Abstract. We present a self-contained analysis of some reduction games, which characterise various natural subclasses of the first Baire class of functions ranging from and into 0-dimensional Polish spaces. We prove that these games are determined, without using Martin’s Borel determinacy, and give precise descriptions of the winning strategies for Player I. As an application of this analysis, we get a new proof of Baire’s lemma on pointwise convergence.

1. Introduction.

A Polish space is a separable and completely metrizable topological space. It is 0-dimensional if it admits a countable basis of clopen (i.e. both open and closed) sets (for basic definitions see [1, 2]). Given two Polish spaces $A$ and $B$, a function $f : A \to B$ is Baire class 1 if $f$ is the pointwise limit of a sequence of continuous functions, i.e., if there is a sequence of continuous functions $(f_n : A \to B)_{n \in \omega}$ such that for all $x \in A$ the sequence $(f_n(x))_{n \in \omega}$ converges to $f(x)$. In 0-dimensional Polish spaces, this is equivalent to saying that the inverse image of every open subset of $B$ is some countable union of closed subsets of $A$. The open subsets of $B$ are denoted by $\Sigma^0_1(B)$, and the countable unions of closed subsets of $A$ are denoted by $\Sigma^0_2(A)$.

An important result concerning the first Baire class that we focus on is the Baire lemma on pointwise convergence – known in France as le grand théorème de Baire – which provides a third characterization of the first Baire class: a function $f : A \to B$ is Baire class 1 iff for every non-empty closed subset $P$ of $A$ the restriction $f|_P$ admits a point of continuity.

Although it makes sense to study the Baire class 1 functions on metrizable spaces in general, we restrict ourselves to some framework that allows to mix games and functions; namely the Polish and 0-dimensional spaces. Any such space is homeomorphic to a closed subspace of the space $\omega^\omega$ of infinite sequences of integers, equipped with the product topology of the discrete topology on $\omega$. This space is called the Baire space, and by misuse of language, its elements are called reals. From this point onwards, we only consider functions from a closed subspace of $\omega^\omega$ into $\omega^\omega$.

Given any subset $X$ of the Baire space, the Gale and Stewart game $G(X)$ is an alternated two-Player infinite game with perfect information, that was introduced by Gale and Stewart in the 1950’s [3]. Player $I$ starts, each Player chooses an integer $x_i$, so that they build the real $x = (x_i)_{i \in \omega}$ in $\omega$ steps, and $I$ wins iff $x$ belongs to the payoff set $X$. A celebrated result due to Martin [4] states that if $X$ is Borel then the Gale-Stewart game with payoff set $X$ is determined, i.e. one of the two Players has a winning strategy.

A different way of looking at this game is to consider that the Players are, instead of constructing a single real $x$, building two different reals $x_I$ and $x_{II}$, where $x_I = (x_{2i})_{i \in \omega}$ and $x_{II} = (x_{2i+1})_{i \in \omega}$. In this setting, a strategy for Player $II$ induces, in a canonical way, a Lipschitz function that maps $x_I$ to $x_{II}$, so this game characterizes
the Lipschitz functions with constant 1, in the sense that Player II has a winning strategy iff the complement of the payoff set, considered as a subset of the product space $\omega^\omega \times \omega^\omega$, has a 1-Lipschitz uniformizing function. These latter games are called reduction games. The idea is then to introduce new rules that give more power to Player II, and thus characterise new classes of functions by the existence of strategies for Player II.

Reduction games stand as a classical tool in descriptive set theory, as far as 0-dimensional spaces are concerned. They played a central role in Wadge’s PhD thesis, as the major tool to define what is now called the Wadge hierarchy of Borel sets [5]. They were also thoroughly used by Duparc to get the full analysis of the Wadge hierarchy for Borel sets of finite rank [6], by Semmes who defined new reduction games to get extensions of the Jayne-Rogers theorem [7], by Motto Ros who wrote a general presentation of reduction games [8].

Our major concern with the first Baire class is to find a natural hierarchy for these functions. One way to obtain such a hierarchy is to assign an ordinal to every Baire class 1 function – using for instance one of the three characterizations mentioned above – as did Kechris and Louveau in the context of Banach spaces [9] or Hertling and Weihrauch [10]. In [9] all ordinal assignments for Baire class 1 functions rank on $\omega_1$, but there is in [10] another one that ranks, as we will see, on $\omega_1 + 1$. Refining van Wesep’s backtrack game, Motto Ros in his PhD thesis [11] found a game characterization of every level of the hierarchy induced by this last ordinal assignment. Although in [10] the authors speak of the level of a function $f$ and in [11] Motto Ros talks about the rank of $f$, we refer to the place of $f$ in the hierarchy as the discontinuity of $f$, because the relation to the three different characterizations is clearer this way.

In this paper, which is a continuation of a conference paper [12], we give a new detailed analysis of some classical reduction games that yields both the determinacy of the games at stake and a precise description of the strategies employed by Player I.

More precisely, we first define the three different kinds of games we will play. The backtrack game, due to van Wesep [13], the eraser game, due to Duparc [6] and the $\alpha$-bounded backtrack game, due to Motto Ros [11].

We then prove that all these games are determined. At this point we could have restricted our attention to the sole Borel functions, in order to use Martin’s result on Borel determinacy. However a careful analysis of the bounded backtrack game gives us a more general determinacy result.

**Theorem 1.1.** Given $f : \omega^\omega \to \omega^\omega$ with closed domain, the eraser game (resp. the backtrack game, resp. the $\alpha$-bounded backtrack game) with parameter $f$ is determined.
We will in fact define the stronger notion of aggressive strategy for $I$ and prove the following more detailed result.

**Theorem 1.2.** Fix $f : \omega^\omega \to \omega^\omega$ with closed domain.

If $II$ has no winning strategy in the backtrack game with parameter $f$, then $I$ has an aggressive winning strategy.

2. The games we play.

2.1. Preliminaries.

2.1.1. Trees. Given any non-empty alphabet $\Sigma$: $\Sigma^{<\omega}$ (resp. $\Sigma^\omega$, resp. $\Sigma^{\leq \omega}$) denotes the set of finite (resp. infinite, resp. any) sequences of elements of $\Sigma$. For $u \in \Sigma^{<\omega}$ and $v \in \Sigma^{\leq \omega}$, we write $u \subseteq v$ when $u$ is a strict initial segment of $v$, $u \subseteq v$ if $u \subseteq v$ or $u = v$. Set $[u] = \{x \in \Sigma^\omega : u \subseteq x\}$ for any $u \in \Sigma^{<\omega}$, and write $\varepsilon$ for the empty sequence. Call $u \cdot v$ the concatenation of $u$ and $v$, and $lg(u)$ or $|u|$ the length of $u$. Note also $u \cdot a$ for $u \cdot (a)$. If $u \neq \varepsilon$ then $lg(u) > 0$ and $d(u)$ denotes the last element of $u$, in other words $u(lg(u) - 1)$. When $A \subseteq \Sigma^{<\omega}$ is linearly ordered by $\subseteq$, we denote by $\bigcup A$ the sequence that is the set-theoretic union of $A$. We denote by $u \cap v$ the largest initial segment shared by $u$ and $v$.

A tree $T$ on $\Sigma$ is a subset of $\Sigma^{<\omega}$ closed under subsequences. If $u \in T$ we say that $u$ is a node of $T$. If $v$ in $T$ is equal to $u \cdot a$ for some $a \in \Sigma$ we say that $v$ is a successor of $u$ in $T$. A node with no successor in $T$ is a terminal node, or leaf, of $T$. A node with more than one successor is a branching node, if it has finitely many successors we say it is a finitely branching node. A tree is finitely branching if every branching node in it is finitely branching. Two nodes $u$ and $v$ are compatible if either $u \subseteq v$ or $v \subseteq u$ holds, which is denoted by $u \perp v$, otherwise $u$ and $v$ are incompatible, which is denoted by $u \parallel v$. A set of pairwise incompatible nodes is an antichain. Let $x \in \Sigma^\omega$, if for all $n \in \omega$, $x|_n \in T$ we say that $x$ is a branch of $T$, and we write $[T]$ for the set of all branches of $T$. If for all $t \in T$ there is an $s \in T$ such that $t \subseteq s$ then we say that $T$ is pruned. Given any subset $\Lambda \subseteq \Sigma^{<\omega}$ we can define $T_\Lambda = \{u \in \Sigma^{<\omega} : \forall v \in \Lambda (u \subseteq v)\}$.

Observe that the set $[u]$ for $u \in \Sigma^{<\omega}$ is the set of branches of a pruned tree on $\omega$. A pruned tree has no leaf. Using the axiom of dependent choice, we have that every pruned tree has a branch. The tree $T_\Lambda$ is the smallest tree containing $\Lambda$.

