrow (p)^n_{\ell,k}$ ", which comes from a strengthening of the (finite) Ramsey theorem. And we look for the version of T 3 which this change in Combinatorics allows to prove.

B. Ramsey revisited

Here we state and prove this theorem in a way which sometimes departs from the current one, in order to lead us to the desired principle $d \rightarrow (p)^n_{\ell,k}$.

Notation

1. For any letter v and for any integer c (standard or not), $\bar{v}_c$ is a shorthand for : $v_1, \ldots, v_c$. On the other hand "$\bar{v}$" denotes $\bar{v}_n$ for some integer n, but a **standard** one. If V is any notation for a sequence we write "$V \subset S$" (or "V **from** S") to mean that each coordinate V_i of V is :

    - an element of S (if S denotes a set);
    - a coordinate of S (if S denotes a sequence).

2. For any set S with a linear order,

$$[S]^n := \{\bar{x}_n \in S^n : x_1 < \cdots < x_n\}.$$

3. Subset X of S is **indiscernible w.r.t.** R (where $R = R(\bar{x}_n)$ is a relation on S) iff R is constant over $[X]^n$. And X is indiscernible w.r.t. a **set** $\Phi = \Phi(\bar{x}_n)$ of such relations if it is indiscernible w.r.t. each one.

Remark The classical terminology is in terms of "homogeneous" rather than indiscernible sets; but the two are equivalent :

- X is indiscernible w.r.t. the ℓ relations R_i iff it is homogeneous w.r.t. the colouring $F : [S]^n \to \exp(\ell)$ defined by $F(\bar{x}_n) = \{i < \ell : R_i(\bar{x}_n)\}$
- conversely, given $F : [S]^n \to \exp(\ell)$ we can define ℓ relations on S such that homogeneous w.r.t. F = indiscernible w.r.t. the ℓ relations.
- Thus the well known partition property $K \to (p)^n_\ell$ roughly means that whenever S has (or is the) cardinal K and $\Phi(\bar{x}_n)$ is a set of relations on S of cardinal at most $|\ell|$ then S contains p indiscernibles w.r.t. $\Phi(\bar{x}_n)$.

The theorem of Ramsey says : $(\forall p, \ell)\exists K\ K \to (p)^n_\ell$. We now sketch a proof which leads to a useful refinment of the result.
Fix a set $\Phi(\bar{x}_n)$ of $|\ell|$ relations on the integer K. Say that X is **pre**indiscernible w.r.to $\Phi(\bar{x}_n)$ if there is a set of relations $\Phi_1(\bar{x}_{n-1})$ such that on X, $\Phi(\bar{x}_n)$ reduces to $\Phi_1(\bar{x}_{n-1})$. Erdos-Rado defined a tree relation on K called the Erdos-Rado tree which is such that all branches of the tree are preindiscernible w.r.t. $\Phi(\bar{x}_n)$; and given $K_1 < \omega$ a counting argument shows that for sufficiently large K the tree has a branch X_1 of cardinal $\geq K_1$. Thus on X_1, $\Phi(\bar{x}_n)$ becomes $\Phi_1(\bar{x}_{n-1})$. And repeating this i times, $i \leq n$, one obtains X_i and $\Phi_i(\bar{x}_{n-i})$ such that on $X_i, \Phi(\bar{x}_n)$ reduces to $\Phi_i(\bar{x}_{n-i})$. In addition the initial value K can be chosen so large that the last set X_n is of cardinal $\geq p$. Thus Φ is constant on X_n, that is X_n is indiscernible w.r.t. Φ. When the sketch is made precise it yields a version with bounds of the theorem, namely: $\forall p \exists K \forall \ell\ (\exp_{n-1}(\ell^K) \to (p)^n_\ell)$.

We next point out a property which the above sets X_i happen to enjoy in addition to the existence of $\Phi_i(\bar{x}_{n-i})$.

Definition - Assume that Φ has a list of "parameters" $\bar{u}$ in addition to n variables: $\Phi = \Phi(\bar{u}, \bar{x}_n)$. Then for any set B,

- indiscernibility **over** B of X w.r.t. $\Phi(\bar{u}; \bar{x}_n)$ means indiscernibility w.r.t. $\Phi(\bar{b}, \bar{x}_n)$ for **every** $\bar{b}$ from B; on the other hand,
- indiscernibility of X w.r.t. $\Phi(\bar{u}; \bar{x}_n)$ (without mentioning over which set B) means: for each $\nu \in X$, the set $\{x \in X : x > \nu\}$ is $\Phi(\bar{u}; \bar{x}_n)$-indiscernible over $\{x \in X : x \leq \nu\}$.

(These notations replace by " ; " the " , " which separates $\bar{u}$ from $\bar{x}_n$ in the argument of Φ, in order to indicate where the "parameters" $\bar{u}$ stop and where the "variables" $\bar{x}_n$ start).

We come back to the above proofsketch of $K \to (p)^n_\ell$, in which Φ only depended on $\bar{x}_n$; it appears that

- a set Y is preindiscernible w.r.t. $\Phi(\bar{x}_n)$ iff it is indiscernible w.r.t. $\Phi(\bar{u}_{n-1}; x_n)$.
- Thus the set X_1 of the proofsketch is indiscernible w.r.t. $\Phi(\bar{u}_{n-1}; x_n)$.
- More generally each set X_i is indiscernible w.r.t. $\Phi(\bar{u}_{n-i}; x_{n-i+1}, ..., x_n)$ - in such case we speak of **parametric** indiscernibility.
- Thus the Erdos-Rado proof of the Ramsey theorem actually yields a finer result, namely the existence of parametric indiscernibles :

Notation - $d \to (p)^n_{\ell,k}$:= for every set $\Phi = \Phi(\bar{w}_k, \bar{x}_n)$ of $|\ell|$ relations on d there is a subset X of d of cardinal p which is indiscernible w.r.t. $\Phi(\bar{w}_k; \bar{x}_n)$. Thus for each $R \in \Phi$:

$$R(\bar{b}_k, \bar{z}_n) \longleftrightarrow R(\bar{b}_k, \bar{y}_n)$$

for all $\bar{b}_k \in X^k$ and all $\bar{z}_n, \bar{y}_n \in [X]^n$ with $b_k < z_1$ and $b_k < y_1$.
Here is the "parametric" refinement of the Ramsey theorem with bounds:

Theorem 1.4

$$(\forall k, p)\exists K \forall \ell \ (\exp_{n-1}(\ell^K) \to (p)^n_{\ell,k}).$$

Remark - Ramsey with bounds follows from T.4 because $d \to (p)^n_{\ell,0}$ and $d \to (p)^n_\ell$ are nearly the same for any ℓ. Parametric indiscernibles were already considered by J. Paris, but in a much stronger sense requiring huge bounds, and with a quite different construction (based on the notion of "relatively large" set).

□ **T 4** As already said the proof is essentially included in the Erdos-Rado proof ; but we have to repeat the proof here both in order to be self contained and to check in an explicit way the bounds that are stated in T 4.
We are given relations $R_i,\ i < |\ell|$, on $d^k \times [d]^n$; for $\bar{a}_m$ in d^m and $a_m < \bar{b}_j \in [d]^j$ with $j \leq n$ we set $h := k + n - j$ and call "type of $\bar{b}_j$ over $\bar{a}_m$" the set

$$\left\{R_i(u_{i_1}, ..., u_{i_h}, \bar{x}_j); i < |\ell| \text{ and } a_{i_1}, ...a_{i_h}, \bar{b}_j \text{ satisfies } R_i(u_{i_1}, ..., u_{i_h}, \bar{x}_j)\right\}$$

There are less than $2^{|\ell| m^h}$ possible such types – called **j**-types – over m elements.

We first prove T 4 for the relations $R_i(\bar{u}_{k+n-1};x)$ hence also in case $n=1$. We call "tree of height m" the tree T_m (ordered by inclusion) of all possible 1-types over $< m$ elements that start with $0,1,\dots,h-1$. The above bound on the number of j-types shows that card $T_m < \ell^{m^{h+1}}$. (By the way this tree of types T_m is isomorphic to the Erdos Rado tree restricted to its m^{th} level)

Next we define the **realization** of T_m, which is a subtree T^*_m of T_m labeled by elements of d: for each node $q \in T_m$ of height 1, label q by $q^* :=$ the smallest $j > k-1$ which has type q over $0,1,\dots,h-1$, – if it exists. For each node $q_1 \in T_m$ of height 2 extending q, label q_1 by $q^*_1 =$ smallest $j \in]q^*, d[$ which has type q_1 over q^* – if it exists. And so on...
If $d > \ell^{p^{h+1}}$ then $d >$ card $T_p \geq$ card T^*_p hence $T^*_{p+1} \backslash T^*_p$ is non empty and T^*_p has a branch of length p. Let X be the set of labels q^* for q in this branch. X is indiscernible w.r.t. $R_i(\bar{u}_h,x)$, $i < |\ell|$ because whenever $a_1 < \dots < a_m < x$ and $a_m < y$ in X, then x and y have same type over $a_1 \dots a_m$, by definition of T_p and T^*_p. Thus $\ell^{p^{h+1}} \to (p)^1_{\ell,h}$ and case $n=1$ (hence $h=k$) is done.
Next for any $n \geq 1$ we inductively assume: $\forall l < \omega \exists k < \omega exp_{n-1}(\ell^\omega) \to (\omega)^n_{\ell,\omega}$ - a statement which we also denote $\exp_{n-1}(\ell^\omega) \to (\omega)^n_{\ell,\omega}$; and we prove $\exp_n(\ell^\omega) \to (\omega)^{n+1}_{\ell,\omega}$. So fix $k < p < \omega$ and relations $R_i(\bar{u}_k;\bar{x}_{n+1})$ on d. Using the inductive hypothesis, let $k' < \omega$ ensure that $\exp_{n-1}(\ell^{k'}) \to (p)^n_{\ell,k+n}$; and set $p' = \exp_{n-1}(\ell^{k'})$. By the above case $n=1$ we also have $k'' < \omega$ such that $\ell^{{p'}^{k''}} \to (p'+1)^1_{\ell,k+n}$; more precisely assume $d \geq \ell^{{p'}^{k''}}$ and let T'_m denote the tree of types of height m of the proof of case $n=1$, for the relations $R_i(\bar{u}_{k+n};x)$, $i < |\ell|$.
This proof showed the existence of a branch of length $p'+1$ in T'^*_m: if X is the set of all labels q^* for q on this branch and whenever in X we have $a_1 < \dots < a_k < b_1 < \dots < b_n < c$ and $b_n < c'$, then by definition of T'_m we conclude that c and c' have same type over $\bar{a}_k\bar{b}_n$ w.r.t. the relations $R_i(\bar{u}_{k+n};x)$, $i < |\ell|$.
Which also means that the type over $\bar{a}_k$ and w.r.t. the relations $R_i(\bar{u}_k;\bar{x}_{n+1})$ of any sequence $c_1 < \dots < c_{n+1}$ with $a_k < c_1$ in X does **not** depend on c_{n+1}. Thus over $X \restriction \{max\ X\}$, let $R''_i(\bar{u}_k;\bar{x}_n)$ denote the relation: "$R_i(\bar{u}_k;\bar{x}_n,x)$ holds for one (hence any) $x > x_n$ in X". Since X has cardinal $p'+1$ and $p' \to (p)^n_{\ell,k}$ we find a set $X' \subset X$ of card p which is indiscernible w.r.t. $R''_i(\bar{u}_k;\bar{x}_n)$, $i < |\ell|$.
Thus if $a_1 < \dots < a_m < z_1 < \dots < z_{n+1}$ and $a_m < y_1 < \dots < y_{n+1}$ in X', then over $\bar{a}_m$, $\bar{z}_{n+1}$ and $\bar{y}_{n+1}$ have same type w.r.t. $R_i(\bar{u}_k;\bar{x}_{n+1})$, $i < |\ell|$. We have shown :

$$\ell^{({p'}^{k''})} \to (p)^{n+1}_{\ell,k}, \text{ where } p' = \exp_{n-1}(\ell^{k'}).$$

And for large enough ℓ, $\ell^{({p'}^{k''})} = 2^{|\ell|.\left[\exp_{n-1}\left(\ell^{k'}\right)\right]^{k''}} < \exp_n\left(\ell^{k'.k''+1}\right)$.

We proved T 4 by induction on $n < \omega$.

T 4 $\square$

Remark
Let us develop the consequences of T 4 for our model N of Peano arithmetic
.

1. T 4 is true in ω, but also inside N when n, k, p remain standard but ℓ gets infinite ; and when the given set Φ of relations on d is **coded** (or definable+bounded) in N. In fact, $d > \exp_{n-1}(\ell^\omega)$ implies $d \to (c)^n_{\ell^c,c}$ for each $c < \omega$. We abbreviate this by saying that N satisfies:

$$\exp_{n-1}(\ell^\omega) \to (\omega)^n_{\ell^\omega,\omega}.$$

2. By overspill in N, $d \to (\omega)^n_{\ell^\omega,\omega}$ implies $d \to (c)^n_{\ell^c,c}$ for some $c > \omega$. We can use this fact to obtain a subset of d which is infinite and which is indiscernible w.r.t. $|\ell|.c$ relations R on d – not only $|\ell|$. In addition the use of dummy variables allows these relations to depend on a varying number k_i of parameters and main variables : $R = R(\bar{u}_k; \bar{x}_i)$ where i and k depend on R, only subject to the conditions $i \le n$ and $k < c/2$.

3. The above theorem is purely combinatorial; the next result shall add a model theoretic touch. We continue to work inside N.

Observe that any coded set Φ of $|\ell|$ relations $R_i = R_i(\bar{x}_n)$ can be turned to a unique coded relation $R(u, \bar{x}_n)$ on $|\ell| \times [d]^n$: namely $R(i, \bar{x}_n) := R_i(\bar{x}_n)$. And indiscernibility w.r.t. Φ is equivalent to indiscernibility **over** $|\ell|$ w.r.t. R.

Notation

1. For any symbol s, $\bar{s}_\omega$ is a shorthand for the ω-sequence $s_1, s_2, \ldots$ And $\bar{s}_\omega - \bar{s}_p := s_{p+1}, s_{p+2}, \ldots.$

2. A sequence $\bar{\nu}_\omega < d$ is **n-indiscernible** over $|\ell|$ if for **every** coded relation R on $|\ell| \times [d]^n$ there is $p = p(R) < \omega$ such that $\bar{\nu}_\omega - \bar{\nu}_p$ is indiscernible over $|\ell|$ w.r.t. R.

Theorem 1.5 *Assume that* $\exp_{n-1}(\ell^\omega) < d$ *inside* N ; *below* d *there is an* ω*-sequence which is* n*-indiscernible over* $|\ell|$.

$\square$**T 5**
Given $\bar{\nu}_\omega < d$ and given a structure M which is a part of N including the set d, assume that

- $\bar{\nu}_\omega$ is n-indiscernible over $|\ell|$, but only for those relations R that are definable in M

- in addition when $R = R(u, \bar{x}_n)$ is defined in M by a formula with parameters $\varphi(\bar{b}, u, \bar{x}_n)$ then $p(R)$ only depends on the parameters $\bar{b}$, not on the pure formula φ.

Then we say that $\bar{\nu}_\omega$ is n-indiscernible over $|\ell|$ **inside M**. Suppose $\exp_2(d) < \beta \subset$M, so that all N-coded relations R on d have their code in M ; then n-indiscernibility of $\bar{\nu}_\omega$ inside M implies n-indiscernibility (and is a bit stronger because of the added uniformity w.r.t. **pure** formulas of M). Thus below we prove a slight strengthening of the theorem by constructing n-indiscernibles inside $N|\beta :=$ the relational structure N truncated to β.

