Determinant versus Permanent: salvation via generalization?
The algebraic complexity of the Fermionant and the Immanant

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1. Algebraic complexity

2. Immanant
\[ f(x, y) = (x + y)^2(z + 3) + 2(x + y)^2 + (z + 3)^2 \]

**Definition**

The *size* of an arithmetic circuit is the number of operational gates.
Algebraic complexity

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A family \( F = (f_n) \) of polynomials is in \( VP \) if there exists a family of circuits \( C_n \) of polynomial size such that for any \( n \)

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Definition
A family \( F = (f_n) \) is in \( VNP \) if there is a family \( G = (g_n) \) in \( VP \) such that

\[ f_n(\bar{x}) = \sum_{\bar{\epsilon} \in \{0,1\}^n} g_n(\bar{\epsilon}, \bar{x}) \]
Definition
Let $S_n$ be the symmetric group on $n$ elements and if $\pi \in S_n$, $c(\pi)$ its number of cycles. The determinant is

$$\det_n(\bar{x}) = (-1)^n \sum_{\pi \in S_n} (-1)^{c(\pi)} \prod_{i=1}^{n} x_{i\pi(i)}$$

Theorem (Valiant 79)
The family $\det = (\det_n)_{n \in \mathbb{N}}$ is in VP
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Let $S_n$ be the symmetric group on $n$ elements and if $\pi \in S_n$, $c(\pi)$ its number of cycles. The permanent is

$$\text{per}_n(\bar{x}) = \sum_{\pi \in S_n} \prod_{i=1}^{n} x_{i\pi(i)}$$

Theorem (Valiant 79)

The family $\text{per} = (\text{per}_n)_{n \in \mathbb{N}}$ is VNP-complete.
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Let $S_n$ be the symmetric group on $n$ elements and if $\pi \in S_n$, $c(\pi)$ its number of cycles. The permanent is

$$\text{per}_n(\bar{x}) = \sum_{\pi \in S_n} \prod_{i=1}^{n} x_{i\pi(i)}$$

**Theorem (Valiant 79)**

*The family $\text{per} = (\text{per}_n)_{n \in \mathbb{N}}$ is VNP-complete*
Conjecture (Valiant hypothesis)

\[ VP \neq VNP \]

Theorem (Bürgisser 2000)

Under Generalized Riemann Hypothesis,

\[ VP = VNP \Rightarrow P/poly = NP/poly \]
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The main approaches of Valiant Hypothesis:
- Geometric Complexity Theory (GCT)
- Lower bounds
- The study of complexity classes (Characterization, complete polynomials)
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\[ VP = VNP \implies P/poly = NP/poly \]

The main approaches of Valiant Hypothesis:

- Geometric Complexity Theory (GCT)
- Lower bounds
- The study of complexity classes (Characterization, complete polynomials)
Definition (informal)

A generalization of the determinant and the permanent is a series of family $F^k = (f_n)^k$ indexed by some $k$ such that

- For certain $k$, $F^k$ are in VP
- For others $k$, $F^k$ are VNP-complete.
Definition
A young diagram is a collection of boxes in left adjusted row with decreasing row length.

Young diagrams

\[
\begin{array}{c}
[4, 4] \\
\begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\end{array}
\quad
\begin{array}{c}
[4, 2] \\
\begin{array}{ccc}
& & \\
& & \\
\end{array}
\end{array}
\]

Definition
Let \( \chi_Y \) be an irreducible character of \( S_n \). Then

\[
\text{im}_\chi(\bar{x}) = \sum_{\pi \in S_n} \chi_Y(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}
\]
**Exemple**

If $Y$ is a single row, then $\text{im}_Y(\bar{x}) = \text{per}(\bar{x})$.

**Exemple**

If $Z$ is a single column, then $\text{im}_Z(\bar{x}) = \text{det}(\bar{x})$.
Theorem (Bürgisser 2000)

If \((Y_n)\) is a family of Young diagrams with only a constant number of boxes at the right of the first column, then

\[(\text{im}_{Y_n})\text{ is in VP}\]
Theorem (Brylinski 2003)

Let $Y_n$ be Young diagrams such that the maximal difference between the size of two consecutive rows is $\Omega(n)$, then

$$(im_{Y_n}) \text{ is VNP-complete}$$
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Theorem (2)

Let \([n, n]\) be the Young diagram with two columns, each with \(n\) boxes. Then

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(\text{im}_{[n,n]}) \text{ is VNP-complete}
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\[(\text{im}_{[n,n]})\text{ is VNP-complete}\]

Theorem (3)

If \((Y_n)\) has a polynomial number of boxes at the right of the first column and a constant number of columns, then

\[(\text{im}_{Y_n})\text{ is VNP-complete}\]
Theorem (2)

Let \([n, n]\) be the Young diagram with two columns, each with \(n\) boxes. Then 

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Theorem (3)

If \((Y_n)\) has a polynomial number of boxes at the right of the first column and a constant number of columns, then 

\[(\text{im}_{Y_n})\text{ is VNP-complete}\]
Theorem (Conclusion)

Let \((Y_n)\) be a family of Young diagrams with a constant number of columns such that \(|Y_n| = \Omega(n)\). Then

- If the number of boxes at the right of the first column is constant \(c\), then \((\text{im}_{Y_n})\) is in \(\text{VP}\).
- If the number of boxes at the right of the first column is logarithmic, then \((\text{im}_{Y_n})\) is not \(\text{VNP}\)-complete.
- If the number of boxes at the right of the first column is polynomial, \((\text{im}_{Y_n})\) is \(\text{VNP}\)-complete.

Perspectives

- Studying the class of polynomials computed by sub-exponentiel circuits.
- Finding a \(\text{VP}\)-complete family!
Theorem (Conclusion)

Let $(Y_n)$ be a family of Young diagrams with a constant number of columns such that $|Y_n| = \Omega(n)$. Then

- If the number of boxes at the right of the first column is constant $c$, then $(im_{Y_n})$ is in $\text{VP}$.
- If the number of boxes at the right of the first column is logarithmic, then $(im_{Y_n})$ is not $\text{VNP}$-complete.
- If the number of boxes at the right of the first column is polynomial, $(im_{Y_n})$ is $\text{VNP}$-complete.

Perspectives

- Studying the class of polynomials computed by sub-exponentiel circuits.
- Finding a $\text{VP}$-complete family!
Thank you!
Definition

If $\pi$ is a permutation, $c(\pi)$ is its number of cycles.

\[
\text{Ferm}_n^k A = (-1)^n \sum_{\pi \in S_n} (-k)^{c(\pi)} \prod_{i=1}^{n} A_{i, \pi(i)}
\]

Let $\text{Ferm}_n^k$ the family of $(\text{Ferm}_n^k)$

- If $k = 1$, then $\text{Ferm}_n^1(\bar{x}) = \det(\bar{x})$
- If $k = -1$, then $\text{Ferm}_n^{-1}(\bar{x}) = \text{per}(\bar{x})$
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Theorem (1)

- $\text{Ferm}^0 = 0$.
- $\text{Ferm}^1$ is in $\text{VP}$
- For $k \in \mathbb{Q}$ different from 0, 1 $\text{Ferm}^k$ is $\text{VNP}$-complete.