2.1.2. Topology. Given a topological space $X$, we denote by $\Pi^0_1(X)$ (resp. $\Sigma^0_1(X)$) the set of closed (resp. open) subsets of $X$, by $\Sigma^0_2(X)$ the set of countable unions of closed sets and $\Pi^0_2(X)$ the set of countable intersections of open sets. A set is in $\Delta^0_2(X)$ iff it is both in $\Sigma^0_2(X)$ and in $\Pi^0_2(X)$.

When no mistake is present we will write $\Gamma$ instead of $\Gamma(X)$, for each $\Gamma(X)$ among the previous classes.

We say that a $A \subseteq X$ is nowhere dense if $A$ has empty interior. A subset $A$ of $X$ is meager if there exists a family $(A_i)_{i \in \omega}$ of nowhere dense sets such that $A = \bigcup_{i \in \omega} A_i$. The space $X$ is a Baire space if every non-empty open set in $X$ is non-meager. The family $\{[u] : u \in \omega^{<\omega}\}$ is a basis for the product topology on $\omega^\omega$, which is so a topological space that we call the Baire space. The map $T \mapsto [T]$ is a bijection between pruned trees on $\omega$ and closed subsets of $\omega^\omega$, so every basic open set of $\omega^\omega$ is also closed, we say that $\omega^\omega$ is 0-dimensional.

The following classical assertions will be used throughout the article.
Proposition 2.1.

(1) Every Polish space is a Baire space [2, Theorem 8.4].

(2) Every countable set is a Polish space for the discrete topology [2, 3.A, Example 2].

(3) The Baire space is canonical amongst 0-dimensional Polish spaces, because any such space is homeomorphic to a closed subset of the Baire space [2, Theorem 7.8].

(4) There are no strictly increasing sequences of open subsets of the Baire space of length $\omega_1$ [2, Theorem 6.9].

In this paper, when we write $f : \omega^\omega \to \omega^\omega$, we mean that $f$ is a partial function, whose domain $\text{dom}(f)$ is a closed subset of $\omega^\omega$, and that ranges in $\omega^\omega$.

Finally, recall that there are no infinitely decreasing sequences of ordinals, so that for every decreasing sequence of ordinals $s$, there exists an integer $l(s)$ such that the sequence $(s(n))_{n > l(s)}$ is constant.

2.2. Definitions. In the sequel, $f$ always denotes a partial function with domain a closed subset of $\omega^\omega$ and ranging into $\omega^\omega$.

The first immediate reduction game that is definable using Gale-Stewart games is: at round $i \in \omega$, I plays an integer $x_i$ then II plays an integer $y_i$. Say II wins iff

- either $x$ is not in $\text{dom}(f)$, or
- (Single Branch Rule, SBR) the tree $T_{(u_i),i<\omega}$ is finitely branching, has exactly one branch $y \in \omega^\omega$, and $f(x) = y$ holds.

Set $T_i = T_{(u_0,...,u_i)}$, so that we have $T_{(u_i),i<\omega} = \bigcup_{i<\omega} T_i$.

The eraser game. We call eraser game of $f$, denoted by $G_e(f)$, the following two-Player infinite game with perfect information. At round number $i \in \omega$, Player I starts and picks up an integer $x_i$, then Player II picks up a finite sequence $u_i \in \omega^{<\omega}$. Let $x = (x_i)_{i<\omega}$ be the real of I’s moves. Player II wins the game iff

- either $x$ is not in $\text{dom}(f)$,
- or (Single Branch Rule, SBR) the tree $T_{(u_i),i<\omega}$ is finitely branching, has exactly one branch $y \in \omega^\omega$, and $f(x) = y$ holds.

The backtrack game. We call backtrack game of $f$, denoted by $G_b(f)$, the following two-Player infinite game with perfect information. At round number $i \in \omega$, Player I picks up an integer $x_i$, then Player II picks up a finite sequence $u_i \in \omega^{<\omega}$. Let $x = (x_i)_{i<\omega}$ be the real of I’s moves. Player II wins the game iff

- either $x$ is not in $\text{dom}(f)$,
or (Strong Single Branch Rule, SSBR) the tree $T_{(u_i)} \in \omega$ is finitely branching, has only finitely many branching nodes, exactly one branch $y \in \omega^\omega$, and $f(x) = y$ holds.

The $\alpha$-bounded backtrack game. It is possible to bind even further the branching possibilities with a countable ordinal $\alpha$ in $\omega_1$. As we will see, it is relevant to look at the game in any basic open set, so take $w$ in $\omega^{<\omega}$. The $\alpha$-bounded backtrack game of $f$ in $w$ denoted by $G_\alpha(f, w)$ is the following game:

At round $i$, Player I chooses an integer $x_i$, then Player II picks up a couple $(\beta_i, u_i) \in (\alpha + 1) \times \omega^\omega$. Let $x = (x_i)_{i \in \omega}$ be the real of I’s moves. Player II wins the game iff

1. either $w \upharpoonright x \not\in \text{dom}(f)$ holds,
2. or both of the following conditions are fulfilled:
   1. (Bound $\alpha$ Rule or BoR) the sequence $(\beta_i)_{i \in \omega}$ is decreasing, and for all integers $i$ and $j$, if $\beta_i = \beta_{i+j}$ then $u_i \cup u_{i+j}$ holds.

We denote by $G_\alpha(f)$ the game $G_\alpha(f, \varepsilon)$, and we call Wadge game of $f$ the game $G_0(f)$. These games were introduced by Motto Ros [11]. We present them here in Semmes’s formalism, but both definition are equivalent.

With these definitions, we allow Player II to build a tree (instead of a real) in which there must be exactly one branch. Since this rule alone is much too general, we restrict further the possibilities for II, but she always will be able to play several times the same sequence, thus we simulate this way the possibility for her to skip.

Branching is the equivalent of erasing. What we mean intuitively is that, if $u_\beta \subseteq u_{i+1} \subseteq \ldots \text{ holds, then } u_i \text{ will be a prefix of the branch } y$, so II has approved what she did at round $i$ all along the way. But if $u_i \perp u_{i+1}$ holds then Player II thinks at round $i+1$ that at least a part of what she did at round $i$ is wrong, and by playing something incompatible with $u_i$ she makes a prefix of $u_i$ become a branching node of $T_{i+1}$.

Definition. To be more specific, we say that II erases $v$ at round $j$ when:

1. $u_i = u \upharpoonright v$ (with $v$ non empty) at round $i$ for some $i < j$,
2. $u_k$ at round $k$ verifies either $u_k \in T_i$ or $u_k \cup u_i$ for every $i < k < j$,
3. $u_j \notin T_i, u \subseteq u_j$, but $u \upharpoonright v(0) \not\subseteq u_j$.

We say that II erases something, or simply that she erases, at round $j$ if there is a $v$ that she erases at round $j$.

A strategy for Player I in the game $G_\alpha(f)$ is a function which tells him what to do at turn $n$ given what II did during the preceding rounds, namely it is a function $\tau : ((\alpha + 1) \times \omega^{<\omega})^{<\omega} \rightarrow \omega$. A strategy for I in $G_e(\varepsilon)$ (or in $G_\varepsilon(f)$) is a function $\tau : (\omega^{<\omega})^{<\omega} \rightarrow \omega$.

Conversely, a strategy for Player II in $G_\alpha(f)$ (resp. $G_e(\varepsilon)$ or $G_\varepsilon(f)$) is a function $\sigma : \omega^{<\omega} \rightarrow (\alpha + 1) \times \omega^{<\omega}$ (resp. $\sigma : \omega^{<\omega} \rightarrow \omega^{<\omega}$). A strategy for II in $G_\alpha(f)$ can be written as a pair $(\sigma_0, \sigma_1)$ of functions, with $\sigma_0 : \omega^{<\omega} \rightarrow (\alpha + 1)$, $\sigma_1 : \omega^{<\omega} \rightarrow \omega^{<\omega}$, and $\sigma(s) = (\sigma_0(s), \sigma_1(s))$.

Two strategies $\tau$ for I and $\sigma$ for II in any of these three games $G$ determine an unique run of $G$ that we denote by $\tau \star \sigma = ((\tau \ast \sigma)_I, (\tau \ast \sigma)_II)$. A strategy $\sigma$ for Player II in $G_e(\varepsilon)$ (resp. $G_\varepsilon(f)$ or $G_\alpha(f)$) is legal if, facing any strategy $\tau$ for I, $(\tau \star \sigma)_II$ respects the SBR (resp. the SSBR or the SSBR and the BoR).

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In any game $G$, if a strategy $\tau$ for $I$ wins against all possible strategies $\sigma$ for $II$ we say that $\tau$ is a winning strategy for Player $I$. Conversely, if $\sigma$ wins against all possible strategies for $I$ we say that $\sigma$ is a winning strategy (or w.s.) for $II$ in $G$. We say that a game $G$ is determined if either Player $I$ or Player $II$ has a w.s. in $G$.

We distinguish some strategies for Player $I$, namely the strategies $\tau_x$ with $x \in \omega^\omega$ that consist in playing $x(i)$ at turn $i \in \omega$, whatever $II$ plays. We observe first some simple and very useful facts.