Let $\bar{\varphi}_\omega$ be a definable enumeration of all formulas of $N|\beta$ with a, d, ℓ as parameters and of the form $\varphi = \varphi(u, \bar{u}_k, \bar{x}_n)$, where $\varphi = \varphi_e$ implies $k \leq e$. For every $r > \omega$, $\bar{\varphi}_r$ is a coded sequence extending $\bar{\varphi}_\omega$. Since $\exp_{n-1}(\ell^\omega) < d$ we can fix r so that $\omega < r$ and $d \to (r)^n_{\ell^r, r}$. Thus there exists $\bar{\nu}_\omega < d$ which is indiscernible w.r.t. each formula $\varphi_e(i, \bar{u}_e; \bar{x}_n)$ with $e < r, i < |\ell|$.

Let M denote the definable closure of $\bar{\nu}_\omega$ in $N|\beta$; $\bar{\nu}_\omega$ is n-indiscernible over $|\ell|$ inside M. (Indeed, for $\bar{b} \in$M fix $p < \omega$ such that $\bar{b}$ belongs to the definable closure of $\bar{\nu}_p$ inside $N|\beta$; then since $\bar{\nu}_\omega - \bar{\nu}_p$ is indiscernible over $\bar{\nu}_p$, it is indiscernible over the definable closure of $\bar{\nu}_p$ hence over $\bar{b}$. And this holds uniformly in $i < |\ell|$)
In addition M is an elementary substructure of $N|\beta$ and the latter is countable recursively saturated by a well known argument ; hence by Theorem 2.3 of [R3], $N|\beta$ is "resplendent": for a boldface Σ^1_1 property P, if P holds in an elementary extension or substructure of $N|\beta$ then it holds in $N|\beta$ itself. In particular the existence below d of n-indiscernibles over $|\ell|$ which holds inside M, then can be transfered to $N|\beta$. This implies the desired conclusion.
T 5□

Remark 1.6 *If N itself is recursively saturated, we could replace $N|\beta$ by N in the preceding proof. This would provide n-indiscernibles inside* **all** *of N (hence with an extra uniformity : if a partition R on d is defined in N using a pure formula φ and parameters $\bar{b}$ then $p(R)$ only depends on $\bar{b}$ and not on φ).*

C. Quantifiers controlled by indiscernibles

Notation

- From now on we set $I = \bar{\nu}_\omega$ where $\bar{\nu}_\omega < d$ is n-indiscernible over $|\ell|$.

- We call Σ_2^I every formula of the structure (N, I) of the form

$$\varphi = \exists \bar{w}^1 \ \exists \bar{v}^1 < I \ \forall \bar{w}^2 \ \forall \bar{v}^2 < I \ \psi$$

  where $\psi = \psi(\bar{u})$ is any formula (of N).

- Then $\varphi^*(\bar{u}; \bar{x}_2)$ denotes

$$\exists \bar{w}^1 \exists \bar{v}^1 < x_1 \forall \bar{w}^2 \forall \bar{v}^2 < x_2 \psi$$

- By allowing n alternations of quantifiers instead of 2 in the above notation, we define the Σ_n^I formulas and for any such formula $\varphi(\bar{u})$ its "control" $\varphi^*(\bar{u}; \bar{x}_n)$.

Lemma 1.7 *: "quantifiers controlled by indiscernibles"*
a) Σ_2^I-control : *let $\varphi(u)$ be a Σ_2^I statement (with parameters). Let $p = p(R) < \omega$, where $R(u; \bar{x}_2) := \varphi^*(u; \bar{x}_2)$ (so $\bar{\nu}_\omega - \bar{\nu}_p$ is indiscernible over $|\ell|$ w.r.t. $\varphi^*(u; \bar{x}_2)$). Then for every $i < |\ell|$:*
$\varphi^*(i; \nu_{p+1}, \nu_{p+2})$ **iff** *(N, I) satisfies $\varphi(i)$.*
b) Δ_3^I-control : *assume 3-indiscernibility, and for $\varphi(u) \in \Sigma_3^I$ set $p := p(\varphi^*(u; \bar{x}_3))$. Then for every $i < |\ell|$:*
$\varphi^*(i; \nu_{p+1}, \nu_{p+2}, \nu_{p+3})$ **implies** *that (N, I) satisfies $\varphi(i)$ (and conversely* **if** *$\neg\varphi$ also is Σ_3^I.*

□L 7
Assume $\neg\varphi^*(i, \nu_{p+1}, \nu_{p+2})$:

$$\forall \bar{w}^1 \forall \bar{v}^1 < \nu_{p+1} \exists \bar{w}^2 \exists \bar{v}^2 < \nu_{p+2} \neg\psi(i);$$

by indiscernibility, for each $q, p < q < \omega$, we have the same statement where q replaces p. And this implies

$$\forall \bar{w}^1 \forall \bar{v}_1 < I \exists \bar{w}^2 \exists \bar{v}^2 < I \ \neg\psi(i).$$

That is, (N, I) satisfies $\neg\varphi(i)$.
Dually, assume $\varphi^*(i, \nu_{p+1}, \nu_{p+2})$:

$$\exists \bar{w}^1 \exists \bar{v}^1 < \nu_{p+1} \forall \bar{w}^2 \forall \bar{v}^2 < \nu_{p+2} \ \psi(i).$$

By indiscernibility, for each $q > p+1$ we can replace ν_{p+2} by ν_q in the above statement. Hence $I < \sigma$, where σ is defined in N as $\max y < d : \exists \bar{w}^1 \exists \bar{v}^1 < \nu_{p+1} \forall \bar{w}^2 \forall \bar{v}^2 < y \ \psi(i)$. And we have $\varphi^*(i; \nu_{p+1}, \sigma)$, which implies : (N, I) satisfies $\varphi(i)$.
Thus (a) is proved; the proof of (b) is similar.

L 7□

From now on 2-indiscernibility of $\bar{\nu}_\omega$ is tacitly assumed unless otherwise stated. Then an easy consequence of Σ_2^I-control is that (N, I) is a model of $\Sigma_2^I - least^{|\ell|}$. For least $y : y = |\ell|$ or $\varphi(y)$ in (N, I) is given inside N by least $y : y = |\ell|$ or $\varphi^*(y; \nu_{p+1}, \nu_{p+2})$. But we can do better :

Theorem 1.8 *(N, I) is a model of $\Sigma_2^I - least^\ell$*

□T 8
Let $\varphi(y) := \exists \bar{w}^1 \exists \bar{v}^1 < I\ \forall \bar{v}^2 < I\ \psi(y)$ be a Σ_2^I formula. Let $\theta(u, \bar{x}_2)$ be the formula : "$1 = u^{th}$ bit of *least* $y : y = \ell$ or $\varphi^*(y; \bar{x}_2)$". For $p = p(\theta)$, $\bar{\nu}_\omega - \bar{\nu}_p$ is indiscernible over $|\ell|$ w.r.to the formula $\theta(u; \bar{x}_2)$. This implies :
For $\mu_1 < \mu_2$ in $\bar{\nu}_\omega - \bar{\nu}_p$ set $z\ :=$ least $y\ [y = \ell$ or $\varphi^*(y; \mu_1, \mu_2)]$. Then z has **constant** value (independent of $\bar{\mu}_2$).
Indeed, the $|\ell|$ bits of z all have constant value.

We show z = least $y : y = \ell$ or $(N, I) \models \varphi(y)$.

1. $(N, I) \models z = \ell$ or $\varphi(z)$
   Indeed assume $z < \ell$; and set $q = p(R)$, where $R(\bar{x}_2)$ is $\varphi^*(z; \bar{x}_2)$. Then $\varphi^*(z; \bar{\mu}_2)$ holds by definition of z. Hence if $\nu_q < \mu_1$ we conclude $(N, I) \models \varphi(z)$.

2. $z' < z \rightarrow (N, I) \models \neg\varphi(z')$
   Indeed, $z' < z \rightarrow z' < \ell$; together with the definition of z it implies : $\neg\varphi^*(z'; \bar{\mu}_2)$. When $\mu_1 > \nu_q$ with $q := p(R)$ where $R(\bar{x}_2) := \varphi^*(z', \bar{x}_2)$ this implies : $(N, I) \models \neg\varphi(z')$.

T 8□

For any formula φ, any easy overspill argument shows that $\forall \bar{w} \exists \bar{v} < I \varphi$ is equivalent to $\exists v' < I \forall \bar{w} \exists \bar{v} < v' \varphi$. Repeating this reduces every **extended** Σ_1^I formula (where quantifiers of N are freely interspersed with quantifiers $\exists v < I$) to a (strict) Σ_1^I one. In view of the Σ_1^I definition of "$x \in D_I(a, c)$", every Σ_1^b formula interpreted in $D_I(a, c)$ is expressible inside (N, I) by a Σ_1^I formula. By the same token, every $E_1(\Sigma_1^b)$ formula interpreted in $D_I(a, c)$ is expressible by a Σ_2^I one. Thus T 8 implies :

Corollary 1.9 *a) $D_I(a, c)$ is a model of $\Sigma_2^{b'} - ind^\ell$.*
b) Assume $\exp(\tau^\omega) < |x|_2$ (for instance $\tau = |x|_4$); then for every Σ_1^b formula φ such that $\forall x \exists y \varphi$ is provable from $Basic + \Sigma_2^{b'}$-ind^τ, there is $K < \omega$ such that ω satisfies

$$\forall x \exists y [\varphi \text{ and } y \in D_{\exp(\tau^K)}(x, x^{|x|^K})].$$

□C 9
If the conclusion does not hold, by compactness we can choose N, a, d, c so that $c > a^{|a|^\omega}$, $d > exp(\tau^\omega)$ and $\neg\exists y \in D_d(a, c)\ \varphi(a, y)$. Then the above model $D_I(a, c)$ satisfies $\neg\exists y\ \varphi(a, y)$; in contrapositive form the proof is done.
C 9□

We want to extend the preceding results to $\Sigma_n^{b'}$ beyond $n = 2$. The first idea which comes to the mind is to use n-indiscernibles for $n \geq 3$. Alas, for all that we know n-indiscernibles only allow to satisfy Δ_3^b-induction : by Δ_3^I-control, if $\bar{\nu}_\omega$ is n-indiscernible over ℓ with $n \geq 3$ then $D_I(a,c)$ satisfies Δ_3^b-induction. This is not enough, hence the

Questions - i) For $n > 2$ does n-indiscernibility (over some given ℓ) imply Σ_n^I-control ; or at least : ii) among all n-indiscernible ω-sequences is there one which ensures Σ_n^I-control (hence $\Sigma_n^I - ind^\ell$) ? (Such a sequence is called **strongly** n-indiscernible over ℓ).

While the answer to (i) most likely is negative, we are planning a paper which positively answers (ii). Using strong n-indiscernibles as in the above case n=2 it is easy to obtain the control of $\Sigma_{n+1}^{b'}$ formulas interpreted in $D_I(a,c)$ by Σ_n^I ones. This would prove T 8 for Σ_{n+1}^I and C 9 for $\Sigma_{n+1}^{b'} - ind^\tau$ whenever $exp_n(\tau) < |x|_2$. But in § 3 below we expose an alternative technique : the **iterated** use of 2-indiscernibles. It obtains induction on a shorter segment than the use of strong n-indiscernibles ; but this induction is obtained for Σ_{n+1}^b and not only $\Sigma_{n+1}^{b'}$ formulas. Thus the "iteration of resources in depth" presented in § 3 is of interest in order to study the evasive difference between the two classes. In addition the naturalness of such iterations calls for their study.

2 The resource of reals

In this section we need an extension of the preceding constructions where the initial resource $D_d(a,c)$ is replaced by a counterfeit analog : one which is contained in some field Q extending N, interpretable in it and such that N is an Integral Part of Q^+ (that is every element of Q^+ lies at distance < 1 from an element of N). These properties are satisfied when Q is the field of fractions of (the ring generated by) N. And the latter case is the only one used in the present paper. So in the sequel we specialize to this case but in order to remind of this new feature, N, Q will be denoted $\mathcal{N}$, $\mathcal{Q}$ in this §2; and for coding purposes we identify each element of $\mathcal{Q}$ with its reduced fraction. When working inside $\mathcal{N}$, $\mathcal{Q}$ plays the role played by the reals in the standard world. Given $a \in \mathcal{N}$ we want to construct "resources of depth d" D_d similar to the resource $D_d(a,c)$ of §1 but that in addition contain some pathology: for instance D_0 includes a divisor $r > 1$ of a even if a is prime in $\mathcal{N}$. That kind of division exists only using rationals or reals, hence it is in $\mathcal{Q}$ that we look for a **counterfeit** resource D_d of this kind. And in order to ensure as before that for any cut $I < d$, $D_I := \cup_{i \in I} D_i \restriction a^{|a|^\omega}$ still gives a model of $Base$ we then require D_d to be **discrete** in the sense : $(D_d - D_d) \cap]0,1[= \emptyset$ (something that automatically holds for a "true" resource $D_d \subset \mathcal{N}$; we used the notation $X - X := \{x - y;\ x, y \in X\}$). The log euclidean chains defined below are counterfeit resources of this kind, with other properties added for technical reasons and/or because they are nice supplements we are able to achieve.

Fact - Notation 2.1 *For x in $\mathcal{Q}^+$, $|x|$:= the unique integer y (of $\mathcal{N}$) such that $2^{y-1} \leq x < 2^y$.*

1. $|x| = ||\lfloor x \rfloor_{\mathcal{N}}|_{\mathcal{N}}$.
2. $|x \cdot y| \leq |x| + |y| + 2$.
3. $|2\,x| = |x| + 1$.
4. $|2\,x+1| = |x|+2 \quad$ *iff* $\quad 0 < 2^{|x|} - x \leq \frac{1}{2}$.
5. *if* $\neg\, 0 < 2^{|x|} - x < 1$ *then* $|2\,x+1| = |x|+1$.

Definition : Sequence $(A_i)_{i \leq \lambda}$ of subsets of $\mathcal{Q}$ is **almost *log*-euclidean** iff :

1. A_λ is discrete (in the above sense).
2. $|a| \cup \{a\} \subset A_0$.
3. Whenever $i < \lambda$

$$A_{i+1} \supset (A_i + A_i) \cup (A_i - A_i) \cup (A_i \cdot A_i) \cup \bigcup_{q \in [1, |a|^{2^i}]} \lfloor \frac{A_i}{q} \rfloor \cup \{2^c;\ c \in |a|^i\} \cup |a|^{2^{i+1}}.$$

The sequence is ***log*-euclidean** if in addition $|A_i| \subset A_{i+1}$ for each $i < \lambda$.
We next list the conventions we will use :

Conventions :
– Let $x \in A_{\lambda-1}$. A_λ being discrete $\lfloor x \rfloor_{A_\lambda}$ is the unique a in A_λ if it exists, such that $a \leq x < a+1$: it is the "integer part of x in A_λ".
– $\lfloor \frac{X}{q} \rfloor = \{\lfloor \frac{x}{q} \rfloor_{A_\lambda}$ (when it exists), $x \in X\}$.
– $A_{i+1} \supset \lfloor \frac{A_i}{q} \rfloor$ is to mean (in addition to the inclusion) : the integer part of $\frac{x}{q}$ in A_{i+1} **does** exist for each x in A_i.
– For $x \in \mathcal{Q}$, $\|x\|$ denotes $max(x, -x)$ (ie. absolute value of x which must be distinguished from its "log" denoted $|x|$).
– $\|X\| = [\,0,\ max\{\ \|x\|\ ,\ x \in X\}\,]$, interval in $\mathcal{Q}$.
– $|X| = [\,0,\ max\{|\ x\ |,\ x \in \|X\|\}\,]$, interval in $\mathcal{N}$.
– $X + X = \{x + y\,;\ x, y \in X\}$ and similarly for $X - X$ and $X \cdot X$.

