MAKING A TREE WITH A FUNCTION?

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Abstract. In [1, 2] the choice of a relevant notion to classify functions is discussed. It should be at the same time simple and topologically relevant. T.Banakh proposed to use trees labelled with Wadge degrees of inverse images, which gives a simple notion by some classical bqo theory. We give here a counter-example to its topological relevance.

1. The question.

The detailed context can be found in [2]. For descriptive set theory, see [3]. For bqo theory see [5].

1.1. Classical background.

Definitions 1.

• The Wadge order on subsets: given $A, B \subseteq \omega^\omega$ we write $A \leq_W B$ if there is a continuous function $f$ such that $A = f^{-1}(B)$. Denote Bor the Borel subsets of $\omega^\omega$.

• Given $s, t \in \omega^{<\omega}$ denote $s \subseteq t$ when $s$ is an initial segment of $t$. A tree$^1$ is a subset of $\omega^{<\omega}$ closed under $\subseteq$. Given two trees $T, T'$, a map $\varphi : T \rightarrow T'$ is a tree-embedding if it is an injection, increasing for $\subseteq$ and $\varphi(s \cap t) = \varphi(s) \cap \varphi(t)$ holds for all $s, t \in T$.

• Given $(Q, \leq_Q)$ a qo, a $Q$-labelled tree is a tree $T$ with a map $l : T \rightarrow Q$. Given two labelled trees $(T, l)$ and $(T', l')$ denote $l \leq_Q l'$ iff there is a tree-embedding $\varphi : T \rightarrow T'$ satisfying $l(s) \leq_Q l'(\varphi(s))$ for all $s \in T$.

Theorem 2.

(Martin) The qo $(\text{Bor}, \leq_W)$ is a bqo.

(Laver [4]) When $Q$ is a bqo, so are the $Q$-labelled trees under $\leq_Q$.

Corollary 3. The Bor-labelled trees are bqo for $\leq_W$.

1.2. The preimage tree of a function.

Definition 4. Given a function $f : \omega^\omega \rightarrow \omega^\omega$ the preimage tree of $f$, denoted $T_f$, is the following Bor-labelled tree:

$$T_f = \{ s \in \omega^{<\omega} \mid f^{-1}([s]) \neq \emptyset \},$$

and $l_f(s) = f^{-1}([s])$.

Where $[s]$ is the basic open set $\{ x \in \omega^\omega \mid s \subseteq x \}$.

$^1$All trees here are on $\omega$. 
By Corollary 3 we have that the order on preimage trees is a bqo on Borel functions, which was in [1] the requirement for being simple. The requirement for topological relevance was to respect the Borel domination on functions.

**Definition 5.** Given a Borel function \( f : \omega^\omega \to \omega^\omega \) the *Borel degree* function of \( f \):

\[
d_f : \omega_1 \to \omega_1
given by \( \alpha \mapsto \inf \{ \beta \text{ such that } f^{-1}(\Sigma^0_\alpha) \subseteq \Sigma^0_\beta \} \).
\]

We then say that \( f \) is *Borel dominated* by \( g \) and write \( f \leq_B g \) iff \( d_f(\alpha) \leq d_g(\alpha) \) for all \( \alpha \).

So the question is:

**Question.** (Banakh) Given \( f \) and \( g \) two Borel functions, if \( T_f \leq_W T_g \) holds, does it imply that \( f \) is Borel dominated by \( g \)?

2. THE ANSWER

We give a pair of simple (as a matter of fact, they are Baire class 1) functions which gives a negative answer to the previous question.

**Definition 6.**

\[
f : \omega^\omega \to \omega^\omega
\]

\[
x \mapsto \begin{cases} 0^\omega & \text{if } \{ i \in \omega : x(i) = 0 \} \text{ is infinite} \\ (0^n)^\omega & \text{if } \{ i \in \omega : x(i) = 0 \} \text{ is of cardinality } n \end{cases}
\]

\[
g : \omega^\omega \to \omega^\omega
\]

\[
x \mapsto \begin{cases} f(x) & \text{if } \{ i \in \omega : x(i) = 0 \} \text{ is of cardinality } \leq x(0) + 1 \\ 0^\omega & \text{otherwise.} \end{cases}
\]

**Proposition 7.** \( T_f \leq_W T_g \) but \( g \) is Borel dominated by \( f \).

**Proof.** Here \( T_f = T_g = \{ 0^n \cup 1^m \mid (n,m) \in \omega^2 \} \). Denote \( O_n \) the open set of reals \( x \) such that \( \{ i \in \omega : x(i) = 0 \} \) is of cardinality at least \( n \) and \( C_n \) the closed set \( \omega^\omega \setminus O_{n+1} \). In particular \( O_0 = \omega^\omega \) and \( C_0 \) is closed. We have

\[
l_f(0^n) = O_n
\]

\[
l_f(0^n \cup 1^m) = O_n \cap C_n \text{ for any integer } m > 0
\]

\[
l_g(0^n) = \bigcup_{i \leq n} [(i)] \cap O_{i+1} \cup \bigcup_{i > n} [(i)] \cap O_n
\]

\[
l_g(0^n \cup 1^m) = \left( \bigcup_{i \geq n} [(i)] \right) \cap C_n \cap O_n \text{ for any integer } m > 0
\]
So for all $n$, both $l_f(0^n)$ and $l_g(0^n)$ are (real) open sets, for all $n$ and $m$ strictly positive both $l_f(0^n \sim 1^m)$ and $l_f(0^n \sim 1^m)$ are intersection of an open set and a closed set, and for $m$ positive and $n$ null, both $l_f(0^n \sim 1^m)$ and $l_f(0^n \sim 1^m)$ are closed sets. Hence $T_f \leq T_g$ holds.

However, we have $f^{-1}(\{0^\omega\})$ is in $\Pi^0_2 \setminus \Delta^0_2$ so $d_f(2) = 3$ and $d_g(2) = 2$ so $g$ is Borel dominated by $f$.

\[\square\]

REFERENCES