**Proposition 2.2.** For all $\alpha \in \omega_1$ and $f : \omega^\omega \rightarrow \omega^\omega$,

1. A w.s. for $II$ in $G_{b}(f)$ is also winning in $G_{e}(f)$.
2. If $\sigma$ is a w.s. for $II$ in $G_{a}(f)$ then $\sigma_1$ is winning in $G_{b}(f)$.
3. If $I$ has a w.s. in $G_{e}(f)$ then he has one in $G_{b}(f)$, and also one in $G_{a}(f)$.
4. A strategy $\sigma$ for $II$ in $G_{e}(f)$ (resp. $G_{b}(f)$, resp. $G_{a}(f)$) is winning iff it wins facing every strategy $\tau_x$ for $I$, with $x \in \omega^\omega$.

**Proof.**

1. The SSBR is a strengthening of the SBR.
2. Any strategy $\tau$ for $I$ in $G_{e}(f)$ or $G_{b}(f)$ induces canonically a strategy for him in $G_{a}(f)$:

$$\tau_\alpha : ((\alpha + 1) \times \omega^\omega)^{<\omega} \rightarrow \omega$$

$$W = (w_0, w_1) \mapsto \tau(w_1)$$

Given any strategy $\tau$ for $I$ in $G_{e}(f)$, if $\sigma$ is winning in $G_{a}(f)$, then it wins against $\tau_\alpha$. Since the SSBR is part of the winning conditions for Player $II$ in $G_{a}(f)$, $\sigma_1$ wins against $\tau$.

3. Suppose $\tau$ is winning for $I$ in the eraser game. Since any (legal) strategy for $II$ in the backtrack game is in particular a (legal) strategy in the eraser game, $\tau$ is winning against all of these, so $\tau$ is winning in the backtrack game. If $\sigma$ is a (legal) strategy for $II$ in the $\alpha$-bounded backtrack game then $\sigma_1$ is a (legal) strategy for $II$ in the backtrack game, and since $\tau$ wins against $\sigma_1$, $\tau_\alpha$ wins against $\sigma$ so $I$ wins all three types of games.

4. If $\sigma$ is winning then it wins facing every $\tau_x$ by definition. Suppose $\sigma$ wins against every $\tau_x$. Let $\tau$ be any strategy for Player $I$, and set $x = (\tau \star \sigma)$. Then $\tau \star \sigma = \tau_x \star \sigma$ holds, so $\sigma$ wins facing $\tau$ since it wins facing $\tau_x$. $\square$

3. **Analyzing games.**

Analyzing a game consists in observing the behaviour of each Player while the function $f$ varies. Since Player $II$ has to guess the image of what $I$ will eventually obtain, it is very hard for her to win the game, so the first analysis is to determine a topological condition on $f$ equivalent to the existence of a winning strategy for $II$. The next step is to establish determinacy, and we get a precise analysis when we are able to refine determinacy by giving precise properties of the winning strategies for at least one of the Players (or, when possible, for both of them).

3.1. **The Wadge game and the eraser game.** The precise analysis of the Wadge game of $f$ is a simple yet very important task, that already appeared in Wadge’s PhD thesis [5].

**Definition 3.1** (Aggressive strategy). Given $f : \omega^\omega \rightarrow \omega^\omega$, $x \in \text{dom}(f)$, and $u \subset x$, 
(1) We say that \( v \in \omega^\omega \) is a winning test for \( x \in \mathbb{G}_0(f, u) \) if \( v \subset f(x) \) and for all integers \( n \) there is \( x_n \in \omega^\omega \) such that \( x|_{\lg(u)+n+1} \subset x_n \) and \( v \not\in f(x_n) \).
(2) Given \( v \in \omega^\omega \) a winning test for \( x \in \mathbb{G}_0(f, u) \), we say that Player I uses \( v \) as a test in \( \mathbb{G}_0(f, u) \) when he plays according to the following strategy \( \tau \):

\[
\tau : (1, \omega^\omega) \rightarrow \omega
\]

\[
W = (v_1, v_2) \rightarrow \begin{cases} 
  x(\lg(u) + i) & \text{if } |W| = i \text{ and } \forall j < i (v_2(j) \subset v) \\
  x(\lg(u) + i) & \text{if } |W| = i \text{ and } a(W) = 0 \\
  x_j(\lg(u) + i) & \text{if } |W| = i \text{ and } a(W) = j + 1.
\end{cases}
\]

with \( a(W) = \begin{cases} 
  0 & \text{if } \forall j < |W| (v_2(j) \subset v) \\
  j + 1 & \text{if } |W| > 0 \text{ and } \exists j < |W| (v_2(j) \not\subset v) \land \forall k < j, v_2(k) \subset v.
\end{cases} \)

(3) If I is using \( v \) as a test, we say that II fails the test if she plays \((0, v')\) for some \( v' \) incompatible with \( v \), and we say she hits the test if she plays \((0, v')\) for some \( v' \) extending \( v \).
(4) A winning test \( v \) for \( x \) of minimal length (among all winning tests for \( x \)) is an aggressive test for \( x \).
(5) We say a strategy \( \tau \) is aggressive if it uses an aggressive test.

**Proposition 3.2.** (Folklore)

(1) If \( f : \omega^\omega \rightarrow \omega^\omega \) is continuous, then II has a w.s. in \( \mathbb{G}_0(f) \).
(2) If \( f : \omega^\omega \rightarrow \omega^\omega \) is discontinuous, then I has an aggressive w.s. in \( \mathbb{G}_0(f) \).

As a consequence, the game \( \mathbb{G}_0(f) \) is determined.

**Proof.**

(1) Suppose \( f : \omega^\omega \rightarrow \omega^\omega \) a continuous function. Consider the following function:

\[
\sigma_f : \omega^\omega \times \omega^\omega \rightarrow 1 \times \omega^\hat{s}
\]

\[
s \rightarrow (0, \bigcap \{ v \in \omega^\hat{s} : f([s]) \cap [v] \neq \emptyset \}).
\]

The function \( f \) being continuous, for all \( x \in \omega^\omega \), for all \( N \in \omega \) there is an \( n \in \omega \) such that for all \( y \in \omega^\omega \), \( x|_n = y|_n \) implies \( f(x)|_N = f(y)|_N \). This implies that \( \sigma_f \) is a w.s. for II in \( \mathbb{G}_0(f) \).

(2) The function \( f \) is discontinuous, so there is at least one point of discontinuity \( x \in \omega^\omega \). The discontinuity of \( f \) in \( x \) implies the existence of a winning test for \( x \) in \( \mathbb{G}_0(f) \). Fix a minimal such test \( v \) for \( x \), and call \( \tau \) the aggressive strategy obtained using \( v \) as a test.

If II neither hits nor fails the test, then she does not respect the SSBR, so she loses. If she does both, then she does not respect the B0R, so she loses as well. Finally if only one of the two cases holds then II loses by definition of the strategy of I, so \( \tau \) is an aggressive w.s. for I in \( \mathbb{G}_0(f) \).

**Fact 3.3.** For any function \( f : \omega^\omega \rightarrow \omega^\omega \), if \( v \) is an aggressive test for some real \( x \in \text{dom}(f) \) in \( \mathbb{G}_0(f) \) and if \( u \) is a strict initial segment of \( v \) then \( f^{-1}([u]) \) contains an open neighborhood of \( x \).
Proof. We have by definition that \( u \subset v \subset f(x) \) holds. If \( f^{-1}(\{u\}) \) does not contain an open neighborhood of \( x \), then \( u \) is a winning test for \( x \) in \( G_0(f) \), which is impossible by minimality of \( v \). \( \square \)

The first analysis of the eraser game is due to Duparc, but its first explicit occurrence and proof are in Semmes' PhD thesis [7]. We give here a slightly different proof.

**Proposition 3.4. (Duparc)**

For any function \( f : \omega^\omega \to \omega^\omega \), \( f \) is of Baire class 1 iff \( II \) has a w.s. in \( G_e(f) \).

**Proof.** If \( f \) is of Baire class 1, fix a sequence \( (f_n)_{n \in \omega} \) of continuous functions such that for all \( n \in \omega \) the domains of \( f \) and \( f_n \) coincide, and for all \( x \in \text{dom}(f) \) the sequence \( (f_n(x))_{n \in \omega} \) converges to \( f(x) \). Fix then, for every integer \( n \), a w.s. \( \sigma_n \) for \( II \) in \( G_0(f_n) \) as in Proposition 3.2.

The strategy \( \sigma \) consists in: follow \( \sigma_0 \) until it gives an answer of length 1, then switch and follow \( \sigma_1 \) until it gives an answer of length 2, and so on. Precisely, for any \( u \in \omega^{<\omega} \), \( \sigma(u) = \sigma_{n_u}(u) \) with \( n_u \) an integer defined by induction on \( |u| \) as follows. Set \( n_e = 0 \), suppose \( n_v \) is defined for \( v \in \omega^{<i} \) with \( i \in \omega \), take \( u \in \omega^{i+1} \) and set

\[
n_u = \begin{cases} 
n_u & \text{if } |\sigma_{n_u}(u)| < n_u + 1 \\
n_u + 1 & \text{otherwise.} \end{cases}
\]

Given \( x \in \text{dom}(f) \) we have \( \sigma(x|_n) \subset f_i(x) \) for some \( i \leq n \), hence the tree \( T \) produced by \( II \) is included in the one generated by all the \( f_n(x) \), which is finitely branching, so \( T \) is finitely branching. Since \( T \) contains \( f_n(x)|_{n+1} \) for all \( n \) it has \( f(x) \) as unique branch.