Remark If $(A_i)_{i \leq \lambda}$ is log-euclidean and $I < \lambda$ is a cut of $\mathcal{N}$ then $A_{<I} := \cup_{i<I} A_i$ and $A_I := A_{<I} \upharpoonright a^{|a|^\omega}$ are models of BASIC. In addition they are "log-euclidean" rings : division by $|x|$ exists for each x in the ring. Finally, their

Σ_0^b diagrams are coded in $\mathcal{N}$; hence $|a|^\omega \subset A_I \subset a^{|a|^\omega}$ implies that A_I is a model of $\Sigma_0^b - ind^{|x|}$.

Definition : Sequence $(B_i)_{i\leq\lambda}$ is **minimal for inclusion** extending $(A_i)_{i\leq\lambda}$ iff :

- $B_0 \supset A_0$.
- $B_{i+1} = A_{i+1} \cup B_i \cup (B_i + B_i) \cup (B_i - B_i) \cup (B_i \cdot B_i) \cup \bigcup_{q\in[1,|a|^{2^i}]} \lfloor \frac{B_i}{q} \rfloor$.

Fact 2.2 *Let k_0, $1 \leq k_0 \leq \lambda$, and $B \subset \mathcal{Q}$.*

1. *If $(A_i)_{i\leq\lambda}$ is almost log-euclidean (resp. log-euclidean) then so is $(A_{i+k_0})_{i\leq\lambda-k_0}$.*
2. *Suppose $(A_i)_{i\leq\lambda}$ is log-euclidean and let $(B_i)_{i\leq\lambda}$ be a chain that is almost log-euclidean, minimal for inclusion extending $(A_i)_{i\leq\lambda}$ with first element $B_0 = A_0 \cup B$. Suppose also $|B| \subset |a|^{2^{k_0}}$.*

   Then $(B_{i+k_0})_{i\leq\lambda-k_0}$ is log-euclidean.

□ **F 2.2**

1. Trivial.
2. **<u>Claim 0</u>** : Let $(C_i)_{i\leq\lambda}$ be a chain almost log-euclidean.

   a) $|C_i| = [0, |\, max\, \|C_i\|\, |\,]$, interval in $\mathcal{N}$.

   b) If $q \in \mathcal{N}^*$ then $|\lfloor \frac{C_i}{q} \rfloor| \subset |C_i|$.

   c) $max\|C_i.C_i\| = (max\|C_i\|)^2$ and $(max\|C_i\|)^2 \in C_{i+1}$.

   d) $max(max\|C_i - C_i\|, max\|C_i + C_i\|) \leq (max\|C_i\|)^2$.

   □ Claim 0.

   c) Let $c \in C_i$ such that $\|c\| = max\|C_i\|$. Then $max\|C_i.C_i\| = c^2$.

   $c^2 \in C_i.C_i$. But $C_i.C_i \subset C_{i+1}$.

   Claim 0. □

   We remember that :

   - $B_0 = A_0 \cup B$.
   - $B_{i+1} = A_{i+1} \cup B_i \cup (B_i + B_i) \cup (B_i - B_i) \cup (B_i \cdot B_i) \cup \bigcup_{q\in[1,|a|^{2^i}]} \lfloor \frac{B_i}{q} \rfloor$

Claim 1 : $max\|B_{i+1}\| \leq max(max\|A_{i+1}\|, (max\|B_i\|)^2)$.

$\square$ Claim 1.

Use Claim 0.

Claim 1. $\square$

Now, to prove that $(B_i)_{k_0 \leq i \leq \lambda}$ is *log*-euclidean, it suffices to show that $|B_i|$ is part of B_{i+1} for $i : k_0 \leq i < \lambda$.

First we show :

Claim 2 : For $i < \lambda$

$$max\|B_i\| \leq max(max\|A_i\|, (max\|B\|)^{2^i}) \quad (*).$$

$\square$ Claim 2.

- $max\|B_0\| = max(max\|A_0\|, max\|B\|)$.

- Suppose it holds for $i < \lambda - 1$.

By Claim 1, $max\|B_{i+1}\| \leq max(max\|A_{i+1}\|, (max\|B_i\|)^2)$.

By induction hypothesis, $max\|B_i\| \leq max(max\|A_i\|, (max\|B\|)^{2^i})$.

Then, $max\|B_{i+1}\| \leq max(max\|A_{i+1}\|, max(max\|A_i\|, (max\|B\|)^{2^i})^2) \leq max(max\|A_{i+1}\|, (max\|A_i\|)^2, (max\|B\|)^{2^{i+1}})$.

By Claim 0, $(max\|A_i\|)^2 \in A_{i+1}$. Then :

$$max\|B_{i+1}\| \leq max(max\|A_{i+1}\|, (max\|B\|)^{2^{i+1}}).$$

and this inductively propagates to each $i < \lambda$.

Claim 2. $\square$

We take logarithms in inequality (*), then we have :

$$|B_i| \subset |A_i| \cup [0, 2^i(|\,max\,\|B\|\,| + 1)]$$

and hence

$$|B_i| \subset |A_i| \cup [0, 2^i(|a|^{2^{k_0}} + 1)]$$

The last intervals being in $\mathcal{N}$.

Finally, it suffices to show that for each $i \geq k_0$, $2^i(|a|^{2^{k_0}} + 1) \subset A_{i+1}$.

For then $|B_i|$ is included in A_{i+1} and B_{i+1}.

Indeed $i \geq k_0 \geq 1$ implies $2^{i+1} = 2^i + 2^i > 2^{k_0} + i$ hence $|a|^{2^{i+1}} \geq 2^i(|a|^{2^{k_0}} + 1)$.

But $|a|^{2^{i+1}} \subset A_{i+1}$.

F 2.2 $\square$

Kernel Lemma 2.3 *If* $|a| \cup \{a\} \subset A_0 \subset \mathcal{N}$ *then for each* $\lambda \in \mathcal{N}$ *there is a log-euclidean chain in* $\mathcal{N}$, $(A_i)_{i\leq\lambda}$ *with card* $A_i < (card\ A_0)^{2^{C\,i}}$ *for some standard* C.

This chain is the " true" kernel from which we shall build counterfeiht log-euclidean chains. Its existence is obvious by induction on λ, because it lies in $\mathcal{N}$, hence the discrete character automatically holds.

Extension of log-euclidean chains

Fact 2.4 *Let* $(A_i)_{i\leq\lambda}$ *be a chain in* $\mathcal{Q}$, *with* $A_{i+1} \supset (A_i + A_i) \ \cup \ (A_i.A_i)$, $0, 1 \in A_0$.
For any $n < 2^\lambda$ *and* $a_1, \ldots, a_n$ *in* A_0, $A_{|n|}$ *includes* $\sum\limits_{i\leq n} a_i$ *and* $\prod\limits_{i\leq n} a_i$.

The notation $\mathcal{Q}[x, a/x]$ allows also nonstandard degrees.

Definitions : Let $P \in \mathcal{Q}[x, a/x]$.

1. Suppose $P(x) = \sum\limits_{i=0}^{m} b_i x^i + \sum\limits_{i=1}^{n} c_i(\frac{a}{x})^i$, then b_i, c_i are called the coefficients of P and $deg(P) := sup\ \{i\,;\, b_i \neq 0\} + sup\ \{i\,;\, c_i \neq 0\}$.
2. We say " $P(x)$ has its coefficients in $A_i/[1, |a|^j]$ " if $P(x) = \sum\limits_{k=0}^{m} \frac{b_k}{p} x^k + \sum\limits_{k=1}^{n} \frac{c_k}{p}(\frac{a}{x})^k$ is such that $b_k, c_k \in A_i$ and $p \in [1, |a|^j]$.

Fact - Notation 2.5 *Let* $(A_i)_{i\leq\lambda}$ *be a log-euclidean chain in* $\mathcal{Q}$ *and* $i_0 \leq \lambda - 3$.
Let $\theta \in \mathcal{Q}$, $P \in \mathcal{Q}[x, a/x]$ *and* $q \in [1, |a|^{2^{i_0}}]$, *so that* $P(\theta)$ *is in* A_{i_0}.
There exists $\nu \in A_{i_0} \cap \mathcal{N}$ *unique,* $0 \leq \nu < q$, *such that* $\lfloor \frac{P(\theta)}{q} \rfloor_{A_{i_0+1}} = \frac{P(\theta)}{q} - \frac{\nu}{q}$.
Denote by the polynomial $\frac{P}{q} - \frac{\nu}{q}$ *the polynomial* $\lfloor \frac{\boldsymbol{P}}{\boldsymbol{q}} \rfloor$.
Then $\lfloor \frac{P}{q} \rfloor(\theta)$ *is in* A_{i_0+1} *and* $P = q \lfloor \frac{\boldsymbol{P}}{\boldsymbol{q}} \rfloor + \nu$.
Moreover if $P(x)$ *has its coefficients in* $A_{i_0}/[1, |a|^{2^{i_0}}]$, *then* $\lfloor \frac{P}{q} \rfloor$ *has coefficients in* $A_{i_0+2}/[1, |a|^{2^{i_0+1}}]$.

□ F 2.5

$P(\theta)$ being in A_{i_0}, the chain $(A_i)_{i\leq\lambda}$ being log-euclidean, $\lfloor \frac{P(\theta)}{q} \rfloor$ exists in A_{i_0+1}.
$\lfloor \frac{P(\theta)}{q} \rfloor \leq \frac{P(\theta)}{q} < \lfloor \frac{P(\theta)}{q} \rfloor + 1$, then $0 \leq P(\theta) - q \lfloor \frac{P(\theta)}{q} \rfloor < q$.

$q \in |a|^{2^{i_0}} \subset A_{i_0}$, $P(\theta) - q\lfloor\frac{P(\theta)}{q}\rfloor \in A_{i_0+3}$, $q \subset |a|^{2^{i_0}} \subset A_{i_0+3}$, A_{i_0+3} being discrete, there exists $\nu \in [0, q-1]$ (unique and ν in $\mathcal{N}$) such that $\nu = P(\theta) - q\lfloor\frac{P(\theta)}{q}\rfloor$, hence $\lfloor\frac{P(\theta)}{q}\rfloor = \frac{P(\theta)}{q} - \frac{\nu}{q}$.
$\lfloor\frac{P}{q}\rfloor(\theta)$ is in A_{i_0+1} because $\lfloor\frac{P}{q}\rfloor(\theta) = (\frac{P}{q} - \frac{\nu}{q})(\theta) = \frac{P(\theta)}{q} - \frac{\nu}{q} = \lfloor\frac{P(\theta)}{q}\rfloor$ wich is in A_{i_0+1}
Observe that ν is in A_{i_0} because $\nu \in |a|^{2^{i_0}} \subset A_{i_0}$.
$P(x)$ having coefficients in $A_{i_0}/[1, |a|^{2^{i_0}}]$, $P(x) = \sum_{i=0}^{n} \frac{b_i}{p} x^i + \sum_{i=1}^{n} \frac{c_i}{p} (\frac{a}{x})^i$ is such that $b_i, c_i \in A_{i_0}$ and $p \in [1, |a|^{2^{i_0}}]$.
Thus $\frac{P(x)}{q} - \frac{\nu}{q} = \sum_{i=1}^{n} \frac{b_i}{pq} x^i + \frac{b_0 - p\nu}{pq} + \sum_{i=1}^{n} \frac{c_i}{pq} (\frac{a}{x})^i$.
Observe that p is in A_{i_0} because $|a|^{2^{i_0}} \subset A_{i_0}$.
b_0, p, ν being in A_{i_0}, $b_0 - p\nu \in A_{i_0+2}$.
It is clear that $pq \in [1, |a|^{2^{i_0+1}}]$.

F 2.5 □

Fact 2.6 *Let $(A_i)_{i \leq \lambda}$ be a log-euclidean chain in $\mathcal{Q}$; let $P, Q \in \mathcal{Q}[x, a/x]$ be of degree $\leq n$ with coefficients in $A_{i_0}/[1, |a|^{2^{j_0}}]$, with $j_0 \leq i_0$. Then :*

1. *$P+Q$ and $P-Q$ are of degree $\leq n$ with coefficients in $A_{i_0+2}/[1, |a|^{2^{j_0+1}}]$.*
2. *PQ is of degree $\leq 2\,n$ with coefficients in $A_{i_0+2\,|n|+4}/[1, |a|^{2^{j_0+1}}]$.*

□ **F 2.6**

Degree properties are trivial. Suppose $p, q \in [1, |a|^{2^{j_0}}]$, $P(x) = \sum_{i=0}^{n} \frac{b_i}{p} x^i + \sum_{i=1}^{n} \frac{c_i}{p} (\frac{a}{x})^i$ and $Q(x) = \sum_{j=0}^{n} \frac{d_j}{q} x^j + \sum_{j=1}^{n} \frac{e_j}{q} (\frac{a}{x})^j$, with $b_i, c_i, d_j, e_j \in A_{i_0}$.

1. $P(x) + Q(x) = \sum_{k=0}^{n} \frac{qb_k + pd_k}{pq} x^k + \sum_{i=1}^{n} \frac{qc_k + pe_k}{pq} (\frac{a}{x})^k$.
   Since $p, q, b_k, c_k \in A_{i_0}$, $qb_k, pd_k \in A_{i_0+1}$ and $qb_k + pd_k \in A_{i_0+2}$. Also $qc_k + pe_k \in A_{i_0+2}$.
   Same for $P - Q$.
   And clearly $pq \in [1, |a|^{2^{j_0+1}}]$.

2. $PQ(x) = \sum_{k=0}^{2n} \frac{s_k}{\mu} x^k + \sum_{k=1}^{2n} \frac{\sigma_k}{\mu} (\frac{a}{x})^k$, with :
   (a) $\mu = pq$.
   (b) $s_k = \sum_{i+j=k} b_i d_j + \sum_{i-j=k} b_i e_j a^j + \sum_{j-i=k} d_j c_i a^i$, for all k, $0 \leq k \leq 2n$.

(c) $\sigma_k = \sum_{i+j=k} c_i e_j + \sum_{j-i=k} b_i e_j a^i + \sum_{i-j=k} d_j c_i a^j$, for all k, $1 \leq k \leq 2n$.

For instance, in the case of s_k :

- $b_i d_j$ in A_{i_0+1}, $k \leq 2n$, $i \leq n$, $j \leq n$, hence the number of terms in $\sum_{i+j=k} b_i d_j$ is at most $n+1$. By Fact 2.4, $\sum_{i+j=k} b_i d_j$ is in $A_{i_0+2+|n|}$.

- Since $b_i e_j \in A_{i_0+1}$ and $a^j \in A_{|n|}$, $b_i e_j a^j \in A_{i_0+|n|+1}$.
Since $i \leq n$, $1 \leq j \leq n$, the number of terms in $\sum_{i-j=k} b_i e_j a^j$ is at most n. By Fact 2.4, $\sum_{i-j=k} b_i e_j a^j$ is in $A_{i_0+1+2\,|n|}$.

- And $\sum_{j-i=k} d_j c_i a^i$ is in $A_{i_0+1+2\,|n|}$.

Thus s_k is the sum of three elements of $A_{i_0+2+2\,|n|}$, and s_k is in $A_{i_0+4+2\,|n|}$.