Take now \( \sigma : \omega^{<\omega} \to \omega^{<\omega} \) is a w.s. for \( II \) in \( G_e(f) \). For all \( n \in \omega \), set \( f_n(x) = \sigma(x|_n)^{\omega^2} \). The functions \( f_n \) are locally constant so continuous, and for all \( x \in \text{dom}(f) \) the sequence \( (f_n(x)) \) converges to \( f(x) \) because \( \sigma \) is winning, so \( f \) is of Baire class 1. \( \square \)

3.2. The \( \alpha \)-bounded backtrack game. The proof of Proposition 3.4 does not say anything about how the winning strategies for \( II \) behave. In particular even a winning strategy can be far from optimal, as it may allow “stupid” moves, for instance moves whose associated basic open set has no intersection with the range of \( f \), those moves being then erased. On the other hand, there is a natural rank on the winning strategies in \( \alpha \)-bounded games that gives a notion of optimality.

This rank has been first defined by Hertling [14], without games, in the study of functions with countable discrete range. Motto Ros redefines it (for arbitrary functions) as a coanalytic rank, and gives a game characterization [11]. Here we proceed the other way around.

Discontinuity, using games. Notice first that if \( II \) has a w.s. in \( G_\alpha(f,w) \) for \( \alpha \) a countable ordinal and \( w \) a finite sequence of integers, then she has also one in \( G_\beta(f',w') \) for every countable ordinal \( \beta \geq \alpha \), finite sequence \( w' \supseteq w \) and function \( f' \subseteq f \). Following Proposition 3.2, looking for the minimal bound \( \alpha \) such that \( II \) wins the \( \alpha \)-bounded game, reveals the level of discontinuity of \( f \).

**Definition 3.5.** Precisely, we define the discontinuity of \( f \) in \( u \):

\[
\text{disc}_f(u) = \min\{\alpha \in \omega_1 : II \text{ has a w.s. in } G_\alpha(f,u)\}
\]
if such an ordinal exists, and set
\[ \text{disc}_f(u) \geq \omega_1 \text{ otherwise.} \]

Write \( \text{disc}(f) \) for \( \text{disc}_f(\varepsilon) \). For any closed subset \( F \) of \( \omega^\omega \), set \( \text{disc}_f(u, F) = \text{disc}_{f|_F}(u) \) and \( \text{disc}_f(F) = \text{disc}(f|_F) \). It is possible to define the discontinuity of a real \( x \) in a closed set \( F \), relatively to \( F \), as the minimal discontinuity of \( f|_F \) in all basic open neighbourhoods \( |u| \) of \( x \). We thus set
\[ \text{disc}_f(x, F) = \min \{ \text{disc}_f(u, F) : u \subset x \} \]
That finally leads us to define, given a function \( f : \omega^\omega \to \omega^\omega \) and a closed subset \( F \) of \( \text{dom}(f) \), the sets \( D_{R\alpha}(f, F) = \{ x \in \omega^\omega : \text{disc}_f(x, F) R \alpha \} \), for \( R \in \{ <, >, =, \leq, \geq \} \) and \( \alpha \leq \omega_1 \).

For short we set \( \text{disc}_f(x) = \text{disc}_f(x, \text{dom}(f)) \) and \( \text{D}_{R\alpha}(f) = \text{D}_{R\alpha}(f, \text{dom}(f)) \), for \( R \in \{ <, >, =, \leq, \geq \} \) and \( \alpha \leq \omega_1 \).

**Proposition 3.6.** Fix \( f : \omega^\omega \to \omega^\omega \), \( u \in \omega^{<\omega} \), and a non-empty closed \( F \subseteq \text{dom}(f) \).

1. The sets \( D_{>\alpha}(f, F) \) and \( D_{\geq \alpha}(f, F) \) are closed in \( F \), \( D_{<\alpha}(f, F) \) and \( D_{\leq \alpha}(f, F) \) are open in \( F \).
2. When \( \text{disc}(f, F) < \omega_1 \), the set \( D_{>\text{disc}(f, F)}(f, F) \) is empty. Otherwise, the set \( D_{\geq \omega_1}(f, F) \) is perfect.
3. If \( \alpha = \text{disc}(f, F) < \omega_1 \), then for all \( \beta < \alpha \) the set \( D_{=\beta}(f, F) \) is non-empty.
   If \( \alpha = \beta + 1 \) we also have \( D_{=\alpha}(f, F) \neq \emptyset \).
4. For all \( \alpha < \omega_1 \), we have \( D_{=0}(f, D_{\geq \alpha}(f, F)) = D_{=\alpha}(f, F) \).

**Proof.**

1. We have that \( D_{<\alpha}(f, F) = \bigcup_{\text{disc}_f(u, F) < \alpha} [u] \) and \( D_{\geq \alpha}(f, F) = (D_{<\alpha}(f, F))^c \)
   (same for \( \leq \) and \( > \)).
2. When \( f|_F \) is of countable discontinuity, by definition \( D_{>\text{disc}(f, F)}(f, F) \) is empty. Otherwise, by (1) and Proposition 2.1, we have that \( D_{=\omega_1}(f, F) \) is not empty. Set \( H = D_{>\omega_1}(f, F) \), and suppose \( x \) is isolated in \( H \). There is a finite prefix \( u \) of \( x \) such that \( [u] \cap H = \{ x \} \), so \( f|_H \) is continuous on \( [u] \), and \( \text{disc}_f(x, F) < \omega_1 \) holds. But we supposed \( x \in H \), whence a contradiction. Hence \( H \) is perfect.
3. Proceed by induction on \( \alpha = \text{disc}(f, F) \). For \( \alpha = 0 \) there is nothing to prove. Suppose that the point is made for \( \alpha \) and that \( \text{disc}(f, F) = \alpha + 1 \). If \( D_{=\alpha+1}(f, F) = \emptyset \) then by definition \( F = D_{\leq \alpha}(f, F) \) which is impossible by minimality of the discontinuity. Using the induction hypothesis, we thus only need to show that \( D_{=\alpha}(f, F) \) is non-empty. Towards a contradiction, suppose it is empty. Then we have \( F = D_{=\alpha+1}(f, F) \cup D_{<\alpha}(f, F) \). Fix a w.s. \( \sigma^0 \) for \( II \) in \( G_{\alpha+1}(f|_F) \). For all finite sequence \( u \) such that \( [u] \cap F \) is contained in \( D_{<\alpha}(f, F) \) there is a minimal prefix \( v_u \) of \( u \) such that \( \text{disc}_f(v_u, F) < \alpha \). For all these \( v_u \) fix a w.s. \( \sigma_{v_u} \) for \( II \) in \( G_{\text{disc}_f(v_u, F)}(f|_{F u}) \). Consider the following strategy for \( II \):
\[
\sigma : \omega^{<\omega} \longrightarrow (\alpha + 1) \times \omega^{<\omega} \\
\begin{cases}
(\alpha, \sigma^0(u)) & \text{if } \sigma^0(u) = \alpha + 1 \\
(\alpha, \varepsilon) & \text{if } \sigma^0(u) = \alpha \\
\sigma_{\varepsilon}(u) & \text{otherwise.}
\end{cases}
\]
Theorem 3.7. For all parameter $a$ function of countable discontinuity. immediate one might be the determinacy of the bounded backtrack game with parameter $a$.

Proof. Proceed by induction on $a$. For $a = 0$, the result is trivial. Assume $a > 0$. For $a > 0$, we have $D_{a+1} = D_a \cup \{0\}$, so $D_a$ is winning on $\omega$. Thus, $D_a$ is winning on $\omega$. For $a > 0$, we have $D_a$ is winning on $\omega$. Hence, $D_a$ is winning on $\omega$. For $a > 0$, we have $D_a$ is winning on $\omega$. Therefore, $D_a$ is winning on $\omega$.

Since $\sigma^0$ is winning, so is $\sigma$ on $D_{\omega+1}(f, F)$. Similarly, $\sigma$ is winning on every $[u] \cap F \subseteq D_{\omega}(f, F)$ because it follows the w.s. $\sigma_{\omega}$. Finally, as $F = D_{\omega+1}(f, F) \cup D_{\omega}(f, F)$ holds, $\sigma$ is a w.s. in $G_\omega(f[F])$, which contradicts the minimality of the discontinuity.

When $a$ is limit, a similar argument shows that for all $\beta < a$ there is an ordinal $\gamma$ such that both $\beta \leq \gamma < a$ and $D_{\gamma}(f, F) \neq \emptyset$ hold. Using the induction hypothesis on $\gamma$ yields then that $D_{a}(f, F) \neq \emptyset$ holds as well.

(4) Fix $\sigma$ a w.s. for $II$ in $G_\omega(f[D_{\omega}(f, F)])$. Consider the following strategy $\sigma'$ for $II$ in $G_\omega(f[D_{\omega}(f, F)])$:

$$\sigma' : \omega^< \omega \rightarrow \{0, \sigma_1(u)\} \text{ if } \text{disc}_f(u, F) = \alpha$$

We show first that $D_\omega(f, D_{\omega}(f, F)) \supseteq D_\omega(f, F)$ holds, so we prove that for all $x \in D_\omega(f, F)$, there is a finite prefix $u$ of $x$ such that $\sigma'$ is winning in $G = G_\omega(f[D_{\omega}(f, F)])$. By definition of discontinuity of a real, there is a prefix $u$ of $x$ such that $\text{disc}_f(u, F) = \alpha$. We now use Proposition 2.2 (4) and show that $\sigma'$ wins against $\gamma$ for all $u \in D_\omega(f, F \cap [u])$. At round $\omega$, $I$ has built $u$ so for all $v \in x$ such that $u \leq v \subseteq z$ we have $\alpha = \text{disc}_f(u, F) \geq \text{disc}_f(v, F) \geq \text{disc}_f(z, F) = \alpha$, so not only $\sigma'(v) = (0, \sigma_1(v))$ holds but also we have $\sigma_0(v) = \alpha$ since $\sigma$ is winning and by minimality of the discontinuity. Hence by definition of the B0R and the B0R, $\sigma'$ is a legal strategy in $G$ and it produces $f(z)$ because $\sigma$ is winning, so she wins.

For the converse, note that by minimality of the discontinuity and Proposition 3.2, $I$ has a w.s. in $G_\omega(f, u)$ for all $u$ such that $\text{disc}_f(u) > \alpha$.

The study of discontinuity has numerous applications, among which the most immediate one might be the determinacy of the bounded backtrack game with parameter a function of countable discontinuity.

**Theorem 3.7.** For all $(\alpha, u, f) \in \omega_1 \times \omega^< \omega \times (\omega^\omega)^\omega$, if $\alpha < \text{disc}_f(u) < \omega_1$ then Player I has a w.s. in $G_\omega(f, u)$.

As a consequence, for all $(\alpha, u, f) \in \omega_1 \times \omega^< \omega \times (\omega^\omega)^\omega$, if $\text{disc}_f(u) < \omega_1$ the game $G_\omega(f, u)$ is determined.

**Proof.** Proceed by induction on $\alpha$. If $\text{disc}_f(u) > \alpha + 1$, apply Proposition 3.6, (3) to find a strict extension $u'$ of $u$ such that $\text{disc}_f(u') = \alpha + 1$. Hence, we can suppose w.l.o.g. that we have both $u = \varepsilon$ and $\text{disc}(f) = \alpha + 1$.

Define a strategy for $I$ in $G_\omega(f)$. By Propositions 3.2 and 3.6, (4), Player I has a w.s. $\tau_0$ in $G_\omega(f[D_{\omega}(f)])$. We modify $\tau_0$ in the following way:

$$\tau_{0, \varepsilon} : (\{\alpha\} \times \omega^< \omega) \rightarrow \omega$$

$$W = (w_0, w_1) \rightarrow \tau_0(\pi_{w_0}, w_1).$$

For all $\beta < \alpha$ and all finite sequences $v$ such that $[v] \cap D_\omega(f) \neq \emptyset$ we have $\beta < \alpha < \text{disc}(v)$, so we can apply the induction hypothesis to $\beta$ and fix a w.s. $\tau_{\beta, v}$ for him in $G_\omega(f, v)$. We want to consider the strategy $\tau$ for Player I in $G_\omega(f)$ that consists in playing according to $\tau_{0, \varepsilon}$ as long as she does not erase, and switch to a convenient strategy $\tau_{\beta, v}$ once she does.
We define formally \( \tau \) by fixing \( \tau(W) = \tau_{\beta_W, v_W}(W) \), for a triple \((\bar{W}, v_W, \beta_W)\) that we define by induction on \(|W|\) for \( W \in ((\alpha + 1) \times \omega^{<\omega})^i \).

Fix \((\bar{v}, v, \beta) = (\bar{\varepsilon}, \varepsilon, 0)\). Suppose \((\bar{W}, v_W, \beta_W)\) is defined on \(((\alpha + 1) \times \omega^{<\omega})^i \) and take \( W = (w_0, w_1) \) in \(((\alpha + 1) \times \omega^{<\omega})^{i+1} \), set \( W|_i = W' \) and \( v = (\tau_{\beta_{v_{W|_i}}, v_{W|_i}}(W|_j))_{j \leq i} \).

\[
(W, \beta_W, v_W) = \begin{cases} 
(\varepsilon, d(w_0), v) & \text{if } \bar{W}' = W', d(w_0) < \alpha \text{ and } v_{W'} = \varepsilon \\
(W', d(W), \beta_{W'}, v_{W'}) & \text{otherwise.}
\end{cases}
\]

Either \( II \) does not erase during the run, so she plays only pairs of the form \((\alpha, u)\) hence \( I \) plays according to \( \tau_0, \varepsilon \) which is winning. Or \( II \) erases and the first time she does so, she plays a pair \((\beta, w)\) with \( \beta < \alpha \). At this round, Player \( I \) has built following \( \tau \) a sequence \( v \), so he starts to follow \( \tau_{\beta, v} \) from that round on, and wins as well. \( \square \)

We will obtain a general determinacy result for the bounded backtrack game as a corollary of the determinacy of the backtrack game itself.

**The topological definition.** We give now the definition introduced by Hertling [14] and by Motto Ros [11]. Given a function \( f \) and \( F \) a closed subset of \( \omega^\omega \), consider the set

\[
D(f, F) := F \setminus D_{\leq 0}(f, F).
\]

This set is the closure of the set of discontinuity points of \( f \) in \( F \) so the map \( D \) is a Borel derivative. It induces in a canonical way (see for instance [2, Chap IV.34] or [1, Chap 4B]), a transfinite sequence of closed sets the following way:

- \( F_0 = F \)
- \( F_{\alpha+1} = D(f, F_\alpha) \)
- \( F_\lambda = \bigcap_{\alpha < \lambda} F_\alpha \) for \( \lambda \) limit.

The \( D \)-rank of \( f \) on \( F \) is the last ordinal after which the sequence \((F_\alpha)_{\alpha \in \omega_1}\) becomes constant if it exists\(^1\), which is the case here by Proposition 2.1, since this sequence is decreasing. Formally:

\[
\text{rk}_D(f, F) = \sup\{ \alpha \in \omega_1 : F_\alpha \neq D(f, F_\alpha) \}.
\]

Set \( \text{rk}_D(f) \) as the \( D \)-rank of \( f \) on \( \omega^\omega \). Here is the link between the two definitions. It was already made by Motto Ros [11, Proposition 107, p.133] but in a very different way.

**Proposition 3.8.** (Motto Ros) Let \( f \) be a function, \( F \) closed in \( \omega^\omega \). If the discontinuity of \( f \) is countable on \( F \) then it is equal to the \( D \)-rank of \( f \) on \( F \).

Moreover, \( \text{disc}(f, F) \) is countable iff \( F_{\text{rk}_D(f, F)+1} \) is empty iff there exists \( \alpha \in \omega_1 \) such that \( F_\alpha \) is empty.

**Proof.** For the first part of the proposition, we prove by induction on \( \gamma \) that \( F_\gamma = D_{\geq \gamma}(f, F) \) holds for all countable \( \gamma \).

By definition we have \( F_0 = D_{\geq 0}(f, F) = F \), so assume now that \( F_\gamma = D_{\geq \gamma}(f, F) \). By Proposition 3.6. (1) and (4), \( D_{=0}(f, D_{\geq \gamma}(f, F)) = D_{=\gamma}(f, F) \) holds, then so does

\[
F_{\gamma+1} = D(f, F_\gamma) = D(f, D_{\geq \gamma}(f, F)) = D_{\geq \gamma}(f, F) \setminus D_{=0}(f, D_{\geq \gamma}(f, F)) = D_{\geq \gamma}(f, F) \setminus D_{=\gamma}(f, F) = D_{>\gamma}(f, F) = D_{\geq \gamma+1}(f, F).
\]

\(^1\)Remark that the \( D \)-rank is usually defined as the ordinal \( \alpha \) such that \( F_\alpha \) is the least fixed point of the sequence \((F_\alpha)_{\alpha \in \omega_1}\). It is either equal to ours or it is the successor of ours.
Suppose that $\gamma$ is limit, and that $F_\xi = D_{\geq \xi}(f, F)$ holds for all $\xi < \gamma$. Then
\[ F_\gamma = \bigcap_{\xi < \gamma} F_\xi = \bigcap_{\xi < \gamma} D_{\geq \xi}(f, F) = D_{\geq \gamma}(f, F), \]
so for all countable $\gamma$, the sets $F_\gamma$ and $D_{\geq \gamma}(f, F)$ are equal, and in particular $rk_D(f, F) = \text{disc}(f, F)$ holds when $f$ is of countable discontinuity.