Clearly $\mu \in [1, |a|^{2^{j_0+1}}]$.

F 2.6 □

Chain patterns : Let $\theta \in \mathcal{Q}$. Let $\mathfrak{T}_{pr}$ be a part of $\mathcal{Q}[x, a/x]$ with degrees bounded by 1, with value in A_0 on θ and with coefficients in A_0.

The interesting cases for us actually put only two elements in $\mathfrak{T}_{pr}$.
Consider the sequence defined by the recurrence : $i_0 = 0$ and

$$i_{k+1} = i_k + 2\,k + 6 \quad [1]$$

and consider the parts of $\mathcal{Q}[x, a/x]$ defined by :

1. $\mathfrak{T}_0 = A_0 \cup \mathfrak{T}_{pr}$.
2. $\mathfrak{T}_{k+1} = A_{k+1} \cup \mathfrak{T}_k \cup (\mathfrak{T}_k + \mathfrak{T}_k) \cup (\mathfrak{T}_k - \mathfrak{T}_k) \cup \mathfrak{T}_k \cdot \mathfrak{T}_k \cup \bigcup_{q \in [1, |a|^{2^k}]} \lfloor \frac{\mathfrak{T}_k}{q} \rfloor$.

This is well defined for all k such that $i_k \leq \lambda$ by the next fact, and is called a *chain pattern* associated to θ, $\mathfrak{T}_{pr}$ and A_0.

Fact 2.7 *Let k such that $i_k \leq \lambda$ and $P \in \mathfrak{T}_k$. Then :*

1. *$P(\theta) \in A_k$.*
2. *The degree of P is $\leq 2^k$.*

[1] $i_k = k^2 + 5\,k$

3. *The coefficients of P belong to* $A_{i_k}/[1,|a|^{2^k}]$.

4. $card\,\mathfrak{T}_k \leq (card\,A_k)^{2^{2k}}$.

□ **F 2.7** : induction on k.
Set $\mathfrak{C}_k = A_{i_k}/[1,|a|^{2^k}]$.
Fact holds for $k = 0$. Suppose it holds for k, we show it for $k+1$. Fix $P, Q \in \mathfrak{T}_k$ and $q \in [1,|a|^{2^k}]$.
By induction hypothesis $P(\theta) \in A_k$, then $q \in [1,|a|^{2^k}]$ and the Fact 2.5 show that $\lfloor \frac{P}{q} \rfloor$ which must occur in $\mathfrak{T}_{k+1}$, is well defined.

1. By induction hypothesis $P(\theta), Q(\theta) \in A_k$. Then $P(\theta)Q(\theta), P(\theta) \pm Q(\theta) \in A_{k+1}$; and $q \in [1,|a|^{2^k}]$ so by Fact 2.5 polynomial $\lfloor \frac{P}{q} \rfloor$ takes value in A_{k+1} on θ.

2. By induction hypothesis $deg(P), deg(Q)$ are $\leq 2^k$. So
   (a) $deg(P \pm Q) \leq max(deg(P), deg(Q)) \leq 2^k$.
   (b) $deg(P\,Q) = deg(P) + deg(Q)) \leq 2^{k+1}$.
   (c) $deg(\lfloor \frac{P}{q} \rfloor) = deg(P) \leq 2^k$.

3. By induction hypothesis the coefficients of P, Q belong to $\mathfrak{C}_k$. By the above Fact :
   (a) Coefficients of $P \pm Q$ are in $A_{i_k+2}/[1,|a|^{2^{k+1}}]$, included in $\mathfrak{C}_{k+1}$.
   (b) Since the degrees of P, Q are $\leq 2^k$, their coefficients belong to $\mathfrak{C}_k$. By Fact 2.6, the coefficients of $P\,Q$ belong to $A_{i_k+2\,|2^k|+4}/[1,|a|^{2^{k+1}}]$ that is $A_{i_{k+1}}/[1,|a|^{2^{k+1}}]$, which is $\mathfrak{C}_{k+1}$.

      Remember $|2^k| = k+1$.
   (c) Since $q \in [1,|a|^{2^k}]$ and the coefficients of P are in $\mathfrak{C}_k$, by Fact 2.5 the coefficients of $\lfloor \frac{P}{q} \rfloor$ are in $A_{i_k+2}/[1,|a|^{2^{k+1}}]$, included in $\mathfrak{C}_{k+1}$.

4. - True for $k = 0$.
   - Suppose $card\,\mathfrak{T}_k \leq (card\,A_k)^{2^{2k}}$.

   The sets $\mathfrak{T}_k$, $\mathfrak{T}_k + \mathfrak{T}_k$, $\mathfrak{T}_k - \mathfrak{T}_k$ and $\mathfrak{T}_k \cdot \mathfrak{T}_k$ have at most $(card\,\mathfrak{T}_k)^2$ elements.

   $\bigcup_{q \in [1,|a|^{2^k}]} \lfloor \frac{\mathfrak{T}_k}{q} \rfloor$ has at most $|a|^{2^k}\,card\,\mathfrak{T}_k$ elements.

   So : $card\,\mathfrak{T}_{k+1} \leq card\,A_{k+1} + 4\,card\,\mathfrak{T}_k{}^2 + card\,A_{k+1}\;card\;\mathfrak{T}_k \leq 2\,card\,A_{k+1}\;card\,\mathfrak{T}_k + 4\,(card\,\mathfrak{T}_k)^2 \leq 2\,card\,A_{k+1}\;(card\,A_k)^{2^{2k}} + 4\,((card\,A_k)^{2^{2k}})^2 \leq 6\,((card\,A_{k+1})^{2^{2k}})^2 \leq (card\,A_{k+1})^{2^{2(k+1)}}$.

F 2.7 □

Notations :

a) $f_{\restriction I}$ is the restriction of f to I.

b) If E is the disjoint sum of a sequence of intervals $(I_k)_{k\leq n}$, $n \in \mathcal{N}$, then $mes(I)$ is the sum of the lengths of this intervals.

c) If I is an interval and A is a finite subset of $\mathcal{Q}$ [2], then $\delta(I, A)$ denotes $inf\{\ \|x-a\|\ ;\ x\in I$ and $a \in A\}$.

Differentiation Lemma 2.8 *Let $f \in \mathcal{Q}[x, \frac{1}{x}]$ be of degree $\mathfrak{d}$. Let $I \subset \mathcal{Q}^*$ be an interval of non-null length and $\mu \in \mathcal{Q}$ such that $\|f_{\restriction I}\| \leq \mu$. For each $k \geq 1$ there is an interval $J \subset I$, with length $\geq \frac{mes(I)}{(3\,(\mathfrak{d}+k))^k}$ such that :*

$$\boxed{mes(I)^k\,\|f^{(k)}_{\restriction J}\| \leq 2^k\,(3\,(\mathfrak{d}+k))^{\frac{k(k+1)}{2}}\,\mu}$$

□ **Differentiation Lemma 2.8** : induction on k.

CASE $k = 1$: we claim that there is an interval $J \subset I$, with length $\geq \frac{mes(I)}{3\,(\mathfrak{d}+1)}$ such that

$$mes(I)\,\|f'_{\restriction J}\| \leq 6\,(\mathfrak{d}+1)\mu.$$

Assume to the contrary that if $J \subset I$ verifies $mes(I)\,\|f'_{\restriction J}\| \leq 6\,(\mathfrak{d}+1)\mu$, then $mes(J) < \frac{mes(I)}{3\,(\mathfrak{d}+1)}$.
Thus $E_0 = \{x \in I\,;\ -6\,(\mathfrak{d}+1)\mu \leq mes(I)\,f'(x) \leq 6\,(\mathfrak{d}+1)\mu\}$ is union of $(\mathfrak{d}+1)$ disjoint intervals, of length at most $< \frac{mes(I)}{3\,(\mathfrak{d}+1)}$, hence $mes(E_0) < \frac{mes(I)}{3}$.

On the other hand the two sets:
- $E_1 = \{x \in I\,;\ mes(I)\,f'(x) > 6\,(\mathfrak{d}+1)\mu\}$
- $E_2 = \{x \in I\,;\ mes(I)\,f'(x) < -6\,(\mathfrak{d}+1)\mu\}$
are union of $\mathfrak{d}+1$ disjoint intervals.
And since $mes(I) = mes(E_0) + mes(E_1) + mes(E_2)$, $mes(E_1) + mes(E_2) > \frac{2\,mes(I)}{3}$.
(E_1 and E_2 being disjoints), $E_1 \cup E_2$ is union of $2(\mathfrak{d}+1)$ disjoint intervals.
This provides an interval J included in E_1 or E_2 such that $mes(J) > \frac{mes(I)}{3\,(\mathfrak{d}+1)}$.
$J = [a, b]$. There is $c \in]a, b[$ such that $\|f(a) - f(b)\| \geq (b-a)\|f'(c)\|$.
Noting that $c \in E_1 \cup E_2$ hence $mes(I)\|f'(c)\| > 6\,(\mathfrak{d}+1)\mu$, we get :

$$2\,\mu \geq \|f(a) - f(b)\| \geq (b-a)\|f'(c)\| > \frac{mes(I)}{3\,(\mathfrak{d}+1)}\|f'(c)\| > 2\,\mu$$

[2] $A = \{a_0, \ldots, a_n\}$, $n \in \mathcal{N}$.

CASE $\boldsymbol{k+1}$: Suppose lemma is true for k, for each term f in x and $\dfrac{1}{x}$ of degree $\mathfrak{d}$. We show it for $\boldsymbol{k+1}$.

By induction hypothesis there is $J_0 \subset I$ of length $\geq \frac{mes(I)}{(3\,(\mathfrak{d}+k))^k}$ such that

$$mes(I)^k\,\|f^{(k)}_{\restriction J_0}\| \leq 2^k\,(3\,(\mathfrak{d}+k))^{\frac{k(k+1)}{2}}\,\mu$$

Set $g = mes(I)^k\, f^{(k)}$ and $\mu_0 = 2^k\,(3\,(\mathfrak{d}+k))^{\frac{k(k+1)}{2}}\,\mu$. Thus $\|g_{\restriction J_0}\| \leq \mu_0$.
Applying case $k=1$ to J_0, g and μ_0 we have an interval $J \subset J_0$, $mes(J) \geq \frac{mes(J_0)}{3\,(deg(g)+1)}$, with

$$mes(J_0)\,\|g'_{\restriction J}\| \leq 6\,(deg(g)+1)\,\mu_0.$$

Suppose $f(x) = \sum\limits_{i=0}^{m} b_i x^i + \sum\limits_{i=1}^{n} c_i (\frac{a}{x})^i$ with $\mathfrak{d} = m+n$.
Observe that : if $k \leq m$ then $deg(g) = \mathfrak{d}$, if $k > m$ then $deg(g) = n+k$, and in both cases $deg(g) \leq \mathfrak{d}+k$.
Thus :

$$mes(J) \geq \frac{mes(J_0)}{3\,(deg(g)+1)} \geq \frac{mes(I)}{3\,(deg(g)+1)\,(3\,(\mathfrak{d}+k))^k} \geq \frac{mes(I)}{(3\,(\mathfrak{d}+k+1))^{k+1}}$$

And $g' = mes(I)^k\, f^{(k+1)}$, $deg(g) \leq \mathfrak{d}+k$, $mes(I) \leq mes(J_0)\,(3(\mathfrak{d}+k))^k$, hence :

$$mes(I)^{k+1}\,\|f^{(k+1)}_{\restriction J}\| = mes(I)\,\underbrace{mes(I)^k\,\|f^{(k+1)}_{\restriction J}\|}_{=\|g'_{\restriction J}\|} \leq mes(J_0)\,(3\,(\mathfrak{d}+k))^k\,\|g'_{\restriction J}\| \leq$$

$$(3\,(\mathfrak{d}+k))^k\,6\,(deg(g)+1)\,\mu_0 \leq (3\,(\mathfrak{d}+k))^k\,6\,(\mathfrak{d}+k+1)\,2^k\,(3\,(\mathfrak{d}+k))^{\frac{k(k+1)}{2}}\,\mu \leq 2^{k+1}\,(3\,(\mathfrak{d}+k+1))^{\frac{(k+1)(k+2)}{2}}\,\mu$$

Differentiation Lemma 2.8 □

First Measure Lemma 2.9 *Let f in $\mathfrak{T}_k \cap \mathcal{Q}[x]$ be of degree $\mathfrak{d}$, let I be an interval and let $\mu \in \mathcal{Q}^+$ such that $\delta(I, A_{i_{k+4}}) > \mu^{\frac{1}{\mathfrak{d}}}\,|a|^{2^{i_{k+3}}}$. Then :*

$$\boxed{mes\{x \in I\,;\ \|f(x)\| \leq \mu\}^{\mathfrak{d}} < \frac{\mu}{|a|^{2^{i_k}}}}$$

□ **First Measure lemma 2.9** : By Fact 2.7, we can set $f(x) = \sum\limits_{i=0}^{\mathfrak{d}} \frac{a_i}{q} x^i$, with a_i in A_{i_k} and $q \in [1, |a|^{2^k}]$.
Set $E = \{x \in I\,;\ \|f(x)\| \leq \mu\}$.
Otherwise we proceed by way of contradiction.
That is to say, we assume $(mes(E))^{\mathfrak{d}} \geq \frac{\mu}{|a|^{2^{i_k}}}$.

E is union of $\mathfrak{d}$ disjoint intervals, let J be one of them with maximal length ; then, $(mes(E))^{\mathfrak{d}} \leq (\mathfrak{d}\, mes(J))^{\mathfrak{d}}$. Hence $mes(J)^{\mathfrak{d}} \geq \dfrac{\mu}{\mathfrak{d}^{\mathfrak{d}}\, |a|^{2^{i_k}}}$.
$J \subset E$, then $\|f_{\restriction J}\| \leq \mu$, and by Differentiation Lemma 2.8 :
- on one hand, there is an interval $J_{\mathfrak{d}} \subset J$ such that : $mes(J)^{\mathfrak{d}}\, \|f^{(\mathfrak{d})}_{\restriction J_{\mathfrak{d}}}\| \leq 2^{\mathfrak{d}}\,(6\,\mathfrak{d})^{\frac{\mathfrak{d}(\mathfrak{d}+1)}{2}}\mu$.
Since $\|f^{(\mathfrak{d})}\| = \mathfrak{d}!\, \|\dfrac{a_{\mathfrak{d}}}{q}\|$, $mes(J)^{\mathfrak{d}} \geq \dfrac{\mu}{\mathfrak{d}^{\mathfrak{d}}\,|a|^{2^{i_k}}}$, and $q \in [1, |a|^{2^k}]$, then :
$\|a_{\mathfrak{d}}\| \leq 2^{\mathfrak{d}}\,(6\mathfrak{d})^{\frac{\mathfrak{d}(\mathfrak{d}+1)}{2}}\mathfrak{d}^{\mathfrak{d}}\,|a|^{2^k}|a|^{2^{i_k}}$, and hence : $a_{\mathfrak{d}} \in [1, |a|^{2^{i_{k+3}}}]$.
(I remind that $\mathfrak{d} \leq 2^k$.)

- On the other hand, there is an interval $J_{\mathfrak{d}-1} \subset J$ such that : $mes(J)^{\mathfrak{d}-1}\, \|f^{(\mathfrak{d}-1)}_{\restriction J_{\mathfrak{d}-1}}\| \leq 2^{\mathfrak{d}-1}\,(3(2\,\mathfrak{d}-1))^{\frac{\mathfrak{d}(\mathfrak{d}-1)}{2}}\mu$.
If we majorize as above, we have : $\|x + \dfrac{a_{\mathfrak{d}-1}}{\mathfrak{d}\, a_{\mathfrak{d}}}\| \leq |a|^{2^{i_{k+3}}}\, \mu^{\frac{1}{d}}$ for all $x \in J_{\mathfrak{d}-1}$.
Since $a_{\mathfrak{d}-1}$ is in A_{i_k}, $a_{\mathfrak{d}} \in [1, |a|^{2^{i_{k+3}}}]$, and $\mathfrak{d} \leq 2^k$, then $\lfloor \dfrac{a_{\mathfrak{d}-1}}{\mathfrak{d}\, a_{\mathfrak{d}}} \rfloor$ is in $A_{i_{k+3}+2}$, and $\delta(I, A_{i_{k+3}+3}) \leq |a|^{2^{i_{k+3}}}\, \mu^{\frac{1}{d}}$, hence $\delta(I, A_{i_{k+4}}) \leq |a|^{2^{i_{k+3}}}\, \mu^{\frac{1}{d}}$.

First Measure Lemma 2.9 $\square$

Second Measure Lemma 2.10 *Suppose* $|a|^{2^{\omega\,\lambda}} < a$. *Let* $f \in \mathfrak{T}_k \backslash \mathcal{Q}[x]$ *(i.e.* f *contains at least one monomial in* $\frac{a}{x}$ *) and be of degree* $\mathfrak{d}$ *; let* I *be an interval such that* $sup(I) \geq 1$, *and let* $\mu \in \mathcal{Q}^+$. *Suppose that* $i_k < \lambda$. *Then :*

$$\boxed{mes\{x \in I\,;\ \|f(x)\| \leq \mu\}^{\mathfrak{d}} < \frac{\mu^{\epsilon}\, sup(I)^{2^{k+2}}}{a^{\frac{1}{5}}}}$$

with ϵ *equal to* 1 *if* $\mu \geq 1$, *and* $\frac{1}{5}$ *otherwise.*

$\square$ **Second Measure Lemma 2.10** : Set $\tilde{\mu} = \dfrac{\mu^{\epsilon}\, sup(I)^{2^{k+2}}}{a^{\frac{1}{5}}}$.
By Fact 2.7, we can set $f(x) = \sum_{i=0}^{m} \frac{b_i}{q} x^i + \sum_{j=1}^{n} \frac{c_j}{q} (\frac{a}{x})^j$, with b_i, c_j in A_{i_k} and $q \in [1, |a|^{2^k}]$.
Set $E = \{x \in I\,;\ \|f(x)\| \leq \mu\}$.
Assume $mes(E) \neq 0$ and $mes(E)^{\mathfrak{d}} \geq \tilde{\mu}$.
E is union of $\mathfrak{d}+1$ disjoint intervals, let J be one of them with maximal length; then $mes(E)^{\mathfrak{d}} \leq ((\mathfrak{d}+1)\, mes(J))^{\mathfrak{d}}$. Hence $mes(J)^{\mathfrak{d}} \geq \dfrac{\tilde{\mu}}{(\mathfrak{d}+1)^{\mathfrak{d}}}$.
$J \subset E$, then $\|f_{\restriction J}\| \leq \mu$, and, by Differentiation Lemma 2.8
Step 1 :

there is an interval $J_0 \subset J$, $mes(J_0) \geq \frac{mes(J)}{(3(\mathfrak{d}+m+1))^{m+1}}$ such that

$$mes(J)^{m+1}\,\|f^{(m+1)}_{\restriction J_0}\| \leq \underbrace{2^{m+1}\,(3\,(\mathfrak{d}+m+1))^{\frac{(m+1)(m+2)}{2}}\mu}_{\mu_0}\ .$$

Observe that : $\mu_0 \leq 2^{2^{3\,k+2}}\mu$ and

$$f^{(m+1)}(x) = \sum_{j=m+2}^{n+m+1} \bar{c}_j\, x^{-j},\ \text{with}\ \bar{c}_{n+m+1} = \frac{(-1)^{m+1}\,c_n\,a^n\,\prod_{j=0}^{m}(n+j)}{q}.$$

Step 2 :
Set $g(t) = mes(J)^{m+1}\,f^{(m+1)}(\frac{1}{t})$ and $\bar{J} = \{\frac{1}{x};\ x \in J_0\}$.
Hence $deg(g) = \mathfrak{d}+1$ and $\|g_{\restriction \bar{J}}\| \leq \mu_0$.

Observe that $mes(\bar{J}) \geq \frac{mes(J_0)}{sup(J_0)^2} \geq \frac{mes(J_0)}{sup(I)^2}$.
By Differentiation Lemma 2.8 , there is $\bar{J}_0 \subset \bar{J}$ such that :

$$mes(\bar{J})^{\mathfrak{d}+1}\,\|g^{(\mathfrak{d}+1)}_{\restriction \bar{J}_0}\| \leq \underbrace{2^{\mathfrak{d}+1}\,(3(2(\mathfrak{d}+1)))^{\frac{(\mathfrak{d}+1)(\mathfrak{d}+2)}{2}}\mu_0}_{\mu_1}\ .$$

Observe that : $\mu_1 \leq 2^{2^{3\,k+4}}\mu$ and $g^{(\mathfrak{d}+1)} = (\mathfrak{d}+1)!\,mes(J)^{m+1}\,\bar{c}_{\mathfrak{d}+1}$.
Since $mes(\bar{J}) \geq \frac{mes(J_0)}{sup(I)^2}$, we have :

$$\frac{mes(J_0)^{\mathfrak{d}+1}}{sup(I)^{2(\mathfrak{d}+1)}}\,mes(J)^{m+1}\,\|\underbrace{(-1)^{m+1}\,c_n\,a^n\prod_{j=0}^{m}(n+j)}_{=\bar{c}_{\mathfrak{d}+1}\cdot q}(\mathfrak{d}+1)\,!\| \leq q\,\mu_1\ .$$

Since $mes(J_0) \geq \frac{mes(J)}{(3(\mathfrak{d}+m+1))^{m+1}}$,

$$\frac{mes(J)^{\mathfrak{d}+1}}{(3(\mathfrak{d}+m+1))^{(m+1)(\mathfrak{d}+1)}sup(I)^{2(\mathfrak{d}+1)}}\,mes(J)^{m+1}\,\|(-1)^{m+1}\,c_n\,a^n\prod_{j=0}^{m}(n+j)\,(\mathfrak{d}+1)\,!\| \leq q\,\mu_1\ .$$

Hence : $\dfrac{mes(J)^{\mathfrak{d}+m+2}}{sup(I)^{2(\mathfrak{d}+1)}}\,\|(-1)^{m+1}\,c_n\,a^n\displaystyle\prod_{j=0}^{m}(n+j)\,(\mathfrak{d}+1)\,!\| \leq (3(\mathfrak{d}+m+1))^{(m+1)(\mathfrak{d}+1)}\,q\,\mu_1$.
On one hand $n \geq 1$, and $c_n \in A_{i_k}$ which is discrete, hence
$a \leq \|(-1)^{m+1}\,c_n\,a^n\prod_{j=0}^{m}(n+j)\,(\mathfrak{d}+1)\,!\|$; on the other hand $mes(J) \geq \dfrac{\tilde{\mu}^{\frac{1}{\mathfrak{d}}}}{(\mathfrak{d}+1)}$,
hence :

$$\frac{\tilde{\mu}^{\frac{\mathfrak{d}+m+2}{\mathfrak{d}}}}{(\mathfrak{d}+1)^{\mathfrak{d}+m+2}\ sup(I)^{2(\mathfrak{d}+1)}}\,a\ \leq (3(\mathfrak{d}+m+1))^{(m+1)(\mathfrak{d}+1)}\,q\,\mu_1\ .$$

So :

$$\frac{\mu^{\frac{\epsilon(\mathfrak{d}+m+2)}{\mathfrak{d}}} sup(I)^{\frac{2^{k+2}(\mathfrak{d}+m+2)}{\mathfrak{d}}}}{(\mathfrak{d}+1)^{\mathfrak{d}+m+2}\ a^{\frac{\mathfrak{d}+m+2}{5\mathfrak{d}}}\ sup(I)^{2(\mathfrak{d}+1)}}\, a \le (3(\mathfrak{d}+m+1))^{(m+1)(\mathfrak{d}+1)}\, q\, \mu_1\ .$$

Hence :

$$\frac{\mu^{\frac{\epsilon(\mathfrak{d}+m+2)}{\mathfrak{d}}} sup(I)^{\frac{2^{k+2}(\mathfrak{d}+m+2)}{\mathfrak{d}}}}{a^{\frac{\mathfrak{d}+m+2}{5\mathfrak{d}}}\ sup(I)^{2(\mathfrak{d}+1)}}\, a \le \underbrace{(\mathfrak{d}+1)^{\mathfrak{d}+m+2}\ (3(\mathfrak{d}+m+1))^{(m+1)(\mathfrak{d}+1)}\, q\, \mu_1}_{\mu_2}\ .$$

Observe that :

- $\mu_2 \le |a|^{2^{3k+6}} \mu$.
- $\mu^{\frac{\epsilon(\mathfrak{d}+m+2)}{\mathfrak{d}}} \ge \mu$, in both cases $\mu \ge 1$ ($\epsilon = 1$), and $\mu < 1$ ($\epsilon = \frac{1}{5}$). (Use the fact that $m \le \mathfrak{d}$ and $\mathfrak{d} \ge 1$.)
- $\frac{a}{a^{\frac{\mathfrak{d}+m+2}{5\mathfrak{d}}}} > a^{\frac{1}{5}}$. Since $1 - \frac{\mathfrak{d}+m+2}{5\mathfrak{d}} = \frac{1}{5} + \frac{\mathfrak{d}-m}{5\mathfrak{d}} + 2\,\frac{\mathfrak{d}-1}{5\mathfrak{d}}$ and $\mathfrak{d} > m$.
- $sup(I)^{\frac{2^{k+2}(\mathfrak{d}+m+2)}{\mathfrak{d}}} > sup(I)^{2(\mathfrak{d}+1)}$. (Use $\mathfrak{d} \le 2^k$.)

Hence $\mu a^{\frac{1}{5}} < |a|^{2^{3k+6}} \mu < |a|^{2^{4\lambda}} \mu$, so $a^{\frac{1}{5}} < |a|^{2^{4\lambda}}$ contradicting the hypothesis that $|a|^{2^{\omega\,\lambda}} < a$.

Second Measure Lemma 2.10 □

Fact 2.11 *Let $(A_i)_{i\le\lambda}$ be a log−euclidean chain. Let $\theta \in \mathcal{Q}$, and $\mathfrak{T}_{pr}$ = a set of polynomials in $\mathcal{Q}[x, a/x]$, of degree ≤ 1, with value in A_0 on θ, and with coefficients in A_0.*
Let $(\mathfrak{T}_i)_{i\le\lambda_0}$ be a chain pattern associated to θ, $\mathfrak{T}_{pr}$, and A_0, with $i_{\lambda_0} < \lambda$.
Suppose b such that :
a) for some integer k_0, $|P(b)| \le |a|^{2^{k_0}}$ for all $P \in \mathfrak{T}_{pr}$.
b) $\neg 0 < \|P(b)\| < 1$ for all $P \in \mathfrak{T}_{\lambda_0}$ of degree non zero.

Then $(\mathfrak{T}_{i+k_0}(b))_{i<\lambda_0-k_0}$ is a log-euclidean chain.

□ **F 2.11**

- $(\mathfrak{T}_i(b))_{i\le\lambda_0}$ is discrete. Indeed :
  – by b) we have $\neg 0 < \|P(b)\| < 1$ for non constant polynomials P in $\mathfrak{T}_{\lambda_0}$.
  – if P is a constant polynomial from $\mathfrak{T}_{\lambda_0}$ and P verifies $0 < \|P(b)\| < 1$, then $0 < \|P(\theta)\| < 1$. But, by Fact 2.7, $P(\theta)$ is in A_{λ_0} which is discrete.

- By definition of chain pattern

$$\mathfrak{T}_{i+1}(b) = A_{i+1} \cup \mathfrak{T}_i(b) \cup (\mathfrak{T}_i(b)+\mathfrak{T}_i(b)) \cup (\mathfrak{T}_i(b)-\mathfrak{T}_i(b)) \cup (\mathfrak{T}_i(b)\cdot\mathfrak{T}_i(b)) \cup \bigcup_{q\in[1,|a|^{2^i}]} \lfloor \frac{\mathfrak{T}_i}{q} \rfloor(b)$$

  To show that $(\mathfrak{T}_i(b))_{i\leq\lambda_0}$ is almost *log*-euclidean, minimal for inclusion extending $(A_i)_{i\leq\lambda_0}$ with first element $A_0 \cup \mathfrak{T}_{pr}(b)$, it suffices to show that

$$\bigcup_{q\in[1,|a|^{2^i}]} \lfloor \frac{\mathfrak{T}_i}{q} \rfloor(b) = \bigcup_{q\in[1,|a|^{2^i}]} \lfloor \frac{\mathfrak{T}_i(b)}{q} \rfloor_{\mathfrak{T}_{i+1}(b)}$$

  - $\subset$ / Let $P \in \mathfrak{T}_i$ and $q \in [1, |a|^{2^i}]$.

  On one hand $\lfloor \frac{P}{q} \rfloor \in \mathfrak{T}_{i+1}$ and $\lfloor \frac{P}{q} \rfloor(b)$ is in $\mathfrak{T}_{i+1}(b)$.

  On the other hand there exists $\nu \in [0, q-1]$ such that $\lfloor \frac{P}{q} \rfloor = \frac{P}{q} - \frac{\nu}{q}$. Then $\lfloor \frac{P}{q} \rfloor(b) \leq \frac{P(b)}{q} < \lfloor \frac{P}{q} \rfloor(b) + 1$ wich means $\lfloor \frac{P(b)}{q} \rfloor_{\mathfrak{T}_{i+1}(b)}$ exists and $\lfloor \frac{P}{q} \rfloor(b) = \lfloor \frac{P(b)}{q} \rfloor_{\mathfrak{T}_{i+1}(b)}$.

  - $\supset$ / Let $c \in \mathfrak{T}_i(b)$ and $q \in [1, |a|^{2^i}]$.

  There exists $P \in \mathfrak{T}_i$ such that $c = P(b)$. By same argument as above $\lfloor \frac{P(b)}{q} \rfloor_{\mathfrak{T}_{i+1}(b)}$ exists and is equal to $\lfloor \frac{P}{q} \rfloor(b)$ wich lies in $\lfloor \frac{\mathfrak{T}_i}{q} \rfloor(b)$.

  Hence $\lfloor \frac{c}{q} \rfloor_{\mathfrak{T}_{i+1}(b)}$ lies in $\lfloor \frac{\mathfrak{T}_i}{q} \rfloor(b)$.

- Then, since $|\mathfrak{T}_{pr}(b)| \subset |a|^{2^{k_0}}$, by Fact 2.2, $(\mathfrak{T}_{i+k_0}(b))_{i\leq\lambda_0-k_0}$ is *log*-euclidean.

F 2.11 $\square$

Integer Part Lemma 2.12 *Let $(A_i)_{i\leq\lambda}$ be a $log-$euclidean chain such that for some $C_0 \in \mathbb{N}$, card $A_k < |a|^{2^{C_0\,k}}$ for all $k < \lambda$. Let $\beta \in \mathcal{Q}$ such that $|\beta| \leq |a|^\omega$.*

Then, there is $b \in \mathcal{Q}$, $\beta - 1 < b \leq \beta$, there is a $log-$euclidean chain $(B_i)_{i\leq\lambda'}$ extending $(A_i)_{i\leq\lambda'}$, such that $B_0 \ni b$, $\lambda' > \sqrt{\frac{\lambda}{2}}$ and for some $C_1 \in \mathbb{N}$, card $B_k < |a|^{2^{C_1\,k}}$ for all $k < \lambda'$.