For the second part of the proposition, we prove that
\[ \text{disc}(f, F) < \omega_1 \Rightarrow F_{rk_D(f,F)+1} = \emptyset \Rightarrow \exists \alpha \in \omega_1(F_\alpha = \emptyset) \Rightarrow \text{disc}(f, F) < \omega_1. \]

Set $\beta = \text{disc}(f, F)$, and suppose it is countable. We have both that $D_{\geq \gamma}(f)$ is non empty for every $\gamma$ in $\beta$, and that $D_{> \beta}(f, F)$ is empty by Proposition 3.6, (2). Use the first part to get the first implication.

Since $rk_D(f, F)$ is always countable, the second implication holds.

Using the first part again we have that if there is an $\alpha < \omega_1$ such that $F_\alpha$ is empty, then so is $D_{\geq \alpha}(f, F)$, and hence $\text{disc}(f, F) < \omega_1$. \hfill $\square$

When the discontinuity is countable, the set $D_{\geq \text{disc}(f,F)+1}(f)$ need not be the first closed set of the sequence to be empty. If the discontinuity is a limit ordinal $\lambda$, the set $D_{\geq \lambda}(f)$ can already be empty.

**Corollary 3.9.** A function $f$ is of countable discontinuity on $F$ a closed subset of $\text{dom}(f)$ iff for every non-empty closed subset $F'$ of $F$, $D(f, F') \neq F'$ holds.

**Proof.** Remark that when $F$ is empty the equivalence is trivial, so suppose it is not. If $f$ is of countable discontinuity $\alpha$ on $F$, then by definition $II$ has a w.s. in $G_\alpha(f|F)$, and hence she has one also in $G_\alpha(f|F')$ for any closed $F' \subset F$. So when $F'$ is not empty, neither is $D_{< \alpha}(f, F')$, which implies that $D(f, F') \neq F'$.

Suppose now that for every non-empty closed $F' \subseteq F$ we have $D(f, F') \neq F'$. Then the only possible fixed point of $(F_\alpha)_{\alpha < \omega_1}$ is the empty set. Since $(F_\alpha)_{\alpha \in \omega_1}$ is a decreasing sequence of closed subsets of the Baire space, it is eventually constant by Proposition 2.1, so it reaches a fixed point of the derivative, it hence means that there is an ordinal $\alpha$ countable such that $F_\alpha$ is empty, which implies by Proposition 3.8 that the discontinuity of $f$ on $F$ is countable. \hfill $\square$

For $X \subseteq \omega^\omega$, set $1_X$ for the characteristic function of $X$, which means $1_X(a) = 1^\omega$ if $a \in X$, $0^\omega$ otherwise. Say then that the discontinuity of $X$ is the discontinuity of its characteristic function.

Now the discontinuity of a set has an immediate equivalent in the hierarchy of topological complexity.

**Proposition 3.10.** (Folklore)

1. A subset $X$ of $\omega^\omega$ is $\Delta_0^0$ iff $\text{disc}(X)$ is countable.
2. If $f : \omega^\omega \to \omega^\omega$ is a function such that $\text{disc}(f) < \omega_1$ holds, then the inverse image of any open set is $\Delta_2^0$.

**Proof.**

1. Let $X$ be a $\Delta_2^0$ subset of $\omega^\omega$. There are closed sets $F_i$ and $F_i^c$ for every integer $i$ such that $X = \bigcup \{F_i : i \in \omega\}$ and $X^c = \bigcup \{F_i^c : i \in \omega\}$. By Corollary 3.9, we only have to show that $D(1_X, F) \neq F$ holds for every nonempty closed set $F$. As any closed subset $F$ of $\omega^\omega$ is a Baire space\(^2\), at least one of the

\(^2\)Indeed $F$ is a Polish space for the topology induced by the product topology on $\omega^\omega$.
closed sets $F$, $F'$ has non empty interior in $F$, otherwise $F$ would be meager. Hence $D(1_X, F) \neq F$ holds.

Suppose now that $\text{disc}(X)$ is countable. It implies that Player $II$ has a w.s. in $\mathbb{G}_e(1_X)$, so $1_X$ is of Baire class 1 by Proposition 3.4, which exactly means that $X$ is $\Delta^0_2$.

(2) Given two functions $h$ and $g$ with the domain of $h$ containing the range of $g$, we first show, following Motto Ros [11, Proposition 110, p 137], how to bound the discontinuity of $h \circ g$ using the discontinuities of both functions. Set $\alpha$ and $\beta$ the discontinuity of $h$ and $g$ respectively, and fix $\sigma = (\sigma_0, \sigma_1)$ and $\tau = (\tau_0, \tau_1)$ two w.s. for $II$, in $\mathbb{G}_a(h)$ and $\mathbb{G}_\beta(g)$ respectively.

Name $\Gamma$ the isomorphism between the set $(\alpha + 1) \times (\beta + 1)$ ordered lexicographically and the ordinal $(\beta + 1) \cdot \alpha + \beta + 1$, and set $\delta = (\beta + 1) \cdot \alpha + \beta$. Then

$$\rho : \omega^{<\omega} \longrightarrow (\delta + 1) \times \omega^{<\omega}$$

$$u \longmapsto (\Gamma(\sigma_0 \circ \tau_1(u), \tau_0(u)), \sigma_1 \circ \tau_1(u))$$

is a w.s. for Player $II$ in $\mathbb{G}_\delta(h \circ g)$.

But now if $U \subseteq \omega^{<\omega}$ is any open set, using this bound, both $\text{disc}(1_{f^{-1}(U)}) = \text{disc}(1_U \circ f)$ and $\text{disc}(1_U \circ f) < \omega_1$ hold. Indeed $1_U$ has discontinuity only 1, and the discontinuity of $f$ is countable. \hfill \Box

The following classical property gives a characterisation of $\Delta^0_2$ sets.

**Corollary 3.11.** (Folklore) A set $A$ is a $\Delta^0_2$ subset of $\omega^{\omega}$ iff for all nonempty closed set $F$ either $A$ or $A^c$ contains a set that is open in $F$.

**Proof.** The set $A$ is $\Delta^0_2$ iff, by Proposition 3.10, $\text{disc}(1_A|_F) < \omega_1$ holds, iff for all $F$ as above we have $D(1_A, F) \neq F$, iff for all $F$ as above, $1_A$ is continuous on an open set $U$ of $F$, which means $1_A|_U$ is constant equal either to $1^\omega$ or $0^\omega$, and so we are done. \hfill \Box

4. Determinacy and Applications

The analysis of the bounded backtrack games implies the following two results of determinacy of the eraser and the backtrack game. Remark that these are stronger than just determinacy, since they both give precise descriptions of winning strategies for Player $I$.

4.1. Determinacy.

**Definition** (Uniform winning test in $\mathbb{G}_e$). Fix a function $f : \omega^{\omega} \rightarrow \omega^{\omega}$ and a closed $F \subseteq \text{dom}(f)$.

- A sequence $v \in \omega^{<\omega}$ is a uniform winning test for $F$ in $\mathbb{G}_e(f)$ if for all $u \in \omega^{<\omega}$, when $[u] \cap F \neq \emptyset$ holds, there are $x^0_n \in [u] \cap F$ and $x^1_n \in [u] \cap F$ such that both $f(x^0_n) \notin [v]$ and $f(x^1_n) \in [v]$ hold.
- We say that a strategy $\tau$ for $I$ in $\mathbb{G}_e(f)$ is uniform if it uses an uniform winning test. Precisely, fix $u_0$ a uniform winning test for $F$ in $\mathbb{G}_e(f)$, and define recursively the following strategy $\tau$.

  Set $\tau(\epsilon) = x^0_0(0)$. Suppose that, after round $i \in \omega$ : $I$ has enumerated $u \in \omega^{i+1}$ such that $u = (x^0_i)_{i+1}$ holds for some $v \in \omega^{<\omega}$ and $\xi \in \{0, 1\}$ ; Player $II$ has played $W \in (\omega^{<\omega})^{i+1}$ with $d(W) = w$.  

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When \( w \) is not an initial sequence of any \( W(j) \) for all \( j < i \): set \( a(W) = 1 \) if \( wT u_0 \) holds and \( a(W) = 0 \) otherwise. Then

\[
\tau(W) = \begin{cases} 
  x^*_i(i+1) & \text{if } w \subseteq W(j) \text{ for some } j < i \text{ or if } a(W) = 1 - \xi \\
  x^{1-\xi}_{i+1}(i+1) & \text{if } a(W) = \xi.
\end{cases}
\]

**Theorem 4.1.** If \( f : \omega^\omega \to \omega^\omega \) is not of the first Baire class, then Player I has a uniform w.s. in \( G_e(f) \).