(Thus we construct b an integer part of β in B_0)

$\square$ **Integer Part Lemma 2.12** : Set $I =]\beta - 1, \beta]$. Let k_0 be such that $|\beta| \leq |a|^{2^{k_0}}$.

Fix λ_0 such that $i_{\lambda_0+4} \leq \frac{2\lambda}{3} < i_{\lambda_0+5}$, and set $\lambda' = \lambda_0 - k_0$. It is clear that $\lambda' > \sqrt{\frac{\lambda}{2}}$.

CASE 1 : $\delta(I, A_{i_{\lambda_0+5}}) = 0$.
It is clear that $i_{\lambda_0+5} + \lambda_0 < \lambda$. So the integer part of β exists in $A_{i_{\lambda_0+5}}$.
Set b equal to this integer part. And set :
- $B_0 = A_0 \cup \{b\}$.
- for all $k \leq \lambda_0$, $B_{k+1} := A_{k+1} \cup (B_k + B_k) \ \cup \ (B_k - B_k) \ \cup \ (B_k \cdot B_k) \ \cup \bigcup_{q \in [1, |a|^{2^k}]} \lfloor \frac{B_k}{q} \rfloor_{A_{\lambda-1}}$.

It is clear that, the B_k's are in $A_{i_{\lambda_0+5}+k}$. Since $i_{\lambda_0+5} + \lambda_0 < \lambda$, they are well defined for all $k \leq \lambda_0$.
Notice that $(B_k)_{k \leq \lambda_0}$ is a minimal almost *log*-euclidean chain extending $(A_k)_{k \leq \lambda_0}$ whose first element is $A_0 \cup \{b\}$. Hence by Fact 2.2, $(B_{k+k_0})_{k \leq \lambda'}$ is *log*-euclidean and verifies the other parts the conclusion.

CASE 2 : $\delta(I, A_{i_{\lambda_0+5}}) > 0$.
Suppose $\mathfrak{T}_{pr} = \{x\}$ and $\theta = 0$. It is clear that the polynomials in $\mathfrak{T}_{pr}$ have values in A_0 on θ and with coefficients in A_0.
By Fact 2.11 since $|\beta| \leq |a|^{2^{k_0}}$, if we show that there exists $b \in]\beta - 1, \beta]$ such that $\neg\, 0 < \|P(b)\| < 1$ for all $P \in \mathfrak{T}_{\lambda_0}$ of degree non zero, then we have : the chain $B_i = \mathfrak{T}_{i+k_0}(b)$, $i \leq \lambda'$, is *log*-euclidean and verifies the Theorem's conclusion.
Observe that hypothesis $\delta(I, A_{i_{\lambda_0+5}}) > 0$ implies $\delta(I, A_{i_{\lambda_0+4}}) > |a|^{2^{i_{\lambda_0+3}}}$; then by First Measure Lemma 2.9 : $mes\{x \in I\,;\ \|P(x)\| \leq 1\}^{\mathfrak{d}} < \frac{1}{|a|^{2^{i_{\lambda_0}}}}$, where $\mathfrak{d}$ is the degree of P.
And since $mes\{x \in I\,;\ 0 < \|P(x)\| < 1\} \leq mes\{x \in I\,;\ \|P(x)\| \leq 1\}$ and $\mathfrak{d} < 2^{\lambda_0}$, then $mes\{x \in I\,;\ 0 < \|P(x)\| < 1\} < \frac{1}{|a|^{2^{i_{\lambda_0}-\lambda_0}}}$.
In addition, by Fact 2.7, $card\, \mathfrak{T}_{\lambda_0} \leq (card\, A_{\lambda_0})^{2^{2\,\lambda_0}} \leq |a|^{2^{(C_0+2)\,\lambda_0}}$.
Since $mes \bigcup_{P \in \mathfrak{T}_{\lambda_0}} \{x \in I\,;\ 0 < \|P(x)\| < 1\} \leq \sum_{P \in \mathfrak{T}_{\lambda_0}} mes\{x \in I\,;\ 0 < \|P(x)\| < 1\}$.
Then
$$mes \bigcup_{P \in \mathfrak{T}_{\lambda_0}} \{x \in I\,;\ 0 < \|P(x)\| < 1\} \leq \frac{card\, \mathfrak{T}_{\lambda_0}}{|a|^{2^{i_{\lambda_0}-\lambda_0}}} \leq \frac{1}{|a|^{2^{i_{\lambda_0}-2\lambda_0}}} < \frac{1}{2}.$$
Hence $mes\{x \in I\,;\ \bigwedge_{P \in \mathfrak{T}_{\lambda_0}} \neg\, 0 < \|P(x)\| < 1\} \geq \frac{1}{2}$. And so there is $b \in]\beta - 1, \beta]$ such that $\neg\, 0 < \|P(b)\| < 1$, for all P in $\mathfrak{T}_{\lambda_0}$.

Integer Part Lemma 2.12 □

Division Lemma 2.13 *Let $a \in \mathcal{N} - \mathbb{N}$ and let $(A_i)_{i \leq \lambda}$ be log−euclidean such that $A_0 \ni a$ and for some $C_0 \in \mathbb{N}$: card $A_k < |a|^{2^{C_0\,k}}$ for all $k < \lambda$,*

and $|a|^{2^{\omega\lambda}} < a$.
Then there is $r \in \mathcal{Q}$ *and* $(B_i)_{i \leq \lambda'}$ *a log−euclidean chain, extending* $(A_i)_{i \leq \lambda'}$*, such that* $B_0 \supset \{r, \dfrac{a}{r}\}$*,* $\lambda' > \sqrt{\frac{\lambda}{2}}$*,* r *is* **odd** *(in* $B_{\lambda'}$*) and for some* $C_1 \in \mathbb{N}$*, card* $B_k < |a|^{2^{C_1 k}}$ *for all* $k < \lambda'$*.*

□ **Division Lemma 2.13** :
Set $\mathfrak{T}_{pr} = \{x\,;\ \dfrac{a}{x}\}$ and $\theta = 1$. It is clear that the polynomials in $\mathfrak{T}_{pr}$ have values in A_0 on θ and with coefficients in A_0.
Let λ' be such that $i_{\lambda'+4} \leq \frac{2\lambda}{3} < i_{\lambda'+5}$. It is clear that $\lambda' > \sqrt{\frac{\lambda}{2}}$ and $i_{\lambda'+5} < \lambda$.
We are going to show that there is $d < a$ such that for all $P \in \mathfrak{T}_{\lambda'}$ we have : $\neg\, 0 < \|P(d)\| < 1$. Then by Fact 2.11 since $|d|$ and $|\dfrac{a}{d}|$ are $\leq |a|$, then $B_i = \mathfrak{T}_i(d)$ $(i \leq \lambda')$ is a *log*-euclidean chain and verifies the conclusion of the Theorem.
Observe that for $P = x$, $P(1) \in A_0$, $\lfloor \dfrac{P(1)}{2} \rfloor_{A_1} = \lfloor \dfrac{1}{2} \rfloor_{A_1} = 0 = \dfrac{P(1)}{2} - \dfrac{\nu}{2}$, with $\nu = 1$. By definition, $\lfloor \frac{P}{2} \rfloor \underset{def}{=} \frac{P}{2} - \frac{\nu}{2}$, hence $\lfloor \frac{x}{2} \rfloor \underset{def}{=} \frac{x}{2} - \frac{1}{2}$. Since $\lfloor \frac{x}{2} \rfloor$ is $\mathfrak{T}_1$, $\lfloor \frac{x}{2} \rfloor(d) \in B_1$, so $\frac{d}{2} - \frac{1}{2} \in B_1$, that is to say d is odd.

Construct a divisor r of a.

Since $|a|^{2^{\omega\lambda}} < a$ there is α non standard, such that $|a|^{2^{2\,\alpha\lambda}} < a$.

<u>**Fact**</u> : There is an interval $I = [b, b+1[$, $b < |a|^{2^{\alpha\lambda}}$, such that $\delta(I, A_{\lambda-1}) > 0$.

□ **Fact**
Otherwise, for all $b \in \mathcal{N}$, $b < \frac{|a|^{2^{\alpha\lambda}}-1}{2}$, the interval $[2\,b, 2\,b+1[$ contains an element of $A_{\lambda-1}$. But these intervals are disjoint, and A_λ discrete. Then *card* $A_{\lambda-1} > \frac{|a|^{2^{\alpha\lambda}}-1}{2}$, contradicting the fact that *card* $A_{\lambda-1} < |a|^{2^{C_0\,\lambda}}$, where C_0 is in $\mathbb{N}$.

Fact □

Let $I = [b, b+1[$ be an interval given by this Fact. Let $P \in \mathfrak{T}_{\lambda'}$ be of degree $\mathfrak{d}$.

Let $\mathfrak{T}^0_{\lambda'}$ be the set of elements of $\mathfrak{T}_{\lambda'}$ which are polynomials in x, and let $\mathfrak{T}^1_{\lambda'}$ be the set of those containing some monomial in $\frac{1}{x}$.

CASE 1 : P is a polynomial in x.

It is clear that by constuction of I, $\delta(I, A_{i_{\lambda'+5}}) > 0$. This implies $\delta(I, A_{i_{\lambda_0+4}}) > |a|^{2^{i_{\lambda_0+3}}}$, and by First Measure Lemma 2.9,

$$mes\{x \in I\,;\ \|P(x)\| \leq 1\}^{\mathfrak{d}} < \frac{1}{|a|^{2^{i_{\lambda'}}}}$$

And since $mes\{x \in I\,;\ 0 < \|P(x)\| < 1\} \leq mes\{x \in I\,;\ \|P(x)\| \leq 1\}$ and $\mathfrak{d} < 2^{\lambda'}$, then $mes\{x \in I\,;\ 0 < \|P(x)\| < 1\} < \frac{1}{|a|^{2^{i_{\lambda'}-\lambda'}}}$.

In addition, by Fact 2.7, $card\,\mathfrak{T}^0_{\lambda'} < card\,\mathfrak{T}_{\lambda'} \leq (card\,A_{\lambda'})^{2^{2\,\lambda'}} \leq |a|^{2^{(C_0+2)\,\lambda'}}$.

And since $mes \bigcup_{P \in \mathfrak{T}^0_{\lambda'}} \{x \in I\,;\ 0 < \|P(x)\| < 1\} \leq \sum_{P \in \mathfrak{T}^0_{\lambda'}} mes\{x \in I\,;\ 0 < \|P(x)\| < 1\}$.

Then $mes \bigcup_{P \in \mathfrak{T}^0_{\lambda'}} \{x \in I\,;\ 0 < \|P(x)\| < 1\} < \dfrac{card\,\mathfrak{T}^0_{\lambda'}}{|a|^{2^{i_{\lambda'}-\lambda'}}} \leq \dfrac{1}{|a|^{2^{i_{\lambda'}-2\lambda'}}} < \dfrac{1}{4}$.

CASE 2 : P contains at least one monomial in $\frac{1}{x}$.

By Second Measure Lemma 2.10, $mes\{x \in I\,;\ \|P(x)\| \leq 1\}^{\mathfrak{d}} < \dfrac{(b+1)^{2^{\lambda'+2}}}{a^{\frac{1}{5}}}$.

Since $(b+1)^{2^{\lambda'+2}} \leq |a|^{2^{(\alpha+1)\lambda}}$ and $\mathfrak{d} < 2^{\lambda'} < 2^{\lambda}$,

$$mes\{x \in I\,;\ 0 < \|P(x)\| < 1\} < \frac{|a|^{2^{(\alpha+1)\lambda}}}{a^{\frac{1}{2^{\lambda+3}}}}.$$

And in same way as above :

$$mes \bigcup_{P \in \mathfrak{T}^1_{\lambda'}} \{x \in I\,;\ 0 < \|P(x)\| < 1\} < \frac{card\,\mathfrak{T}^1_{\lambda'}\,|a|^{2^{(\alpha+1)\lambda}}}{a^{\frac{1}{2^{\lambda+3}}}} \leq \frac{|a|^{2^{(\alpha+1)\lambda+1}}}{a^{\frac{1}{2^{\lambda+3}}}} < \frac{1}{4}.$$

(Use the fact $|a|^{2^{2\,\alpha\lambda}} < a$.)

Hence, Cases 1 and 2 give : $mes \bigcup_{P \in \mathfrak{T}_{\lambda'}} \{x \in I\,;\ 0 < \|P(x)\| < 1\} < \dfrac{1}{2}$.

Then $mes\{x \in I\,;\ \bigwedge_{P \in \mathfrak{T}_{\lambda'}} \neg\, 0 < \|P(x)\| < 1\} \geq \frac{1}{2}$. And hence, there is $r \in I$ such that $\neg\, 0 < \|P(r)\| < 1$, for all P in $\mathfrak{T}_{\lambda'}$.

Division Lemma 2.13 $\square$

We next apply the constructions of a model inside a resource in depth D_d made in §1, when $d^{\omega} < |a|_2$ and $D_d(a, c)$ is replaced by a log-euclidean chain $(B_i)_{i \leq d}$ as obtained in the preceding lemmas. For any cut $I < d$ we set $B_I = a^{|a|^{\omega}} \cap (\cup_{i<I} B_i)$ equipped with the operations $+, \times, |x|, 2^{|x||y|}$ induced by $\mathcal{Q}$ and $\mathcal{N}$.

Lemma 2.14 *Let* $(B_i)_{i<d}$ *be a log-euclidean chain with* $d < |a|$

a) If B_0 *has a counterfeiht element* $r \notin \mathcal{N}$*, then* B_I *does* **not** *satisfy "* $\forall i < |r| \; \lfloor r/2^i \rfloor$ *exists ".*

b) B_I *satisfies Basic +* $\forall x, y \; \lfloor y/|x| \rfloor$ *exists +* $\Sigma_0^b - ind^{|x|}$*. And if* $\ell^\omega < d$*, we can choose* I *so that* B_I *satisfies* $\Sigma_1^b - ind^\ell$ *; in particular if* $|a|_3^\omega < d$ *we can ensure* $\Sigma_1^b - ind^{|x|_3}$*.*

c) If $I = \bar{\nu}_\omega$ *for some 2-indiscernible sequence* $\bar{\nu}_\omega$ *over* $|\ell|$ *then* B_I *satisfies* $\Sigma_2^{b'} - ind^\ell$*. Hence if* $\exp(\tau^\omega) \prec |x|_2$ *we can ensure* $\Sigma_2^{b'} - ind^\tau$*.*

□ **Lemma 2.14**

(a) Assume B_I satisfies " $\forall i < |r| \; \lfloor r/2^i \rfloor$ exists ". Then there is a sequence $(q_i, r_i)_{i<|r|}$ coded in $\mathcal{N}$ with q_i and r_i in B_{d-3}, such that $r = q_i 2^i + r_i$ and $0 \leq r_i < 2^i$.
And in $\mathcal{N}$ we can prove that $r_i \in \mathcal{N}$ by induction on $i < |r|$. In particular $r \in \mathcal{N}$, contradicting the counterfeiht character of r. Indeed :

– For $i = 0$, $r = r2^0 + 0$, then $r_0 = 0$ in $\mathcal{N}$.
– Assume $r_i \in \mathcal{N}$.
We have $r_{i+1} - r_i = (q_i - 2q_{i+1})2^i$.

> Case 1 : $r_{i+1} < 2^i$. Then $-1 < q_i - 2q_{i+1} < 1$. B_d is discrete, hence $q_i = 2q_{i+1}$, $r_{i+1} = r_i$ which are in $\mathcal{N}$.
>
> Case 2 : $r_{i+1} \geq 2^i$. Then $0 \leq q_i - 2q_{i+1} < 2$. B_d is discrete, then $q_i = 2q_{i+1}$ or $q_i = 2q_{i+1} + 1$. $r_i \in \mathcal{N}$ assumed, then in the first case $r_{i+1} = r_i$ is $\mathcal{N}$, in the second case $r_{i+1} = r_i + 2^i$ is in $\mathcal{N}$, since $r_i \in \mathcal{N}$ assumed, and $2^i \in \mathcal{N}$.

This proved $r_i \in \mathcal{N}$, for all $i < |r|$.
Hence, for $i = |r| - 1$, since $2^{|r|-1} \leq r < 2^{|r|}$, then $r = 2^{|r|-1} + r_{|r|-1}$ and in this way $r \in \mathcal{N}$ - contradicting the hypothesis $r \notin \mathcal{N}$.

(b) Clearly B_I is log-euclidean, that is, satisfies $\forall x \forall y \; \lfloor y/|x| \rfloor$ exists. It satisfies *Basic* - the one delicate axiom is : $\forall x \; |2x + 1| = |x| + 1$.
Let $x \in B_I$. Then $2^{|x|} \in B_I$. B_I is discrete, $\neg 0 < 2^{|x|} - x < 1$, hence $|2x + 1| = |x| + 1$ by **Fact 2.1** and the axiom is true.

In addition B_I satisfies $\Sigma_0^b - least^{|x|}$. For the interpretability of $\mathcal{Q}$ in $\mathcal{N}$ implies that the whole chain B_d is coded in $\mathcal{N}$. This implies that the simple diagram of the structure B_I is the restriction of a set coded in $\mathcal{N}$. Then the fact that B_I and $\mathcal{N}$ coincide up to $|a|^\omega$ implies that every Σ_0^b formula with parameters ϕ interpreted inside B_I has its restriction X to $|a|^\omega$ that is coded in $\mathcal{N}$. Hence X has a least element which inside B_J is *least* $x : \phi(x)$. The remaining results are proved just as in §1.

(c) Since the chain B_d is coded in $\mathcal{N}$ it is easy to prove that every $E_1(\Sigma_1^b)$ formula interpreted in B_I is equivalent to a Σ_2^I formula. Then satisfaction of $E_1(\Sigma_1^b) - ind^\ell$ in B_I follows from $\Sigma_2^I - ind^\ell$. The remaining results are proved just as in Ch 1.

Lemma 2.14 □

Theorem 2.15 *Let T (resp.: T') denote $\Sigma_0^b - ind^{|x|} + \forall x \forall y \lfloor x/|y| \rfloor$ exists, extended by $\Sigma_1^b - ind^\tau$ (resp.: by $\Sigma_2^{b'} - ind^\tau$).*

a) If $\tau^\omega < |x|_2$ then T does **not** *prove any of the following statements :*

1. *every divisor of a power of 2 is itself a power of 2*
2. *$\lfloor x/2^i \rfloor$ exists for every $i < |x|$*
3. *S_2^1 and EBASIC*
4. *the set of primes is NP*
5. *NP=co-NP.*

b) If $\exp(\tau^\omega) < |x|_2$ then T' does not prove the above statements, nor for **any** *$p < \omega$:*
6. $\Sigma_p^b = \Sigma_{p+1}^b$.

□ **Theorem 2.15**
(b) Let d be as in above Lemma 2.14, apply the Kernel Lemma 2.3 to get a chain $(A_i)_{i \le \lambda}$ with $\lambda = d^2$ and apply to it Division Lemma 2.13 to get a log-euclidean chain $(B_i)_{i \le d}$ which includes $a, r, a/r$ for some $r \in \mathcal{Q}$. By the preceding Lemma 2.14 we obtain a model M of T' such that $a, r, a/r \in$M.

1. If we took a to be a power of 2 in $\mathcal{N}$, the odd divisor r shows that M falsifies statement (1)

2+3. By Lemma 2.14.(a), M falsifies (2), hence (3) since S_2^1 implies (2) (as well as (1)), and since EBASIC implies (2).

4. Let $\varphi(u) = \exists v < t(u)\varphi_0$ be any Σ_1^b formula where φ_0 is Σ_0^b, and assume that a is prime in N. If $\neg\varphi(a)$ holds, even Peano's Arithmetic does not prove that $\varphi(u)$ defines primality. So assume $\exists v < t(a)\varphi_0(a)$ and that the set B_0 was chosen so as to involve b such that $\varphi_0(a, b)$ holds. No longer is a

prime inside M, yet b witnesses that $\varphi(a)$ holds.

Theorem 2.15 $\square$

5. Since M falsifies S^1_2 it also falsifies $\Sigma^b_1 = \Sigma^b_2$ because otherwise $\Sigma^b_1 - ind^{|x|_3}$ (which holds in M if we took $\tau \geq |x|_4$) would imply S^1_2.

6. The next section's T 3.4 constructs a model M which in addition to the properties used in (5) satisfies for each $p < \omega$ some amount of Σ^b_p-induction; then one argues as in (5) but one also takes advantage of this additional property of M.

(a) Similar proof, only simpler.

Theorem 2.15 $\square$

3 Iterated resources in depth

By abuse of notation we speak of a resource D_d, to mean a resource of depth d, $D_0 \subset \ldots \subset D_d$. We consider the resource B_d and the model B_I used in the last proof of the last theorem ; they were constructed inside $\mathcal{N}$, now we want to make a similar construction inside B_I. This is impossible as long as $B_I \subset a^{|a|^\omega}$, because an element c above this cut was indispensible. So we make a preliminary modification of B_d, B_I - but a mild one: we choose $r \in \mathcal{N}$ infinite but arbitrarily small compared to a, c, d ; we include the set $|a|^r$ in B_0, accordingly changing the B_i's and we set $B_I = a^{|a|^{r\omega}} \cap (\cup_{i \in I} B_i)$. This new B_I is included in the old B_d hence it is discrete ; it satisfies $\Sigma^b_0 - least^{|x|}$ by the same proof as in L 2.19.b ; it has the same counterfeit pathology as the former B_I because the old set B_0 is included in the new one ; and it satisfies $E_1(\Sigma^b_1 - ind^\tau)$ by the same proof as before. In addition it has an element above $a^{|a|^\omega}$.

Definition - A **Γ-resource** of **nonstandard** depth d is a coded chain $D_0 \subset \ldots \subset D_d$ such that $a \in D_0$ and that for all cuts $J < J' < d$, $D_J := \cup_{j<J} D_j \cap a^{|a|^{r\omega}}$ is a Γ **(-elementary)** submodel of $D_{J'}$; if $d - \omega$ denotes the largest cut $< d$ it amounts to say that D_J is a Γ submodel of $D_{d-\omega}$ for every cut $J < d$.

Example
a) B_d is a nonstandard Σ^b_0-resource of depth d: for every cut $I < d$, B_I is a Σ^b_0 submodel of $B_{d-\omega}$ since $|a|^{r.\omega} \subset D_{d-\omega}$. The same applies to the resource $D_d(a,c)$ of §1 if we modify the definition of $D_I(a,c)$ as done above for B_I so that $D_I(a,c)$ has an element $c' > a^{|a|^\omega}$.

b) We let $\bar{\varphi}_r$ be an $\mathcal{N}$-coded sequence such that $\bar{\varphi}_\omega$ enumerates all standard Σ^b_n formulas $\varphi = \varphi(x,u)$. Define D_i by $D_0 := \ |a|^r \cup \{a\}$, $D_{i+1} := \ c \cap$

[$\mathcal{L}_B D_i \cup \{f_e(x);\ x \in D_i,\ e < r\}$] where f_e is a **witnessing function** for (φ_e *and* $u < c$), that is: $(\exists u < c\varphi_e(x,v))$ implies $f_e(x) < c$ and implies $\varphi_e(x, f_e(x))$. Then D_d is a Σ^b_n-resource since D_I is a Σ^b_n submodel of $\mathcal{N}$ for every cut $I < d$.

Definition We call an integer C a code **for the Γ-diagram of M restricted to X**, if the restriction of the set coded by C to the standard sentences of Γ with parameters from X precisely consists of the sentences of this kind that hold in the structure M.
The above D_d is a resource such that D_I is for any cut $I < d$ a Σ^b_n submodel of $\mathcal{N}$ and not only of $D_{d-\omega}$; in such a case we speak of a Σ^b_n-resource **with respect to** $\mathcal{N}$. Then clearly the Σ^b_n diagram of $D_{d-\omega}$ – which is also the Σ^b_n-diagram of $\mathcal{N}$ restricted to $D_{d-\omega}$ – is coded in $\mathcal{N}$. This allows D_I to satisfy induction for Σ^b_n formulas (and more than that).

D_d is not counterfeit and this greatly limits usefulness (beyond giving an instructive example) ; in order to have counterfeit analogs we have to imitate the construction of D_d when the ground model $\mathcal{N}$ is replaced by a counterfeit one M such as $M = B_J$ from § 2. This is done by L 2 below, using two ideas to make up for the weakness of M:
- to construct Φ-resources for every **finite** subset Φ of Σ^b_{n+1}, and then to go by resplendence from these Φ-resources to a Σ^b_{n+1} one
- to construct Σ^b_{n+1}-resources that **relative to** M are **only** Σ^b_n-resources.

We always assume that M is a model of *Base* with operations $+, \times, |x|, 2^{|x||y|}$ induced by those of $\mathcal{Q}$ (where $|x|$ is defined by: $|x| \in \mathcal{N}$ *and* $2^{|x|-1} \leq x < 2^{|x|}$) ; in addition M has elements $a, c' > a^{|a|^\omega}$ and is included in some $\mathcal{N}$-coded discrete subset of $\mathcal{Q}$. Example: $M = B_I$.

Proposition 3.1 *Assume that* $\ell \subset M$ *which satisfies* $E_1(\Sigma^b_n) - ind^\ell$ *for some* $n, 0 < n < \omega$
a) Inside M *the* Φ*-diagram of* M *restricted to* X *is coded, for every finite subset* Φ *of* Σ^b_n *closed under subformulas and for any set* X *bounded by* a *and coded by a sequence of length*$< \ell^\omega$.
b) For every $C \in M$ *which inside* M *codes a sequence of length* ℓ *there is* C' *which inside* $\mathcal{N}$ *codes the same sequence ; and if inside* M *the elements* C_i *coded by* C *are themselves codes of sequences of length* $\leq \ell$ *then there is* C'' *which inside* $\mathcal{N}$ *codes the same sequence of sequences.*
c) $E_1(\Sigma^b_n) - ind^\lambda$ *implies for each* $\ell < \lambda$ *and each* C *the following form of the pigeonhole principle:* $[\forall i < \ell\ (C)_i < \ell - 1] \longrightarrow [\exists i < j < \ell\ (C)_i = (C)_j]$.

□ P 1
(a) Consequence of $E_1(\Sigma^b_n)$-induction on the cardinal of X.
(b) We only have to show the definability (in $\mathcal{N}$) of the sequence coded by C; remember that ℓ is the same set in M as in $\mathcal{N}$. And that $(C)_i := \lfloor C/2^{|a|i} \rfloor - \lfloor C/2^{|a|(i+1)} \rfloor 2^{|a|}$ is definable in M from C, i, a, ℓ. Given $x, y \in M$

if $\lfloor x/y \rfloor$ exists in M then it is definable in $\mathcal{N}$ as the unique element z of D such that $yz \leq x < y(z+1)$. Thus the definition of $(C)_i$ in M can be translated in $\mathcal{N}$. The other part has a similar proof.
(c) The classical proof is by induction on ℓ , and for $\ell < \lambda$ it is naturally formalized in $E_1(\Sigma_n^b) - ind^\lambda$.

P 1 □

Lemma 3.2 *Under the assumptions of P 1 if $exp_2(d.\omega) < \ell$ then for every finite subset Φ of Σ_{n+1}^b, for every subset $A_0 < a$ of M which inside M is coded by a sequence of length$< exp_2(d.\omega)$:*
a) there is a Φ-resource $A_d \in M$ which w.r.t. M is a Φ^--resource, Φ^- being the restriction to $\Sigma_n^b \cup \Pi_n^b$ of Φ.
b) In addition if $\beta \in A_0$ and M satisfies $\Phi - least^\beta$ we can ensure that A_I satisfies $\Phi - least^\beta$ for any cut $I < d$.
c) If $\ell' \in N, \ell' \subset A_0$ and J is the sup of a subsequence of d which in $\mathcal{N}$ is 2-indiscernible over $|\ell'|$ then A_J satisfies $E_1(\Sigma_1^b(\Phi)) - least^{\ell'}$ – hence an appropriate choice of Φ can ensure that A_J satisfies any given finite part of $E_1(\Sigma_{n+2}^b) - least^{\ell'}$.

□**L 2**

In M we fix $c' > a^{|a|^\omega}$ and we call **evaluation** any coded map $w : T \longrightarrow W$ where $T \subset T_d$ and $W \subset c'$, such that
i) T is closed under subterms and $w(t_i) = w(s_i)$ for $i = 1, 2$ implies $w(f(t_1, t_2)) = w(f(s_1, s_2))$
ii) $w(b) = b$ for each $b \in A_0$; $w(t + t') = max(c' - 1, w(t) + w(t'))$ and similarly for the other operations of $\mathcal{L}_B$
iii) $e \leq s \longrightarrow \varphi_e(w(f_e(t), w(t))$ is satisfied (in M)
iv) $s < e \leq s' \longrightarrow w(f_e(t)) \in W$ and $\varphi_e(w(f_e(t)), w(t))^W$ is satisfied – where φ^W denotes the formula φ in which the outermost existential quantifiers are relativized to the set W ; agreeing that a quantifier $\exists$ of the prefix is "outermost" if every universal quantifier which precedes $\exists$ is sharply bounded.