As a consequence, the game \( G_e(f) \) is determined for every \( f : \omega^\omega \to \omega^\omega \).

**Proof.** Let \( u_0 \in \omega^{<\omega} \) be such that \( A = f^{-1}([u_0]) \) is not \( \Delta_0^e \). According to Corollary 3.11, both \( A \) and \( A^c \) are dense in a closed subset \( F \) of \( \omega^\omega \), so \( u_0 \) is a uniform winning test for \( F \) in \( G_e(f) \), so call \( \tau \) the uniform strategy that uses \( u_0 \).

Given a strategy \( \sigma \) for Player II, there are three possibilities.

First, from round \( i \) on, \( II \) plays \( u_j \) compatible with \( u_0 \) or included in one of her previous moves, then by definition \((\tau \star \sigma)_I \) is not in \( f^{-1}([u_0]) \), so \( I \) wins; or vice-versa \( II \) plays \( u_j \) incompatible with \( u_0 \) for \( j > i \), but in this case \((\tau \star \sigma)_I \) is in \( f^{-1}([u_0]) \) and \( I \) also wins.

The third and final possibility is that neither of the first two cases holds, but then \( II \) does not respect the SBR, and \( I \) wins.

The strategy \( \tau \) that we just defined has an interesting property. Indeed, not only Player I has a winning test \( v_x \) for every real \( x \), but he has an uniform winning test \( u_0 \) that is worth for all \( x \). That is why we say that \( \tau \) is a uniform w.s. for \( I \) in \( G_e(f) \).

When \( f \) is of the first Baire class, the construction of Theorem 4.1 is not possible, since the existence of an uniform winning test for Player I would contradict Proposition 3.4. It is however possible to adapt the same idea to get a strategy for \( I \) which will be weaker since it is not uniform, but strong enough to be winning in the backtrack game.

**Definition 4.2** (Winning test in \( G_b \), agressive strategy). Fix a function \( f : \omega^\omega \to \omega^\omega \), a closed \( F \subseteq \text{dom}(f) \) and name \( T_F \) the pruned sub-tree of \( \omega^{<\omega} \) such that \( F = [T_F] \). For all finite sequences \( u \in T_F \) call \( \mathcal{T}_u \) the set of pairs \((x,v)\) such that \( v \) is a winning test for \( x \in [u] \) in \( G_b(f) \), as in Definition 3.1, (1). Say that the family \( \{ \mathcal{T}_u : u \in T_F \} \) is a winning test for \( F \) in \( G_b(f) \) iff

1. the set \( \mathcal{T}_u \) is non-empty for all \( u \in T_F \),
2. for all \( u, u' \in T_F \), and all \( (x,v) \in \mathcal{T}_u \), if \( u \subset u' \subset x \) holds, then there is some \((x',v') \in \mathcal{T}_{u'} \) such that \( v \) and \( v' \) are incompatible.

Given \( \{ \mathcal{T}_u : u \in \omega^{<\omega}, u \in T_F \} \) a winning test for \( F \), we say that a strategy \( \tau \) in \( G_b(f) \) defined as follows uses the winning test.

We define inductively on the length of \( W \in (\omega^{<\omega})^{<\omega} \) a triple \((x,W,u,W) \in (\omega^\omega \times \omega^{<\omega} \times \omega^{<\omega}) \) such that both \((x,W,u,W) \in \mathcal{T}_{uw} \) and \( uW \subseteq xW|_{\text{lg}(W)} \) hold. We set then \( \tau(W) = xW|_{|W|} \).

Fix \((x,v) \in \mathcal{T}_{x} \), and suppose that the construction is made on \((\omega^{<\omega})^i \) for some \( i \in \omega \). Take \( W \) in \((\omega^{<\omega})^{i+1} \) with \( d(W) = w \). Set \( W' = W|_{i} \), \((x',u',v') = (xW,uW,vW) \) and \( u = x'|_{i+1} \). By induction hypothesis on \( W' \) we have \( u' \subseteq x'|_{i} \subset u \subset x' \). Since
we have a winning test we can choose \((x, v) \in \mathcal{T}_u\) such that \(v \perp v'\) holds. Define now
\[
(x_W, u_W, v_W) = \begin{cases} 
(x, u, v) & \text{if } w \supseteq v', \\
(x', u', v') & \text{otherwise}. 
\end{cases}
\]

With the same notations than the previous definition, if \(v_W\) is chosen of minimal length among the possible sequences \(v\) such that \((x_W, v)\) is in \(\mathcal{T}_{uw}\), for all \(W \in (\omega_{<\omega}) \cup \omega\), then we say that the strategy \(\tau\) is aggressive.

**Remark 4.3.** The previous strategy \(\tau\) consists in an iterative use of winning tests \(v\) for reals \(x\) in \(\mathbb{G}_0(f)\). Player \(I\) uses a test \(v_0\), waits for \(II\) to hit or fail it, and if she hits it, he throw \(v_0\) away and starts to use a new test \(v_1\) incompatible with \(v_0\), and so on.

As a consequence, suppose that \(II\) respects the SBR and produces a real \(y\) that is precisely the image of the real \(x\) produced by \(I\). This implies, by definition of \(\tau\), that he has used infinitely many times some incompatible tests, and that she has hit all of them so she does not respect the SSBR.

**Theorem 4.4.** For all functions \(f : \omega^\omega \to \omega^\omega\), if \(\text{disc}(f) \geq \omega_1\), then \(I\) has an aggressive w.s. in \(\mathbb{G}_b(f)\).

As a consequence, the game \(\mathbb{G}_b(f)\) is determined.

**Proof.** Suppose \(\text{disc}(f) \geq \omega_1\), then \(F = D_{\omega_1}(f)\) is not empty, so for all \(u \in T_F\) (for \(T_F\) the same as above), there is a real \(x \in [u] \cap F\) such that \(x\) is a discontinuity point of \(f \mid F\).

Hence, for all \(u \in T_F\), \(\mathcal{T}_u\) is not empty. Take now \(u, u' \in T_F\), fix \((x, v) \in \mathcal{T}_u\), and suppose that \(u \subset u' \subset x\) holds. We have to find \((x', v')\) in \(\mathcal{T}_{u'}\) with \(v\) and \(v'\) incompatible. We now prove that there is a discontinuity point \(x'\) of \(f' = f \mid F \cap [w]\) such that \(f(x') \notin [v]\) holds. Suppose the contrary, so all discontinuity points of \(f'\) are mapped into \([v]\). But by definition of \(F\) discontinuity points of \(f'\) are dense in \([u'] \cap F\), so \([v]\) being closed the image of continuity points of \(f'\) are in \([v]\) as well and finally \([u']^\ast\) is entirely mapped into \([v]\), which contradicts the fact that \(v\) is a test for \(x\).

Any sufficiently long test \(v'\) for \(x'\) gives a convenient \((x', v')\), moreover if \(v\) was aggressive, i.e. of minimal length in \(\mathcal{T}_u\), since \(\mathcal{T}_{u'} \subset \mathcal{T}_u\) holds, any test \(v'\) for \(x'\) is incompatible with \(v\) and hence gives a convenient \((x', v')\).3

Finally \(\{\mathcal{T}_u : u \in T_F\}\) is a winning test for \(F\) in \(\mathbb{G}_b(f)\). Call \(\tau\) the aggressive strategy of Player \(I\) that uses this test, it remains to prove that \(\tau\) is winning for \(I\) in \(\mathbb{G}_b(f)\). There are two possibilities.

Either \(II\) decides to stop erasing at round \(i\) for some \(i \in \omega\), i.e. at round \(j \geq i\) she plays \(v_j\) with \(v_j\) extending \(v_i\) or in \(T_i\). Call \(u\) the sequence enumerated by \(I\) at round \(i\) following \(\tau\), so from round \(i\) onwards, the run is equivalent to a run in \(\mathbb{G}_0(f, u)\).

By definition of \(\tau\) there are \(u' \subset u\) and \(v' \in \mathcal{T}_{u'}\) such that he is using \((x', v')\) as a test at round \(i\). If she hits \(v'\) at round \(i\) by playing \(v_i \supseteq v'\), then \(I\) starts using some \(v \in \mathcal{T}_u\) incompatible with \(v'\) as a test in \(\mathbb{G}_0(f, u)\) and wins, otherwise he goes on using \(v'\) but she did not hit it at round \(i\) so he wins anyways.

If \(II\) never decides to stop erasing she also loses by failing the strong single branch rule.

We get, as announced, the full determinacy of the bounded backtrack game.

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3This argument is reminiscent of [15, Lemma 3.4.3, p.16] and [16, Lemma 2.2].
Corollary 4.5. For all \((\alpha, u, f) \in \omega_1 \times \omega^{<\omega} \times (\omega^{<\omega})^{\omega^\omega}\), the game \(G_\alpha(f, u)\) is determined.

Proof. There are two possibilities. Either \(\text{disc}_f(u) < \omega_1\), then we can use Theorem 3.7 to get the determinacy of \(G_\alpha(f, u)\). Or \(\text{disc}_f(u) \geq \omega_1\), then by Theorem 4.4 Player I has a winning strategy in \(G_b(f|_u)\), and hence by Proposition 2.2, (3) he also has one in \(G_\alpha(f, u)\) for all \(\alpha < \omega_1\), so the game \(G_\alpha(f, u)\) is trivially determined. \(\square\)

4.2. Baire’s Lemma. In the proof of Theorem 4.1 we gave a description of the w.s. for Player I. It reveals the behavior of functions of the first Baire class, and it allows us to give an alternative, simple and purely game-theoretical proof of the following very classical theorem, originally proved by René Baire.

Theorem 4.6. (Baire)

A function \(f : \omega^\omega \to \omega^\omega\) is Baire class 1 iff for every closed set \(A \subseteq \text{dom}(f)\), \(f|_A\) has a continuity point.

Proof. \((\Rightarrow)\): Suppose \(f\) is Baire class 1. It is enough to prove that \(f\) has a continuity point, since for any closed set \(A\), \(f|_A\) is still of the first Baire class.

If \(D_{\omega_1}(f)\) is non-empty, since it is open it contains a basic open set \([u]\) for some \(u \in \omega^{<\omega}\). Then \(II\) has a w.s. in \(G_\alpha(f, u)\) for some \(\alpha < \omega_1\), so \(D_{\omega_0}(f, [u]) \neq \emptyset\). Since it is open in \([u]\) we can pick \(v \supseteq u\) such that \(f|_v\) is continuous, so any point \(x \in [v]\) is a continuity point of \(f|_v\). But \([v]\) is an open neighborhood of \(x\), so \(x\) is a continuity point of \(f\). Hence we can suppose that \(\text{dom}(f) = D_{\omega_1}(f)\).

By Theorem 4.4, \(I\) has an aggressive strategy \(\tau\) in \(G_b(f)\). Take also \(\sigma\) a w.s. for \(II\) in \(G_e(f)\). Set \(x = (\tau \ast \sigma)_I\), and let \(v\) be any arbitrary finite initial segment of the real \(y\) such that \(\{y\} = [T_{(\tau \ast \sigma)}]_I\). It remains to prove that \(f^{-1}([v])\) contains an open neighborhood of \(x\), or equivalently that \(v\) is not a winning test for \(x\) in \(G_0(f)\). Following Remark 4.3, there is a sequence \((x_n, v_n)_{n \in \omega}\), each \(v_n\) being an aggressive test for \(x_n\) in \(G_0(f)\) that is used by \(I\) and hit by \(II\) during the run, say at round \(k_n\). This implies that the sequence \((x_n)_{n \in \omega}\) converges to \(x\), and since she hits all the tests, \((f(x_n))_{n \in \omega}\) converges to \(y\), and \(y = f(x)\) holds since \(\sigma\) is winning. Hence there is an integer \(N\) such that for all \(n \geq N\) the test \(v_n\) strictly extends \(v\). Since \(\tau\) is aggressive, \(v_N\) is of minimal length in \(T_{x_N|k_N} = T_{x|k_N}\), among those which are incompatible with \(v_{N-1}\) if \(N \geq 1\), hence \(v\) is not a test for \(x\).

\((\Leftarrow)\): As in the proof of Theorem 4.1, there is a closed set \(F\) and an uniform winning test \(u\) for \(F\) in \(G_e(f)\), so \(f|_F\) has no continuity point. \(\square\)

This proof induces a game-theoretical condition on continuity points. Let \(\tau\) be an aggressive winning strategy for \(I\) in \(G_b(f)\) and \(P\) the set of reals \(x\) that he produces against \(II\) when he follows \(\tau\). Let \(\sigma\) be a strategy for \(II\).

Corollary 4.7. Let \(x = (\tau \ast \sigma)_I\). If \(\sigma\) is winning in \(G_e(f)\) then \(f|_P\) is continuous on \(x\).

4.3. Towards the result of Jayne and Rogers? Another result, due to Jayne and Rogers, concerns a fragment of the first Baire class \([17, 18, 15, 16, 19]\). Its 0-dimensional version states that a function \(f : A \to B\) is \(\Sigma^0_1\)-preserving\(^4\) iff there is a countable partition \((A_i)_{i \in \omega}\) of \(A\) in closed sets such that for each integer \(i\) the function \(f|_{A_i}\) is continuous, we say that \(f\) is piecewise continuous.

\(^4\)The inverse image of a \(\Sigma^0_2\) subset of \(B\) is a \(\Sigma^0_2\) subset of \(A\).
In an earlier accepted version of this paper was included a proof of this theorem. We found unfortunately a gap in this proof, which we have not been able to fill so far. We think however that this idea should work and provide an insightful proof, and we give here a motivating example.

The idea is to take a closer look at her strategies. We obtained indeed Baire’s Theorem by only looking at his strategies, for what was needed there, was a characterization of a continuity point.

We lack of a strong notion of optimality for strategies of Player II. In the bounded backtrack game, optimality was guaranteed by the definition of discontinuity which introduces minimality. After Theorem 4.4, we know that this notion is worth for the general backtrack game as well. In the eraser game, we can not base optimality on Player II’s behavior while following her winning strategy, for a minimality notion is missing. Player II could indeed erase at the wrong moment or play outside I’s image for instance. Here is the illustration.

**Example 4.8.** We call $\text{count}_0$ the following function from and into $\omega^\omega$:

$$
\text{count}_0 : x \mapsto \begin{cases} 0^\omega & \text{if } \{i \in \omega : x(i) = 0\} \text{ is infinite} \\
(0^n)^\omega & \text{if } \{i \in \omega : x(i) = 0\} \text{ is of cardinality } n.
\end{cases}
$$

**Fact 4.9.** The function $\text{count}_0$ is Baire class 1 and not $\Sigma^0_2$-preserving.

**Proof.** In the game $G_e(\text{count}_0)$, Player II can play $1^i$ as long as I does not play any 0’s, and when he plays the first 0, she erases by playing $0^i1$ and she can continue. Hence she has a w.s. and $\text{count}_0$ is Baire class 1.

But now Player I has an aggressive w.s. in $G_b(\text{count}_0)$ using the fact that the sequence $(0^i)^\omega$ is a minimal (hence aggressive) test in the set $T_0$ for all integers $i$. When he follows this aggressive strategy and she follows the strategy we described above in the eraser game, then she wins and she plays in the closed set $\{0^\omega\}$ whose inverse image is the set of all sequences with infinitely many 0’s. In this way we find a closed set whose inverse image is $\Pi^0_2$-complete, witnessing that $\text{count}_0$ is not $\Sigma^0_2$-preserving. \(\Box\)

What are the nice properties of II’s strategy that we used to find $\{0^\omega\}$? It is noticeable that II, if she follows the strategy we described, avoids the following situations.

She avoids losing moves. For instance she never plays $2^i$ which is not a prefix of the image of any real.

She also does not erase too much. For instance it would be winning to start playing $0^i$ for some integer $i$ against an aggressive strategy. Player I’s reaction (see Definition 4.2) would be to continue playing non-null integers, so that she should first erase $0^i$ and play $1^i$, only to erase it once again and play 0 as soon as I triggers the aggressive test.

Another way to describe this feature is to ask II to be trustworthy, so when she erases something, she should not be allowed to play it again later. We thus call a strategy for Player II trustworthy if moves that have been erased by Player II cannot be played again later. Unfortunately such a rule is too strong for II to still have a winning strategy in general, as shown by the next example.

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We call $[\omega]^1$ the set of all singletons, seen as elements of $2^{\omega}$:

$$[\omega]^1 = \{(0^n)^c1^{\omega} : n \in \omega\}.$$ 

Using this set we can define the following modification of $count_0$:

$$count'_0 : x \mapsto \begin{cases} 
0^\omega & \text{when } x = 0^\omega \\
1^\omega & \text{when } x \in [\omega]^1 \\
count_0(x) & \text{otherwise.}
\end{cases}$$

In $G_e(count'_0)$, Player II has a winning strategy but not a trustworthy winning strategy. Any winning strategy is indeed non-trustworthy if $I$ uses first (0) as a test for $0^\omega$ and then (1) as a test for some $(0^n)^c1^{\omega}$:

suppose Player I starts by playing 0 at each round; any winning strategy for Player II eventually (say, after $n$ 0 of Player I) has to respond with a 0, otherwise, she will lose against the constant sequence $0^\omega$. After Player II has responded by 0, Player I plays a 1 and then 0 at each round again. Now eventually, Player II has to erase the played 0 and play a 1, otherwise she will lose against the sequence $(0^n)^c1^{\omega}$. After this has happened, Player I plays a second 1, thus playing outside of $[\omega]^1$. Now Player II needs to revise the first bit once more from 1 to 0.

The technic for the proof of Fact 4.9 remains however correct for the function $count'_0$, and can be easily modified for many other functions.

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