Note that an evaluation is always partial on T_d because the conclusion of (iii,iv) cannot be always satisfied, and if it is failing the term $f_e(t)$ cannot belong to T. Assume that $t_1, t_2 \in T \cap T_{d-1}$ but $t_1 + t_2 \notin T$; let w' denote the map extending w to $t_1 + t_2$ with $w'(t_1 + t_2) := w(t_1) + w(t_2)$. Then w' still is an evaluation: (i) to (iii) clearly continue to be satisfied by w' ; and so does (iv) if we observe that $\varphi_e(w(f_e(t), w(t))^W$ implies $\varphi_e(w(f_e(t), w(t))^{W'}$ for any extension W' of W (because only outermost existential quantifiers are relativized to W). This also works if we replace the operation $+$ on t_1, t_2 by any other operation $f \in \mathcal{L}_\mathcal{B}$; it shows that if the evaluation w happens to be **maximal** (w.r.t. inclusion) and if we set $A_i = im\ w|T_i$ then A_d is a resource of depth d. In this argument we can also replace the term $t_1 + t_2$ by $f_e(t)$ with

$e \leq s$ provided we set $w'(f_e(t))$ equal to an $b < c'$ such that $\varphi_e(b, w(f_e(t))$ is satisfied ; it shows that for every cut $I < d$ the formulas $\varphi_e, e \leq s$ are witnessed in A_I as soon as they are witnessed in M. In addition A_I is a model of *Base* since A_d is a resource and this implies for $\bar{a} \subset c'$ and $B=$ the code of $\bar{a}$ that $B \in A_I$ implies $\bar{a} \subset A_I$. Thus the translation of each $\varphi' \in \Phi^-$ by some $\varphi_e, e \leq s$ is valid inside A_I and Φ^- too is witnessed in A_I just as in M. Thus (for maximal w) A_I is a Φ^- submodel of M and the above resource A_d is a Φ^--resource w.r.t. M. Finally, a similar argument with $e > s$ shows that A_d is also a Φ-resource. Towards a proof of (a) by contradiction we assume that there is no resource A_d with all these properties - hence every evaluation w has a strict extension w', starting with $w_0 :=$ identity over A_0. Then there is a strictly increasing sequence of evaluations of length ℓ: this follows from $E_1(\Sigma_n^b) - DC^\lambda$ using P 0.4.a, because the above definition of an evaluation is easily seen to be a formula of $E_1(\Sigma_n^b)$ extended by "small" quantifiers, $\forall i < \ell^k$. Since T_d has an enumeration of length $< \ell$ it contradicts the pigeonhole principle (P 1.c). This contradiction ends the proof of (a). The same proof also establishes (b) if in case that $e \leq s$ and that M satisfies $\varphi_e - least^\beta$ the condition (iii) on w is set to require that $w(f_e(t))$ is the **canonical** witness *least* $u < \beta : \varphi_e(u, w(t))$.
We prove (c): from the resource A_d we obtain by P 1.(b+c) a code $C \in N$ of the Φ-diagram of $A_{d-\omega}$, because for $\varphi(\bar{u}) \in \Phi$ and $\bar{a} \subset A_{d-\omega}$ this diagram contains $\varphi(\bar{a})$ iff M satisfies $\varphi(\bar{a})^{A_d}$. Then Σ_2^I formulas define with the help of the parameter C the interpretation in $A_{d-\omega}$ of all $\Sigma_{n+1}^b(\Phi)$ formulas ; so that the argument of § 1 which proves that $D_J(a, c)$ satisfies $E_1(\Sigma_1^b) - ind^\ell$ here proves $E_1(\Sigma_1^b(\Phi)) - ind^{\ell'}$ to hold in A_J.

From a model with (some) $E_1(\Sigma_n^b)$-induction the lemma allows to deduce a model with $E_1(\Sigma_{n+1}^b)$-induction ; thus ω iterations of the lemma starting with $M := B_I$ (from § 2) yield models with $E_1(\Sigma_n^b)$-induction for all $n < \omega$. But there are some delicate points :
1. the current model M of these iterations may be too weak to construct 2-indiscernibles inside it ; the solution is to have A_0 contain an initial segment ℓ of $\mathcal{N}$ and to construct **inside** $\mathcal{N}$ an indiscernible subsequence of ℓ
2. This being decided, we have two opposite constraints to conciliate because on the one hand, the maximum size which the initial resource A_0 can have decreases at each iteration ; and on the other hand, we just decided to put a whole initial segment ℓ of $\mathcal{N}$ in A_0, of respectable size. We have to ensure the compatibility of these two constraints ; this is done by the Fact below.
3. Here is another difficulty: starting from a model M of $\theta_n := E_1(\Sigma_p^b) - ind^{\ell_p}$ for suitable values (depending on n) of p and ℓ_p, part (c) of the lemma provides a model M' of θ_{n+1} but θ_{n+1} does not imply θ_n (indeed, θ_{n+1} is induction for stronger formulas, yet on a much smaller segment than θ_n). So this new model M' is stronger than M in some respects but weaker in others - making it unclear how to keep only benefits from its construction. The

solution is to consider the scheme $\alpha_n := \Sigma^b_p - ind^{exp(\ell_p)}$; α_n is a consequence (by "divide and conquer") of θ_n which being of smaller complexity goes over from M to M' - with the help of the lemma (b). Thus iteration n times of the lemma will provide a model of $\alpha_1 + ... + \alpha_n$ although it does not provide one of $\theta_1 + ... + \theta_n$. Note that α_{n+1} does not imply α_n, and its truth inside M is not strong enough to allow for the construction of a model M' of θ_{n+1} nor α_{n+1}. Thus we need to go through the θ_n's, although only α_n's occur in the final result – T 4 below. Also we need § 1's construction of M from indiscernibles and not only the pigeonhole principle, in order to start from $\theta_1 := E_1(\Sigma^b_1) - ind^{\ell}$; and we need the construction in lemma (a) of a Σ^b_{n+1} resource, not only of a Σ^b_n resource w.r.t. M....

Fact 3.3 *If $exp(\tau^\omega) < |x|_2$ then inside $\mathcal{N}$ we can find a sequence $(\ell_n, d^n)_{n<\omega}$ so that $\tau(a)^\omega < d^0, d^{1^\omega} < |a|_2, exp(\ell_n^\omega) < d^n, exp_2(d^{n+1}.\omega) < \ell_n$ and $\ell_n > |a|_n + 3$.*

□ **F 3** For fixed $n, K < \omega$ it is easy to satisfy the inequalities from 0 to n in which the exponent ω is replaced by K ; hence by overspill on K it remains possible to find l_i, d^i satisfying the same inequalities for some $K > \omega$. In such a case the original inequalities (:= with ω in place of K) are satisfied ; finally overspill on n allows to satisfy all inequalities at once.

F 3 □

We fix a sequence $(\ell_n, d^n)_{0<n<\omega}$ as in the Fact; by § 2 there is a discrete resource B_d such that $d = d^0$ and $|a|$ is included in B_0 which introduces some counterfeit pathology: say $2^{|a|}, r, 2^{|a|}/r$ belong to B_0 while $r/2$ does not belong to B_d. By induction on $n < \omega$ we show the existence of a model $M_n \subset B_d$ such that d^n is initial segment of M_n which also has $\ell_i, i \leq n$ as elements and which satisfies $Base + \Sigma^b_0 - least^{|a|} + \Sigma^b_{2i-1} - least^{\ell_i}$ $(0 < i < n) + E_1(\Sigma^b_{2n-1}) - ind^{\ell_n}$; for $n = 0$ we take $M_n = B_J$ as constructed in § 2. Inductively assuming that M_n is constructed for some n we let M denote M_n and we apply the preceding lemma (a) with n, ℓ, d, replaced by $2n-1, \ell_{n+1}, d^{n+1}$ and with $A_0 := \{a, r, 2^{|a|}/r\} \cup d^{n+1} \cup F$ where F is any finite subset of M including $\ell_i, i < n$. We obtain a Φ resource $A_{d^{n+1}}$ which w.r.t. M is a Φ^- resource ; this for any finite $\Phi \subset \Sigma^b_{2n+1}$. Using the resplendence of $N|\beta$, from the existence of $A_{d^{n+1}}$ for all these sets Φ we can infer the existence of $A_{d^{n+1}}$ even when $\Phi = \Sigma^b_{2n+1}$ - so that $A_{d^{n+1}}$ is a full Σ^b_{2n} resource which w.r.t. M is a Σ^b_{2n-1} resource. Now we apply (c) of the same lemma ; or rather, we apply its proof in case Φ is the whole infinite set Σ^b_{2n+1} - so that $\Phi^- = \Sigma^b_{2n}$: it yields a cut $J < d^{n+1}$ such that A_J satisfies $\theta_{n+1} := E_1(\Sigma^b_{2n+1}) - ind^{\ell_{n+1}}$; we apply in addition (b), observing that M satisfies $\alpha_n := \Sigma^b_{2n-1} - least^{exp(\ell_n^\omega)}$: it yields that $A_J =: M_{n+1}$ still satisfies α_n. In addition M_{n+1} satisfies $Base + \Sigma^b_0 - least^{|a|} + \Sigma^b_{2i-1} - least^{\ell_i} (0 < i < n)$ since M satisfies this Σ^b_{2n} theory and since A_J is a Σ^b_{2n} submodel of M (note

that this ultimate point still adds an idea not mentioned in the preliminary sketch we made before F 3).
Thus M_n exists for each $n < \omega$ by induction on n ; using again the resplendence of $N|\beta$ it implies the existence of a model M_ω of $Base + \Sigma_0^b - least^{|a|} + \Sigma_1^b - ind^{exp(\tau(a)^\omega)}$ plus (for every $n, 2 < n < \omega$) $\Sigma_{2n-1}^b - least^{exp(\ell_n^\omega)}$. Thus :

Theorem 3.4 *If* $\exp(\tau^\omega) < |x|_2$*, there is* d *such that* $d^\omega < |a|_2$ *and for every finite subset* F *of* B_0 *there is a model* M' *of* $Base + \Sigma_1^b - ind^e xp(\tau^\omega) + \{\Sigma_{2i-1}^b - ind^{|x|_{3i}}; i < \omega\}$ *such that* $\{a\} \cup, F \subset M \subset B_d$.

4 BOOTSTRAPPING (Conclusion)

A. Resources in depth
1. We constructed counterfeit resources in depth which are optimal in two ways: their pathology is maximal; their depth cannot be any larger.
2. Inside these resources we expect to find strong n-indiscernibles which allow Σ_n^I control and thus allow to construct models of $\Sigma_n^{b'} - ind^\tau$ whenever $exp_{n-1}(\tau^\omega) < |x|_2$; such models yield witnessing and independence results for these theories.
3. But here, we proved the results of (2) only in case $n = 1, 2$. In case $n > 2$ the iteration of resources in depth gave results which are weaker than those indicated in (2) - except that they use Σ_k^b in place of $\Sigma_k^{b'}$; and this advantage is not offered by other methods such as the use of strong n-indiscernibles.
4. It is plausible that one can win one more exponential and prove the same independence results for each n, but when the bound $\tau^\omega < |x|_2$ is replaced by $\tau^\omega < |x|_1$. This would be optimal because the results become false for $\tau = x$; but the proof should contain a novel ingredient for otherwise the witnessing results which we obtained by the same method would also be improved – to a point that seems unlikely.
5. A long term dream is to build a resource in depth that is defined not in $\mathcal{N}$ but in some o-minimal non standard structure extending the reals. Then one would benefit of the Vapnik Tchervonenkis property which holds in o-minimal structures. The latter results in a polynomial lower bound for the Ramsey theorem applied to partitions induced by $\mathcal{R}$-definable relations. This could lead to construction of models of full Bounded Induction. We think this perspective can really be adressed if one is willing to use conjectures rather results, on the asymptotic upper bounds of the Vapnik Tchervonenkis property. Taking up such an ambition leads to beautiful problems **simultaneously** about o-minimality and about non standard models of Arithmetic. A precise question to address in order to start research on this theme is - given any formula $\phi(\bar{x}) \in \mathcal{L}$ - whether we can construct a discrete substructure (A, x^y) of a non standard model R_{exp} of T_{exp} such that ϕ_A is the restriction to A of some definable relation of R_{exp}.

B) Non standard resources in general

1. So far we dealt with one kind of resource : resources in DEPTH. Such a resource is a way to build models of the most basic axioms - $BASIC$ in our case. Then combinatorics are used to turn them to models of much stronger axioms (from an algebraical point of view; alas still not from a computational point of view). Quite similar features are satisfied by other kinds of resources. The first example is the Kirby-Paris work called "Indicator theory", connected to Jeff Paris's proof of the Paris-Harrington independence result. Their resource is simply (unary) SIZE : it is a sufficiently large segment of the integers. This gave rise to many variations, always using size as a resource but considerably varying the combinatorics. An example of a different kind is given in Wilkie's proof of the Buss witnessing theorem for Σ_1^b formulas in S_2^1. See also the extension given by Chinchilla [C]. There the resource is TIME : it is all elements computable from a given one a, using programs of essentially standard size and using polynomial or subexponential time (w.r.t. a). Other parameters of a computation lead to related resources which could yield new models and results ; to begin with, SPACE or a combination of space and time ; and also (program) LENGTH: length of the programs used to compute the elements of the resource, when it is not limited to quasi standard lengths as in the Wilkie or Chinchilla examples. These resources yield models of longer induction hence stronger systems than here ; on the other hand and for this very reason they are exclusively true resources, not counterfeit ones. Our use of "parametric Ramsey" is perhaps not applicable to resources in time and space ; but length could work and give a new valuable resource in connection with our indiscernibility arguments.

2. There still is an existing resource to be mentioned : inspired by the Kirby-Paris work, [M-R] made a parallel of Indicator Theory where size is taken in an infinitary sense : large initial segments of $\mathcal{N}$ were replaced by large segments V_α of the universe. The combinatorics used was : an infinitary version of the Paris-Harrington theorem, which is consequence of iterated Mahlo cardinals. The result was philosophically interesting : it constructed a mathematical bridge from the consistency of iterated Mahlo cardinals to that of weakly compact ones. (An unbridgeable gap is undesirable, it would indicate a possible inconsistency already at the weakly compact level). We expect that a parallel use of non standard models of Set Theory could be developed using our parametric indiscernibles and the control it allows. Such a development suggests to take the $L'_\alpha s$ (segments of the constructible universe L) as a resource ; and the Erdos Rado theorem would replace the Ramsey theorem in order to get the indiscernibles. Here it is time to point out that the Erdos Rado theorem has a parametric version exactly as the Ramsey theorem and with the same proof. But we did not work on its use, we do not even know what kind of results could be obtained in this way.

The work of Enayat [E] is also related to [M-R] but develops a connection with another Set Theory : Quine's "New Foundation" NF.
In this set theoretic context the name of Harvey Friedman must be cited. His "true unprovable" results [F] rest on unexplicit and subtle "resources", again in the sense : tools to satisfy the most basic axioms of Arithmetic or of Set Theory (such as the axioms of discrete ordered rings or those of Finite Set Theory). It should be rewarding to make these resources become explicit.

3. The above scattered examples of resources ought to become part of a systematic study of resources and resource management, which among others should say more about a presently mysterious questions : what are all possible resources ?

4. The combinatorics used in connection with resources in depth should be varied systematically (whereas we only used Pigeonhole Principles, and Indiscernibles). This should be imitated with the other arithmetical kinds of resources. Such variations go together with variation of the basic language and axioms (for instance L_B and *Base* can be replaced by their analogs from Rudimentary Arithmetic). The wide work done in the framework of Paris-Kirby Indicator Theory is full of variations of this kind in the case of strong theories. They should be heuristically useful, suggesting imitations in the weak context.

C. Quantifier control and Indiscernibles
Remark : Quantifier control (in the sense of L.1.7 or related senses) is a notion which seems to us to have a future. Note that the case where $N \restriction X$ is a Γ submodel of N for some set Γ of formulas is another instance of quantifier control of a structure S $= (N, X)$: the formula controlling φ is φ itself ! Not a new example but an instructive one. For if the quantifier complexity of Γ is bounded it shows that quantifier control is quite different from quantifier elimination: "elimination" automatically propagates to higher quantifier complexities but not "control". Thus we shall have structures $(\mathcal{Q}, N, J)$ with one part (N, J) that enjoys quantifier control, and the other part $\mathcal{Q}$ which can be assumed to be real closed, hence to enjoy quantifier elimination ...

One may look for the appearance of a new type of quantifier control obtained by use of stability theoretic ideas in place of partition properties. For quantifier control of a structure S by a structure N is about N-definability of S-definable sets. This is dual to S-definability of N-definable sets (restricted to S), which is the central notion of stability (under the name "stable embeddability"). The duality is asymetrical because in Stability one almost always assumes that S is elementarily embedded in N - which is only the routine case of quantifier control. Perhaps we should define stable embeddability in case S is **not** elementarily embedded in N, and relate it to non trivial quantifier control.

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